HAESE MATHEMATICS

Mathematics

Applications and interpretation SL 2

Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with
IB Diploma Programme

HAESE MATHEMATICS
Specialists in mathematics education

Mathematics

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IB Diploma
Programme

MATHEMATICS: APPLICATIONS AND INTERPRETATION SL

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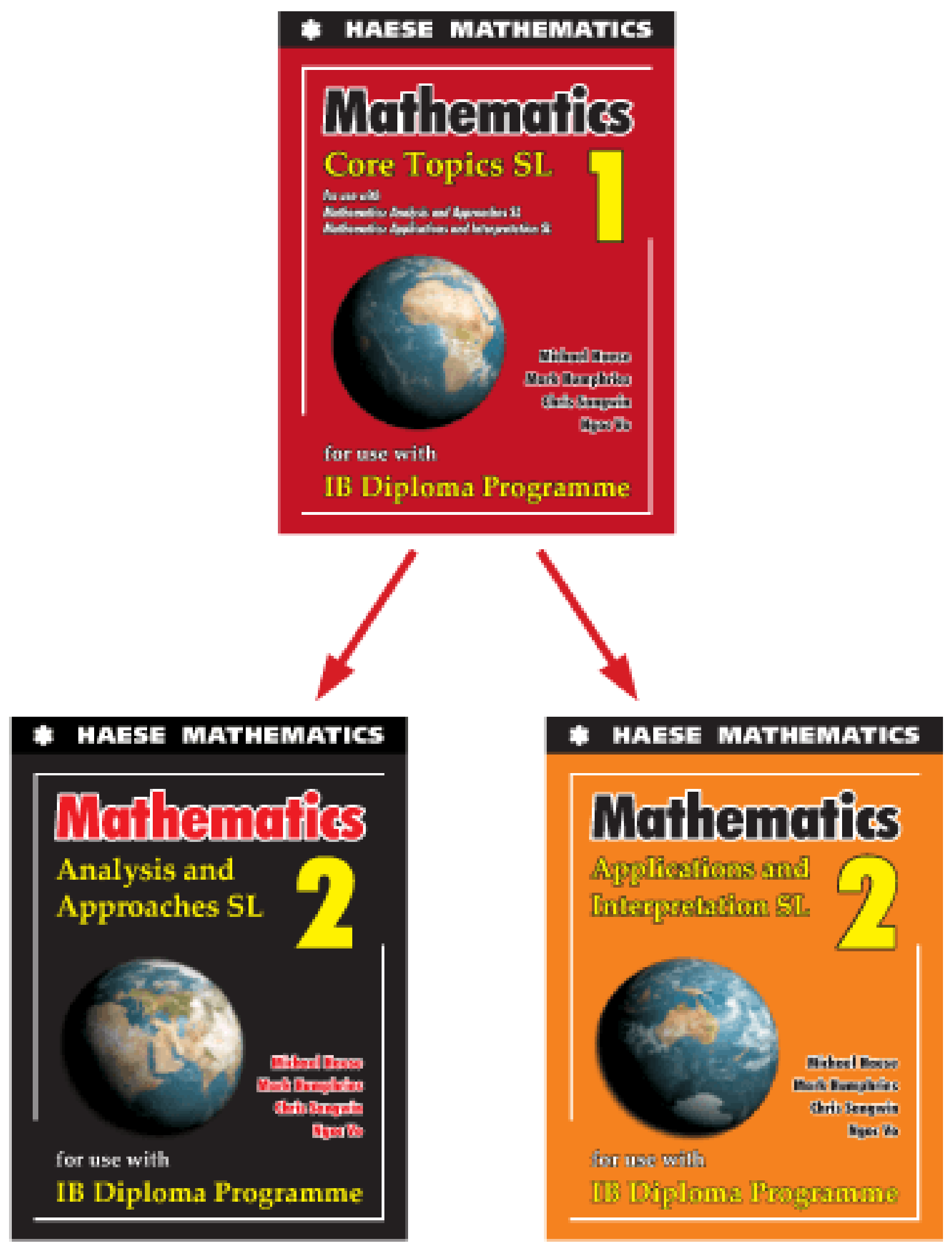
FOREWORD

This book has been written for the International Baccalaureate Diploma Programme course *Mathematics: Applications and Interpretation SL*, for first teaching in August 2019, and first assessment in May 2021.

SL Mathematics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1
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This book is designed to complete the course in conjunction with the **Mathematics: Core Topics SL** textbook. It is expected that students will start using this book approximately 6-7 months into the two-year course, upon the completion of the Mathematics: Core Topics SL textbook.

The *Mathematics: Applications and Interpretation* courses have a focus on technology, and the book has been written with this focus in mind. Where appropriate, graphics calculator screenshots and instructions have been provided to help students use technology to solve problems. An algebraic approach to solving the problem may be included for completeness, and to help students enhance their understanding. The material is presented in a clear, easy-to-follow style, free from unnecessary distractions, while effort has been made to contextualise questions so that students can relate concepts to everyday use.

Each chapter begins with an Opening Problem, offering an insight into the application of the mathematics that will be studied in the chapter. Important information and key notes are highlighted, while worked examples provide step-by-step instructions with concise and relevant explanations. Discussions, Activities, Investigations, and Research exercises are used throughout the chapters to develop understanding, problem solving, and reasoning.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

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PMH, MAH, CS, NV

ACKNOWLEDGEMENTS

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The cartoon on page 392 is by Randall Munroe (xkcd), and is available at https://xkcd.com/882/. Licensing terms can be viewed at https://xkcd.com/license.html.

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SELF TUTOR

Simply 'click' on the **Self Tutor** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Example 2 **Self Tutor**

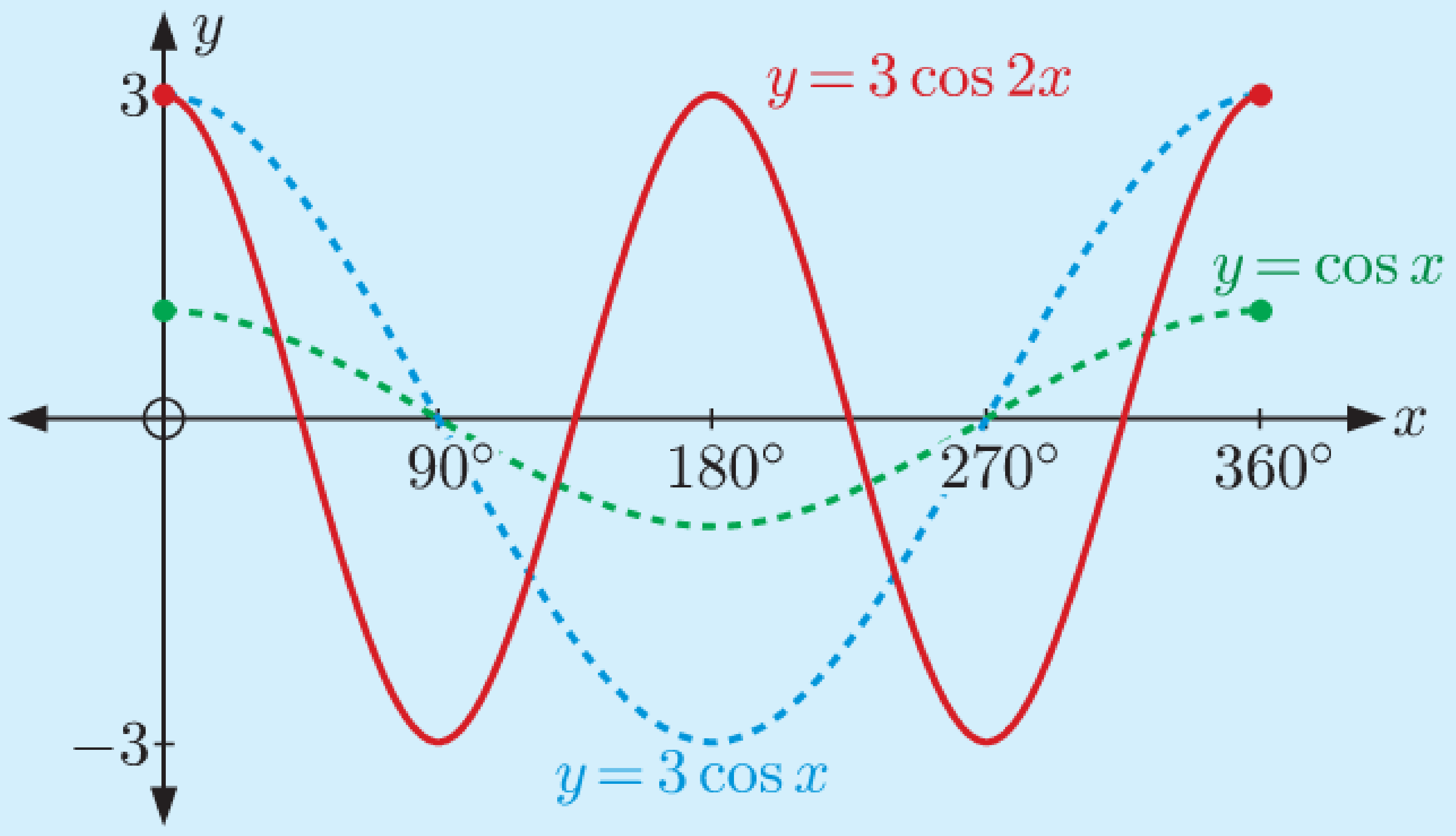
Sketch the graph of $y = 3\cos 2x$ for $0° \leqslant x \leqslant 360°$.

$a = 3$, so the amplitude is $|\,3\,| = 3$.

$b = 2$, so the period is

$$\frac{360°}{b} = \frac{360°}{2} = 180°.$$

We stretch $y = \cos x$ vertically with scale factor 3 to give $y = 3\cos x$, then stretch $y = 3\cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = 3\cos 2x$.



See **Chapter 9**, **Trigonometric functions**, p. 229

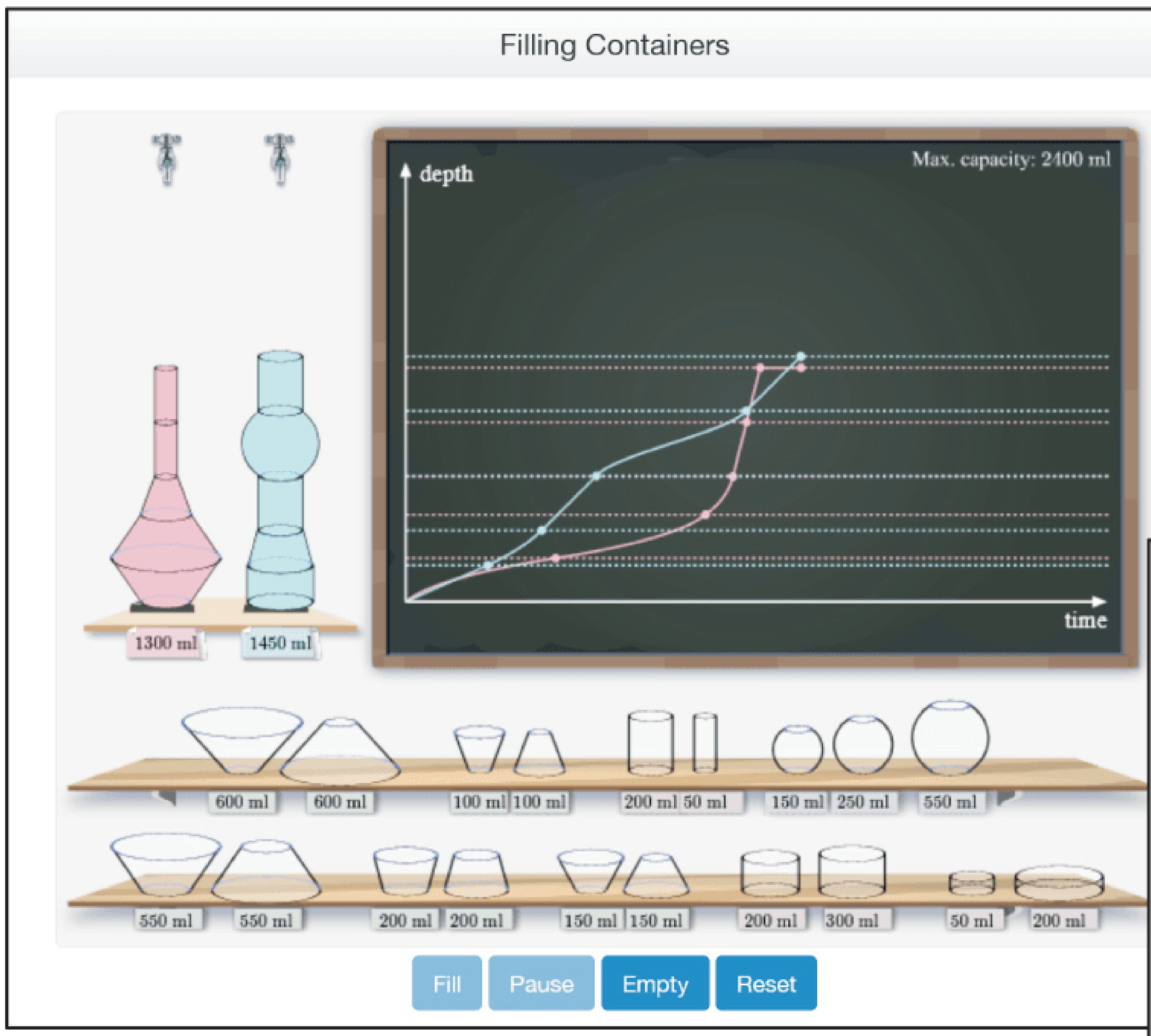
INTERACTIVE LINKS

Interactive links to in-browser tools which complement the text are included to assist teaching and learning.

Icons like this will direct you to:

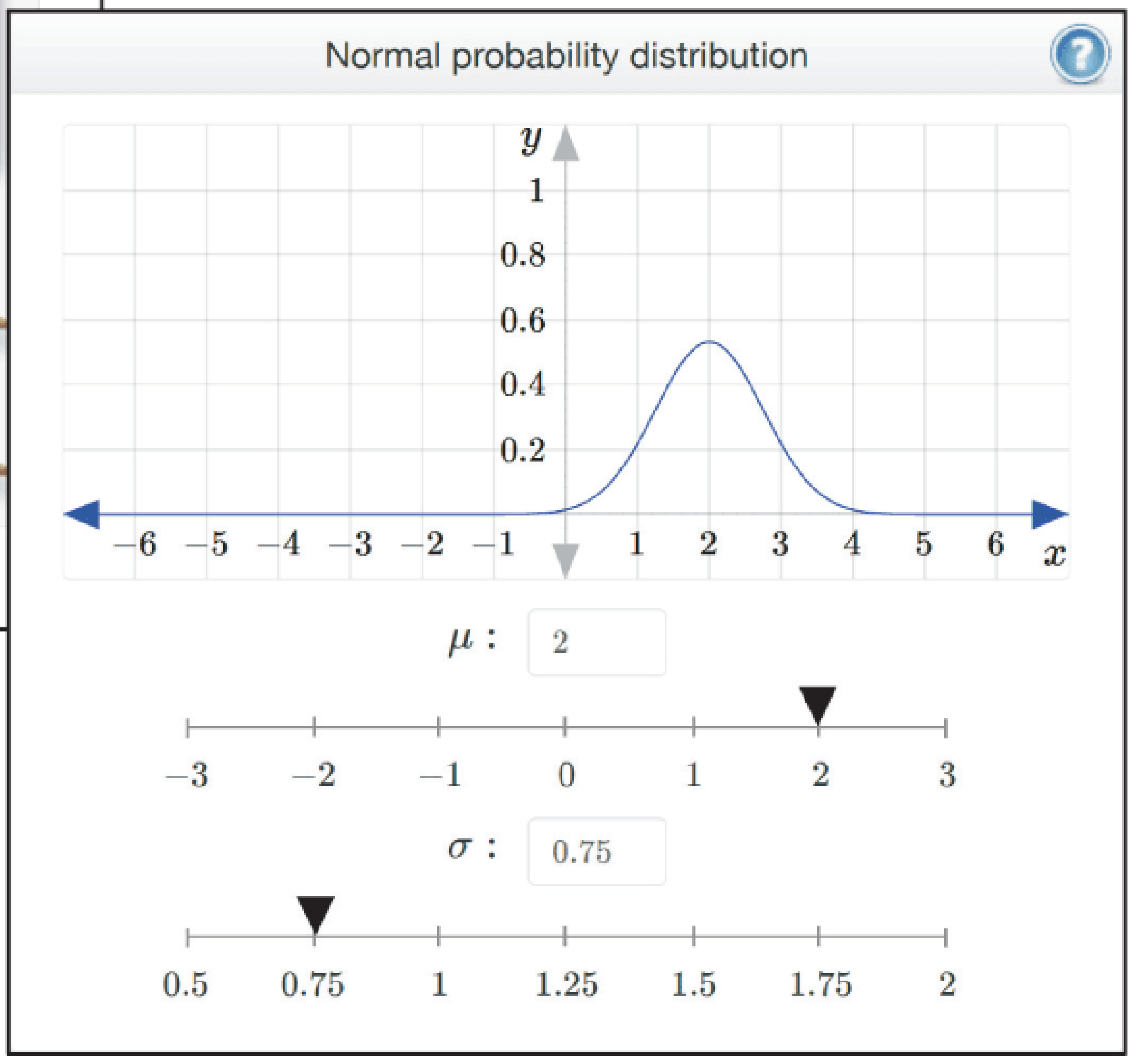
ICON

- interactive demonstrations to illustrate and animate concepts
- games and other tools for practising your skills
- graphing and statistics packages which are fast, powerful alternatives to using a graphics calculator
- printable pages to save class time.


Filling Containers
depth
Max. capacity: 2400 ml
time
1300 ml
1450 ml
600 ml
600 ml
100 ml
100 ml
200 ml
50 ml
150 ml
250 ml
550 ml
550 ml
550 ml
200 ml
200 ml
150 ml
150 ml
200 ml
300 ml
50 ml
200 ml
Fill
Pause
Empty
Reset

See **Chapter 10**, **Differentiation**, p. 242

Save time, and make learning easier!


Normal probability distribution
y
1
0.8
0.6
0.4
0.2
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
x
−3
−2
−1
0
1
2
3
0.5
0.75
1
1.25
1.5
1.75
2

μ : 2

σ : 0.75

See **Chapter 15**, **The normal distribution**, p. 365

GRAPHICS CALCULATOR INSTRUCTIONS

Graphics calculator instruction booklets are available for the **Casio fx-CG50**, **TI-84 Plus CE**, **TI-*n*spire**, and the **HP Prime**. Click on the relevant icon below.

CASIO
fx-CG50

TI-84 Plus CE

TI-*n*spire

HP Prime

When additional calculator help may be needed, specific instructions are available from icons within the text.

GRAPHICS
CALCULATOR
INSTRUCTIONS

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SYMBOLS AND NOTATION USED IN THIS COURSE

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}'$	the set of irrational numbers
$\mathbb{R}$	the set of real numbers
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}},\ \sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$)
$a^{\frac{1}{2}},\ \sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$)
$\lvert x \rvert$	the modulus or absolute value of x $\lvert x \rvert = \begin{cases} x \text{ for } x \geqslant 0 & x \in \mathbb{R} \\ -x \text{ for } x < 0 & x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to

$\ngtr$	is not greater than
$\nless$	is not less than
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$n!$	$n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1$
$\binom{n}{r}$ or ${}^nC_r$	the r^{th} binomial coefficient, $r = 0, 1, 2, ...$ in the expansion of $(a+b)^n$
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$\lim\limits_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\dfrac{dy}{dx}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\int y\, dx$	the indefinite integral of y with respect to x
$\int_a^b y\, dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log x$	the logarithm in base 10 of x
$\ln x$	the natural logarithm of x, $\log_e x$
sin, cos, tan	the circular functions
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$	the inverse circular functions

A(x, y)	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
PB(A, B)	the perpendicular bisector of [AB]
$\widehat{A}$	the angle at A
$\mathrm{C\widehat{A}B}$	the angle between [CA] and [AB]
$\triangle$ABC	the triangle whose vertices are A, B, and C
$\parallel$	is parallel to
$\perp$	is perpendicular to
P(A)	probability of event A
P(A')	probability of the event 'not A'
P$(A \mid B)$	probability of the event A given B
$x_1, x_2, ...$	observations of a variable
$f_1, f_2, ...$	frequencies with which the observations $x_1, x_2, x_3, ...$ occur
$p_1, p_2, ...$	probabilities with which the observations $x_1, x_2, x_3, ...$ occur
P$(X = x)$	the probability distribution function of the discrete random variable X
$P(x)$	the probability mass function of a discrete random variable X

E(X)	the expected value of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
s^2	sample variance
s	standard deviation of the sample
B(n, p)	binomial distribution with parameters n and p
N(μ, σ^2)	normal distribution with mean μ and variance σ^2
$\sim$	is distributed as
r	Pearson's product-moment correlation coefficient
H_0	the null hypothesis
H_1	the alternative hypothesis
$T \sim t_{n-1}$	the random variable T has the Student's t distribution with $n-1$ degrees of freedom
χ^2	chi-squared
χ^2_{calc}	calculated chi-squared value
χ^2_{crit}	critical value of the chi-squared distribution
f_{obs}	observed frequency
f_{exp}	expected frequency

THEORY OF KNOWLEDGE

Theory of Knowledge is a Core requirement in the International Baccalaureate Diploma Programme.

Students are encouraged to think critically and challenge the assumptions of knowledge. Students should be able to analyse different ways of knowing and areas of knowledge, while considering different cultural and emotional perceptions, fostering an international understanding.

The activities and discussion topics in the below table aim to help students discover and express their views on knowledge issues.

Chapter 4: Modelling p. 85	**MOVEMENT OF THE PLANETS**
Chapter 5: Bivariate statistics p. 121	**MATHEMATICAL EXTRAPOLATION**
Chapter 7: Direct and inverse variation p. 168	**MEASURES OF TEMPERATURE**
Chapter 8: Exponentials and logarithms p. 210	**IS MATHEMATICS AN INVENTION OR A DISCOVERY?**
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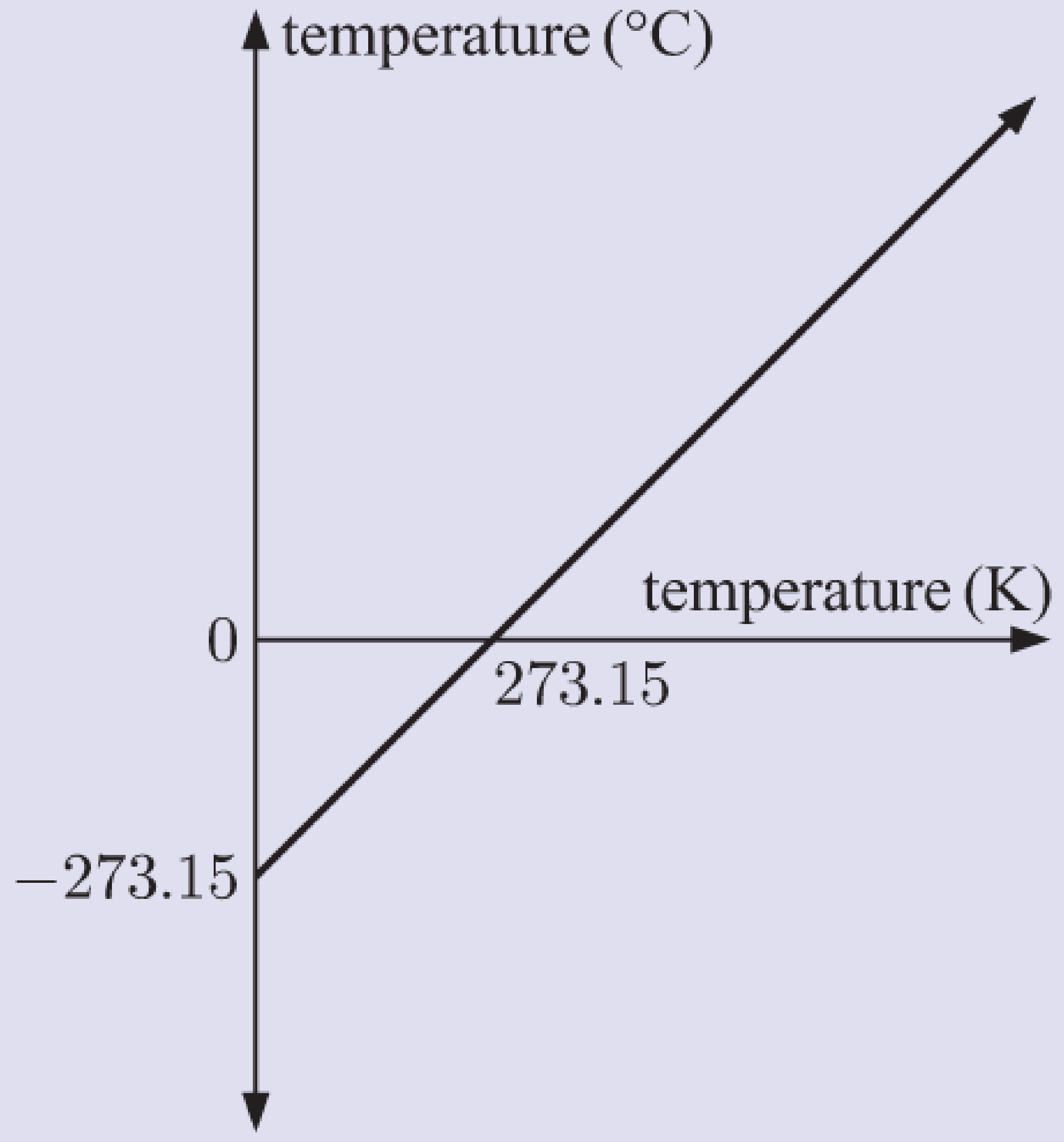
Chapter 16: Hypothesis testing p. 391	**SIGNIFICANCE LEVELS IN RESEARCH**
Chapter 16: Hypothesis testing p. 405	**STATISTICAL FALLACIES**

THEORY OF KNOWLEDGE

In 1848, **William Thomson** (1824 - 1907), also known as **Lord Kelvin**, proposed the need for a temperature scale starting at *absolute zero*. His idea stemmed from research showing a proportional relationship between the kinetic energy of a system and its temperature. By extrapolating his results to a point where the kinetic energy of a system was zero, Thomson was able to predict *absolute zero* as about -273°C.



The SI unit for temperature is the kelvin (K), named in Thomson's honour. *Absolute zero* is regarded as 0 kelvin, and is defined as -273.15°C. An increase of 1 kelvin corresponds to an increase of 1°C, so 0°C is equivalent to 273.15 K, and 100°C is equivalent to 373.15 K.

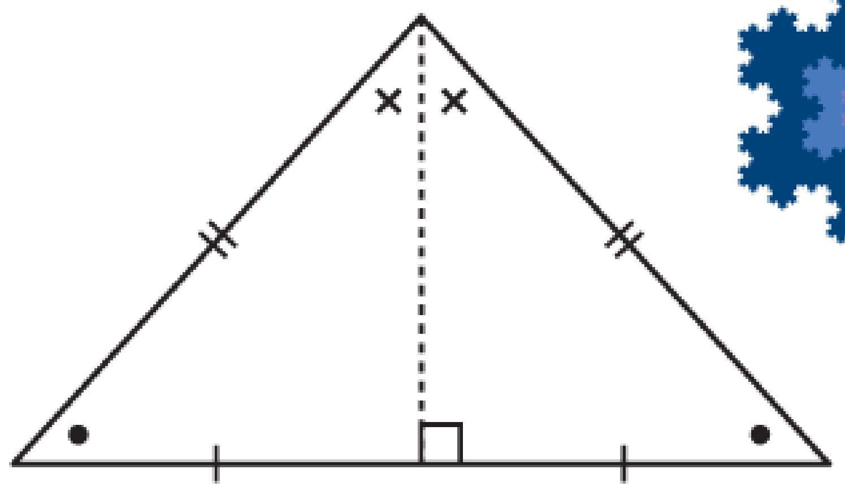
1. In what ways is it useful to use variables in direct proportion?
2. Which measure of temperature is most convenient?
3. What is the most *natural* measure of temperature?

See **Chapter 7**, **Direct and inverse variation**, p. 168

GEOMETRIC FACTS

TRIANGLE FACTS

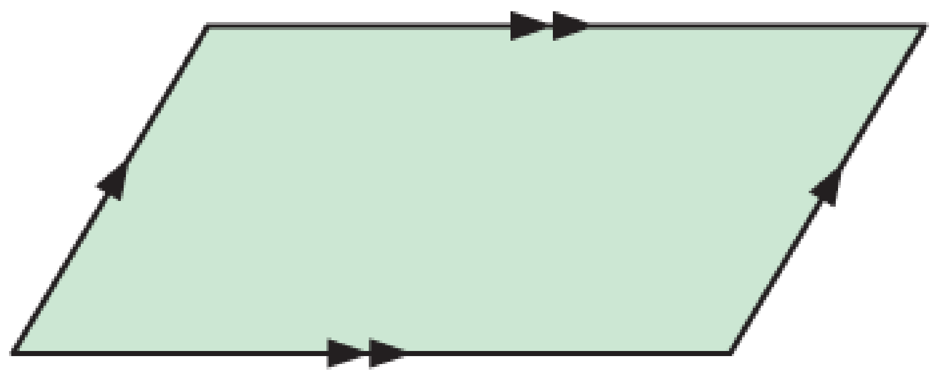
- The sum of the interior angles of a triangle is 180°.
- In any isosceles triangle:
    - the base angles are equal
    - the line joining the apex to the midpoint of the base bisects the vertical angle and meets the base at right angles.



GEOMETRY PACKAGE

QUADRILATERAL FACTS

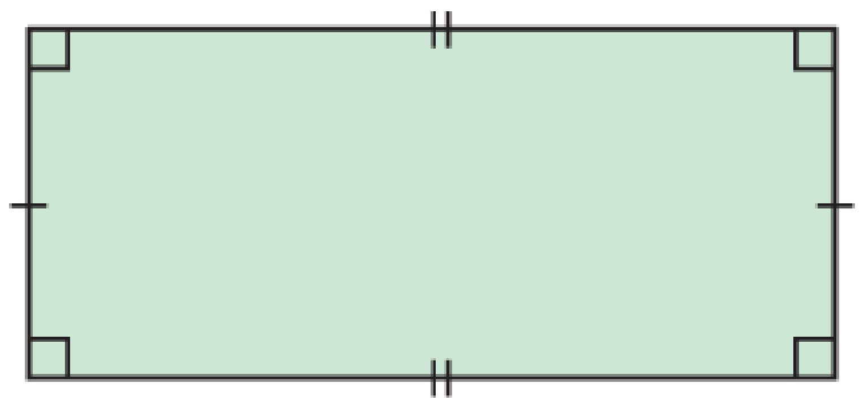
- The sum of the interior angles of a quadrilateral is 360°.
- A **parallelogram** is a quadrilateral which has opposite sides parallel.



Properties:
- opposite sides are equal in length
- opposite angles are equal in size
- diagonals bisect each other.

GEOMETRY PACKAGE

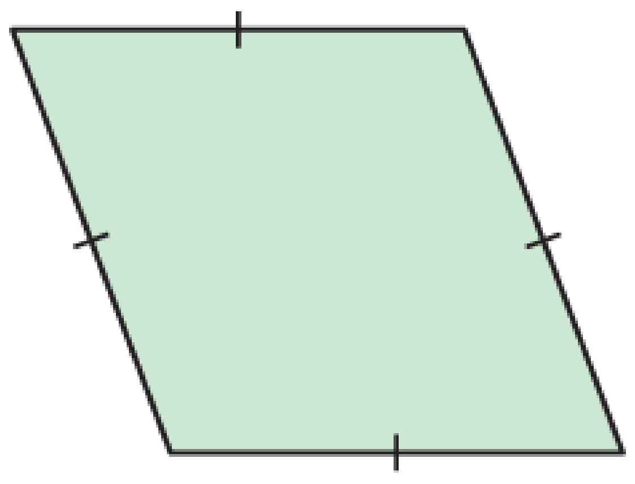
- A **rectangle** is a parallelogram with four equal angles of 90°.



Properties:
- opposite sides are parallel and equal
- diagonals bisect each other
- diagonals are equal in length.

GEOMETRY PACKAGE

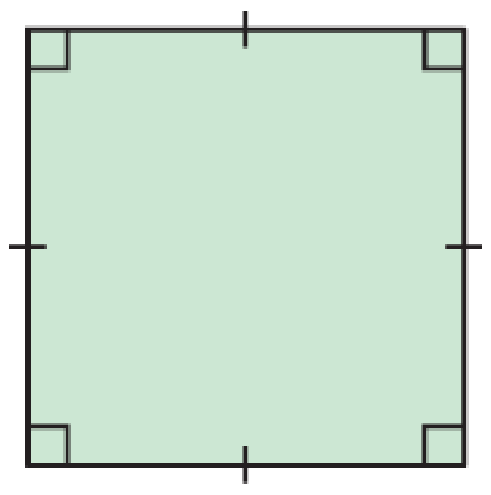
- A **rhombus** is a parallelogram in which all sides are equal in length.



Properties:
- opposite sides are parallel
- opposite angles are equal in size
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex.

GEOMETRY PACKAGE

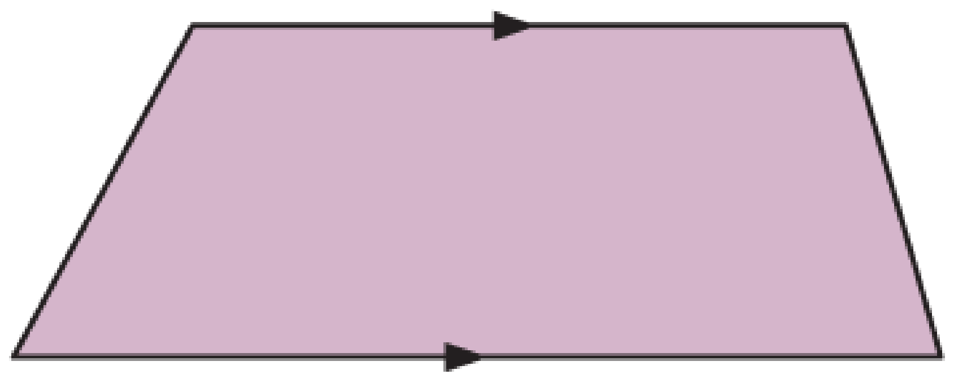
- A **square** is a rhombus with four equal angles of 90°.



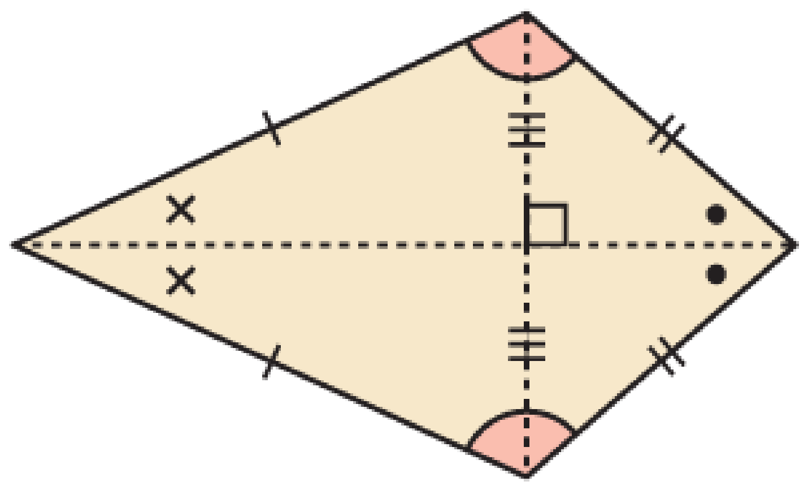
Properties:
- opposite sides are parallel
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex
- diagonals are equal in length.

GEOMETRY PACKAGE

- A **trapezium** is a quadrilateral which has a pair of parallel opposite sides.

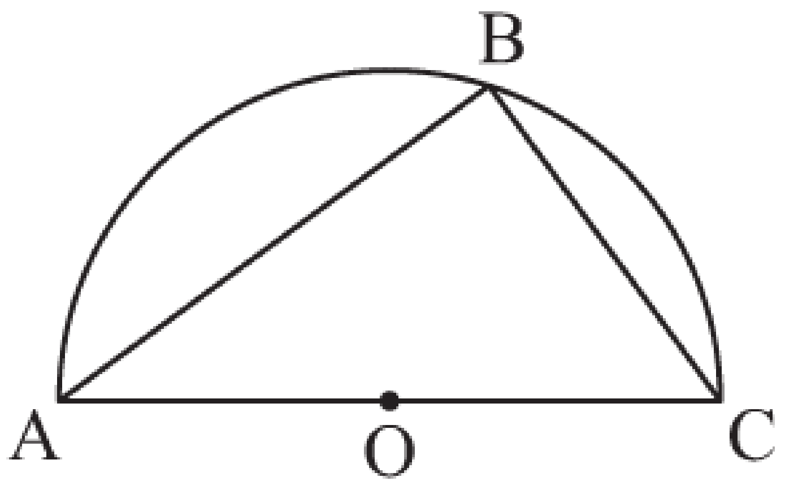
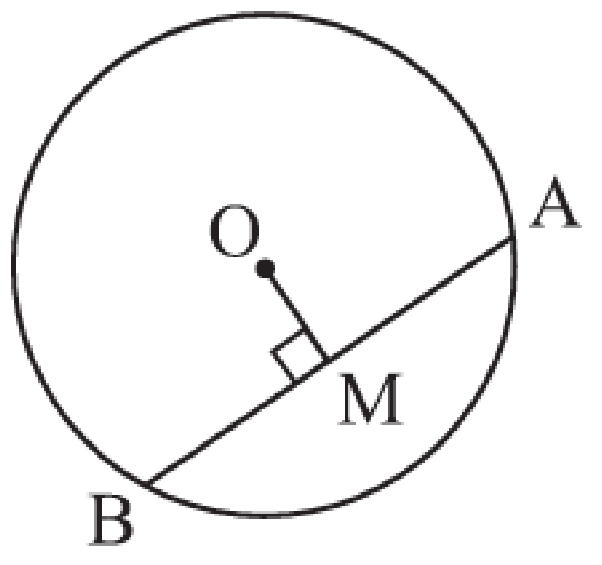
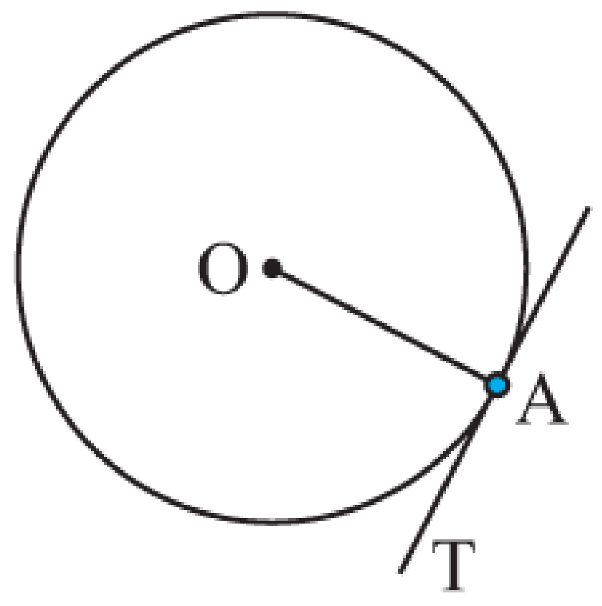
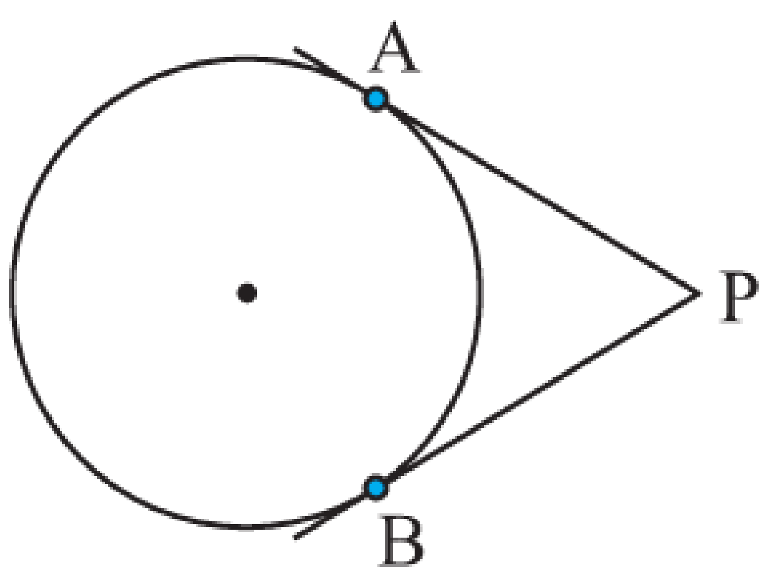
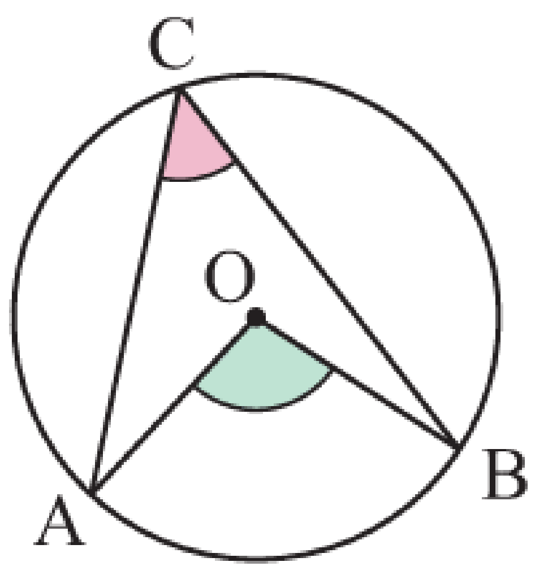
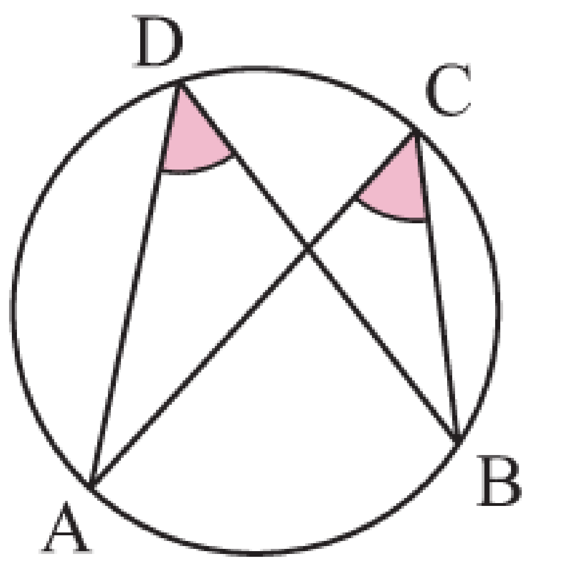
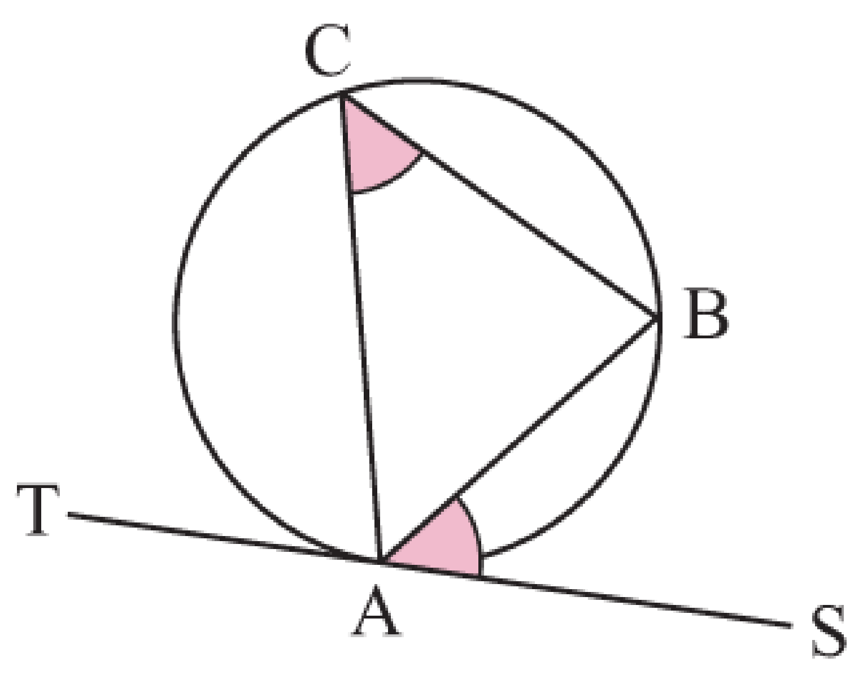


- A **kite** is a quadrilateral which has two pairs of adjacent sides equal in length.



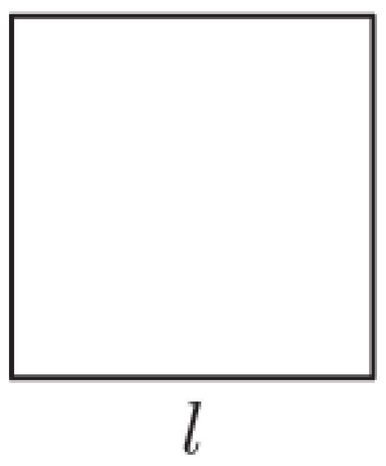
Properties:
- one diagonal is a line of symmetry
- one pair of opposite angles are equal
- diagonals cut each other at right angles
- **one** diagonal bisects **one** pair of angles at the vertices
- one of the diagonals bisects the other.

CIRCLE FACTS

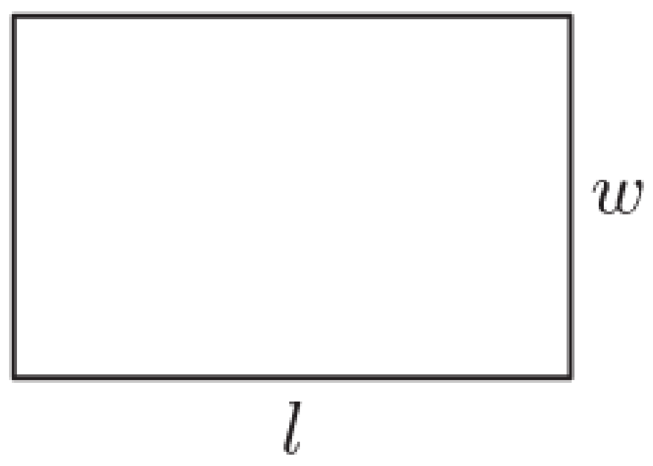
Name of theorem	*Statement*	*Diagram*
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$A\hat{B}C = 90^\circ$ GEOMETRY PACKAGE 
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	$AM = BM$ GEOMETRY PACKAGE 
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	$O\hat{A}T = 90^\circ$ GEOMETRY PACKAGE 
Tangents from an external point	Tangents from an external point are equal in length.	$AP = BP$ GEOMETRY PACKAGE 
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	$A\hat{O}B = 2 \times A\hat{C}B$ GEOMETRY PACKAGE 
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	$A\hat{D}B = A\hat{C}B$ GEOMETRY PACKAGE 
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	$B\hat{A}S = A\hat{C}B$ GEOMETRY PACKAGE 

USEFUL FORMULAE

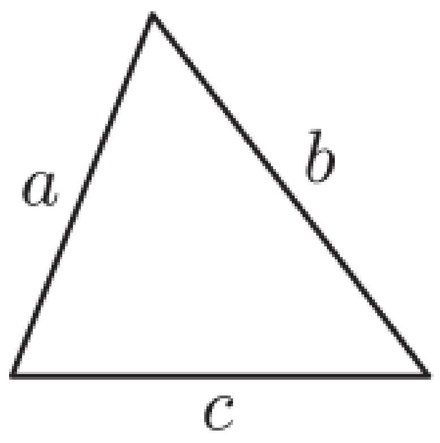
PERIMETER FORMULAE



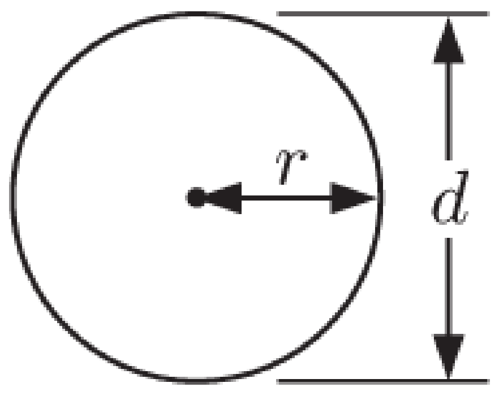
square
$P = 4l$



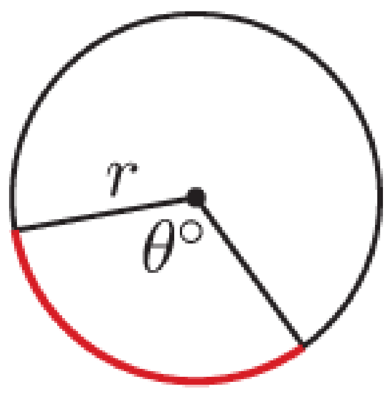
rectangle
$P = 2(l + w)$



triangle
$P = a + b + c$

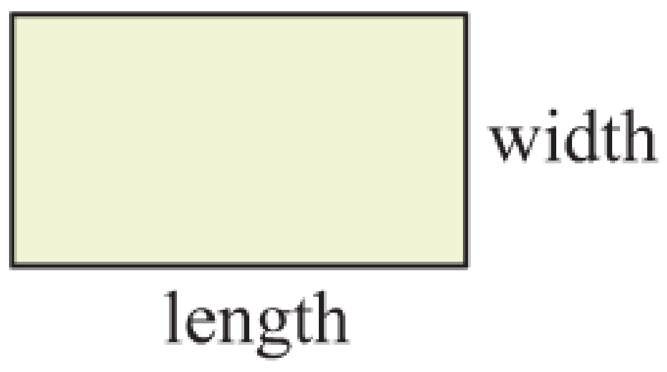
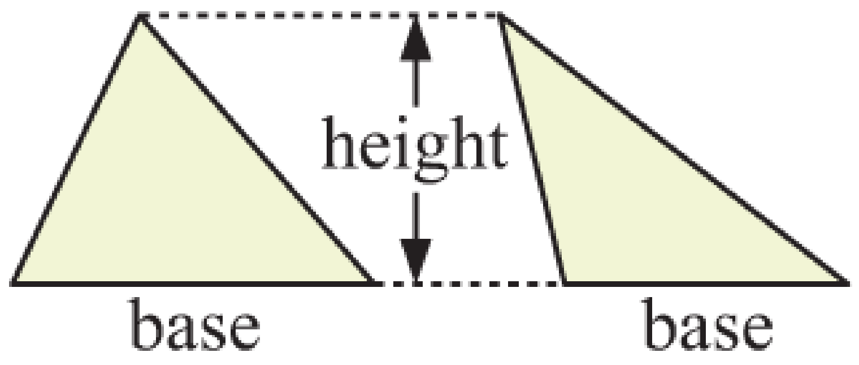
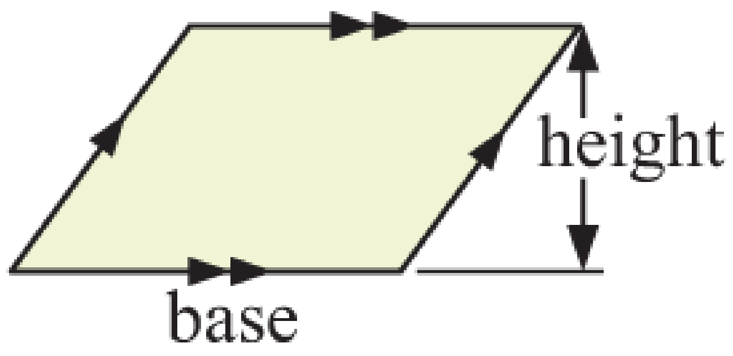
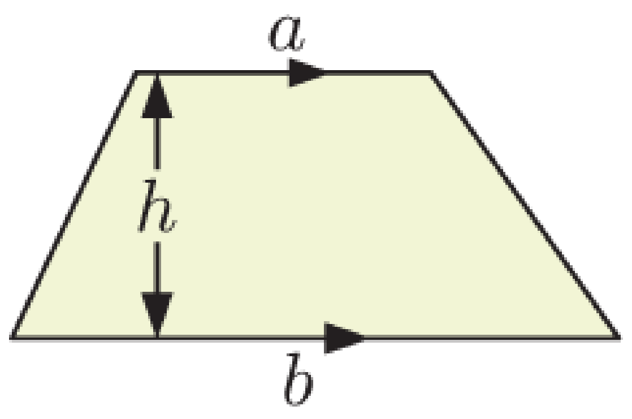
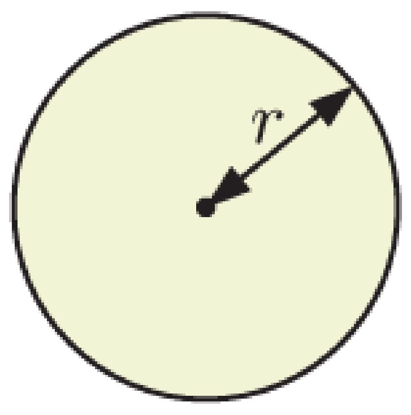
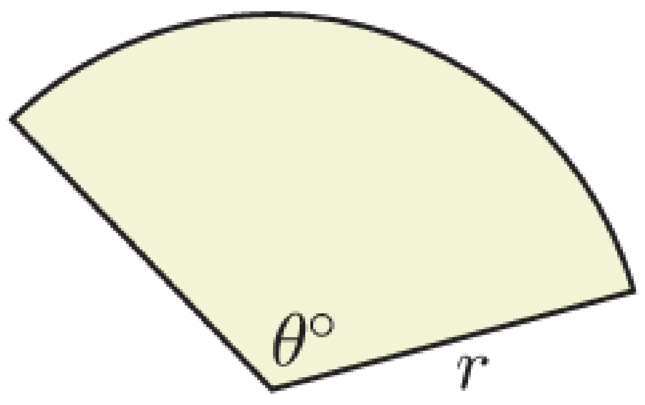


circle
$C = 2\pi r$
or $C = \pi d$



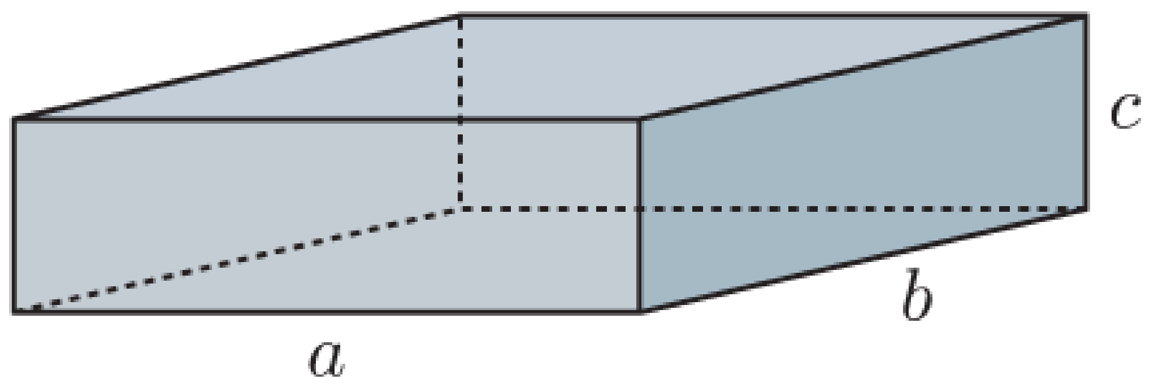
arc
$l = \left(\frac{\theta}{360}\right) 2\pi r$

AREA FORMULAE

Shape	*Diagram*	*Formula*
Rectangle	 width length	$A = \text{length} \times \text{width}$
Triangle	 height base base	$A = \frac{1}{2} \times \text{base} \times \text{height}$
Parallelogram	 height base	$A = \text{base} \times \text{height}$
Trapezium or **Trapezoid**		$A = \left(\frac{a+b}{2}\right) \times h$
Circle		$A = \pi r^2$
Sector		$A = \left(\frac{\theta}{360}\right) \times \pi r^2$

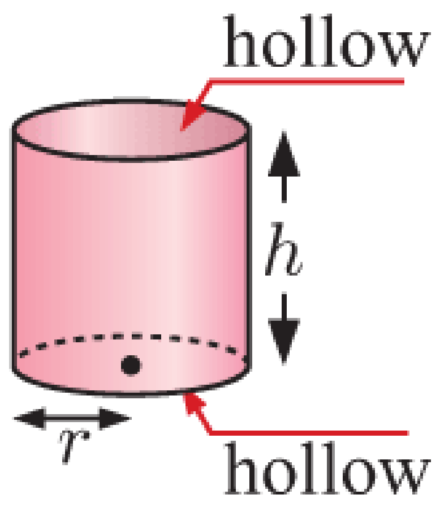
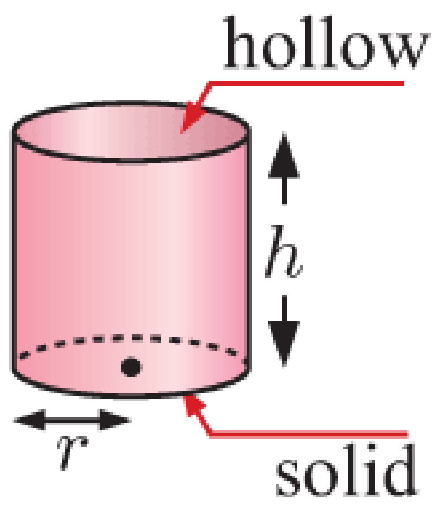
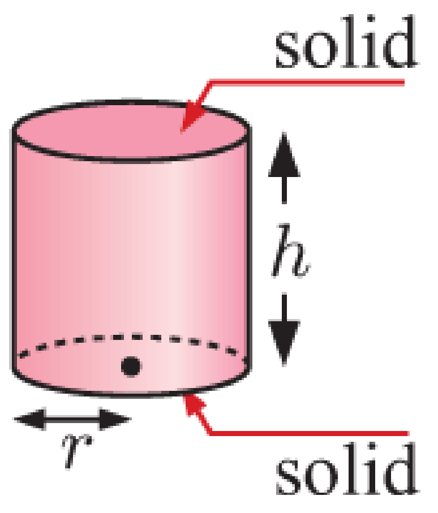
SURFACE AREA FORMULAE

RECTANGULAR PRISM

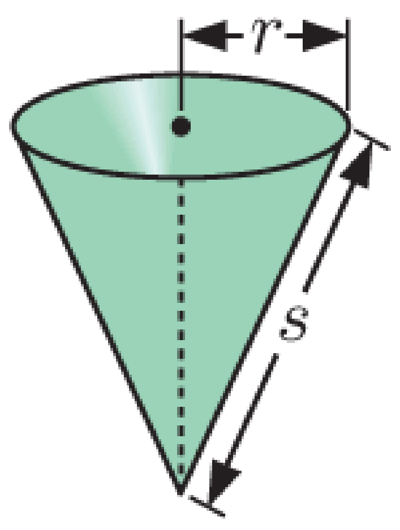
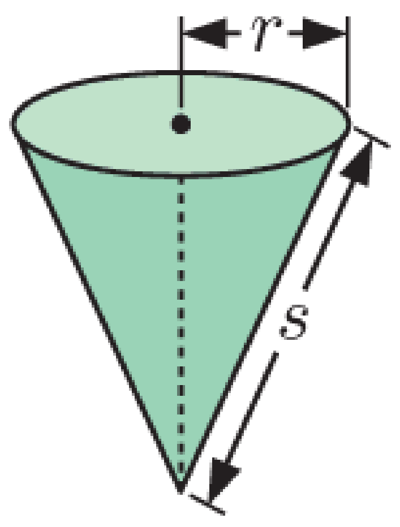


$$A = 2(ab + bc + ac)$$

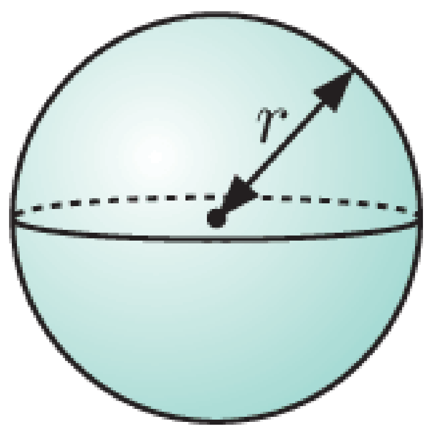
CYLINDER

Object	*Outer surface area*
Hollow cylinder  hollow h r hollow	$\boldsymbol{A = 2\pi rh}$ (no ends)
Open cylinder  hollow h r solid	$\boldsymbol{A = 2\pi rh + \pi r^2}$ (one end)
Solid cylinder  solid h r solid	$\boldsymbol{A = 2\pi rh + 2\pi r^2}$ (two ends)

CONE

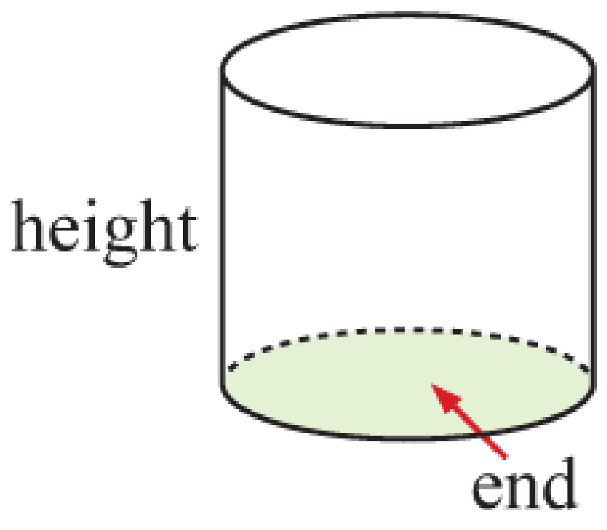
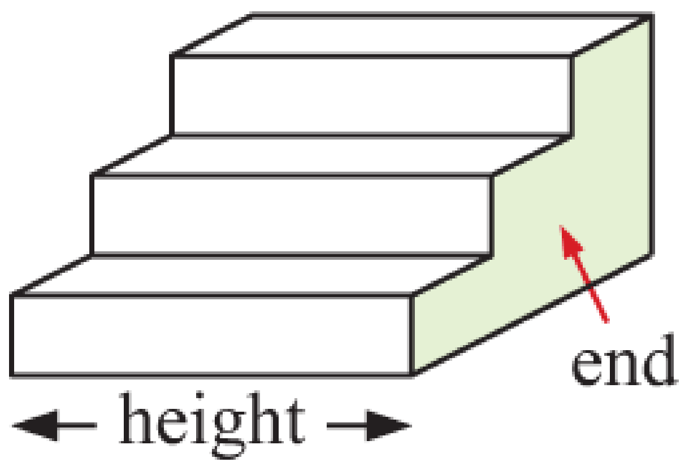
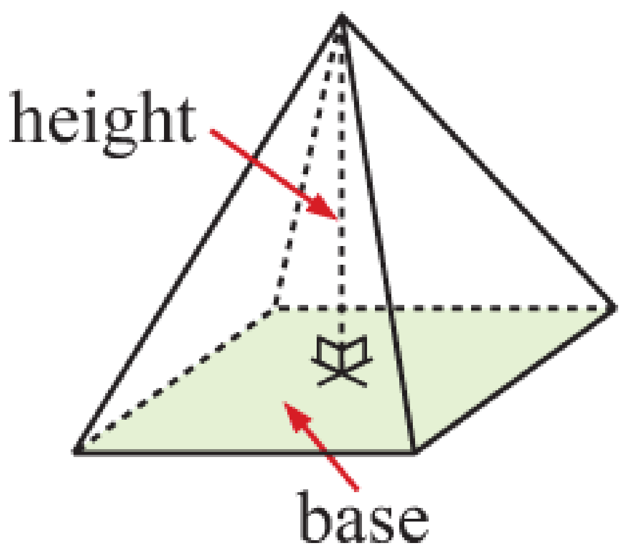
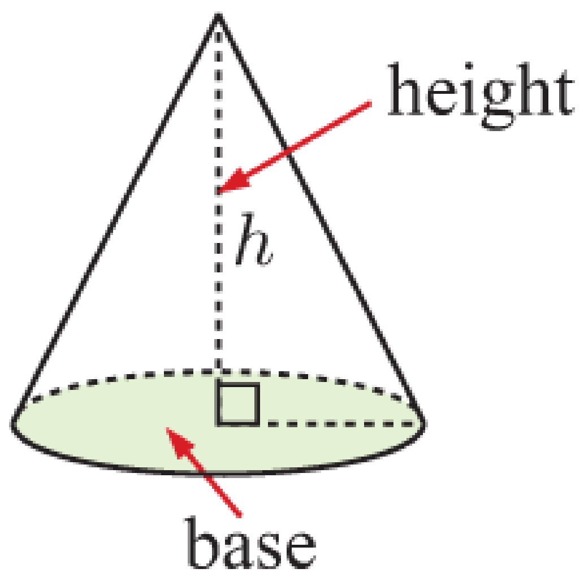
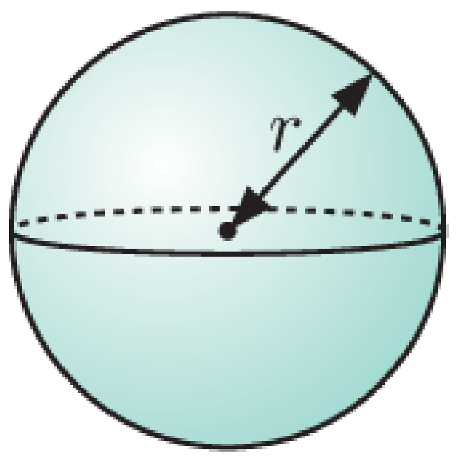
Object	*Outer surface area*
Open cone 	$\boldsymbol{A = \pi rs}$ (no base)
Solid cone 	$\boldsymbol{A = \pi rs + \pi r^2}$ (solid)

SPHERE



$$A = 4\pi r^2$$

VOLUME FORMULAE

Object	*Diagram*	*Volume*
Solids of uniform cross-section	 	$\boldsymbol{V = \text{area of end} \times \text{length}}$
Pyramids and cones	 height base  height h base	$\boldsymbol{V = \frac{1}{3}(\text{area of base} \times \text{height})}$
Spheres		$\boldsymbol{V = \frac{4}{3}\pi r^3}$

Chapter 1

Approximations and error

Contents:

A Rounding numbers
B Approximations
C Errors in measurement
D Absolute and percentage error

OPENING PROBLEM

Three friends are looking at the Eureka Tower in Melbourne, Australia. They are trying to work out how tall it is, but they keep on arguing.

- Andreas tries to carefully count the floors. He thinks there are 92. He then estimates the average height of a floor to be 3 metres. So, his estimate for the height is $92 \times 3 = 276$ m.
- Bernadette tries to carefully count the floors too, but she is less sure of the exact number. She thinks it is somewhere from 89 to 93, so she decides to use 90 as her estimate. She is sure that each floor is more than 3 m but less than 4 m. She therefore estimates 3.5 m for each floor. Her estimate for the height is $90 \times 3.5 = 315$ m.
- Carlos knows most about buildings. He is confident the height of a floor is about 3.3 m. He is too lazy to try to count each individual floor, so he counts them in lots of about 10. He is fairly sure 90 is the best guess, so his estimate for the height is $90 \times 3.3 = 297$ m.

Things to think about:

a Who do you think had the best *method*?

b Who was wisest in the way they rounded numbers?

c The Eureka Tower is in fact about 297.3 m tall.
  i Whose estimate was the most accurate?
  ii What was the percentage error in Andreas' estimate?

A ROUNDING NUMBERS

There are many occasions when it is sensible to give an **approximate** answer.

For example, it is unreasonable to give the exact distance between the Earth and the Sun, because it is continually changing. The distance varies from its *perihelion*, about 146 million km, to its *aphelion*, about 152 million km.

We use the symbol $\approx$ or sometimes $\doteqdot$ to show that an answer has been approximated.

RULES FOR ROUNDING OFF

- If the digit after the one being rounded off is **less than 5** (0, 1, 2, 3, or 4) we round **down**.
- If the digit after the one being rounded off is **5 or more** (5, 6, 7, 8, 9) we round **up**.

Example 1 **Self Tutor**

Round off:

a 436 to the nearest 10 **b** 716 to the nearest 100
c 1050 to the nearest 100 **d** 19 628 to the nearest 1000.

a $436 \approx 440$ {round up, as 6 is greater than 5}
b $716 \approx 700$ {round down, as 1 is less than 5}
c $1050 \approx 1100$ {5 is rounded up}
d $19\,628 \approx 20\,000$ {round up, as 6 is greater than 5}

EXERCISE 1A.1

1 Round off to the nearest 10:
a 86 **b** 81 **c** 85 **d** 128 **e** 162
f 104 **g** 635 **h** 1822 **i** 699 **j** 3045

2 Round off to the nearest 100:
a 215 **b** 264 **c** 3750 **d** 3950 **e** 26 341

3 Round off to the nearest 1000:
a 8365 **b** 3500 **c** 19 210 **d** 19 650 **e** 114 823

4 Round off to the accuracy given:
a The height of Mt Everest is 8848 m. (to the nearest 10 m)
b The surface area of Lake Baikal in Russia is 31 722 km^2. (to the nearest 1000 km^2)
c The population of New Zealand in 2018 was 4 749 598. (to the nearest 1000)
d The attendance at an English football match was 85 512 people. (to the nearest 100 people)
e The area of Australia is 7 692 000 km^2. (to the nearest 100 000 km^2)
f An average weight of an adult African elephant is 5443 kg. (to the nearest 100 kg)
g The distance between Paris and Sydney is 16 950 km. (to the nearest 100 km)
h The average distance from Earth to the Moon is 384 400 km. (to the nearest 100 000 km)
i The population of South America in 2018 was 428 240 515. (to the nearest 1 000 000)

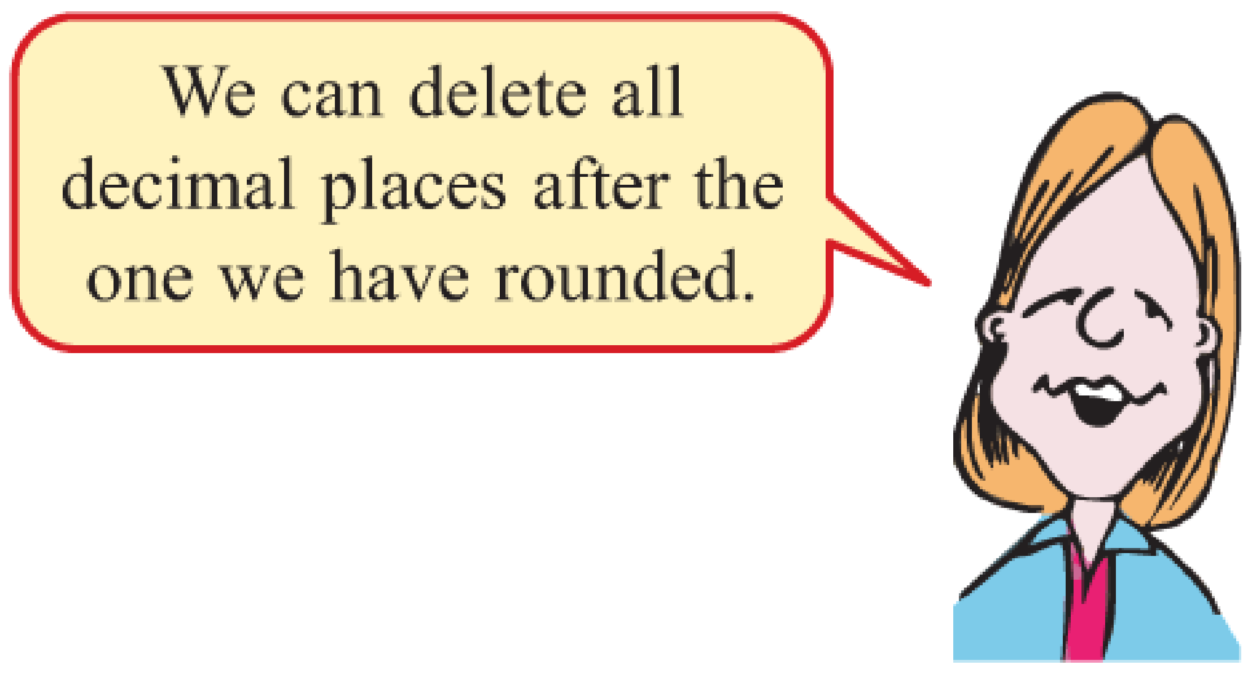
ROUNDING DECIMAL NUMBERS

A survey found that a total of 6428 flights were taken by 825 people last year. However, it is not sensible to give the average number of flights per person as 7.791 515 152. An approximate answer of 7.8 is more appropriate.

Example 2 **Self Tutor**

Round: **a** 8.43 to one decimal place
b 3.5169 to two decimal places.

a $8.43 \approx 8.4$ {round down, as $3 < 5$}
b $3.5169 \approx 3.52$ {round up, as $6 > 5$}


We can delete all decimal places after the one we have rounded.

EXERCISE 1A.2

1 Round the following to the number of decimal places stated in brackets.

a 6.181 [1] **b** 6.181 [2] **c** 3.25 [1] **d** 17.403 [2]

e 2.131 58 [3] **f** 0.1940 [1] **g** 0.0972 [2] **h** 102.382 [2]

2 In 2009 Usain Bolt ran 100 m in 9.58 seconds. Round this time to 1 decimal place.

3 The average height of children in a class is 1.435 m. Round this height to 2 decimal places.

4 The thickness of a sheet of paper is 0.012 cm. Round this thickness to 2 decimal places.

5 The number π is a mathematical constant. The first six digits of π are 3.141 59. Round π to:

a 1 decimal place **b** 3 decimal places **c** 4 decimal places.

6 The fraction $\frac{5}{19} \approx 0.263\,157\,895$. Round this number to:

a 1 decimal place **b** 2 decimal places **c** 6 decimal places.

7 Evaluate, giving your answer to 3 decimal places.

a $\sqrt{2}$ **b** $\sqrt{5}$ **c** $\sqrt{23}$ **d** $\sqrt[3]{4}$ **e** $\sqrt[3]{-15}$ **f** $\sqrt[3]{450}$

8 Calculate the following, rounding your answers to 2 decimal places:

a $(16.8 + 12.4) \times 17.1$ **b** $16.8 + 12.4 \times 17.1$ **c** $127 \div 9 - 5$

d $127 \div (9 - 5)$ **e** $37.4 - 16.1 \div (4.2 - 2.7)$ **f** $\dfrac{16.84}{7.9 + 11.2}$

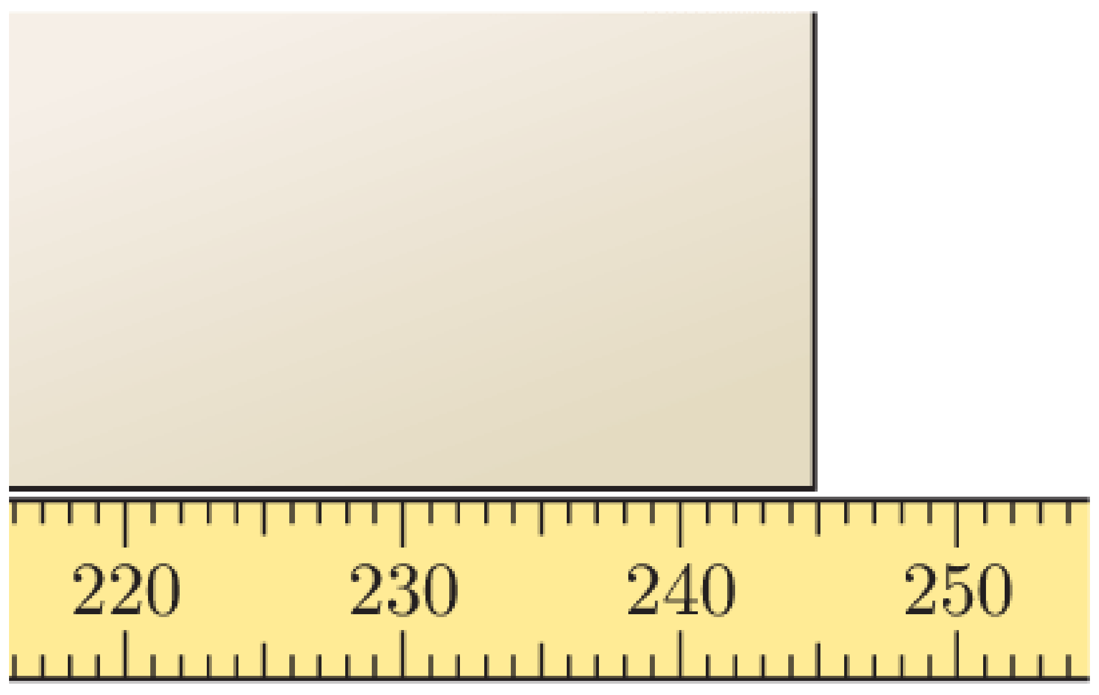
g $\dfrac{27.4}{3.2} - \dfrac{18.6}{16.1}$ **h** $\dfrac{27.9 - 17.3}{8.6} + 4.7$ **i** $\dfrac{0.0768 + 7.1}{18.69 - 3.824}$

9 Over a 23 game water polo season, Kerry scored 40 goals for her team. Find Kerry's average number of goals per game, correct to 2 decimal places.

10 Wang used a tape measure to check the length of his kitchen bench. After viewing the tape, he recorded the length as 2.45 m, which he rounded up to 2.5 m.

When his mother asked him about the length, he said it was about 3 m.

Explain what Wang has done wrong, and discuss why we need to be careful when we make approximations.



DISCUSSION

$\sqrt{61}$ is approximately $7.810\,249\,676\ldots$.

Mateo rounded this number to 2 decimal places, giving $\sqrt{61} \approx 7.81$

Hiba rounded this number to 3 decimal places, giving $\sqrt{61} \approx 7.810$

Would it be fair to say that Hiba's estimate is "more accurate", even though both estimates have the same value?

ROUNDING OFF TO SIGNIFICANT FIGURES

The first **significant figure** of a decimal number is the first (left-most) non-zero digit.

For example:
- the first significant figure of 4567 is 4
- the first significant figure of 0.012 34 is 1.

Every digit to the right of the first significant figure is regarded as another significant figure.

To round off to a number of significant figures:

> Count off the specified number of significant figures then look at the next digit.
> - If the digit is less than 5, round **down**.
> - If the digit is 5 or more, round **up**.
>
> Delete all figures following the significant figures, replacing with 0s where necessary.

Example 3 **Self Tutor**

Round:
- **a** 3.461 to 2 significant figures
- **b** 0.007 24 to 2 significant figures
- **c** 708 to 1 significant figure
- **d** 20.158 to 3 significant figures.

a $3.\underset{\uparrow}{4}61 \approx 3.5$ {2 significant figures}

This is the 2nd significant figure, so we look at the next digit which is 6.
The 6 tells us to round the 4 up to a 5 and delete the remaining digits.

b $0.\underset{\uparrow}{00}7\,24 \approx 0.0072$ {2 significant figures}

These zeros at the front are place holders and so must stay.
The first significant figure is 7, and the second significant figure is 2.
The 4 tells us to leave the 2 as it is and delete the remaining digits.

c $\underset{\uparrow}{7}08 \approx 700$ {1 significant figure}

7 is the first significant figure so it has to be rounded.
The 0 tells us to keep the original 7 in the hundreds place, so we convert the 08 into 00.
These two zeros are place holders. They are not significant figures, but they need to be there to make sure the 7 has value 700.

d $2\underset{\uparrow}{0}.158 \approx 20.2$ {3 significant figures}

This 0 is significant as it lies between two non-zero digits.
The third significant figure is 1.
The 5 tells us to round the 1 up to a 2 and delete the remaining digits.

In IB examinations you are expected to give answers to 3 significant figures unless otherwise specified in the question.

EXERCISE 1A.3

1 Round to 2 significant figures:

a 128 **b** 8342 **c** 2.568 **d** 0.0134

e 163 870 **f** 1.086 **g** 3958 **h** 6.611

2 Round to 3 significant figures:

a 83 064 **b** 10 044 **c** 0.105 26 **d** 31.695

e 70.707 **f** 4.0007 **g** 0.036 71 **h** 19.989

3 Round to 4 significant figures:

a 16.382 **b** 438.207 **c** 6 873 681 **d** 0.028 885

4 The exact crowd size at a rock concert was 96 257 people. Round the crowd size to:

a 1 significant figure **b** 2 significant figures **c** 3 significant figures.

5 Evaluate the following, giving your answers to 3 significant figures:

a $\sqrt{7}$ **b** 2π **c** $36 \div 17$ **d** 517×3802

e $(0.986)^5$ **f** $\dfrac{16.3 - 2.68}{3.1}$ **g** $\sqrt{5.4 - 2.18}$ **h** $\dfrac{9.58}{\sqrt{2.8}}$

6 A theatre has 32 rows with 28 seats in each. Find the total number of seats, rounding your answer to 2 significant figures.

7 A ballroom has dimensions 30.1 m × 8.5 m. Find its area, rounding your answer to 3 significant figures.

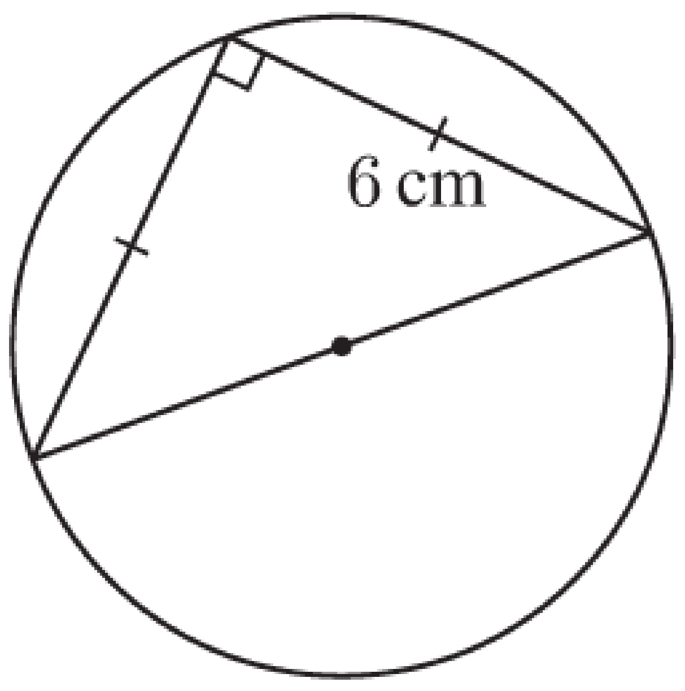
8 The proceeds of a garage sale were $752.25, and this was shared equally between 4 people. Calculate the amount each person received:

a to 3 significant figures **b** to 5 significant figures.

9 The speed of sound in dry air at 20°C is 343 m s^{-1}. Calculate how many metres sound travels in one hour, giving your answer to two significant figures.

10 To calculate the area of this circle, Eric performed the following calculations:

$$\begin{aligned} &\text{Diameter of circle} = \sqrt{6^2 + 6^2} = \sqrt{72} \approx 8.49 \text{ cm} && \{3 \text{ significant figures}\} \\ &\therefore \ \text{radius of circle} = 8.49 \div 2 \approx 4.25 \text{ cm} && \{3 \text{ significant figures}\} \\ &\therefore \ \text{area of circle} = \pi \times 4.25^2 \approx 56.7 \text{ cm}^2 && \{3 \text{ significant figures}\} \end{aligned}$$



a Explain why Eric's final answer of 56.7 cm² may not be accurate to 3 significant figures.

b Find the correct area of the circle, rounded to 3 significant figures.

B APPROXIMATIONS

A fast way of estimating a calculation is to perform a **one figure approximation**:

- Leave single digit numbers as they are.
- Round all other numbers to one significant figure.
- Perform the calculation.

Example 4 Self Tutor

Estimate:

a 872×52 **b** $61\,812 \div 384$ **c** 4.37×0.482

a
$$\begin{aligned} &872 \times 52 \\ \approx\ &900 \times 50 \\ \approx\ &45\,000 \end{aligned}$$

b
$$\begin{aligned} &61\,812 \div 384 \\ \approx\ &60\,000 \div 400 \\ \approx\ &600 \div 4 \\ \approx\ &150 \end{aligned}$$

c
$$\begin{aligned} &4.37 \times 0.482 \\ \approx\ &4 \times 0.5 \\ \approx\ &2 \end{aligned}$$

DEMO

EXERCISE 1B

1 Estimate using a one figure approximation:

a 32×6
b 58×7
c 81×30
d 207×3
e 487×50
f 6117×4
g 48×23
h 61×42
i 103×47
j 3125×18
k 422×307
l 3818×27
m 2.7×1.15
n 5.36×0.68
o 28.37×6.13

2 Estimate using a one figure approximation:

a $86 \div 3$
b $64 \div 5$
c $512 \div 21$
d $610 \div 43$
e $4182 \div 19$
f $78\,638 \div 82$
g $318 \div 62$
h $47\,320 \div 193$
i $0.628 \div 3$
j $46.1 \div 5.2$
k $631.7 \div 0.29$
l $18.7 \div 3.86$

3 Estimate, using one figure approximations, the cost of:

a 8 kg of apples at €2.80/kg
b 3 airline tickets at $213 each
c 7 theatre tickets at $87.30 each
d 55 L of fuel at £1.49/L.

4 Estimate using one figure approximations:

a the distance travelled if Brodie drives for 4.2 hours at 63 $\text{km}\,\text{h}^{-1}$
b the number of days in 14 years
c the average weight carried per truck if 423 tonnes of cargo is divided equally between 18 trucks
d the total pay for a part-time worker who works on average 18.2 hours per week for 12 weeks, and earns €21.50 per hour.

DISCUSSION

Suppose there are 19 biscuits in a packet, and 32 packets are packed in a carton. Using a one figure approximation, we estimate there are $20 \times 30 = 600$ biscuits in the carton. This is close to the actual number of 608 biscuits.

Will one figure approximations always be this close to the actual value? What is it about these particular numbers that makes the estimate close? By comparison, you might consider 24 packets each containing 14 biscuits.

ACTIVITY 1 ESTIMATION

An **estimation** is a value which has been found by judgement or prediction instead of carrying out a more accurate measurement.

We often estimate when it is difficult for us to measure. Whenever we do this it is important to round numbers to sensible accuracy.

In order to make reasonable estimations we often appeal to our previous experience.

Work in pairs for this Activity.

What to do:

PRINTABLE WORKSHEET

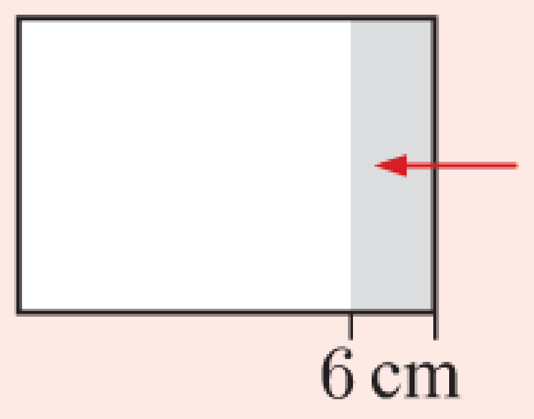
1 For each of the following objects:

- **i** Each person should *estimate* the quantity without measuring.
- **ii** As a pair, take measurements and hence find a good approximation for the quantity.
- **iii** Discuss the errors in your estimates.

a the height of your classroom doorway
b the length of your classroom
c the temperature in your classroom
d the perimeter of the basketball court
e the area of the basketball court
f the capacity of a large container.

2 **a** *Estimate* the area of the school car park without measuring.
b Discuss how you could determine an upper bound for what the area could be.
c Discuss how you could obtain a more accurate approximation for the area.

3 **a** Measure in mm the length and width of a sheet of 80 gsm A4 photocopying paper.
b What is its area in m^2 and how many sheets make up 1 m^2?
c 80 gsm means 80 grams per square metre. What is the mass of one sheet of A4 paper?
d What is the approximate mass of this part of the sheet?



e Crumple the 6 cm strip into your hand and feel how heavy it is.

C ERRORS IN MEASUREMENT

When we take measurements, we are usually reading some sort of scale.

The scale of a ruler may have centimetres marked on it, but when we measure the length of an object, it is likely to fall between two divisions. We **approximate** the length of the object by recording the value at the nearest centimetre mark. In doing so our answer may be inaccurate by up to a half a centimetre.

A measurement is accurate to $\pm\frac{1}{2}$ of the smallest division on the scale.

ACTIVITY 2 MEASURING DEVICES

Examine a variety of measuring instruments at school and at home. Make a list of the names of these instruments, what they measure, what their units are, and the degree of accuracy to which they can measure.

For example:

0 cm 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
mm 0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300

A ruler measures length. In the Metric System it measures in centimetres and millimetres, and can measure to the nearest millimetre. Its accuracy is $\pm\frac{1}{2}$ mm.

Example 5 Self Tutor

Ling uses a ruler to measure the length l of her pencil case. She records the length as 18.7 cm.

Find the range of values in which the length may lie.

18.7 cm is 187 mm, so the measuring device must be accurate to the nearest half mm.

$\therefore$ the range of values is $187 \pm \frac{1}{2}$ mm

The actual length is in the range $186\frac{1}{2}$ mm to $187\frac{1}{2}$ mm.

$\therefore$ $18.65 \text{ cm} < l < 18.75 \text{ cm}$.

We do not include the endpoints of the interval because the length can never be *exactly* these values.

EXERCISE 1C

1 State the accuracy of the following measuring devices:
  - **a** a tape measure marked in cm
  - **b** a measuring cylinder with 1 mL graduations
  - **c** a beaker with 100 mL graduations
  - **d** a set of scales with marks every 500 g
  - **e** a thermometer with marks every 0.1°C.

2 Roni checks his weight every week using scales with 1 kg graduations. This morning he recorded a weight of 68 kg. In what range of values does Roni's actual weight w lie?

3 Find the range of possible values corresponding to the following measurements:

a 27 mm **b** 38.3 cm **c** 4.8 m

d 1.5 kg **e** 25 g **f** 3.75 kg

4 Tom's digital thermometer said his temperature was 36.4°C. In what range of values did Tom's actual temperature T lie?

5 Joanne's exercise watch displays the distance she has run to 3 significant figures. State the *least* distance Joanne could have run, if the watch displays:

a 1.06 km **b** 9.72 km **c** 10.1 km

Comment on the accuracy of the watch.

6 Four students measured the width of their classroom using the same tape measure. The measurements were 6.1 m, 6.4 m, 6.0 m, 6.1 m.

- **a** Which measurement is likely to be incorrect? Explain your answer.
- **b** What answer would you give for the width of the classroom? Explain your answer.
- **c** What graduations do you think were on the tape measure?

7 Hasan has many lengths of rope. He has measured each length to be 2.4 m.

- **a** In what range of values does the actual length of a rope l lie?
- **b** If Hasan carefully places n of his ropes end to end, in what range of values will the total length of rope L lie?

8 In the 800 m race at the sports carnival, the times recorded for Jiao and Liang were 2 min 8 s and 2 min 13 s respectively.
Find the range of possible values for the time t by which Jiao beat Liang.

Example 6 **Self Tutor**

A rectangular board was measured as 78 cm by 24 cm. Find the boundary values for its perimeter.

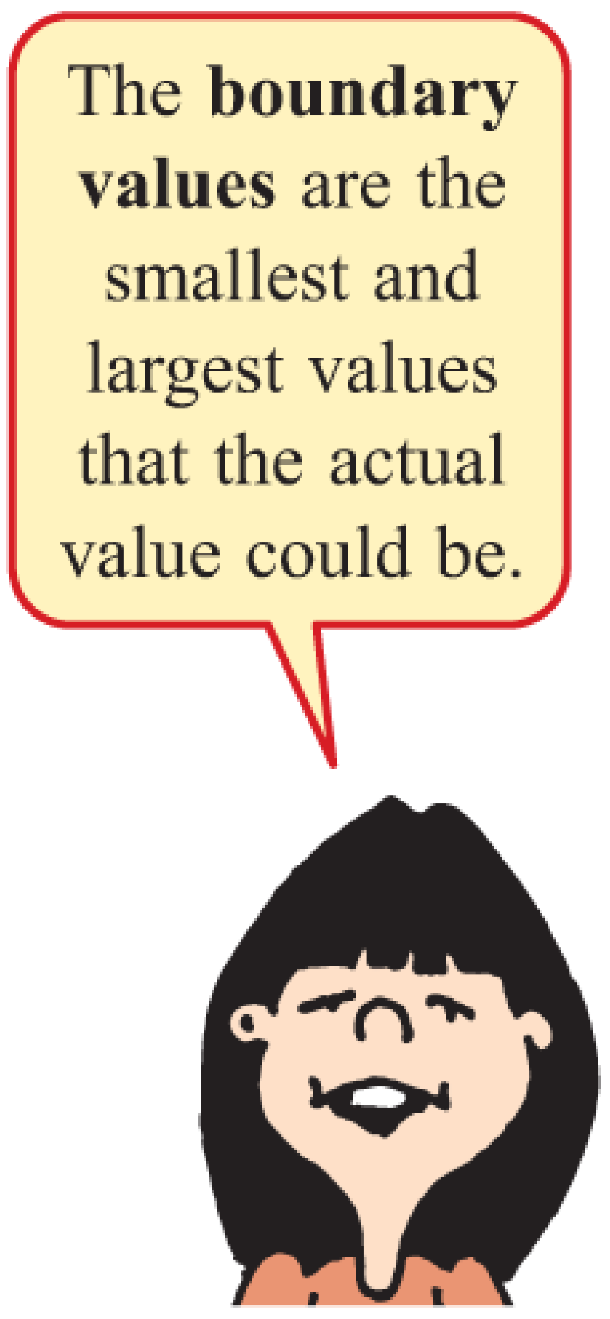
The length of the board could be from $77\frac{1}{2}$ cm to $78\frac{1}{2}$ cm.

The width of the board could be from $23\frac{1}{2}$ cm to $24\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the perimeter is $2 \times 77\frac{1}{2} + 2 \times 23\frac{1}{2} = 202$ cm

and the upper boundary of the perimeter is $2 \times 78\frac{1}{2} + 2 \times 24\frac{1}{2} = 206$ cm

The perimeter is between 202 cm and 206 cm, which is 204 ± 2 cm.


The **boundary values** are the smallest and largest values that the actual value could be.

9 A rectangular bath mat was measured as 86 cm by 38 cm. Find the boundary values of its perimeter.

10 A rectangular garden bed is measured as 252 cm by 143 cm. Find the range of possible values for the total length of edging l required to border the garden bed.

Example 7 **Self Tutor**

A paving brick is measured as 18 cm × 10 cm. What are the boundary values for its actual area?

The length of the paving brick could be from $17\frac{1}{2}$ cm to $18\frac{1}{2}$ cm.

The width of the paving brick could be from $9\frac{1}{2}$ cm to $10\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $17\frac{1}{2} \times 9\frac{1}{2} = 166.25$ cm^2

and the upper boundary of the area is $18\frac{1}{2} \times 10\frac{1}{2} = 194.25$ cm^2.

The area is between 166.25 cm^2 and 194.25 cm^2.

This could also be represented as $\dfrac{166.25 + 194.25}{2} \pm \dfrac{194.25 - 166.25}{2}$ cm^2
which is 180.25 ± 14 cm^2.

11 A rectangle is measured to be 6 cm by 8 cm. Find:

- **a** the largest area it could have
- **b** the smallest area it could have.

12 Find the boundary values for the actual area of a glass window measured as 42 cm by 26 cm.

13 The base of a triangle is measured as 9 cm and its height is measured as 8 cm. In what range of values does its actual area A lie?

14 Find the boundary values for the actual volume of a box measuring 4 cm by 8 cm by 6 cm.

15 Find the range of values in which the actual volume V of a house brick measuring 21.3 cm by 9.8 cm by 7.3 cm must lie.

16 A cylinder is measured to have radius 5 cm and height 15 cm. Find the boundary values for the cylinder's volume.

17 A cone is measured to have radius 8.4 cm and height 4.6 cm. Find the boundary values for the cone's volume.

18 Eko measures the diameter of a ball to be 18.2 cm. Do you expect the rounding in Eko's measurement to have more effect on a calculation of the ball's surface area, or a calculation of its volume? Explain your answer.

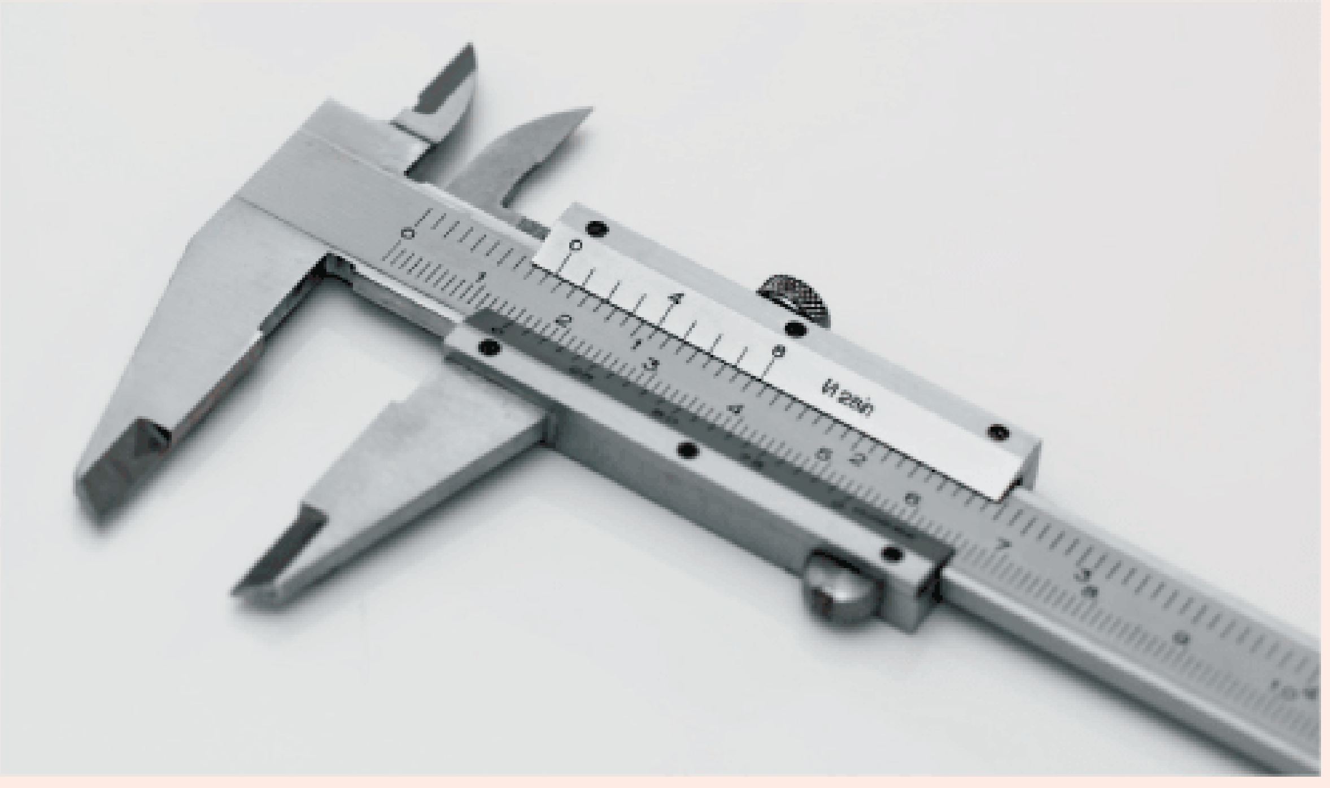
19 Rachel measures the base side lengths of a square-based pyramid to be 4.6 cm, and its height to be 5.2 cm. Find the boundary values for the pyramid's:

- **a** volume
- **b** surface area.

RESEARCH — VERNIER SCALES

A **vernier scale** is used to measure the length of objects with a high degree of accuracy.

Research how vernier scales work.



D ABSOLUTE AND PERCENTAGE ERROR

Whenever we measure a quantity there is almost always a difference between our measurement and the actual value. We call this difference the **error**.

The *size* or *magnitude* of the error, whether the measured or estimated value is too high or too low, is called the **absolute error**.

If the actual or exact value is V_E and the approximate value is V_A then the

$$\textbf{absolute error} = |V_A - V_E|$$

Error is often expressed as a percentage of the exact value:

$$\textbf{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

Example 8 **Self Tutor**

Suppose you estimate a fence's length to be 70 m, whereas its true length is 78.3 m. Find, correct to one decimal place, the:

a absolute error **b** percentage error.

a absolute error $= |V_A - V_E|$
$= |70 - 78.3|$ m
$= |-8.3|$ m
$= 8.3$ m

b percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{8.3}{78.3} \times 100\%$
$\approx 10.6\%$

EXERCISE 1D

1 Find the absolute error and percentage error in rounding:

a the yearly profit of €1 367 540 made by a company to €1.37 million
b a population of 31 467 people to 31 000 people
c a retail sales figure of \$458 110 to \$460 000
d the number of new cars sold in a year from 2811 to 3000.

2 Find the absolute error and percentage error if you estimate:

a the mass of a brick to be 5 kg when its actual mass is 6.238 kg
b the perimeter of a property to be 100 m when its actual perimeter is 97.6 m
c the capacity of a container to be 20 L when its actual capacity is 23.8 L
d the time to write a computer program to be 50 hours when it actually takes 72 hours.

Example 9 **Self Tutor**

Alan wants to lay carpet on his 4.2 m by 5.1 m lounge room floor. He estimates the area of the lounge room by rounding each measurement to the nearest metre.

a Find Alan's estimate of the lounge room area.
b The carpet costs \$39 per square metre. Find the cost of the carpet using Alan's estimate of the area.
c Find the actual area of Alan's lounge room.
d Find the percentage error in Alan's estimation.
e Will Alan have enough carpet to cover his lounge room? How should he have rounded the measurements?

a Area ≈ 4 m $\times$ 5 m
≈ 20 m^2

b Cost $= 20 \times \$39$
$= \$780$

c Actual area $= 4.2 \times 5.1$
$= 21.42$ m^2

d Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{|20 - 21.42|}{21.42} \times 100\%$
$\approx 6.63\%$

e Alan only has 20 m^2 of carpet, so he will not have enough to cover his lounge room.
He should have rounded the measurements *up* to make sure he had enough carpet.

3 Jon's apartment is a 10.3 m by 9.7 m rectangle.

a Find the actual area of the apartment.

b Estimate the floor area by rounding each length to the nearest metre.

c Find the absolute error and percentage error in your estimate.

4 The cost of freight for a parcel is dependent on its volume. Justine lists the dimensions of a parcel as 24 cm by 15 cm by 9 cm on the consignment note. The actual dimensions are 23.9 cm × 14.8 cm × 9.2 cm.

a Calculate the actual volume of the parcel.

b Estimate the volume using the dimensions given on the consignment note.

c Find the absolute error and percentage error in the calculation using the consignment note.

5 A hotel wants to cover an 8.2 m by 9.4 m rectangular courtyard with synthetic grass. The manager estimates the area by rounding each measurement to the nearest metre.

a Find the manager's estimate of the area.

b The synthetic grass costs \$85 per square metre. Find its cost using the manager's estimate.

c Find the actual area of the rectangle.

d Calculate the percentage error in the manager's estimate.

e Will the hotel have enough grass to cover the courtyard?

f Find the cost of the grass if the manager had rounded each measurement *up* to the next metre.

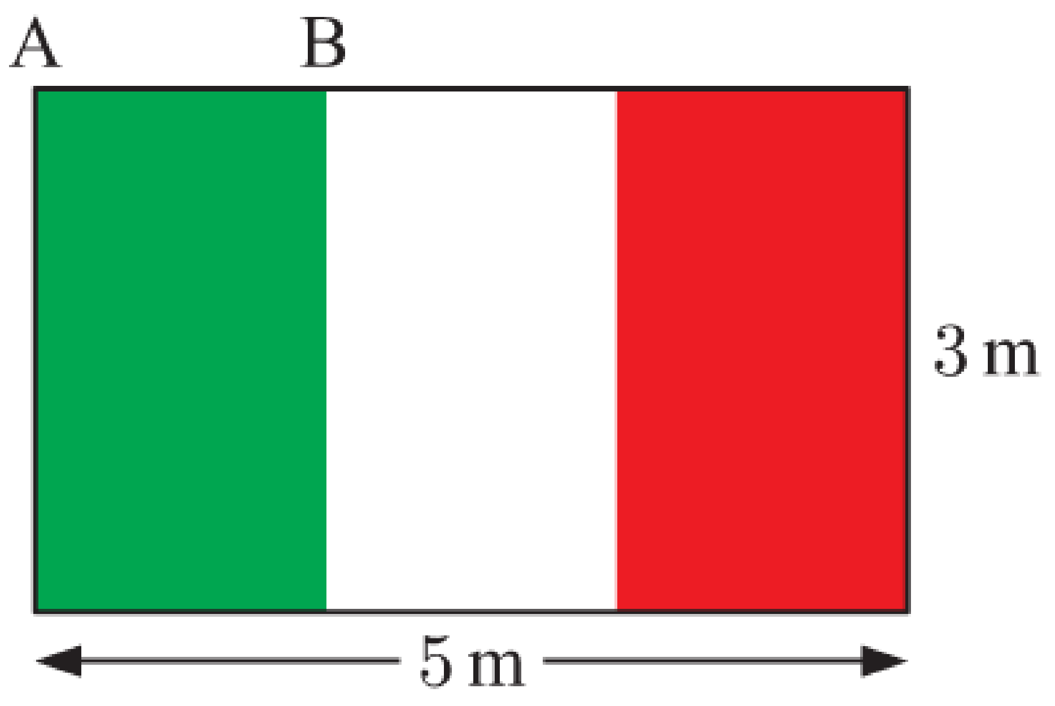
6 The Italian flag has three different regions of equal size. Consider the flag alongside.

a Find the area of the green section exactly.

b Find the length AB correct to 1 decimal place.

c Use your rounded value in **b** to estimate the area of the green section.

d Find the percentage error in your estimate.



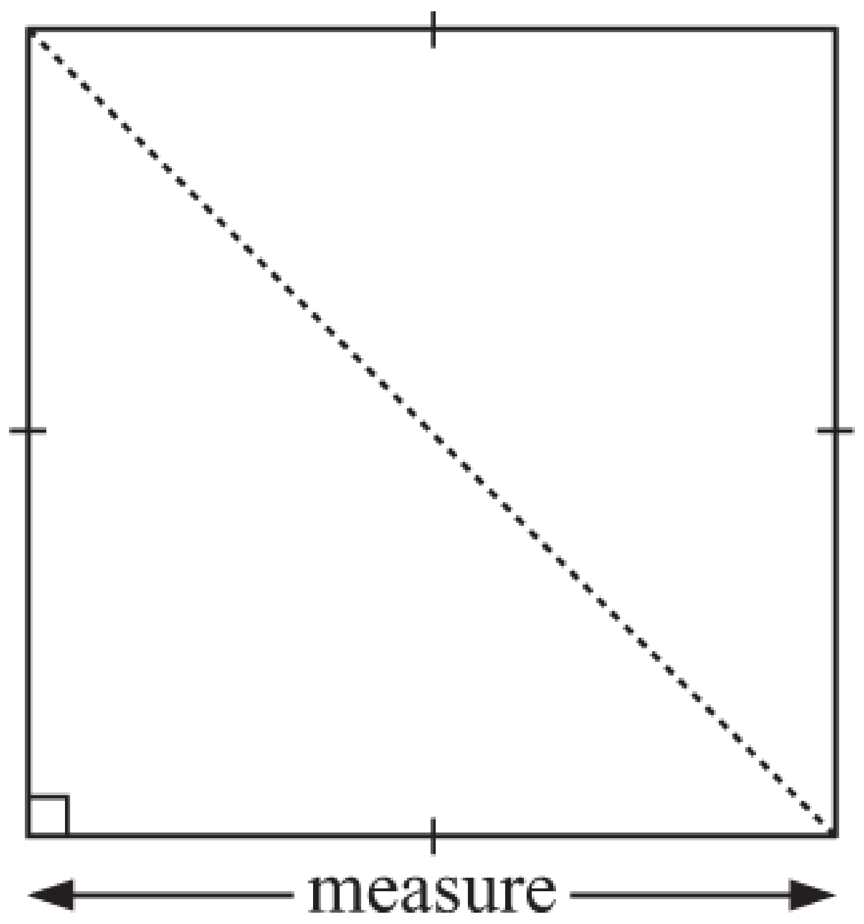
7 Hemi estimates that she can drive at an average speed of 70 km h^{-1} between her house and the beach, 87 km away. One particular journey took her 1 hour and 20 minutes.

a Calculate Hemi's average speed for this journey.

b Find the absolute error and percentage error in her estimate.

8 The sentence below is translated from an ancient Indian text, the *Śulba sūtra*:

> *The measure is to be increased by its third and this (third) again by its own fourth less the thirtyfourth part (of that fourth); this is (the value of) the diagonal of a square (whose side is the measure).*



a Use Pythagoras' theorem to show that the diagonal of a square is $\sqrt{2}$ times the measure of its side.

b Hence show that the text estimates the value of $\sqrt{2}$ as $\frac{577}{408}$.

c Find the percentage error for this estimate, giving your answer in scientific notation.

Example 10 **Self Tutor**

The side length of a square is measured as 22 cm, rounded to the nearest centimetre.

a Use this measurement to estimate the area of the square.

b Find the boundary values for the area of the square.

c Hence find the maximum percentage error in the estimate.

a Area $\approx 22 \text{ cm} \times 22 \text{ cm} \approx 484 \text{ cm}^2$

b The side length of the square could be from $21\frac{1}{2}$ cm to $22\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $21\frac{1}{2} \times 21\frac{1}{2} = 462.25 \text{ cm}^2$

and the upper boundary of the area is $22\frac{1}{2} \times 22\frac{1}{2} = 506.25 \text{ cm}^2$.

c If the exact area V_E was 462.25 cm^2, the

$$\begin{aligned}\text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|484 - 462.25|}{462.25} \times 100\% \\ &\approx 4.71\%\end{aligned}$$

If the exact area V_E was 506.25 cm^2, the

$$\begin{aligned}\text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|484 - 506.25|}{506.25} \times 100\% \\ &\approx 4.40\%\end{aligned}$$

$\therefore$ the maximum percentage error in the estimate $\approx 4.71\%$.

9 The side lengths of a rectangle are measured as 2.3 m and 1.4 m, rounded to one decimal place.

a Use these measurements to estimate the area of the rectangle.

b Find the boundary values for the area of the rectangle.

c Hence find the maximum percentage error in the estimate.

10 Jasper measured the dimensions of his cylindrical can of tuna. He found the radius was 4 cm and the height was 5 cm, rounded to the nearest centimetre.

a Use these measurements to estimate the volume of the can.

b Find the boundary values for the volume of the can.

c Hence find the maximum percentage error in the estimate.

11 Carolina completed a 250 km car trip (rounded to the nearest km). The GPS in her car displays the average speed for the trip as 56.8 km h^{-1} (rounded to 1 decimal place).

a Estimate the time it took Carolina to complete the trip.

b Find the maximum possible:

i absolute error

ii percentage error in the estimate.

DISCUSSION

Why is it important to understand errors?

What things can go wrong if people measure inaccurately or round off incorrectly?

You may wish to consider the examples in the previous Exercise, and also cases such as:

- If a pilot flies off-course by 0.1° for 1000 km, how far away from his target will he be?
- What happens to a patient if a doctor injects 2 mg of a drug instead of 2 μg?

REVIEW SET 1A

1 Round off to the nearest 100:

a 7423 **b** 32 191 **c** 10 543 **d** 408 961

2 The mathematical constant $e \approx 2.718\,281\,828\,459\ldots$. Round this value to:

a 2 decimal places **b** 5 decimal places **c** 8 decimal places.

3 Evaluate the following, rounding your answers to 3 significant figures:

a $\sqrt{27}$ **b** $\dfrac{2.3 \times 9.4}{1.3}$ **c** 0.307^3

4 Estimate using a one figure approximation:

a 47×7 **b** 89×16 **c** $267 \div 48$

5 **a** Estimate $5877 \div 32$ using a one figure approximation.

b Do you think the exact value of $5877 \div 32$ is greater or less than your estimate? Explain your answer.

6 A triangular garden is measured to have sides of length 8 m, 12 m, and 14 m, rounded to the nearest metre. Find the range of possible values for the perimeter P of the garden.

7 **a** How accurate is a tape measure marked in cm?

b Find the range of possible values for a measurement of 36 cm.

c If the sides of a square are measured to be 36 cm, in what range of values must the actual area A lie?

8 Find the absolute error and percentage error if you:

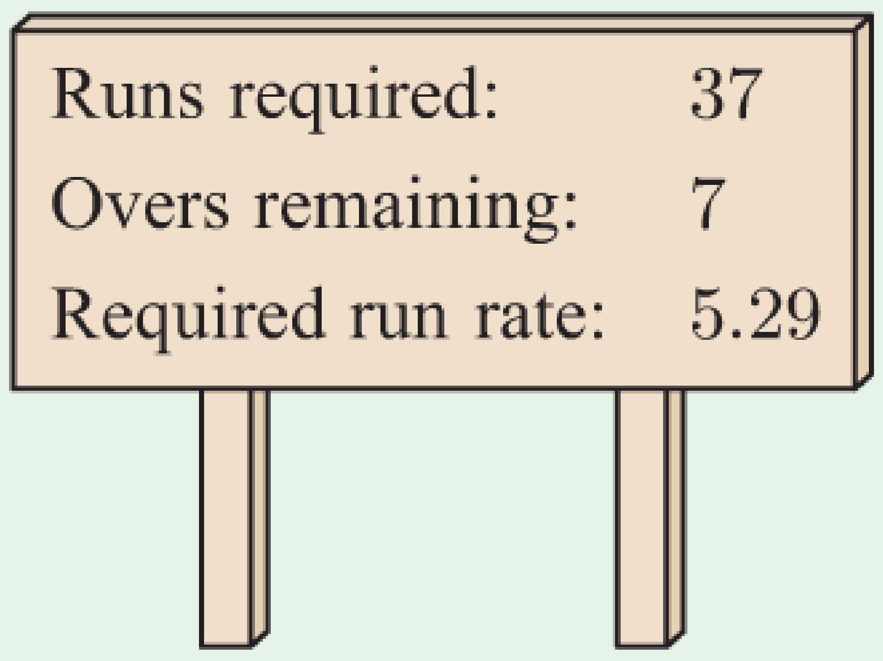
a estimate your credit card balance to be $2000 when it is $2590

b round 26.109 cm to 26 cm

c estimate the number of people at a music festival to be 4000 when there are 4386 in attendance.

9 In limited overs cricket, teams must score as many runs as possible within a particular number of overs. For the team batting second, the *required run rate* is found by dividing the number of *runs required* by the number of *overs remaining*.

On this scoreboard, the required run rate has been rounded to 2 decimal places. Find the percentage error in the calculation.



10 A circular gazebo has diameter 2.8 m. In calculating the area of the gazebo, the diameter is rounded to the nearest metre.

a Find the actual area of the gazebo.

b Find the calculated area of the gazebo.

c Find the absolute error and percentage error in the calculation.

REVIEW SET 1B

1 Round 74 815 to:

a the nearest 10 **b** the nearest 100 **c** the nearest 1000.

2 A self-service car wash received 248 customers in the last 13 days. Calculate the average number of customers per day, rounded to 1 decimal place.

3 Evaluate $\sqrt{5.4 \times 7.6}$, rounding your answer to:

a 2 decimal places **b** 4 significant figures.

4 Estimate using a one figure approximation:

a 28×74 **b** 5.84×8.09 **c** $57.9 \div 23.5$

5 Lucinda and Daniel bought 88 wedding invitations costing £4.90 each.

- **a** Find the exact total cost of the invitations.
- **b** Use a one figure approximation to estimate the total cost of the invitations.
- **c** Find the percentage error in your estimate.

6 Dafne competed in the 100 m sprint at her school sports carnival. Her time for the race, rounded to 1 decimal place, was recorded as 14.9 seconds.
Find the range of possible values for:

a her time t seconds **b** her average speed s $\text{m}\,\text{s}^{-1}$.

7 A box has dimensions 5 cm by 7 cm by 10 cm, rounded to the nearest centimetre. Find the boundary values for the surface area A of the box.

8 Edward measures the width and height of a television screen to the nearest 10 cm. The width is approximately 150 cm and the height is approximately 90 cm.

- **a** Use Edward's measurements to estimate the length of the diagonal of the screen.
- **b** Given that the diagonal has length 177.8 cm, find the absolute error and percentage error of the estimate.

9 An architect designs a support beam to be $\sqrt{5}$ metres long. The builder working from the architect's plans converts this length to a decimal number.

- **a** Write down the length of the support beam correct to the nearest:
  - **i** metre **ii** centimetre **iii** millimetre.
- **b** For each answer in **a**, write down how many significant figures were specified.
- **c** The architect insists that there be no more than 1% error. Which of the approximations in **a**, if any, will satisfy this?

10 In order to quickly estimate areas and volumes, people use various approximations for π.
Find the percentage error if π is approximated by:

a 3 **b** 3.1 **c** 3.14 **d** $\frac{22}{7}$ **e** $\frac{355}{113}$

11 The radius of a circle is measured as 3.5 cm, rounded to 1 decimal place.

- **a** Use this measurement to estimate the area of the circle.
- **b** Find the boundary values for the area of the circle.
- **c** Find the maximum percentage error in the estimate.

Chapter 2

Loans and annuities

Contents:
- **A** Loans
- **B** Annuities

OPENING PROBLEM

2018
2043
£
ANNUITY

Diane is about to retire at the age of 68. She has accumulated £600 000 in savings over her career, which she places in an *annuity* fund paying 4% p.a. interest compounded monthly. Her plan is to make monthly withdrawals from the account to live on for the next 25 years.

Things to think about:

- **a** What is meant by an annuity fund? How does it differ from the standard compound interest investments we have studied previously?
- **b** Diane reasons that for her money to last for 25 years or 300 months, she should be able to withdraw at least $\frac{£600\,000}{300} = £2000$ each month. Is Diane correct? Explain your answer.
- **c** If Diane's investment is to be completely used up by regular withdrawals over 25 years, how much can she withdraw each month?

Loans and **annuities** play an important role in modern life. Understanding these processes allows you to make informed financial decisions.

HISTORICAL NOTE

In 14th century Italy, merchants would set up stalls in local markets and lend money to customers. Interest was applied, and the borrower was expected to repay the money in regular intervals. The word "bank" is derived from the Italian word *banca*, which describes the benches on which the merchants sat.

If a banker suffered a series of bad loans and was unable to repay his own creditors, the creditors would break up his bench, showing that the banker was no longer in business. This was known as *banca rotta*, meaning "broken bench". Over time, the Italian word for "broken" was replaced with the Latin word *rupta*, giving *banca rupta*, the basis of the modern term "bankrupt".

A LOANS

A common way to borrow money to finance larger purchases such as houses, cars, renovations, education expenses, and share portfolios, is to take out a **personal loan**. The financial institution lends an amount of money to the borrower, who repays the amount plus interest by a series of regular repayments over a given time period.

The process of repaying a loan with a series of regular repayments is called **amortisation**.

Interest is calculated on the **reducing balance** of the loan, so the interest gradually reduces as the loan is repaid.

Because of the interest, the total repayment will be greater than the amount originally borrowed. The difference between these values is the interest paid.

interest paid = total repayments − amount borrowed

CALCULATING REPAYMENTS

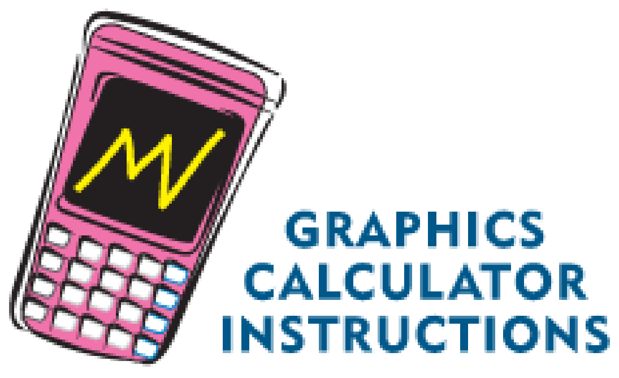
We can use a **table of repayments** to work out the repayments required for a loan. The table below shows the amount you need to repay each month for every $1000 borrowed.

Table of monthly repayments per $1000 borrowed

Loan term (years)	*Annual interest rate* 4%	4.5%	5%	5.5%	6%	6.5%	7%
1	85.1499	85.3785	85.6075	85.8368	86.0664	86.2964	86.5267
1.5	57.3314	57.5557	57.7805	58.0059	58.2317	58.4581	58.6850
2	43.4249	43.6478	43.8714	44.0957	44.3206	44.5463	44.7726
3	29.5240	29.7469	29.9709	30.1959	30.4219	30.6490	30.8771
4	22.5791	22.8035	23.0293	23.2565	23.4850	23.7150	23.9462
5	18.4165	18.6430	18.8712	19.1012	19.3328	19.5661	19.8012
10	10.1245	10.3638	10.6066	10.8526	11.1021	11.3548	11.6108
15	7.39688	7.64993	7.90794	8.17083	8.43857	8.71107	8.98828
20	6.05980	6.32649	6.59956	6.87887	7.16431	7.45573	7.75299

We can also calculate the monthly repayments using the TVM solver on a calculator.

In this case, you are receiving the amount loaned, so PV is positive. PMT is the amount to be repaid each time period, so it is negative.



Example 1 **Self Tutor**

Erica takes out a personal loan of $16 500 to buy a car. She negotiates a term of 4 years at 5.5% p.a. interest.

a Calculate the monthly repayments using the table. Check your answer using technology.

b Hence calculate: **i** the total repayment **ii** the total interest charged.

a From the table, the monthly repayments on each $1000 for 4 years at 5.5% p.a. = $23.2565

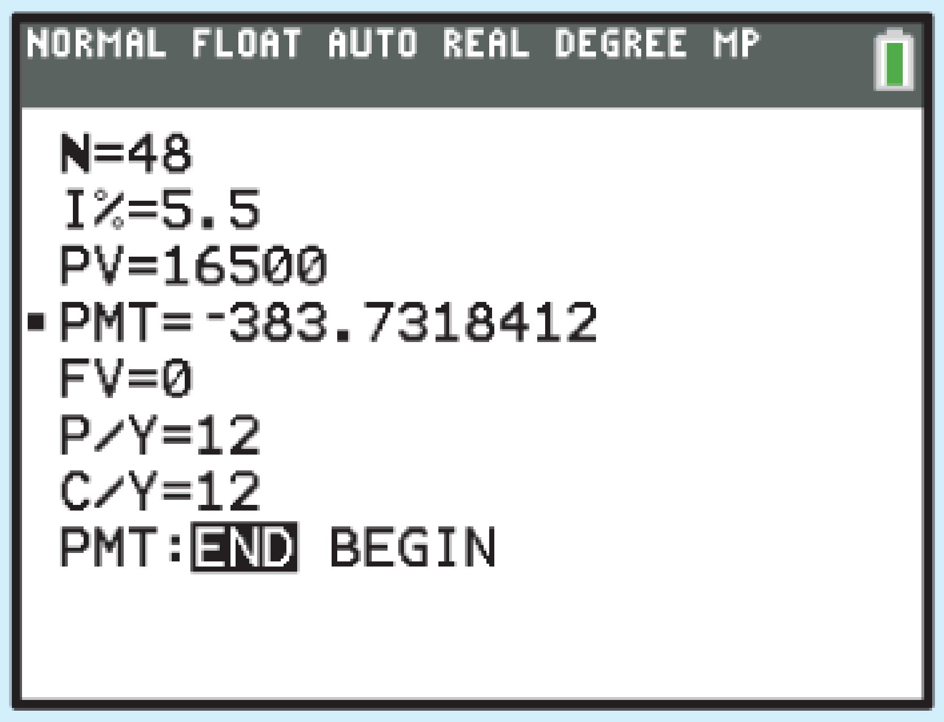
$\therefore$ repayments on $16 500

= $23.2565 × 16.5 {16.5 lots of $1000}

= $383.732

The repayments are $383.74 per month.

TI-84 Plus CE



```
N=48
I%=5.5
PV=16500
▪PMT=-383.7318412
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

The monthly repayment
is always rounded up.

b **i** Total repayment

= monthly repayment × number of months

= $383.74 × 48

= $18 419.52

ii Interest

= total repayment − amount borrowed

= $18 419.52 − $16 500

= $1919.52

In the following Exercise, calculate monthly repayments using your calculator. Where appropriate, check your answers using the table.

EXERCISE 2A

1 Don takes out a personal loan for \$12 000 to fund his wedding. He will repay it over 5 years at 6% p.a. Calculate the:

a monthly repayments **b** total repayment **c** interest charged.

2 Jay and Penni need £9500 to fund house renovations. They take out a personal loan over 3 years at 4.5% p.a. Calculate the:

a monthly repayments **b** total repayment **c** interest charged.

3 Binh-vu needs \$15 000 to buy a boat. His bank offers him a personal loan at 6.5% p.a. Calculate the total interest he will pay if he repays the loan with monthly repayments over:

a 2 years **b** 5 years.

4 Mimi takes out a personal loan of €10 000. She will repay the loan over 4 years at 6% p.a. She is charged a €150 application fee and a €10 monthly service fee.
Calculate the total cost of taking out the loan.

5 Becky wants to borrow \$25 000 to purchase some shares. She has the following options:

Balance Bank: 4.5% p.a. over 5 years
Cash Credit Union: 5% p.a. over 3 years.

a Which loan has the smaller monthly repayments?

b Which loan charges less interest in total?

c Which loan would you recommend for Becky? Explain your answer.

6 The spreadsheet alongside shows the progress of a \$30 000 loan taken out over 5 years at 8% p.a. interest compounded monthly.

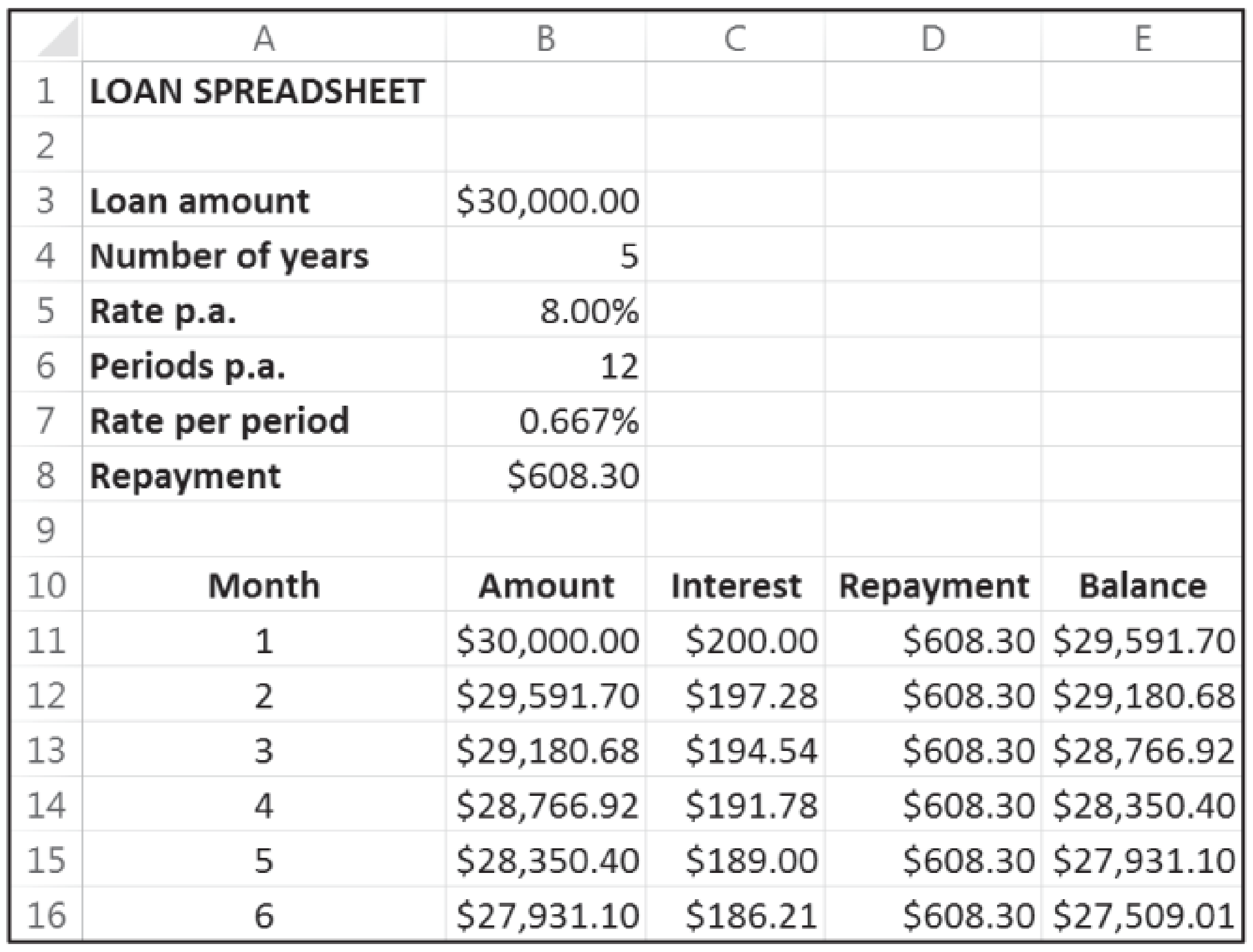
a Check that the monthly repayment has been correctly calculated.

b What is the account balance after six months?

c How much interest is paid in:

i month 1 **ii** month 6?

Explain the differences between these amounts.



	A	B	C	D	E
1	**LOAN SPREADSHEET**				
2					
3	**Loan amount**	\$30,000.00			
4	**Number of years**	5			
5	**Rate p.a.**	8.00%			
6	**Periods p.a.**	12			
7	**Rate per period**	0.667%			
8	**Repayment**	\$608.30			
9					
10	**Month**	**Amount**	**Interest**	**Repayment**	**Balance**
11	1	\$30,000.00	\$200.00	\$608.30	\$29,591.70
12	2	\$29,591.70	\$197.28	\$608.30	\$29,180.68
13	3	\$29,180.68	\$194.54	\$608.30	\$28,766.92
14	4	\$28,766.92	\$191.78	\$608.30	\$28,350.40
15	5	\$28,350.40	\$189.00	\$608.30	\$27,931.10
16	6	\$27,931.10	\$186.21	\$608.30	\$27,509.01

d Click on the spreadsheet icon.
Fill row 16 down to the end of the 60th month.

i How much interest is paid in the 60th month?

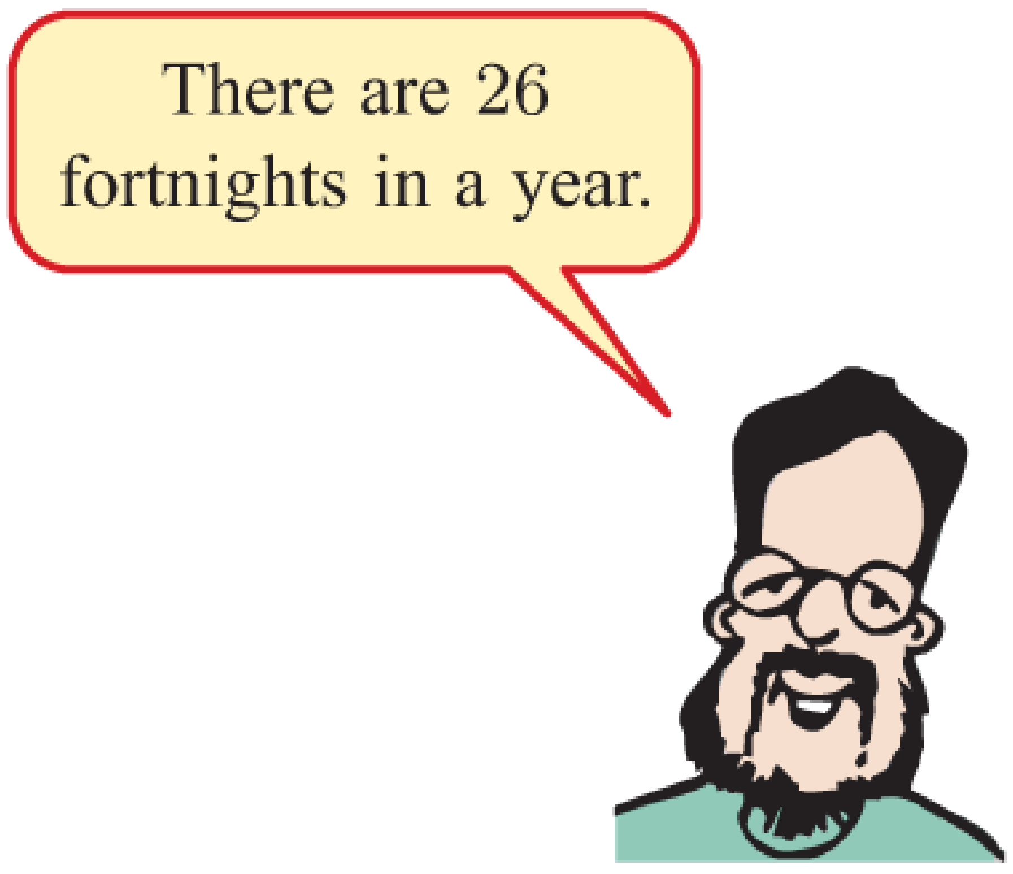
ii How much interest is paid in total?

SPREADSHEET

e Explain why the final payment is slightly less.

7 Grace takes out a loan of \$7000 to buy a jet ski. She will repay the loan over 2 years at 9.9% p.a. interest compounded fortnightly.

- **a** Calculate Grace's fortnightly repayments.
- **b** Find the total amount of interest charged on the loan.


There are 26 fortnights in a year.

Example 2 **Self Tutor**

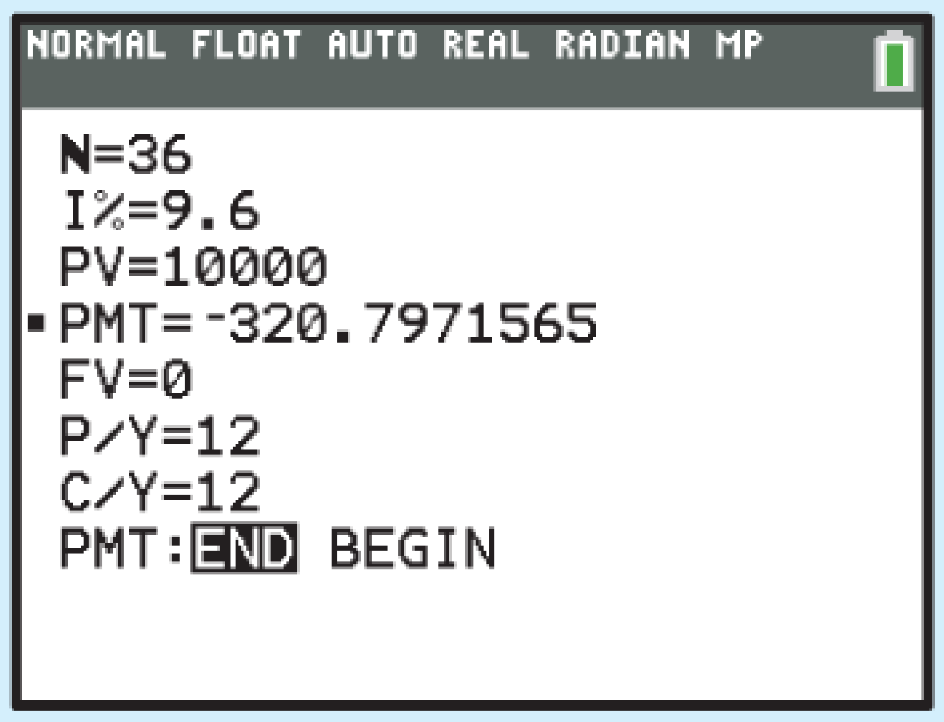
Ryan takes out a \$10 000 loan to be repaid over three years at 9.6% p.a. interest compounded monthly.

- **a** Calculate the monthly repayments.
- **b** Find the outstanding balance on the loan after 1 year of repayments.

a $N = 3 \times 12 = 36$, $I\% = 9.6$, $PV = 10\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

$\therefore \quad PMT \approx -320.80$

The monthly repayment is \$320.80 .

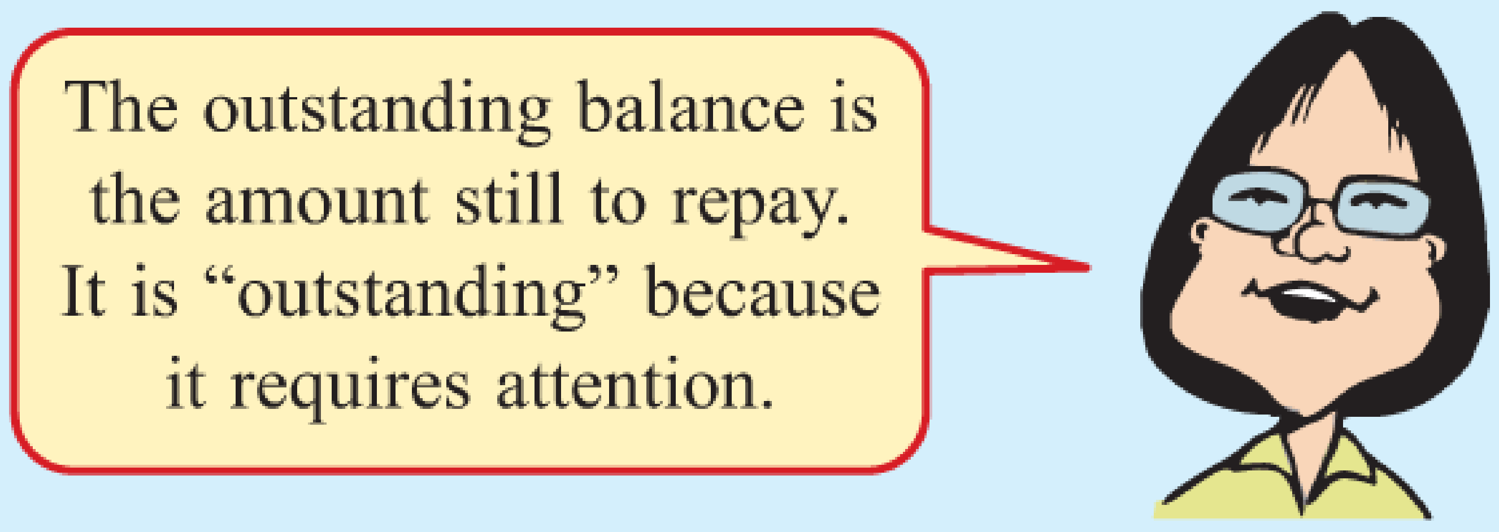


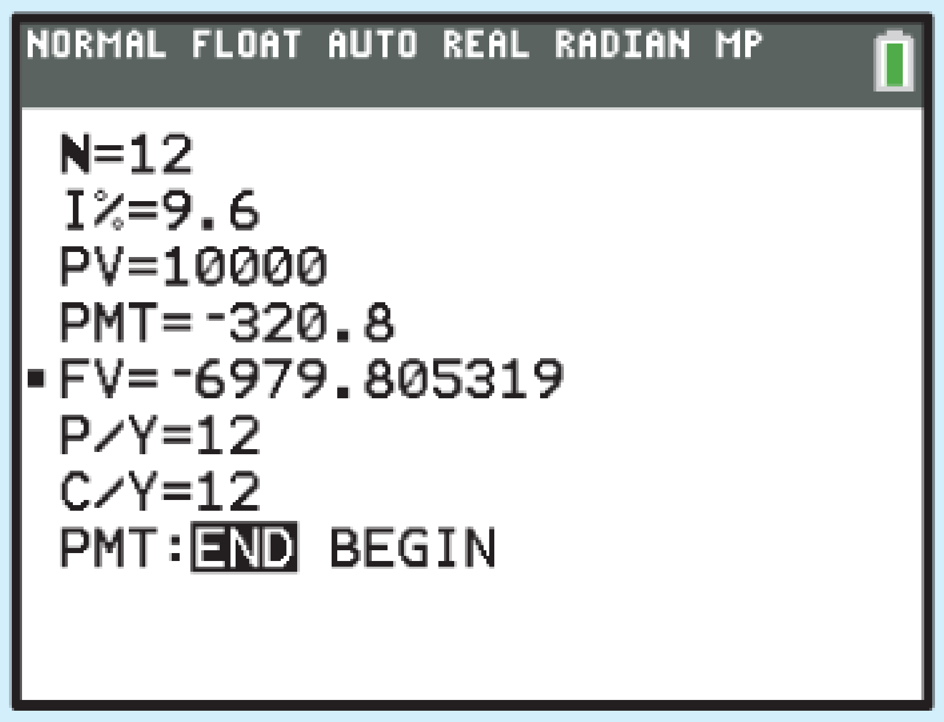
```
NORMAL FLOAT AUTO REAL RADIAN MP
N=36
I%=9.6
PV=10000
▪PMT=-320.7971565
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

b $N = 1 \times 12 = 12$, $I\% = 9.6$, $PV = 10\,000$, $PMT = -320.80$, $P/Y = 12$, $C/Y = 12$

$\therefore \quad FV \approx -6979.81$

The outstanding balance on the loan after 1 year is \$6979.81 .


The outstanding balance is the amount still to repay. It is "outstanding" because it requires attention.



```
NORMAL FLOAT AUTO REAL RADIAN MP
N=12
I%=9.6
PV=10000
PMT=-320.8
▪FV=-6979.805319
P/Y=12
C/Y=12
PMT:END BEGIN
```

8 Pieter takes out a loan of £20 000 to renovate his home. He agrees to repay the loan over 4 years at 8.25% p.a. interest compounded monthly. Find:

- **a** Pieter's monthly repayments
- **b** the outstanding balance on the loan after 1 year of repayments.

9 Jacob wants to buy a piano valued at \$12 000. He has \$3500 in savings, and will borrow the remaining money.

- **a** How much will Jacob borrow?
- **b** Jacob will repay the loan over 5 years at 7.9% p.a. interest compounded quarterly.
  - **i** Calculate Jacob's quarterly repayments.
  - **ii** Calculate the total interest charged on the loan.
  - **iii** Find the outstanding balance on the loan after 2 years.

10 Simon took out a loan of \$7000 to pay for a holiday. He will repay the loan over 4 years with monthly repayments of \$165.36 .

- **a** What annual interest rate, compounded monthly, is being charged?
- **b** Calculate the total interest Simon will pay in the first year.

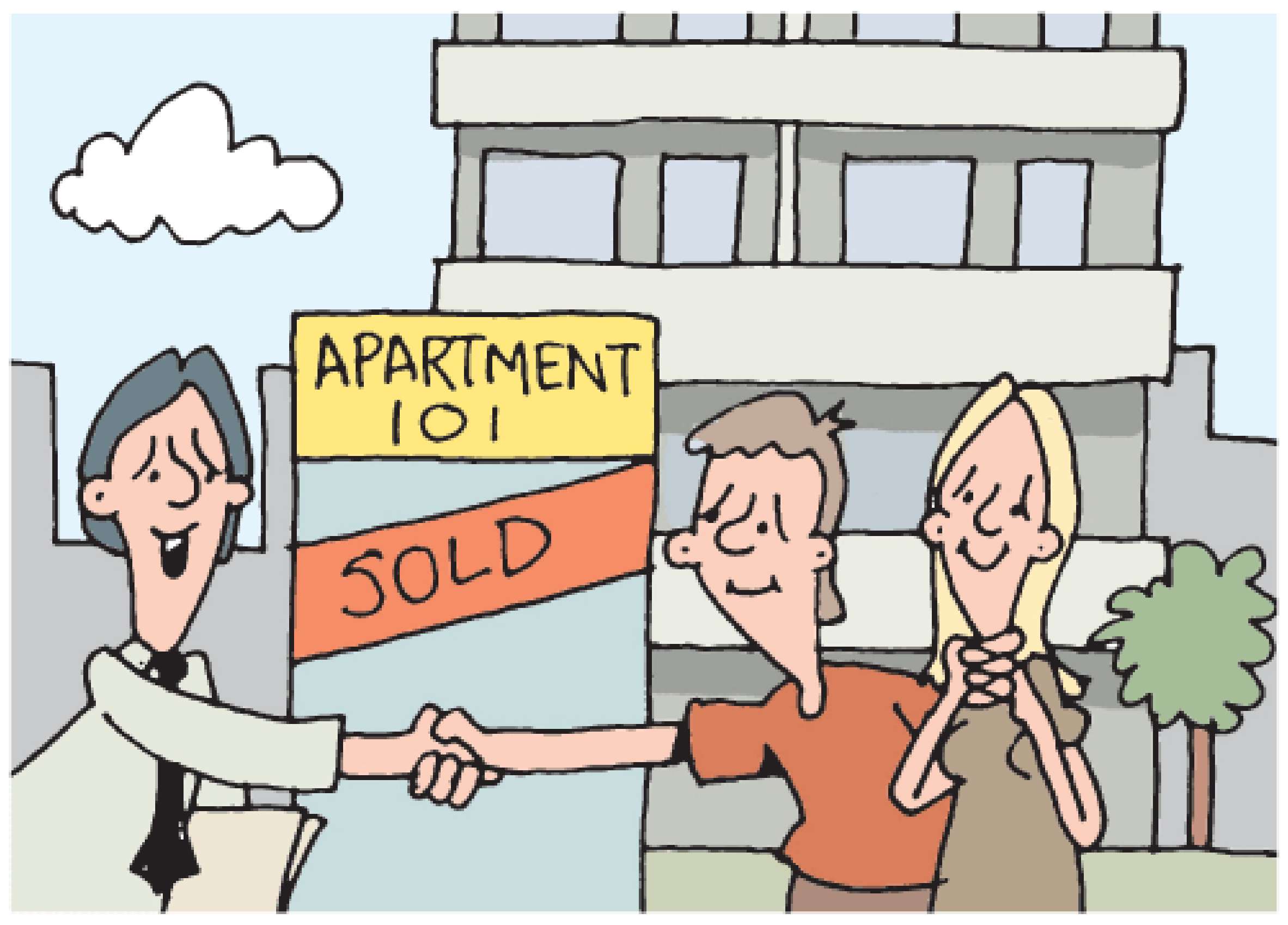
11 Consider a loan of \$25 000 at 8.5% p.a. interest compounded monthly.

- **a** Calculate the monthly loan repayments if the loan is taken out for:
  - **i** 3 years
  - **ii** 5 years
  - **iii** 7 years.
- **b** Which loan charges the least interest in total? Explain your answer.

12 Ally takes out a loan to buy a car for €18 000 over 5 years at 10.5% p.a. interest compounded monthly.

- **a** Find the monthly repayments.
- **b** Calculate the total interest Ally will pay on the loan.
- **c** Find the outstanding loan balance after $2\frac{1}{2}$ years of repayments.
- **d** Explain why Ally still has more than half the loan to pay off when half the loan period has passed.

13 Shane and Julie have taken out a \$250 000 loan to purchase an apartment. They will repay the loan over 20 years at 6.25% p.a. fixed interest compounded monthly.



- **a** Calculate the monthly repayments.
- **b** Calculate the total interest charged.
- **c** Find the outstanding balance on the loan after 10 years.
- **d** After 10 years, the couple can afford to increase their monthly repayments. They aim to completely repay the loan over the next 5 years.
  - **i** Calculate their monthly repayments for the next 5 years.
  - **ii** Calculate the total interest that would be charged over the 15 years.
  - **iii** How much interest will the couple save by repaying their loan sooner?

ACTIVITY 1

Interest rates have changed a great deal over time. They have a big impact on what homes a family can afford.

MONTHLY REPAYMENTS

PRINTABLE TABLE

What to do:

1. Suppose you have saved \$50 000 as deposit for a new house. The house is valued at \$450 000 (including all fees), so you will need a loan for \$400 000.

   Copy and complete the tables following with the monthly repayments needed, and the total interest paid, for different interest rates and loan durations. You will need to click on the "Monthly repayments" icon and use the table of monthly repayments given. Consult the websites of some banks to find the current interest rates.

Monthly repayments

	4%	6%	8%	10%	12%	*Current interest rates*
5 years						
10 years						
15 years						
20 years						

Total interest paid

	4%	6%	8%	10%	12%	*Current interest rates*
5 years						
10 years						
15 years						
20 years						

2 Discuss the impact of higher interest rates on the total interest that needs to be paid over the course of a loan. Include a discussion of the benefit of paying off a loan over a shorter time period, and what effect interest rates have on a family's ability to do this.

ACTIVITY 2 THE LOAN REPAYMENTS FORMULA

In the previous Exercise we used technology and a table of values to find the necessary repayments for a loan. In this Activity we explore the calculations leading to these values. We will use our knowledge of geometric sequences and series.

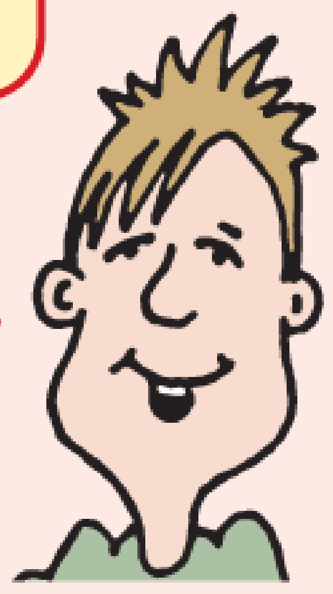
Consider a loan for which PV is the amount originally borrowed
n is the number of repayment periods
i is the interest rate *per period*, and
p is the repayment per period.

After each time period, interest is charged on the loan by multiplying the balance by $(1+i)$. The repayment p is then subtracted from the balance.

So, the balance after:

- 0 repayment periods $= PV$
- 1 repayment period $= PV(1+i) - p$
- 2 repayment periods $= [PV(1+i) - p](1+i) - p$
  $= PV(1+i)^2 - p(1+i) - p$

$PV(1+i)^n$ is the compound interest formula for the balance if no repayments were made.



What to do:

1 Show that the balance of the loan after:

  a 3 repayment periods $= PV(1+i)^3 - p(1+i)^2 - p(1+i) - p$

  b 4 repayment periods $= PV(1+i)^4 - p(1+i)^3 - p(1+i)^2 - p(1+i) - p$

2 Following the pattern in the formulae, the balance of the loan after n repayment periods will be

$$\begin{aligned} & PV(1+i)^n - p(1+i)^{n-1} - p(1+i)^{n-2} - \ldots. - p(1+i) - p \\ &= PV(1+i)^n - p[(1+i)^{n-1} + (1+i)^{n-2} + \ldots. + (1+i) + 1] \\ &= \underbrace{PV(1+i)^n}_{\substack{\text{compound interest formula} \\ \text{for balance if no repayments} \\ \text{were made at all.}}} - \underbrace{\sum_{k=0}^{n-1} p(1+i)^k}_{\substack{\text{reduction due to regular} \\ \text{repayments}}} \\ &= PV(1+i)^n - \frac{p[(1+i)^n - 1]}{(1+i)-1} \qquad \{\text{sum of a geometric series formula}\} \\ &= PV(1+i)^n - p\,\frac{(1+i)^n - 1}{i} \end{aligned}$$

After n repayments, the loan must be completely repaid, so $PV(1+i)^n - p\,\dfrac{(1+i)^n - 1}{i} = 0$.

Rearrange this equation to show that the regular repayment must be $p = \dfrac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$.

3 By setting $PV = 16\,500$, $i = \dfrac{0.055}{12}$, and $n = 4 \times 12 = 48$, verify the result in **Example 1**.

4 Use the formula $p = \dfrac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$ to check the repayments you calculated in **Exercise 2A**.

5 Jade takes a loan of \$20 000 to be repaid over 5 years at 5% p.a. interest compounded monthly. Use the formula to find her monthly repayments. Check your answer using technology.

6 Apply the formula in a spreadsheet to generate the table of values for loan repayments shown on page **33**.

7 Are there any other advantages to knowing a formula to find an answer, rather than relying on a calculator's built-in solver?

DISCUSSION

Do you think it is better to *borrow* money to purchase an item you want, or to *save* your money until you can afford to purchase the item?

In your discussion, consider:

- the costs associated with borrowing
- the benefits associated with having the item sooner rather than later
- the effect of inflation on the cost of the item.

B ANNUITIES

An **annuity** is an investment where an individual makes a lump-sum deposit, and then makes regular withdrawals over a fixed time period. The investment earns interest according to the balance of the annuity each time period.

In effect, an annuity is the reverse of a personal loan. The person loans the bank their money, and the bank makes regular repayments over time. As the balance in the annuity fund decreases, so does the interest it earns. However, the regular repayments remain the same.

Annuities are most commonly used as a source of regular income for people who have retired from work. Money saved during their working life is "rolled over" into an annuity fund. The regular withdrawals provide money for the person to live on.

DISCUSSION

In some countries, all workers receive **superannuation** payments as part of their salaries. They do not have access to this money until they reach retirement age, so it is a form of compulsory saving.

Why might a government make superannuation compulsory?

When performing calculations with annuities:

- The person is depositing a lump-sum with the bank, so PV is negative.
- The person receives regular repayments, so PMT is positive.

Example 3 **Self Tutor**

Heather has just retired at age 65. She has \$900 000 in her savings fund. She "rolls over" the money into an annuity fund which returns 4% p.a. compounded monthly.

a If Heather withdraws \$5400 per month to live on, how long will it take for the money in the fund to run out?

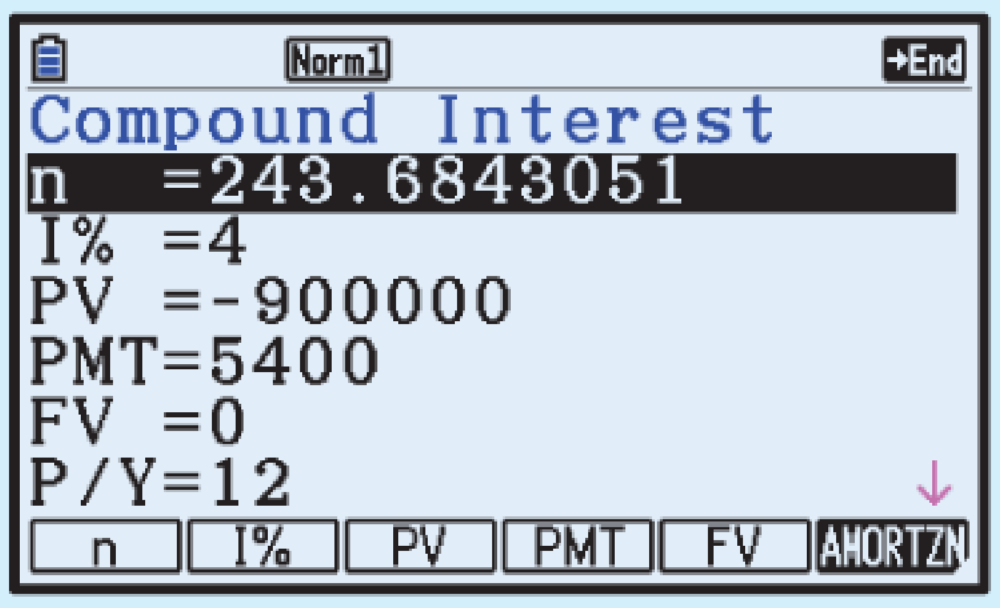
b If Heather wants the money to last until she is 90 years old, how much can she afford to withdraw each month?

a We need to find how long it will take for the future value to fall to \$0.

$I\% = 4,\ \ PV = -900\,000,\ \ PMT = 5400,\ \ FV = 0,$
$P/Y = 12,\ \ C/Y = 12$

$\therefore\ \ N \approx 243.7$

It will take 244 months (or 20 years 4 months) for the money in the fund to run out.



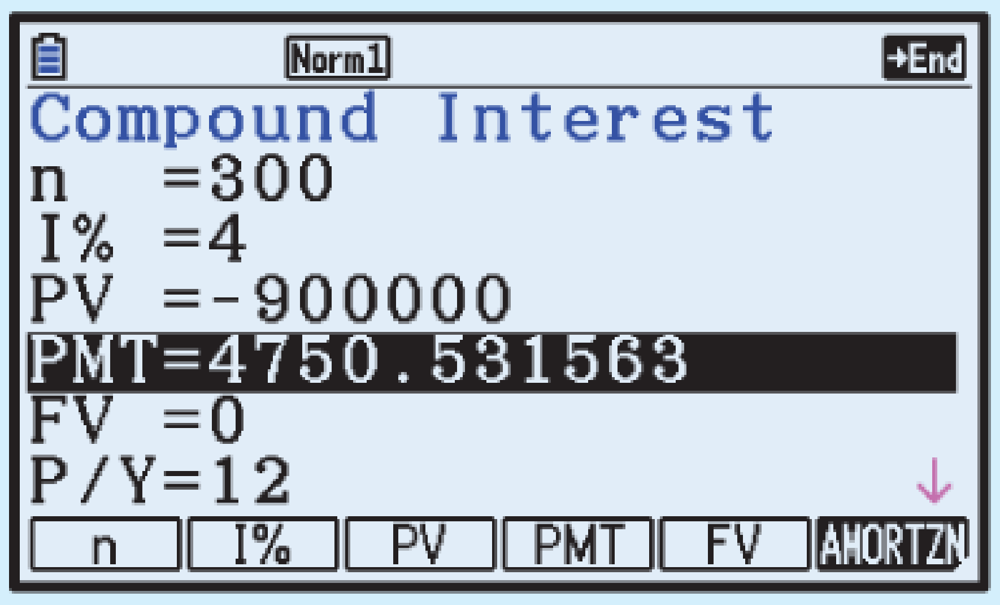
```
Compound Interest
n   =243.6843051
I%  =4
PV  =-900000
PMT=5400
FV  =0
P/Y=12
```

b Heather wants the money to last $90 - 65 = 25$ years.

$N = 25 \times 12 = 300,\ \ I\% = 4,\ \ PV = -900\,000,\ \ FV = 0,$
$P/Y = 12,\ \ C/Y = 12$

$\therefore\ \ PMT \approx 4750.53$

Heather can afford to withdraw \$4750.53 each month.



```
Compound Interest
n   =300
I%  =4
PV  =-900000
PMT=4750.531563
FV  =0
P/Y=12
```

EXERCISE 2B

1 Sue has achieved her target of \$700 000 in her savings fund for her retirement at age 55. She rolls the money into an annuity account earning 4.85% p.a. compounded monthly.

  a How long will Sue's money last if she withdraws \$4000 per month?

  b If Sue wants her money to last for 30 years, how much can she afford to withdraw per month?

2 Célia deposits €500 000 in an annuity fund which earns 4.5% p.a. interest compounded monthly. Célia wants the money to last for 20 years.

- **a** How much can Célia afford to withdraw each month?
- **b** Find the outstanding balance of her fund after 5 years.

3 Terrence has $800 000 in his savings fund. He rolls his money into an annuity fund which earns 4.7% p.a. compounded monthly. He wants to withdraw $7000 each month to live on.

- **a** How long will his money last?
- **b** How much *longer* would his money last if he only withdrew $6000 each month?

4 Henry has retired at age 68 with £830 000 in his savings fund. He rolls the money into an annuity fund that earns 4% p.a. compounded monthly.

- **a** If Henry wants the money to last until he is 85, how much can he afford to withdraw each month?
- **b** Henry currently lives on £6000 per month. Will Henry be able to maintain his current standard of living? Explain your answer.

5 Tamsyn invests $750 000 in a savings account which pays 3.8% p.a. interest compounded quarterly.

- **a** How much money will be in the account when she retires in 10 years' time?
- **b** When she retires, Tamsyn rolls the money into an annuity fund which pays 4.9% p.a. interest compounded monthly. Tamsyn would like the money to last for 15 years.
  - **i** How much can Tamsyn withdraw each month?
  - **ii** How long will it take for the balance of the fund to fall to $300 000?

6 After retiring at age 55, Danny rolls his £600 000 into an annuity account earning 5.9% p.a. compounded quarterly.

- **a** If he wants his money to last for 25 years, how much can he withdraw each quarter?
- **b** How much money will be left in Danny's annuity account when he is 68?
- **c** How much *more* could Danny withdraw each quarter if his money only needed to last 20 years?

7 When Maggie retires at 70, she will deposit her savings in an annuity account which pays 6.2% p.a. interest compounded monthly. She wants to withdraw $4500 per month from the annuity account until she is 90.

- **a** Show that Maggie will withdraw a total of $1 080 000 from her annuity account.
- **b** Explain why Maggie does not need $1 080 000 in savings at the time when she retires.
- **c** Calculate the amount Maggie will need in savings when she retires.

8 Abby has carefully saved $1 000 000 during her career. She rolls it into an annuity fund which pays 5% p.a. interest compounded monthly. If Abby withdraws $4000 each month, how long will Abby's money last?

Hint: Find the balance of the fund after 1 year.

9 Igor has saved €550 000. He wants to put the money in an annuity fund where he can withdraw €5000 per month for the next 15 years.

- **a** What interest rate, compounded monthly, will Igor require?
- **b** The best interest rate Igor can find is 4.9% p.a. compounded monthly.
  - **i** If he still wants to withdraw €5000 per month, how much *less* time will the money last?
  - **ii** If he still wants the money to last 15 years, how much *less* must his withdrawals be each month?

10 Luke rolls his $700 000 of savings into an annuity fund which earns 4.5% p.a. interest compounded monthly. He wants the money to last for 16 years.

- **a** How much money can Luke withdraw each month?
- **b** Find the outstanding balance of the fund after 10 years.
- **c** After 10 years, Luke receives an inheritance of $100 000 which he adds to the annuity fund. How much is Luke now able to withdraw each month for the remaining 6 years?

ACTIVITY 3 PERPETUITIES

A **perpetuity** is a special type of annuity in which the regular payments continue indefinitely. This is achieved by ensuring the interest earned by the investment matches the payments taken out.

The **price** we would expect to pay for a perpetuity is therefore the principal required to give interest equal to the regular repayment.

So, the price or **present value** of a perpetuity is given by

$PV = \dfrac{PMT}{r}$ where PMT is the payment received each year and
r is the annual interest rate.

For example, for a perpetuity which pays $2000 per year indefinitely, with an interest rate of 4% p.a., you would expect to pay $PV = \dfrac{PMT}{r} = \dfrac{\$2000}{0.04} = \$50\,000$.

What to do:

1 Find the price you would expect to pay for a perpetuity of:

- **a** $300 per year with an interest rate of 5% p.a.
- **b** $4000 per year with an interest rate of 3.2% p.a.
- **c** $18 000 per year with an interest rate of 4.5% p.a.

2 Find the annual payment you would expect to receive from a perpetuity which has:

- **a** present value €1000 and interest rate 3.5% p.a.
- **b** present value €25 000 and interest rate 4.8% p.a.

3
- **a** Suppose you had $500 000 in savings, and used it to purchase a perpetuity with interest rate 4% p.a. What payment would you expect to receive each year?
- **b** Suppose $500 000 is deposited in a pension fund which pays 4% p.a. interest compounded annually. How much can be withdrawn from the fund each year so that the money lasts for:

  i 10 years **ii** 20 years **iii** 50 years **iv** 200 years **v** 1000 years?
- **c** Comment on your answers to **a** and **b**.

ACTIVITY 4 GROWING ANNUITIES

In the previous Exercise, we have assumed that the amount withdrawn each time period does not change over time.

This is unrealistic for several reasons:

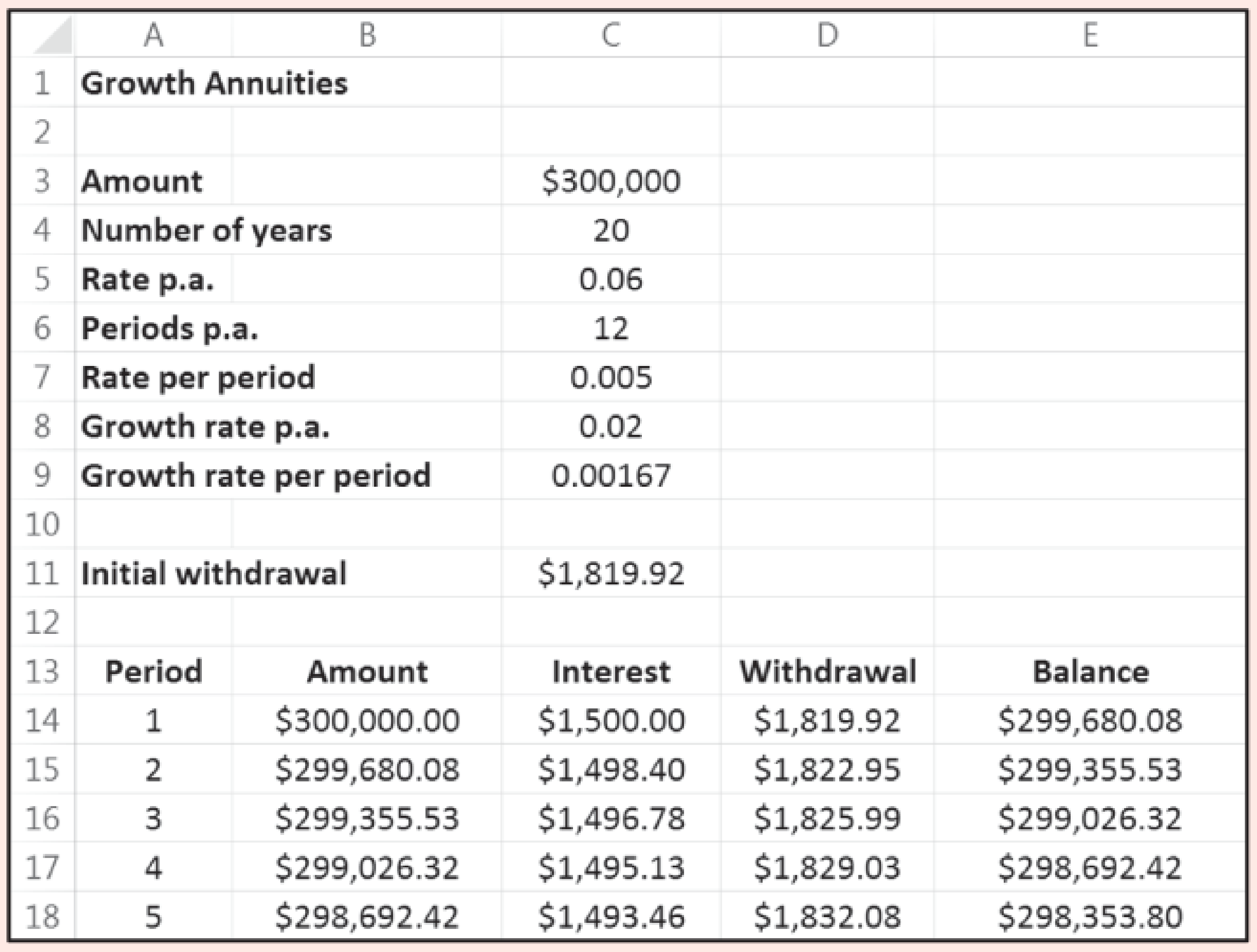
- The prices of goods and services increase over time with inflation, so the cost of living at the same standard increases.
- As we age, we inevitably need more health care and domestic services, so these expenses increase.

In a **growing annuity**, the amount withdrawn increases by a certain percentage, called the **growth rate**, each year.

What to do:

1 **a** Click on the icon to open a spreadsheet for a growing annuity fund. $300 000 is deposited into the fund which earns 6% p.a. interest compounded monthly for 20 years. We assume a growth rate of 2% p.a.
The initial withdrawal is calculated as $1819.92.

SPREADSHEET



	A	B	C	D	E
1	**Growth Annuities**				
2					
3	**Amount**		$300,000		
4	**Number of years**		20		
5	**Rate p.a.**		0.06		
6	**Periods p.a.**		12		
7	**Rate per period**		0.005		
8	**Growth rate p.a.**		0.02		
9	**Growth rate per period**		0.00167		
10					
11	**Initial withdrawal**		$1,819.92		
12					
13	**Period**	**Amount**	**Interest**	**Withdrawal**	**Balance**
14	1	$300,000.00	$1,500.00	$1,819.92	$299,680.08
15	2	$299,680.08	$1,498.40	$1,822.95	$299,355.53
16	3	$299,355.53	$1,496.78	$1,825.99	$299,026.32
17	4	$299,026.32	$1,495.13	$1,829.03	$298,692.42
18	5	$298,692.42	$1,493.46	$1,832.08	$298,353.80

b Discuss what is happening to the withdrawals over time.

c Find the amount withdrawn for the final time period.

d Use your calculator to find the regular withdrawal you would make for this investment. Compare this value with the initial and final withdrawals in the growing annuity.

e How could you calculate the regular withdrawal found in **d** by adjusting the spreadsheet?

2 Suppose $500 000 is deposited into a growing annuity fund which earns 4% p.a. interest compounded monthly for 25 years. Assume a growth rate of 3% p.a.

a Use the spreadsheet to find:

i the initial monthly withdrawal **ii** the total amount withdrawn over 25 years.

b Hence find the total interest earned by the annuity fund. Check your answer by finding the sum of the interest payments in the spreadsheet.

3 Predict the effect on the initial withdrawal of:

a increasing the amount originally deposited

b increasing the duration of the annuity

c increasing the interest rate

d increasing the growth rate.

Use the spreadsheet to check your answers.

4 For a growing annuity over n periods with initial deposit PV, interest rate i per period, and growth rate g per period, the initial withdrawal w is given by $w = \dfrac{PV \times (i-g) \times (1+i)^n}{(1+i)^n - (1+g)^n}$.

- **a** Verify that this formula gives the correct initial withdrawals for the scenarios in **1** and **2**.
- **b** What happens to this formula when $g = 0$? Compare your answer with the formula found in **Activity 2** on page **38**.

REVIEW SET 2A

1 Alberto takes out a personal loan for $23 000 at 7% p.a. over 5 years. Calculate:

- **a** his monthly repayments
- **b** the total of the repayments
- **c** the total interest charged.

2 Alexandra takes out a loan of €2000 to pay for an emergency vet bill. She will repay the loan over 6 months at 8.12% p.a. interest compounded fortnightly. Calculate:

- **a** Alexandra's fortnightly repayments
- **b** the outstanding balance on the loan after 6 fortnights.

3 Simone borrows $410 000 to purchase a house. She agrees to repay the loan over 25 years at 6.95% p.a. interest compounded monthly.

- **a** Find Simone's minimum monthly repayments.
- **b** Show that the total interest Simone will pay is greater than the amount she originally borrowed.

4 Yasmin retires at age 60 with $500 000 in her savings fund. She rolls the money into an annuity fund earning 5.25% p.a. interest compounded monthly.

- **a** How long will Yasmin's money last if she withdraws $6000 per month?
- **b** If Yasmin wants her money to last for 25 years, how much can she afford to withdraw per month?

5 After retiring at age 65, Vasili rolls €1 400 000 of savings into an annuity account earning 5.4% p.a. interest compounded monthly. He wants his money to last for 30 more years.

- **a** How much can Vasili withdraw per month?
- **b** How long will it take for the balance of the fund to fall below €1 000 000?
- **c** How much of Vasili's annuity will be left after 20 years?

6 When Scott retires at 68, he will deposit his savings in an annuity fund which pays 5.4% p.a. interest compounded monthly. He wants to be able to withdraw $6000 per month from the fund until he is 85.

- **a** Calculate the amount Scott will need to have in savings when he retires.
- **b** How much money will be left in Scott's account when he is 80?

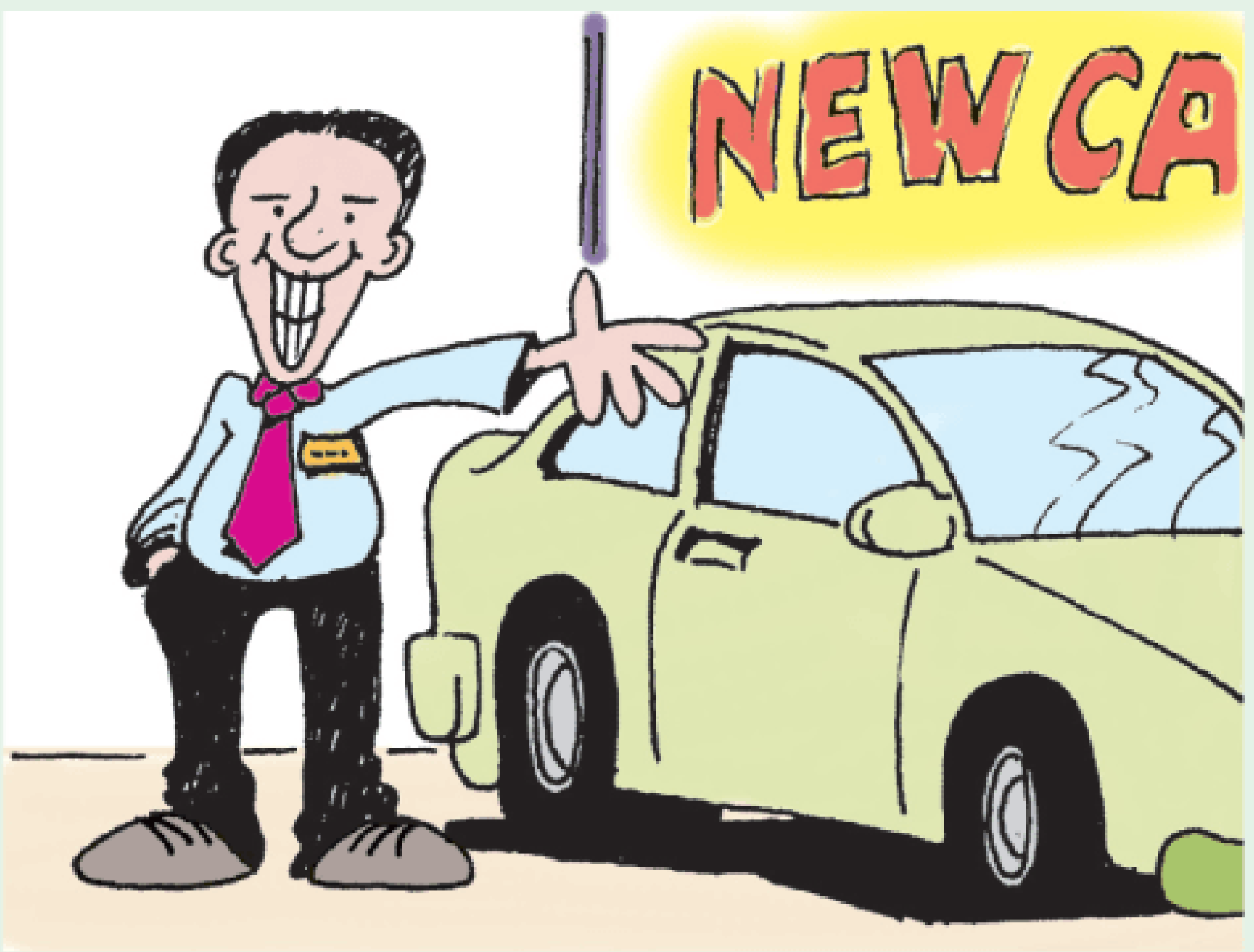
REVIEW SET 2B

1 Nicola and Hamish take out a personal loan of \$12 000 to pay for their wedding. The loan is to be repaid over 4 years at 5.5% p.a. Calculate:

- **a** their monthly repayments
- **b** the total interest charged on the loan.

2 Peter has saved \$4500 towards buying a car. The car he wants to buy is valued at \$22 000, so he will borrow the remaining money. He is able to get a loan for 4 years at 6.9% p.a. interest, compounded quarterly.

- **a** How much will Peter borrow?
- **b** Calculate Peter's quarterly repayments.
- **c** Calculate the total interest charged on the loan.
- **d** Find the outstanding balance after 2 years.



3 A loan of 500 000 pesos is taken out at 6% p.a. interest compounded monthly.

- **a** Calculate the monthly loan repayments if the loan is taken out for:
  - **i** 4 years
  - **ii** 6 years.
- **b** Which loan charges the least total interest? Explain your answer.

4 Answer the **Opening Problem** on page **32**.

5 Pia retires at age 62 with €350 000 in her savings fund. She rolls this money into an annuity fund earning 5.8% p.a. interest compounded monthly.

- **a** How much will she be able to withdraw each month if her money is to last another 2 decades?
- **b** How much *more* will she be able to withdraw each month if her money was to only last 15 years?

6 Harold rolls his £800 000 of savings into an annuity account. He wants the money to last for 15 years.

- **a** Given that Harold can withdraw £6284.75 each month, find the annual interest rate, compounded monthly, of the account.
- **b** How much longer would Harold's money last if he withdrew only £5000 each month?

Chapter 3

Functions

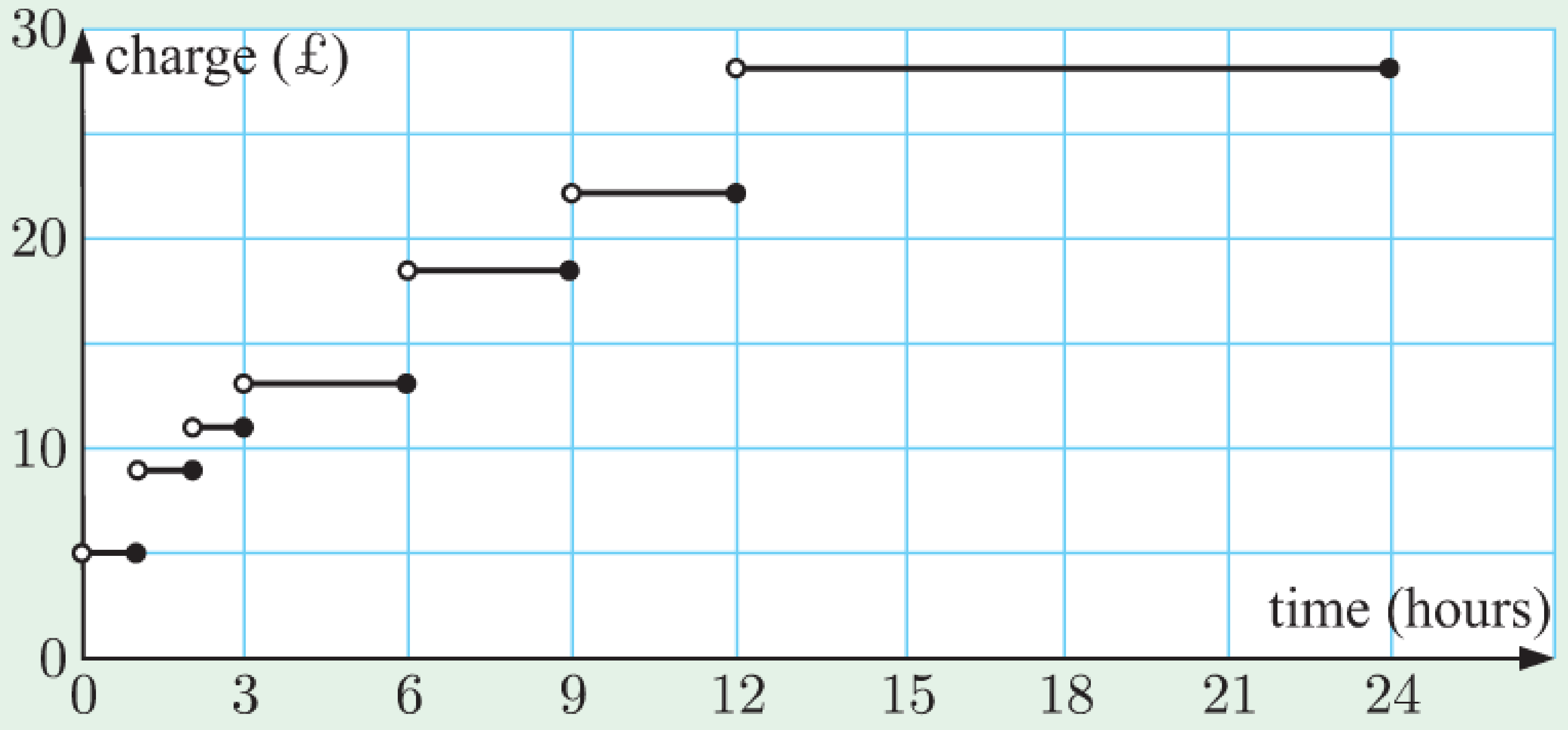
Contents:

A Relations and functions
B Function notation
C Domain and range
D Graphs of functions
E Sign diagrams
F Transformations of graphs
G Inverse functions

OPENING PROBLEM

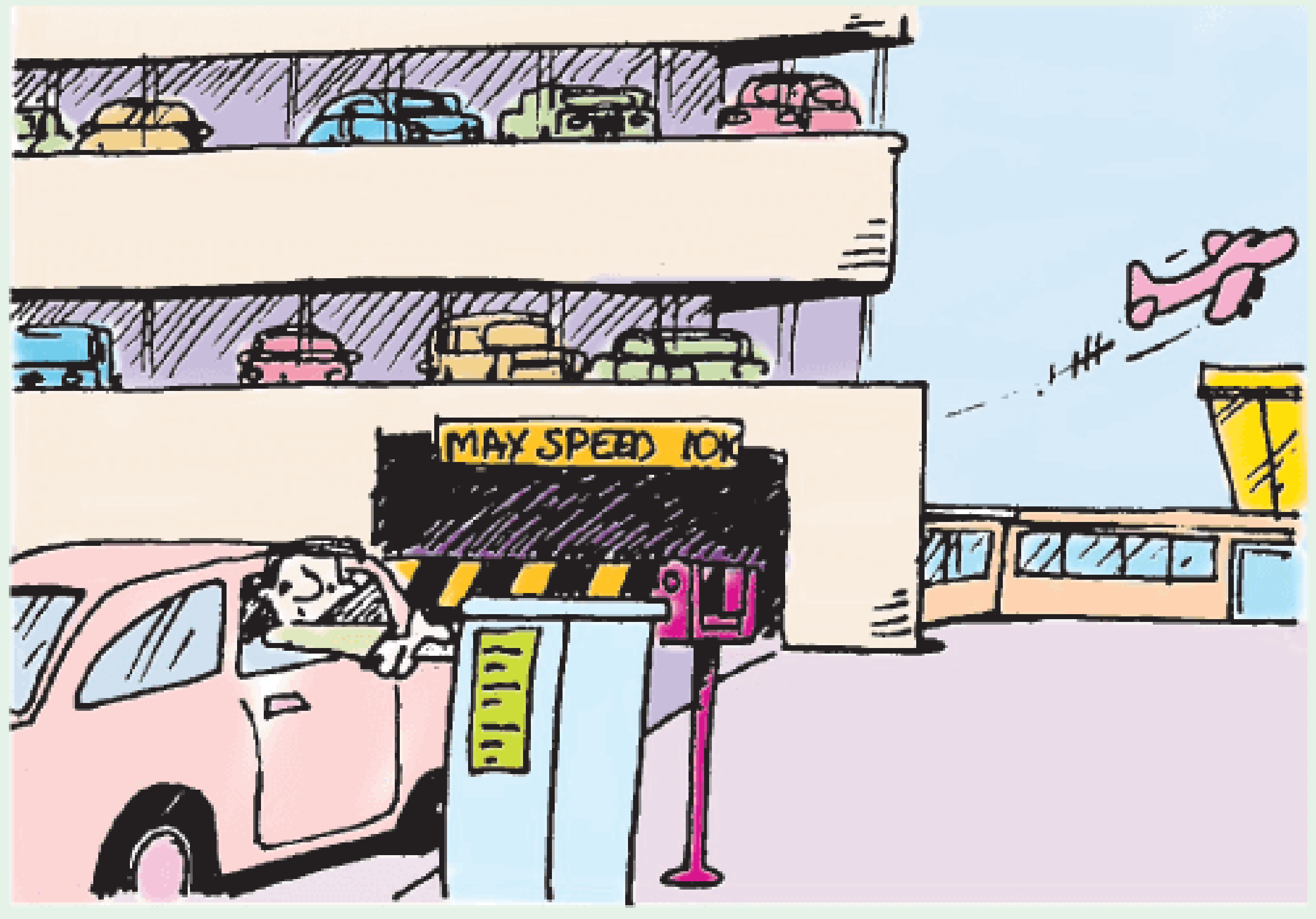
The charges for parking a car in a short-term car park at an airport are shown in the table and graph below. The total charge is *dependent* on the length of time t the car is parked.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$	£5.00
$1 < t \leqslant 2$	£9.00
$2 < t \leqslant 3$	£11.00
$3 < t \leqslant 6$	£13.00
$6 < t \leqslant 9$	£18.00
$9 < t \leqslant 12$	£22.00
$12 < t \leqslant 24$	£28.00



Things to think about:

- **a** What values of *time* are illustrated in the graph?
- **b** What are the possible charges?
- **c** What feature of the graph ensures that there is only one charge for any given time?



In the **Opening Problem** we see a relationship between the two variables *time* and *charge*.

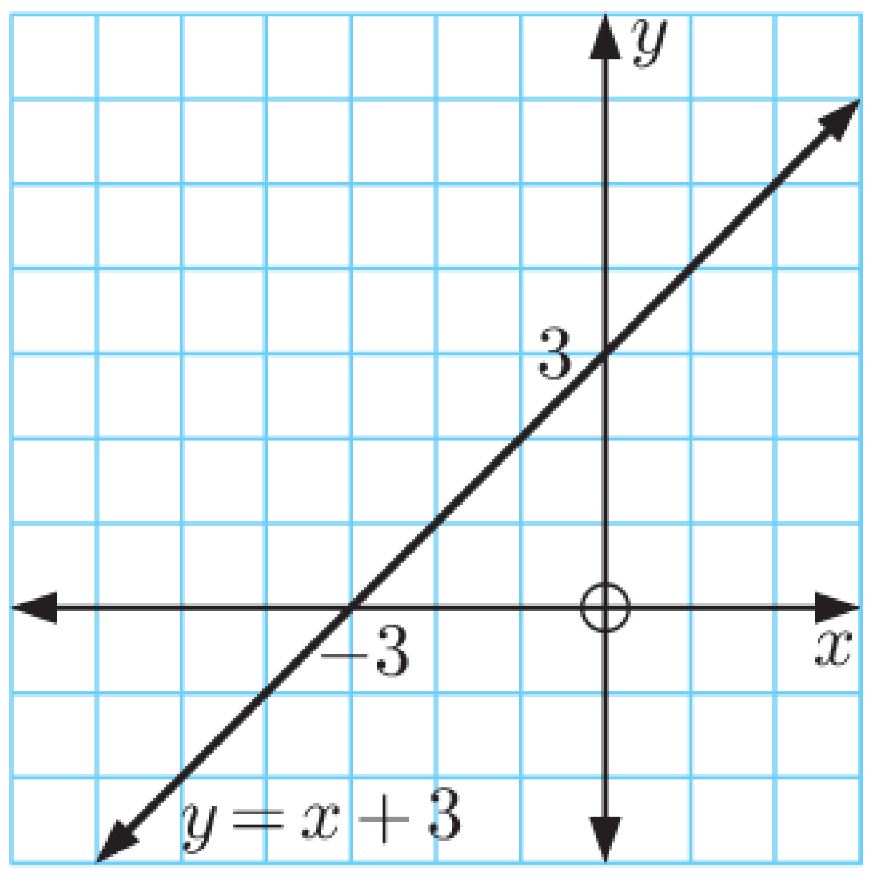
In this Chapter we explore what it really means for the relationship between two variables to be called a **function**. We will then explore properties of functions which will help us work with and understand them.

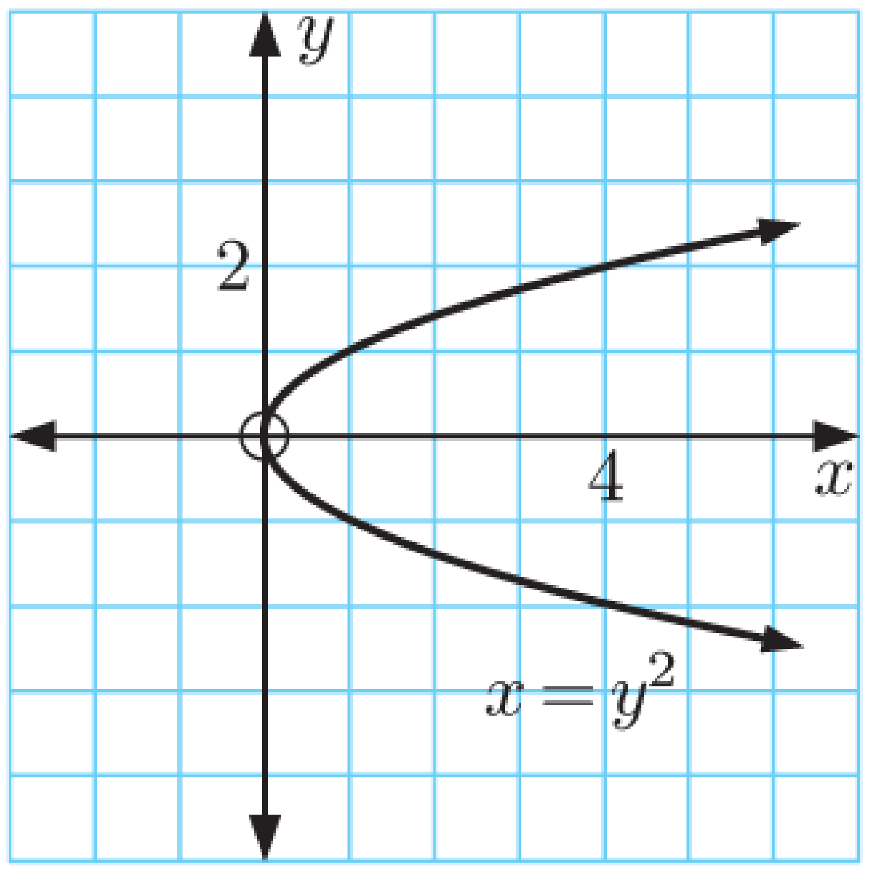
A RELATIONS AND FUNCTIONS

A **relation** between variables x and y is any set of points in the (x, y) plane. We say that the points *connect* the two variables.

A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

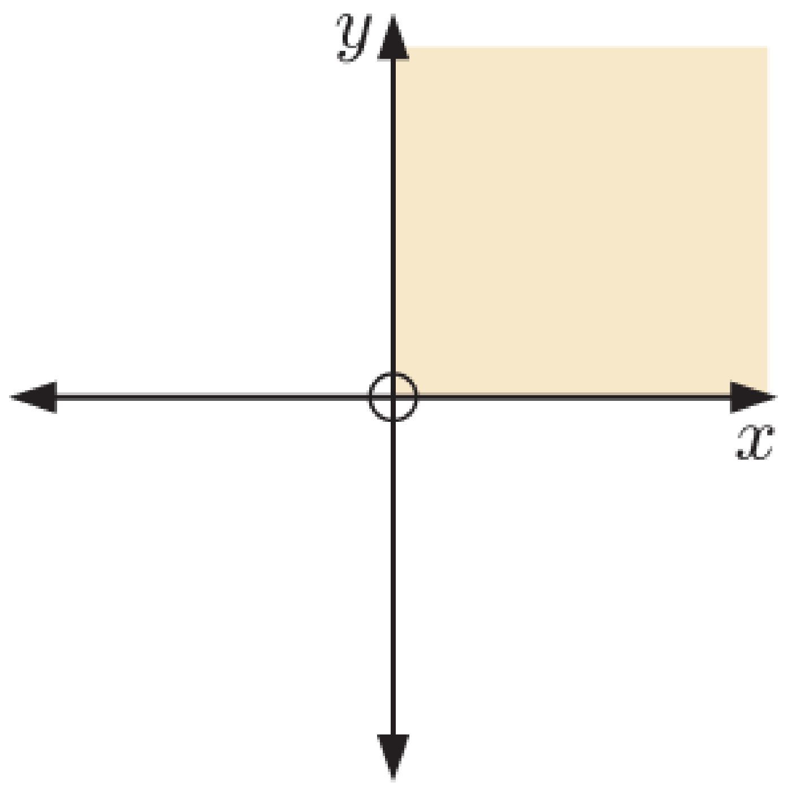
For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:





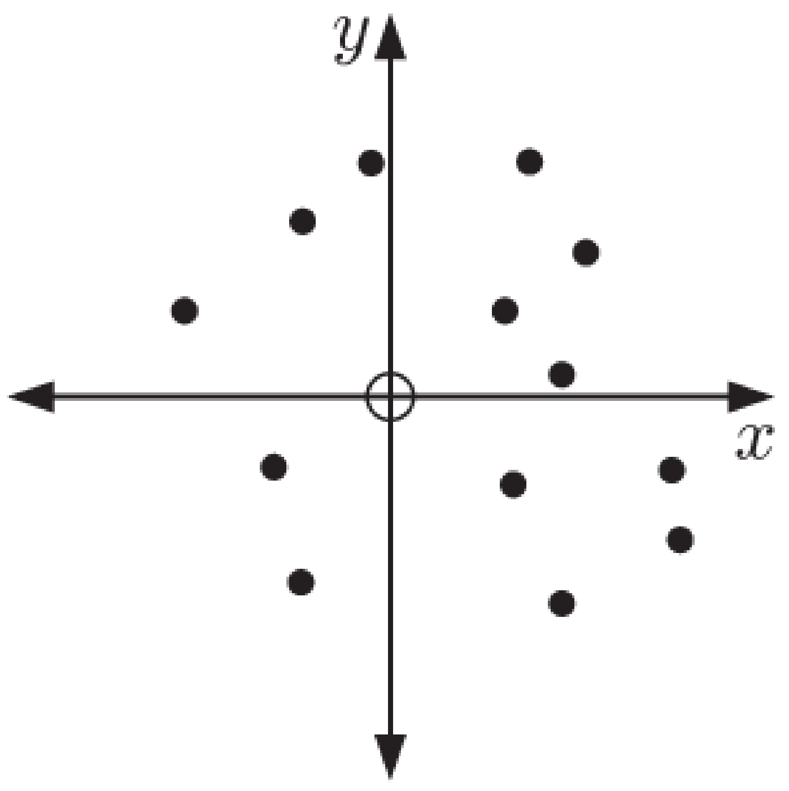
However, not all relations can be defined by an equation. Below are two examples:

(1)



The set of all points in the first quadrant is the relation $x > 0$, $y > 0$.

(2)



These 13 points form a relation. It can be described as a finite set of points, but not by an equation.

FUNCTIONS

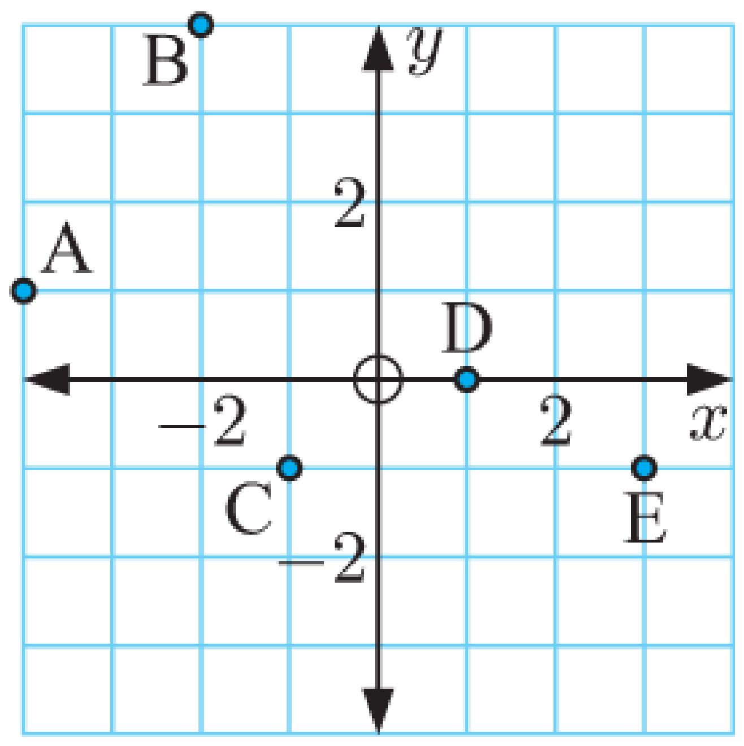
A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from this definition that a function is a special type of relation.

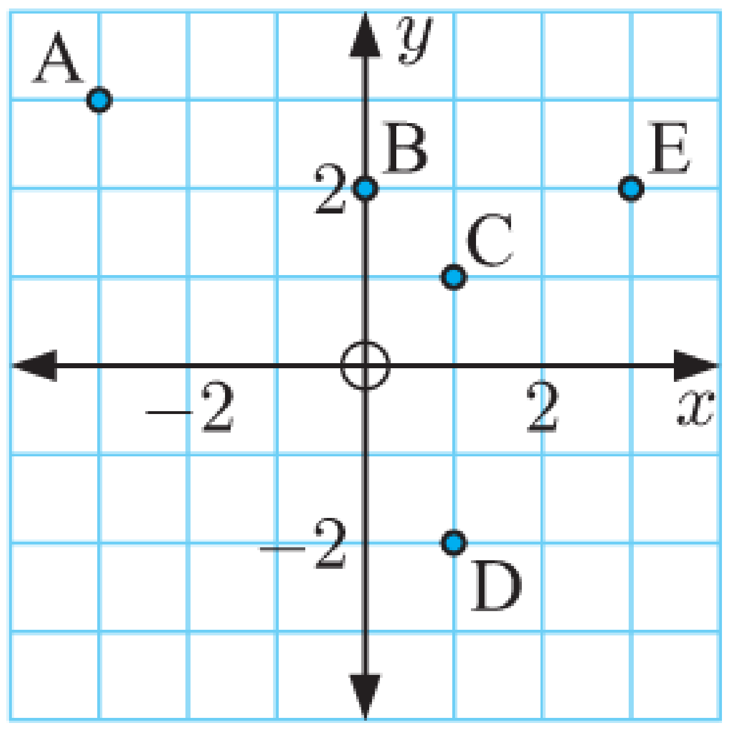
Every function is a relation, but not every relation is a function.

For example:

- The relation formed by these points is a function, since no two points have the same x-coordinate.



- The relation formed by these points is *not* a function, since the points C and D have the same x-coordinate.



ALGEBRAIC TEST FOR FUNCTIONS

Suppose a relation is given as an equation. If the substitution of any value for x results in at most one value of y, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of x there is only one corresponding value of y.
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

GEOMETRIC TEST OR VERTICAL LINE TEST FOR FUNCTIONS

Suppose we draw all possible vertical lines on the graph of a relation.

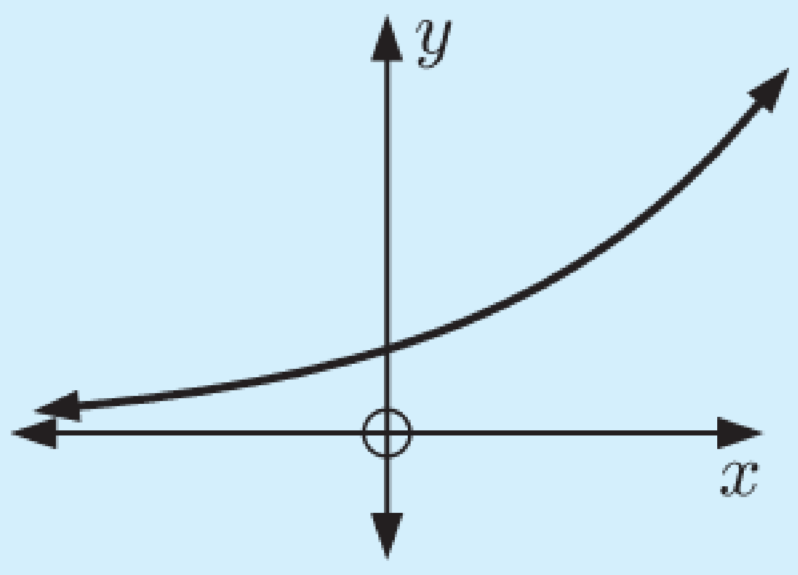
- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is not a function.

Example 1

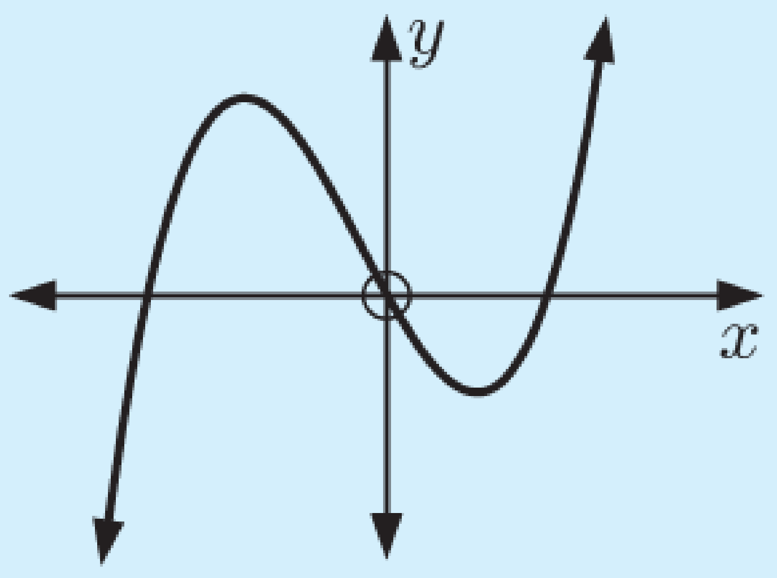
Self Tutor

Which of the following relations are functions?

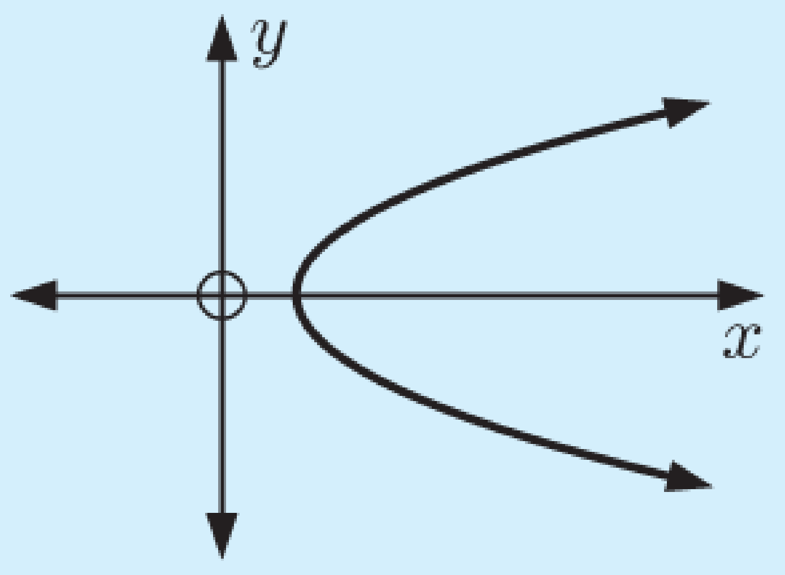
a



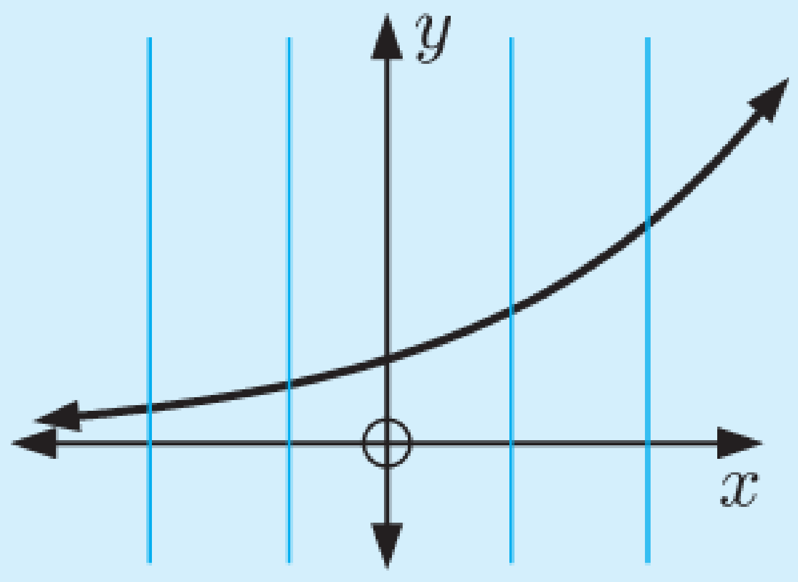
b



c

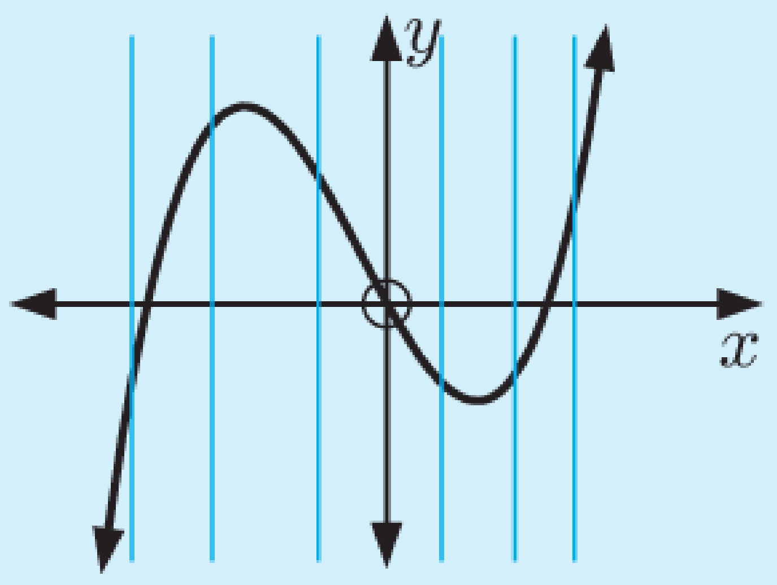


a



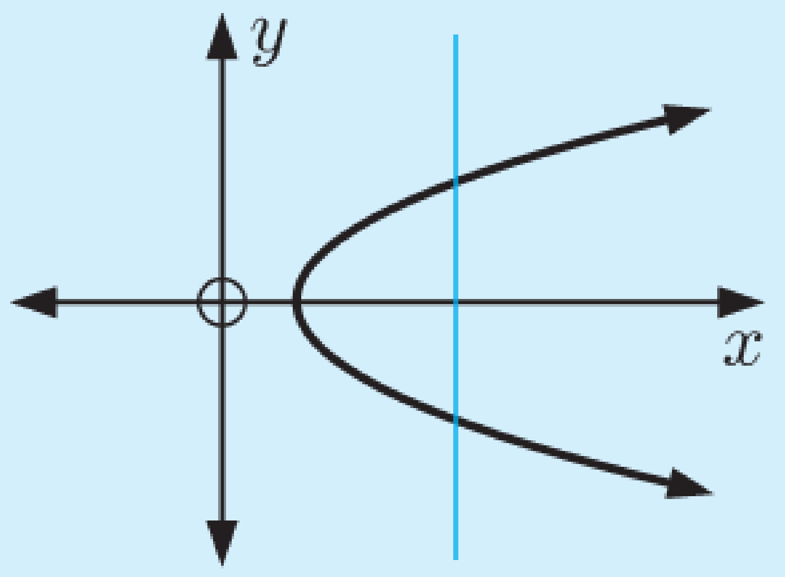
a function

b



a function

c



not a function

DEMO

GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —○— , this point is **not included**.
- If a graph contains a small **filled-in circle** such as ——● , this point is **included**.
- If a graph contains an **arrowhead** at an end such as ——► , then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 3A

1 Which of the following sets of ordered pairs is a function? Explain your answers.

a $\{(1,\ 3),\ (2,\ 4),\ (3,\ 5),\ (4,\ 6)\}$

b $\{(1,\ 3),\ (3,\ 2),\ (1,\ 7),\ (-1,\ 4)\}$

c $\{(2,\ -1),\ (2,\ 0),\ (2,\ 3),\ (2,\ 11)\}$

d $\{(7,\ 6),\ (5,\ 6),\ (3,\ 6),\ (-4,\ 6)\}$

e $\{(0,\ 0),\ (1,\ 0),\ (3,\ 0),\ (5,\ 0)\}$

f $\{(0,\ 0),\ (0,\ -2),\ (0,\ 2),\ (0,\ 4)\}$

2 Use algebraic methods to decide whether these relations are functions. Explain your answers.

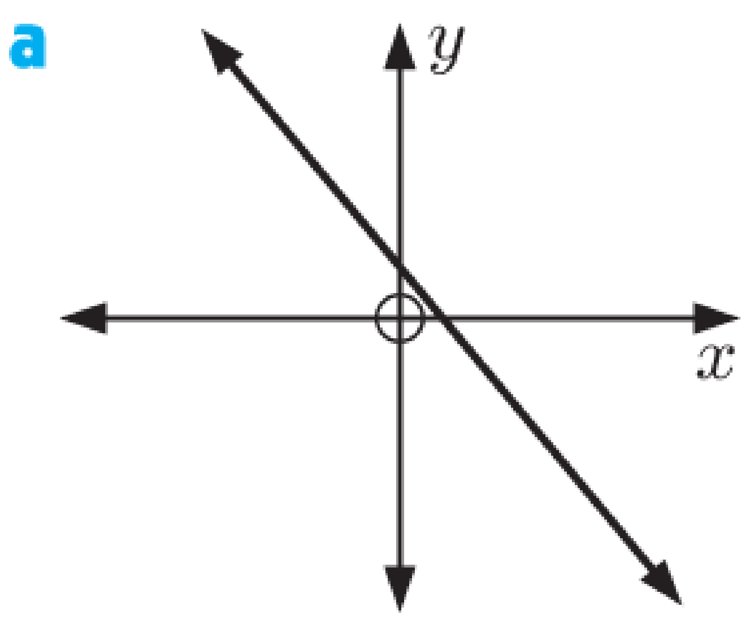
a $y = x^2 - 9$

b $x + y = 9$

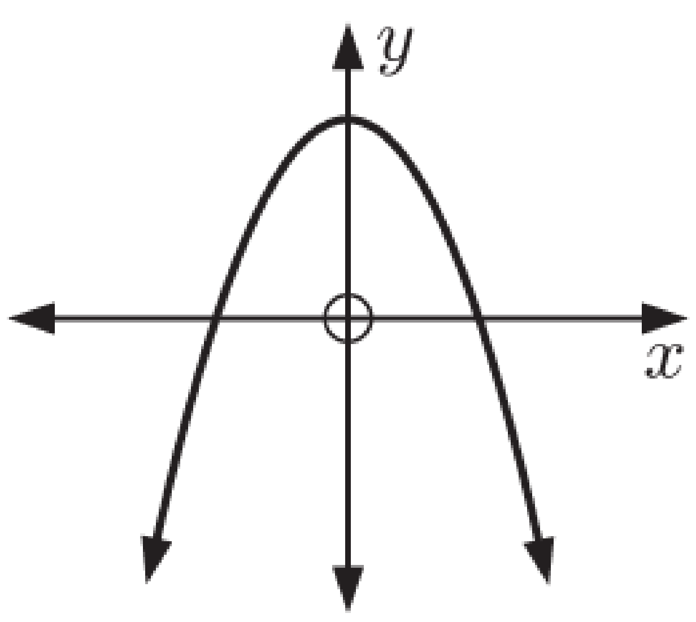
c $x^2 + y^2 = 9$

3 Use the vertical line test to determine which of the following relations are functions:

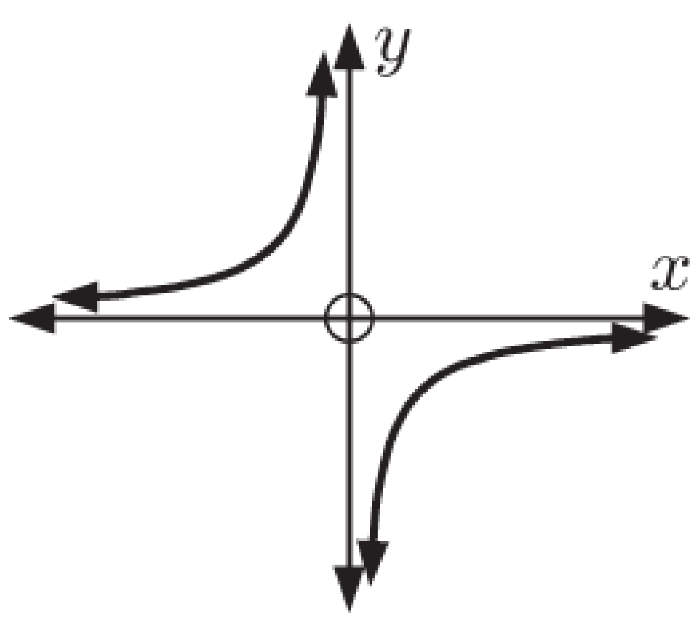
a



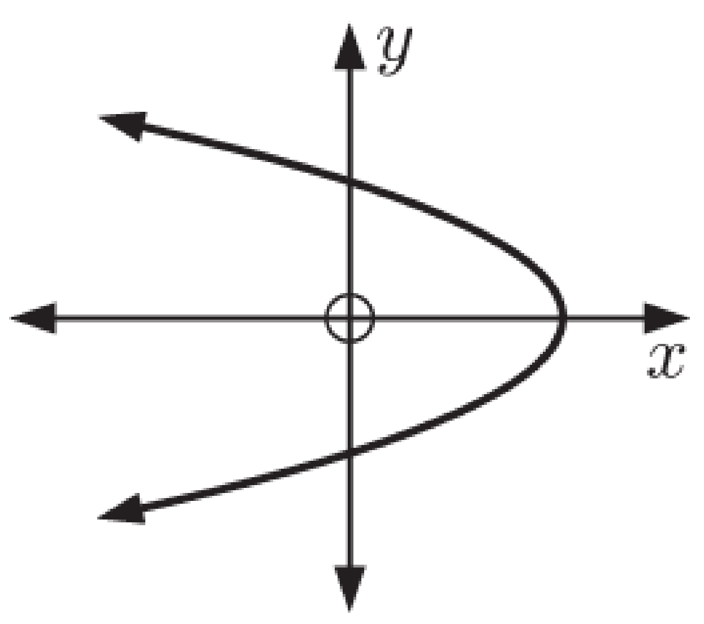
b



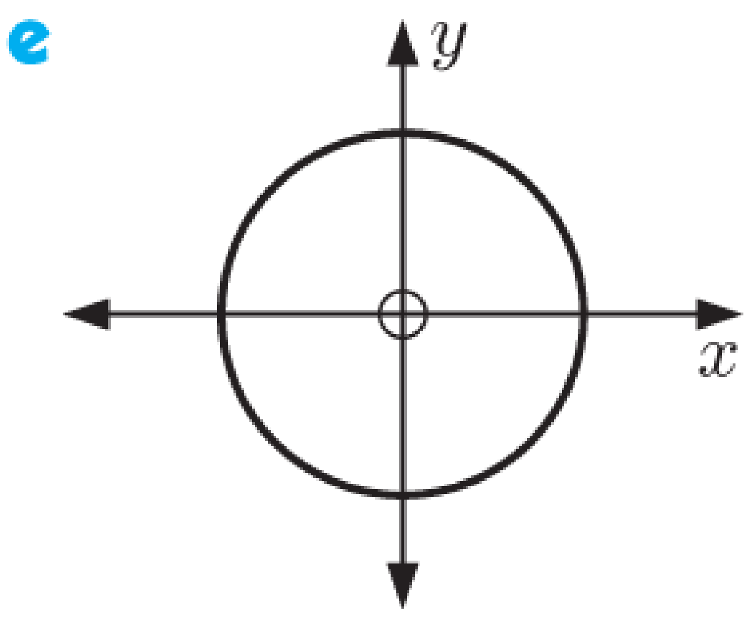
c



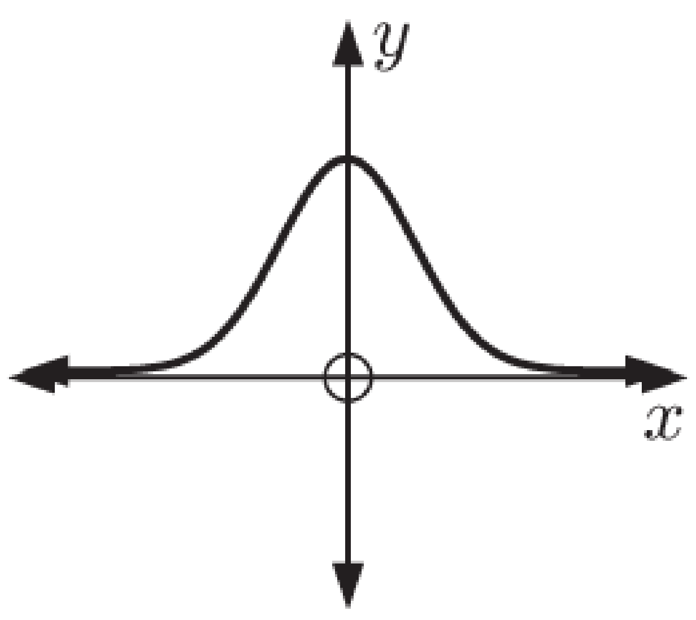
d



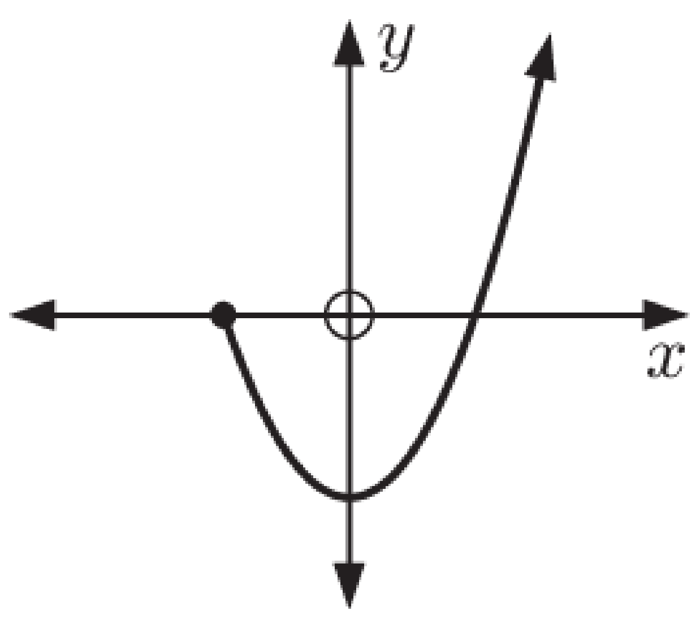
e



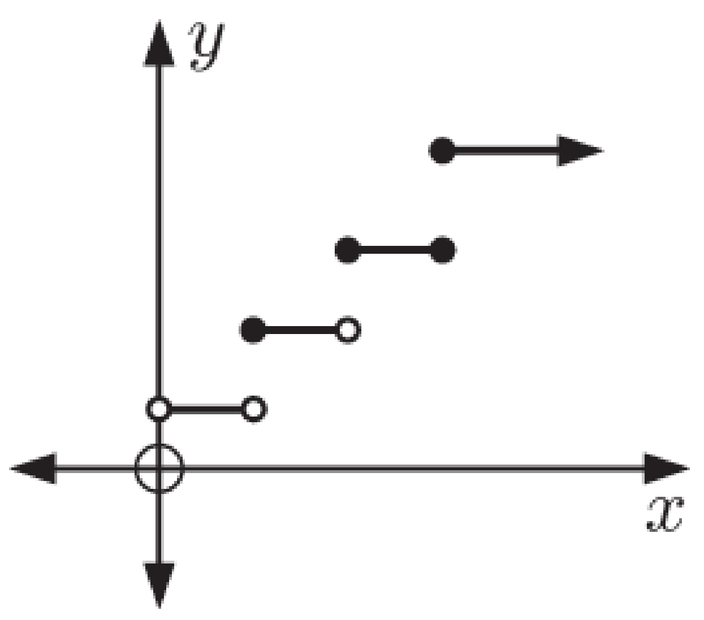
f



g



h



4 The managers of a new amusement park are discussing the schedule of ticket prices. Maurice suggests the table alongside. Explain why this relation between *age* and *cost* is not a function, and discuss the problems that this will cause.

Age	*Cost*
0 - 2 years (infants)	\$0
2 - 16 years (children)	\$20
16+ years (adults)	\$30

5 Is it possible for a function to have more than one y-intercept? Explain your answer.

6 Is the graph of a straight line always a function? Give evidence to support your answer.

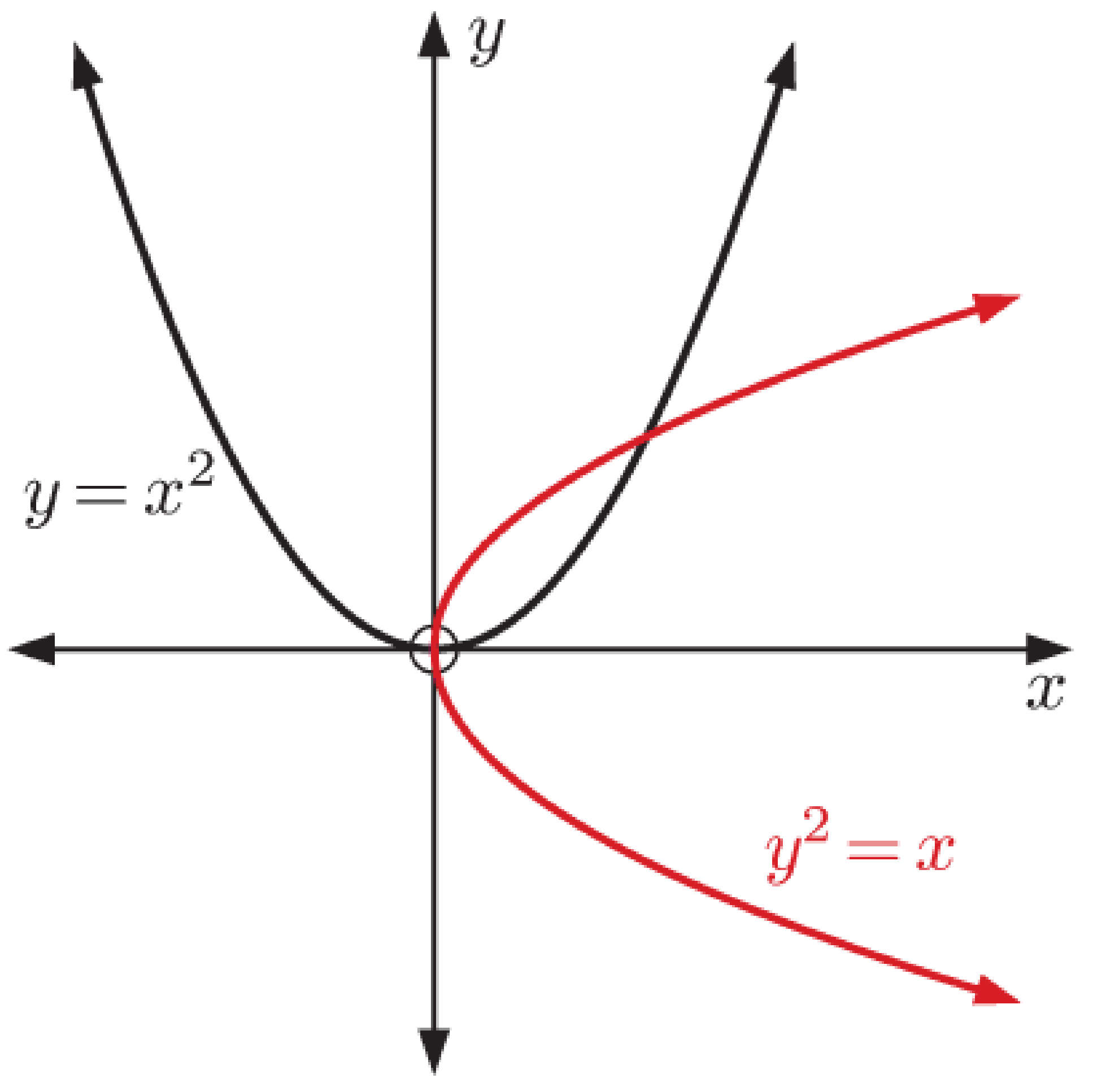
7 The graph alongside shows the curves $y = x^2$ and $y^2 = x$.

a Discuss the similarities and differences between the curves, including whether each curve is a function. You may also consider what transformation(s) map one curve onto the other.

b Using $y^2 = x$, we can write $y = \pm\sqrt{x}$.

i What part of the graph of $y^2 = x$ corresponds to $y = \sqrt{x}$?

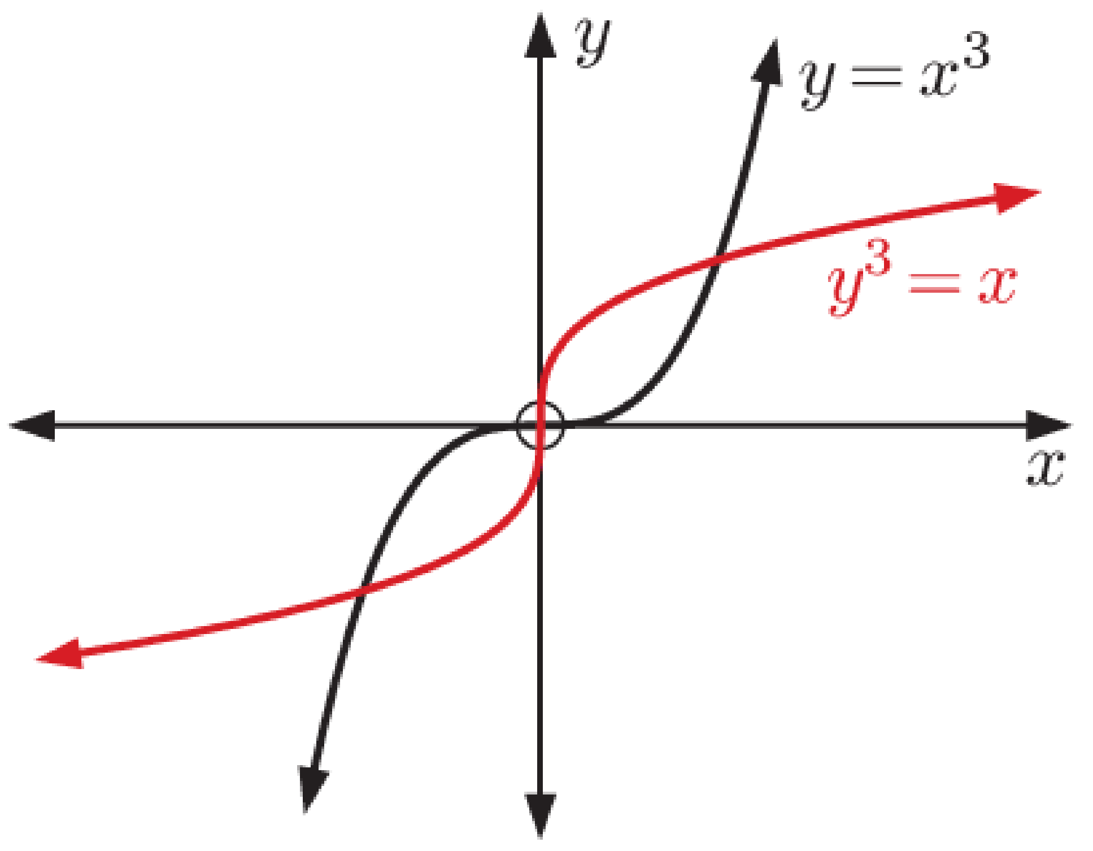
ii Is $y = \sqrt{x}$ a function? Explain your answer.



8 The graph alongside shows the curves $y = x^3$ and $y^3 = x$.

a Explain why both of these curves are functions.

b For the curve $y^3 = x$, write y as a function of x.



DISCUSSION

In the **Opening Problem**:

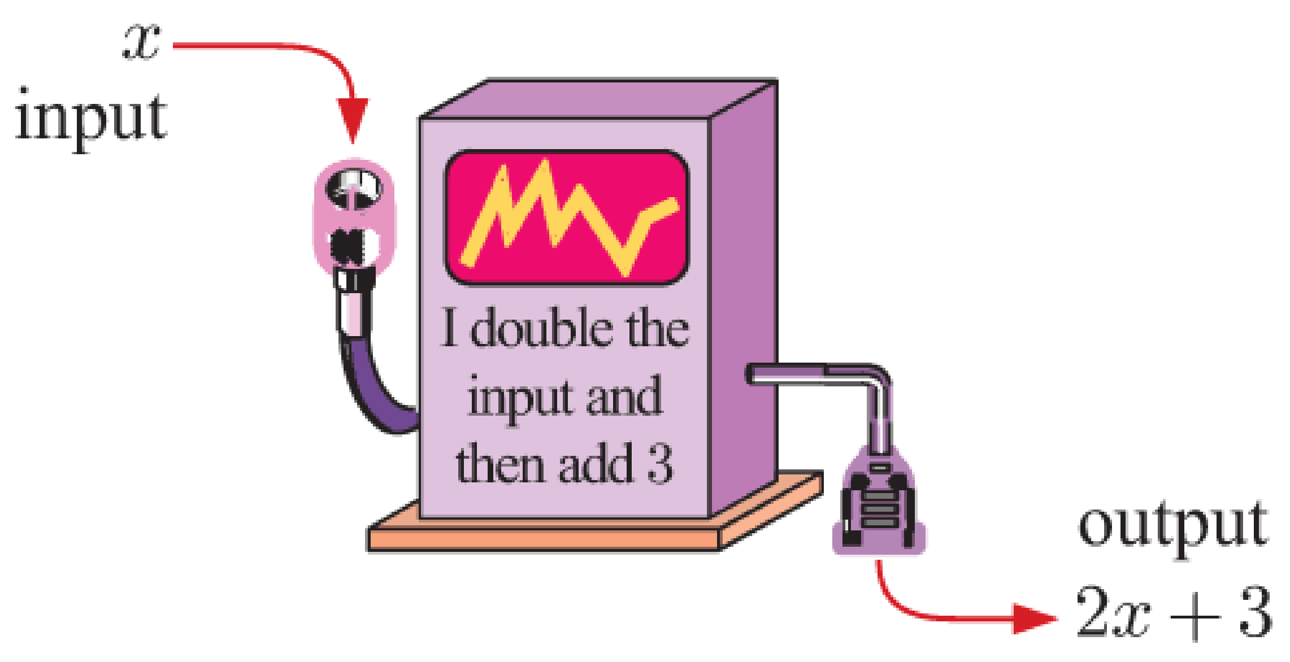
- Is the relation describing the car park charges a function?
- If we know the *time* somebody parked for, can we determine the exact *charge* they need to pay?
- If we know the *charge* somebody pays, can we determine the exact *time* they have parked for?

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.

This "machine" has been programmed to perform a particular function.

If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

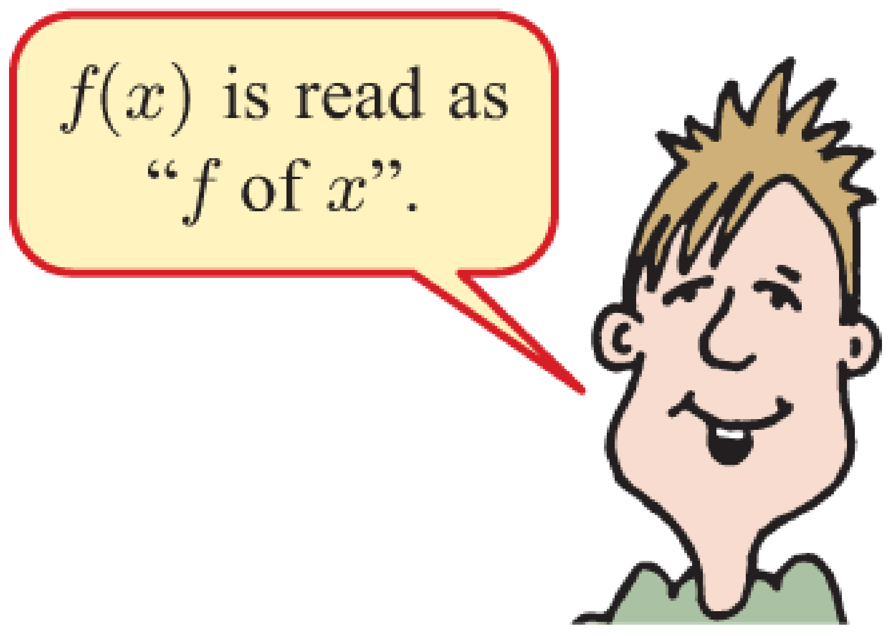

x
input
I double the input and then add 3
output
$2x+3$

If we use f to represent that particular function, we can write "f is the function that will convert x into $2x+3$."

So, f would convert 2 into $2(2)+3=7$ and
-4 into $2(-4)+3=-5$.

In **function notation**, we write $f(x)=2x+3$ or $y=2x+3$.

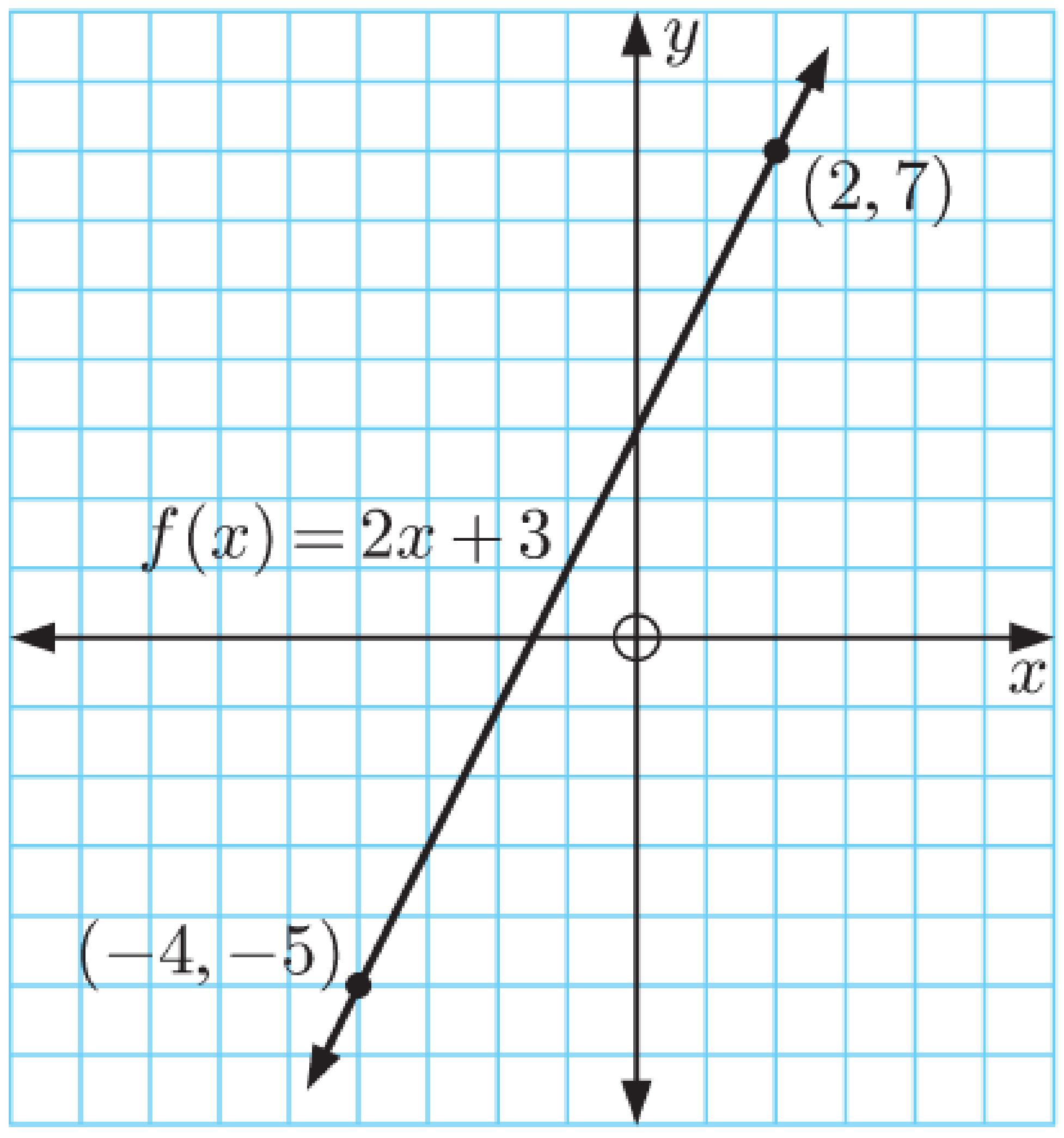
$f(x)$ is the value of y for a given value of x, so $y=f(x)$.


$f(x)$ is read as "f of x".

$y=f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x)=2x+3$:

- $f(2)=2(2)+3=7$
  $\therefore$ the point $(2, 7)$ lies on the graph of the function.
- $f(-4)=2(-4)+3=-5$
  $\therefore$ the point $(-4, -5)$ also lies on the graph.



Example 2 **Self Tutor**

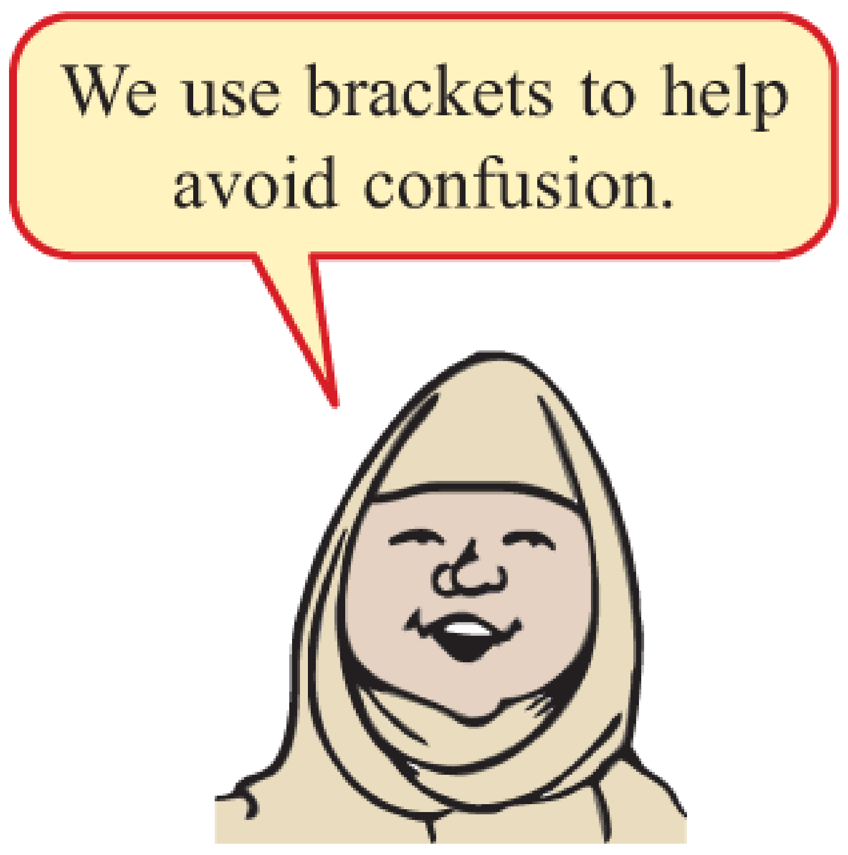
If $f(x)=2x^2-3x$, find the value of:

a $f(5)$ **b** $f(-4)$

$f(x)=2x^2-3x$

a $f(5)=2(5)^2-3(5)$ {replacing x with (5)}
$=2\times25-15$
$=35$

b $f(-4)=2(-4)^2-3(-4)$ {replacing x with (-4)}
$=2(16)+12$
$=44$


We use brackets to help avoid confusion.

EXERCISE 3B

1 If $f(x)=3x+2$, find the value of:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $f(x)=3x-x^2+2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

3 If $g(x) = x - \frac{4}{x}$, find the value of:

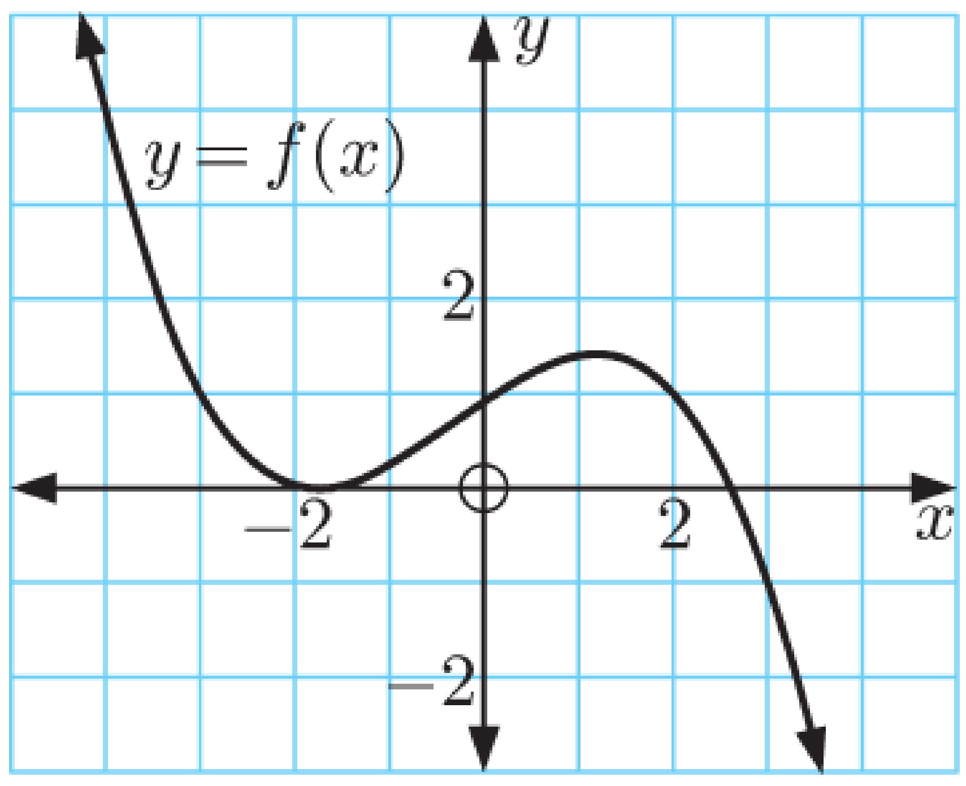
a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

4 The graph of $y = f(x)$ is shown alongside.

a Find:

i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.



5 Suppose $G(x) = \frac{2x+3}{x-4}$.

a Evaluate:

i $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find x such that $G(x) = -3$.

6 Suppose $f(x) = 1 - 3x$ and $g(x) = \sqrt{x+5}$.

a Show that $f(-1) = g(11)$. **b** Find x such that $f(x) = g(4)$.

Example 3 **Self Tutor**

If $f(x) = 5 - x - x^2$, find in simplest form:

a $f(-x)$ **b** $f(x+2)$ **c** $f(x-1) - 5$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}

$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}

$= 5 - x - 2 - [x^2 + 4x + 4]$

$= 3 - x - x^2 - 4x - 4$

$= -x^2 - 5x - 1$

c $f(x-1) - 5 = (5 - (x-1) - (x-1)^2) - 5$ {replacing x with $(x-1)$}

$= 5 - x + 1 - (x^2 - 2x + 1) - 5$

$= -x^2 + x$

7 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$

d $f(2a)$ **e** $f(x+2)$ **f** $f(x+h)$

8 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$

d $F(x^2)$ **e** $F(3x)$ **f** $F(x+h)$

9 If $f(x) = x^2$, find in simplest form:

a $f(3x)$ **b** $f\left(\frac{x}{2}\right)$ **c** $3\,f(x)$ **d** $2\,f(x-1)+5$

10 If $f(x) = \frac{1}{x}$, find in simplest form:

a $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2\,f(x)+3$ **d** $3\,f(x-1)+2$

11 f represents a function. Explain the difference in meaning between f and $f(x)$.

12 On the same set of axes, draw the graphs of three different functions $f(x)$ such that $f(2)=1$ and $f(5)=3$.

13 Find a function $f(x) = ax+b$ for which $f(2)=1$ and $f(-3)=11$.

14 Samantha is filling her car with petrol. The amount of petrol in the tank after t minutes is given by $P(t) = 5+10t$ litres.

a Find $P(3)$, and interpret your answer.

b Find t when $P(t)=50$, and explain what this represents.

c How many litres of petrol were in the tank when Samantha started to fill it?

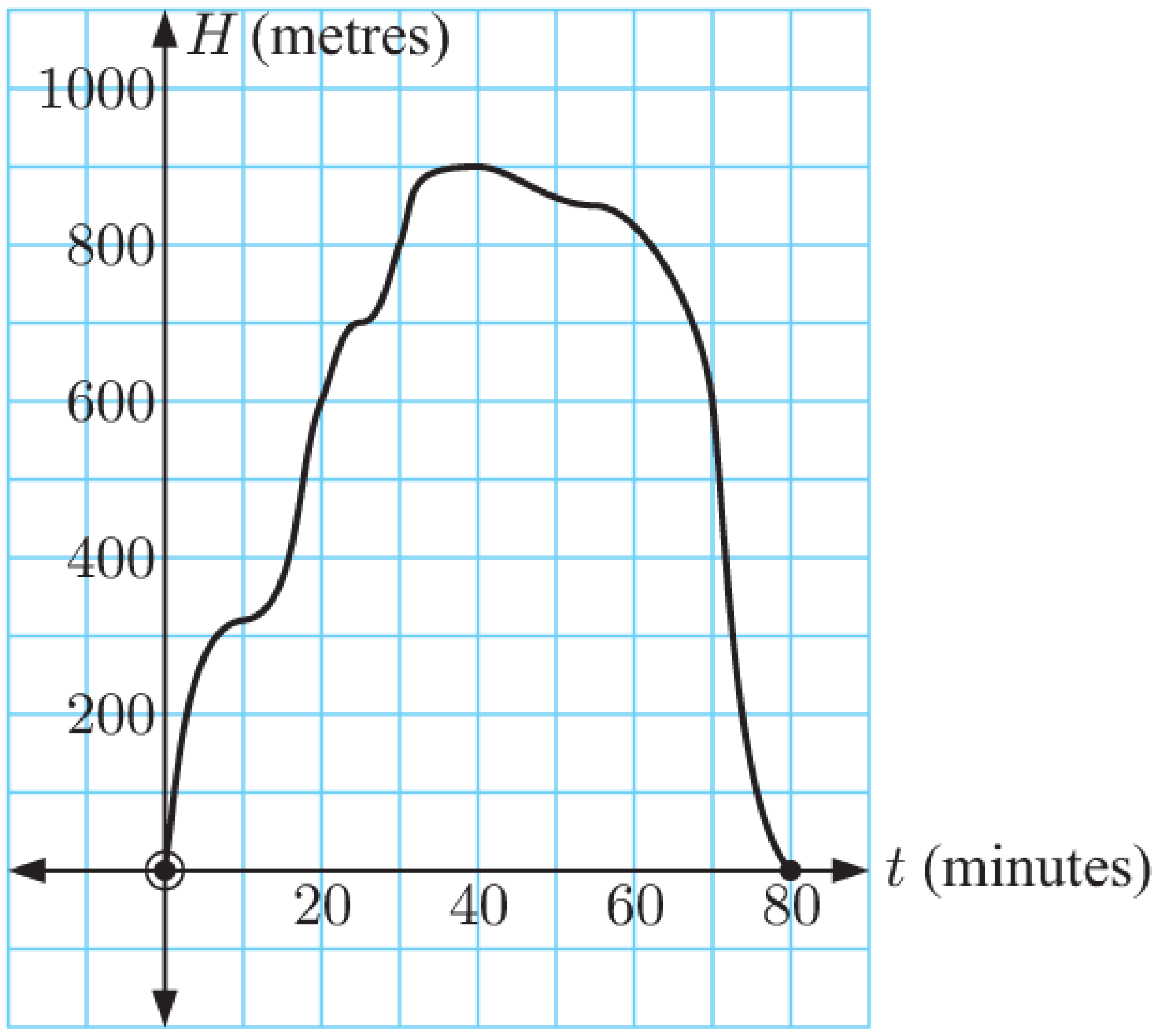
15 For a hot air balloon ride, the function $H(t)$ gives the height of the balloon after t minutes. Its graph is shown alongside.

a Find $H(30)$, and explain what your answer means.

b Find the values of t such that $H(t) = 600$. Interpret your answer.

c For what values of t was the height of the balloon recorded?

d What range of heights was recorded for the balloon?



16 Given $f(x) = ax + \frac{b}{x}$, $f(1)=1$, and $f(2)=5$, find constants a and b.

17 The function $T(x) = ax^2+bx+c$ has the values $T(0)=-4$, $T(1)=-2$, and $T(2)=6$. Find a, b, and c.

18 The value of a photocopier t years after purchase is given by $V(t) = 9000-900t$ pounds.

a Find $V(4)$, and state what $V(4)$ means.

b Find t when $V(t)=3600$, and explain what this means.

c Find the original purchase price of the photocopier.

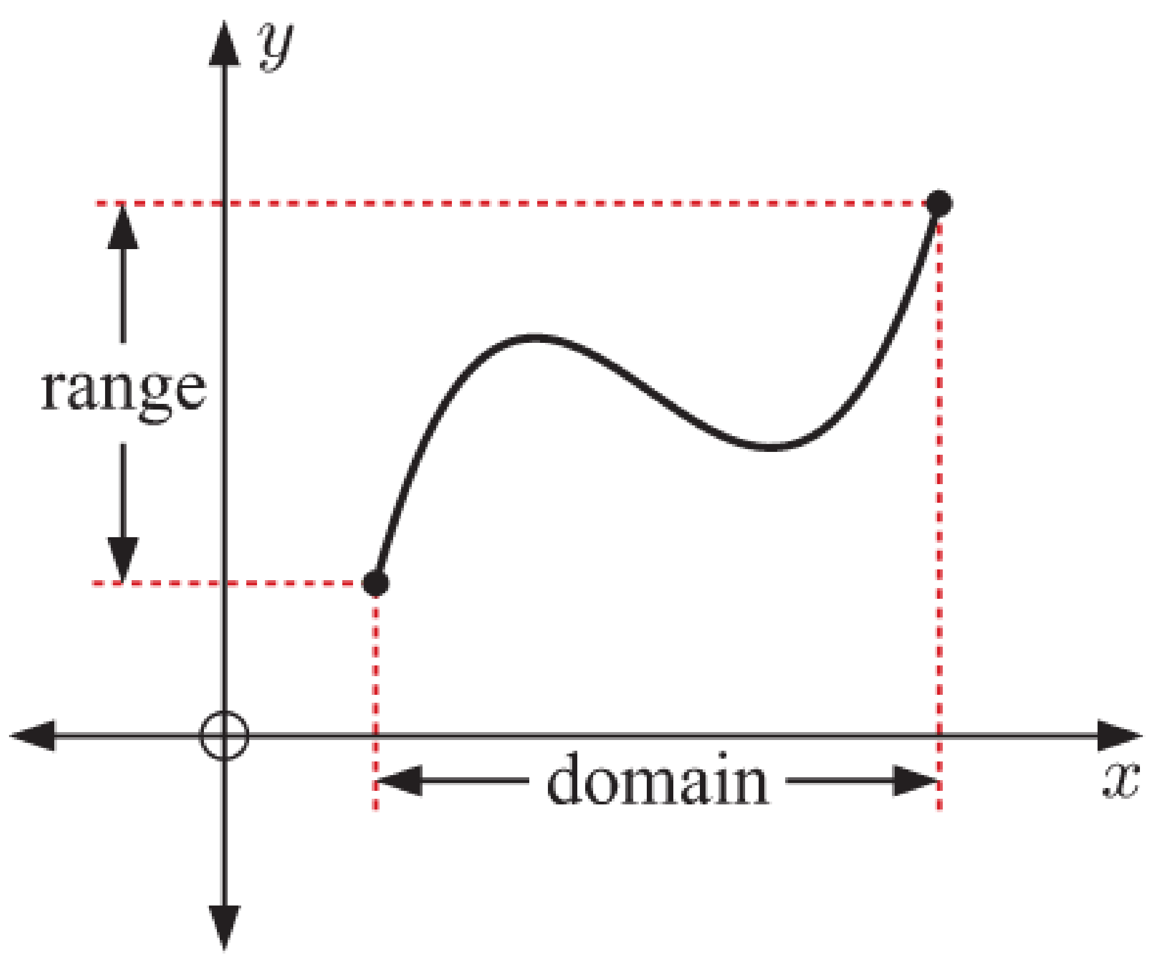
d For what values of t is it reasonable to use this function?

C DOMAIN AND RANGE

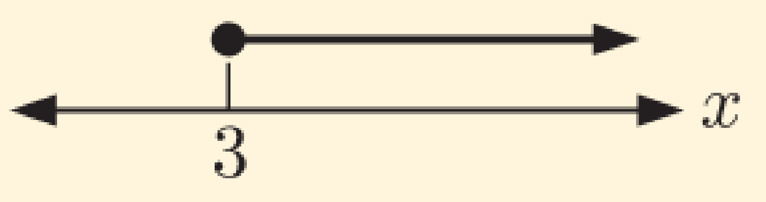
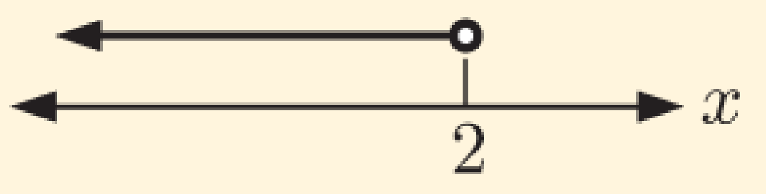
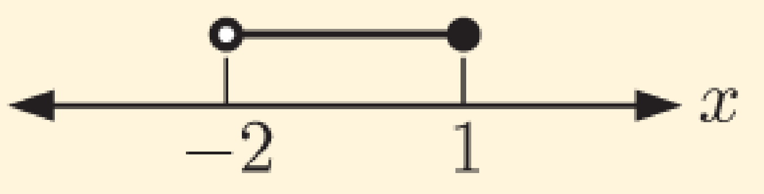
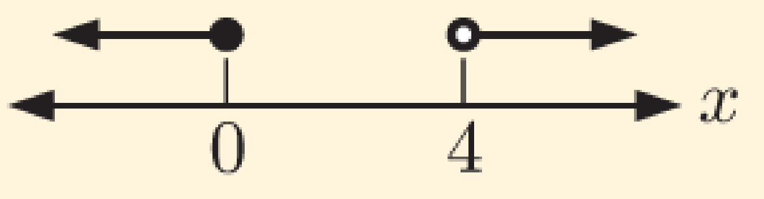
We have seen that a relation is a set of points which connects two variables.

The **domain** of a relation is the set of values which the variable on the horizontal axis can take. This variable is usually x.

The **range** of a relation is the set of values which the variable on the vertical axis can take. This variable is usually y.



The domain and range of a relation can be described using **set notation**, **interval notation**, or a **number line graph**. For example:

Set notation	*Interval notation*	*Number line graph*	*Meaning*
$\{x \mid x \geqslant 3\}$	$x \geqslant 3$		the set of all x such that x is greater than or equal to 3
$\{x \mid x < 2\}$	$x < 2$		the set of all x such that x is less than 2
$\{x \mid -2 < x \leqslant 1\}$	$-2 < x \leqslant 1$		the set of all x such that x is between -2 and 1, including 1
$\{x \mid x \leqslant 0 \text{ or } x > 4\}$	$x \leqslant 0$ or $x > 4$		the set of all x such that x is less than or equal to 0, or greater than 4

To find the domain and range of a function, we can observe its graph.

For example:

(1) Consider again the car park charges in the **Opening Problem**.

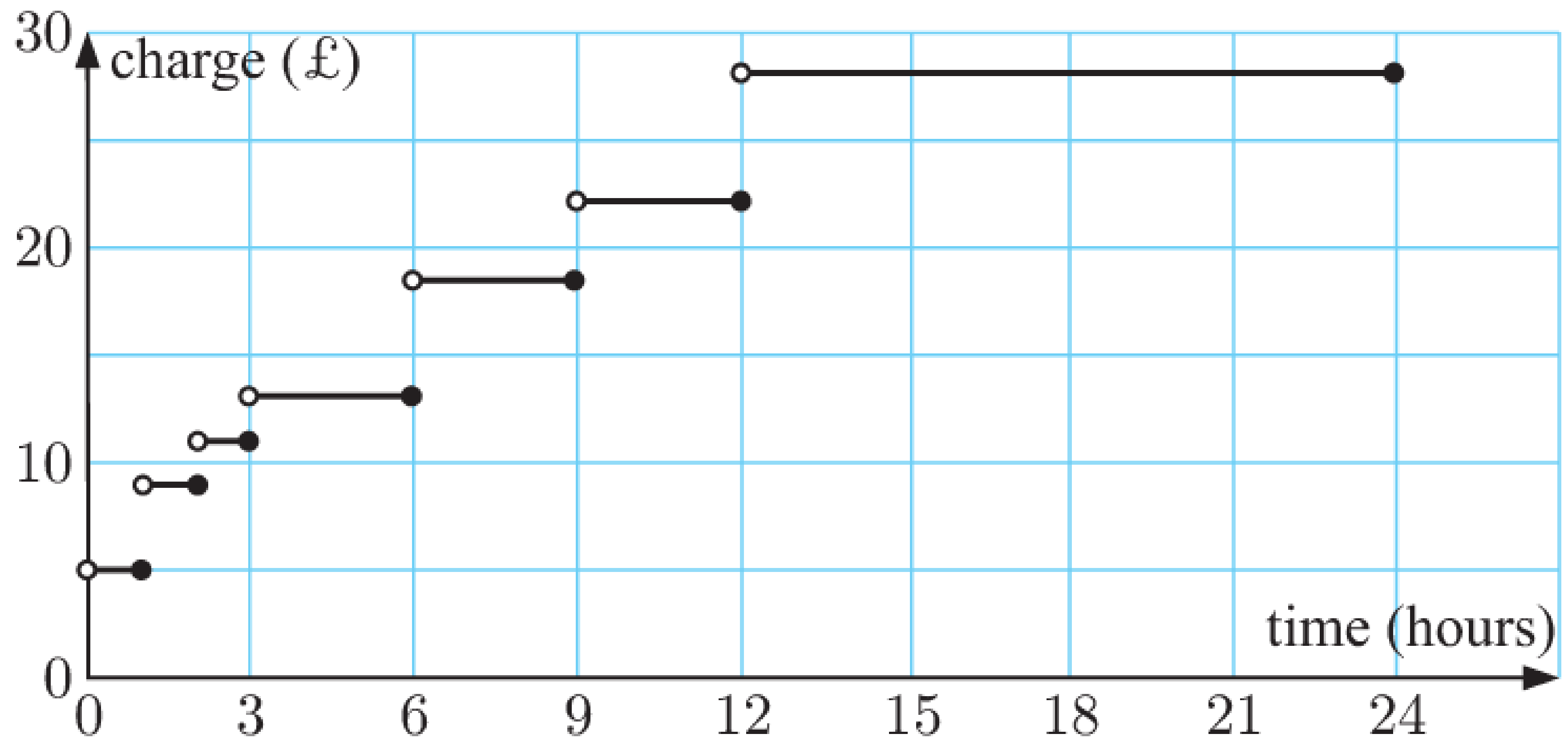
The function is defined for all times t such that $0 < t \leqslant 24$.

$\therefore$ the domain is $\{t \mid 0 < t \leqslant 24\}$.

The possible charges are £5, £9, £11, £13, £18, £22, and £28.

$\therefore$ the range is

$\{5, 9, 11, 13, 18, 22, 28\}$.



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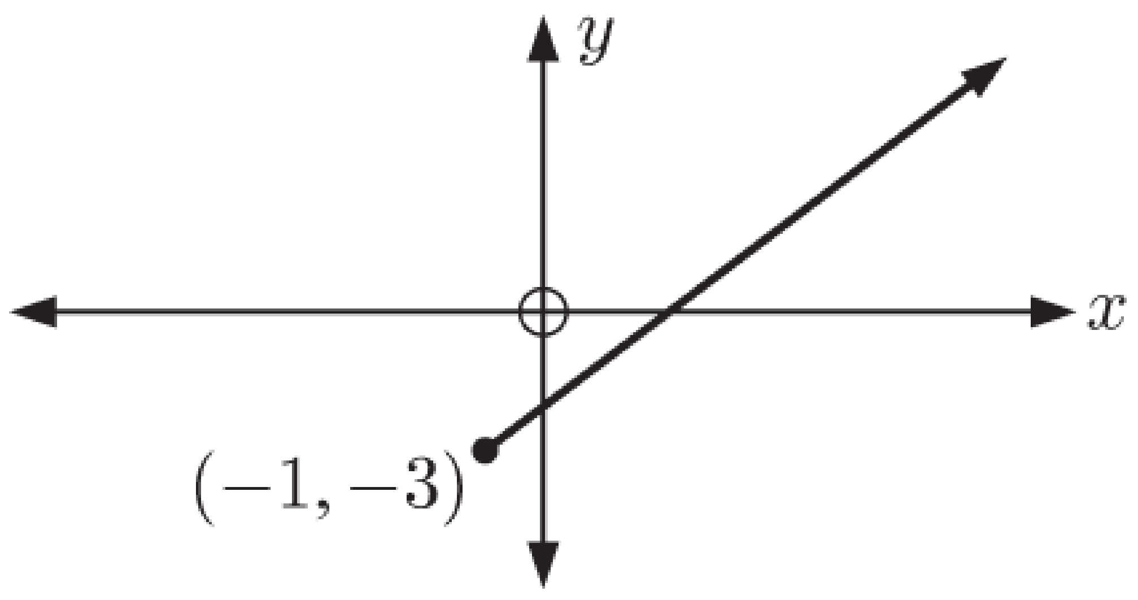
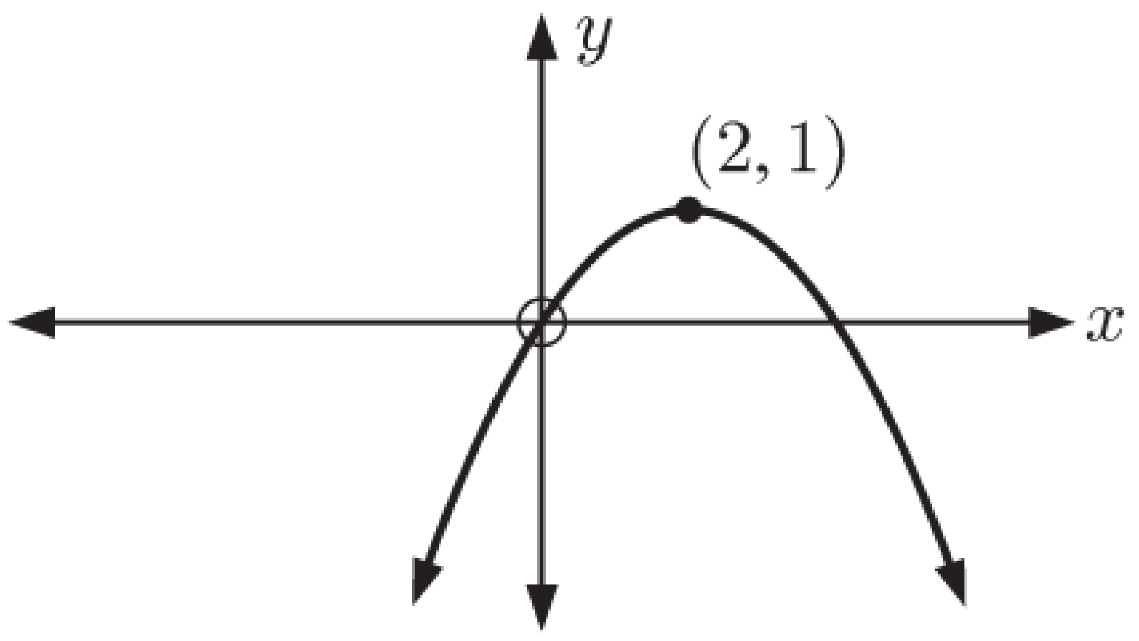


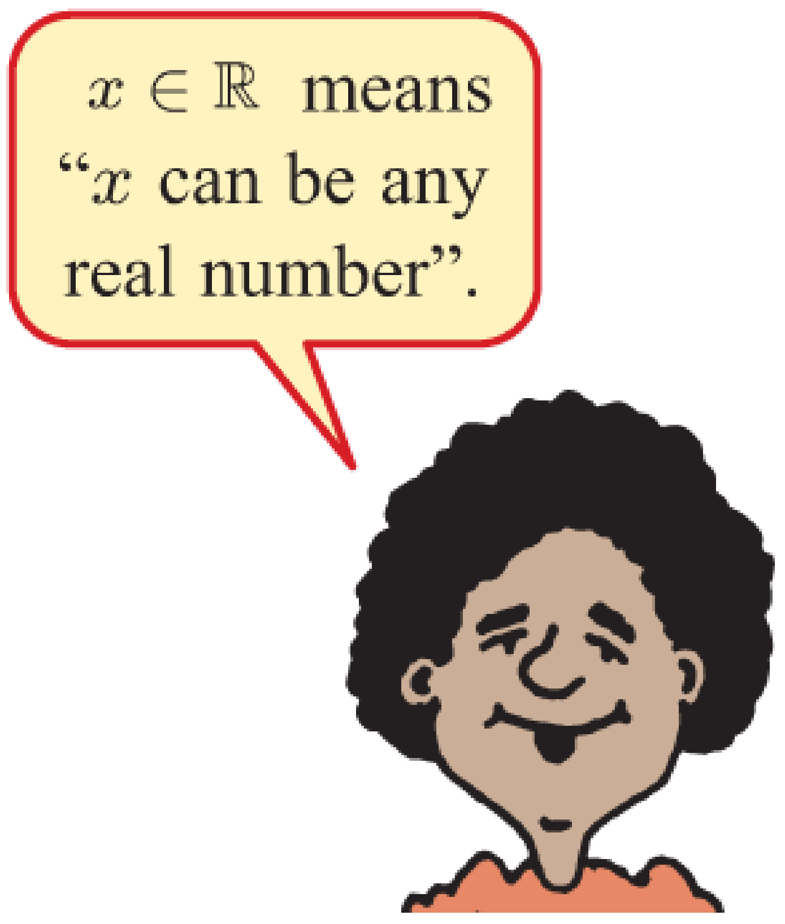
All values of $x \geqslant -1$ are included,
so the domain is $\{x \mid x \geqslant -1\}$.
All values of $y \geqslant -3$ are included,
so the range is $\{y \mid y \geqslant -3\}$.

(3)

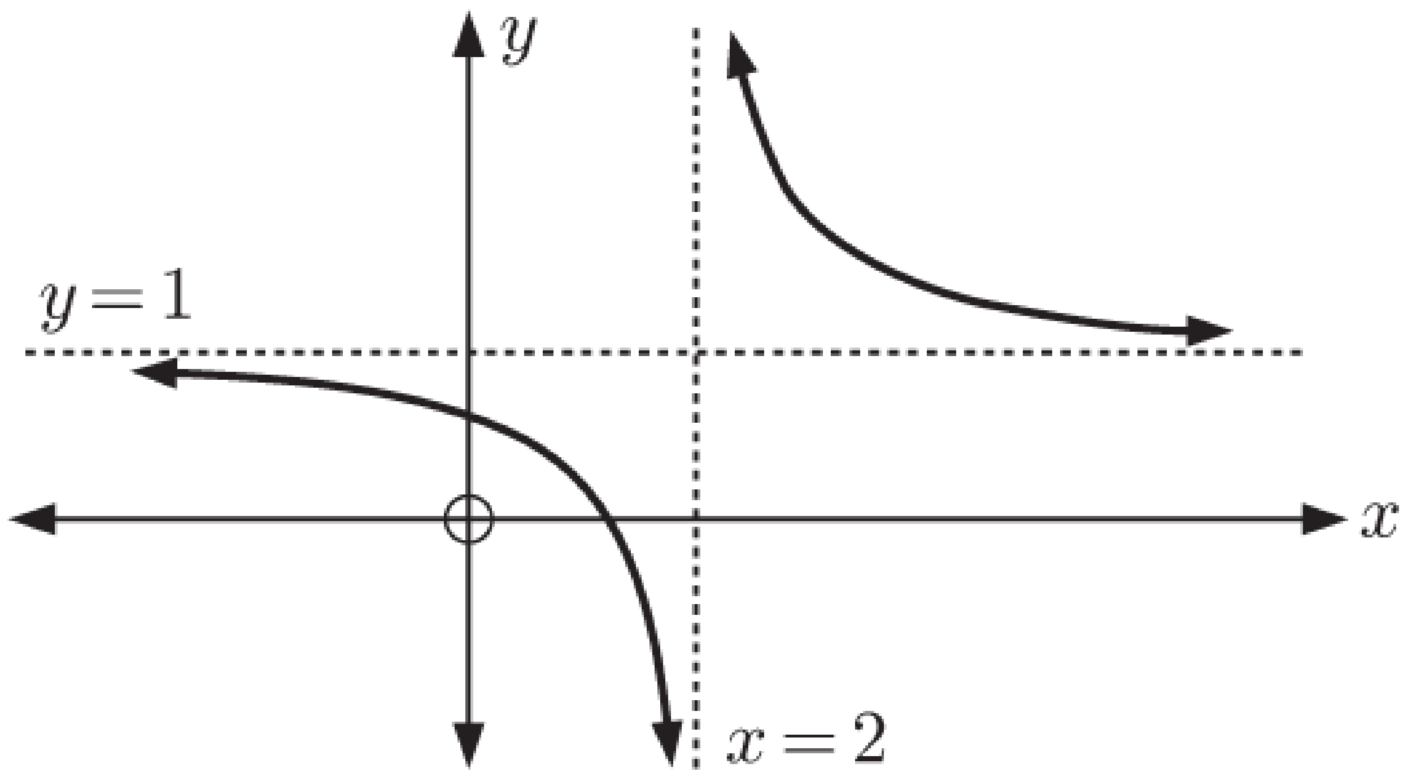


x can take any value,
so the domain is
$\{x \in \mathbb{R}\}$ or $x \in \mathbb{R}$.

y cannot be > 1,
so the range is $\{y \mid y \leqslant 1\}$.


$x \in \mathbb{R}$ means “x can be any real number”.

(4)



x can take all values except 2,
so the domain is $\{x \mid x \neq 2\}$ or $x \neq 2$.

y can take all values except 1,
so the range is $\{y \mid y \neq 1\}$ or $y \neq 1$.

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

Some examples of natural domains are shown in the table opposite.

Click on the icon to obtain software for finding the natural domain and range of different functions.

DOMAIN AND RANGE

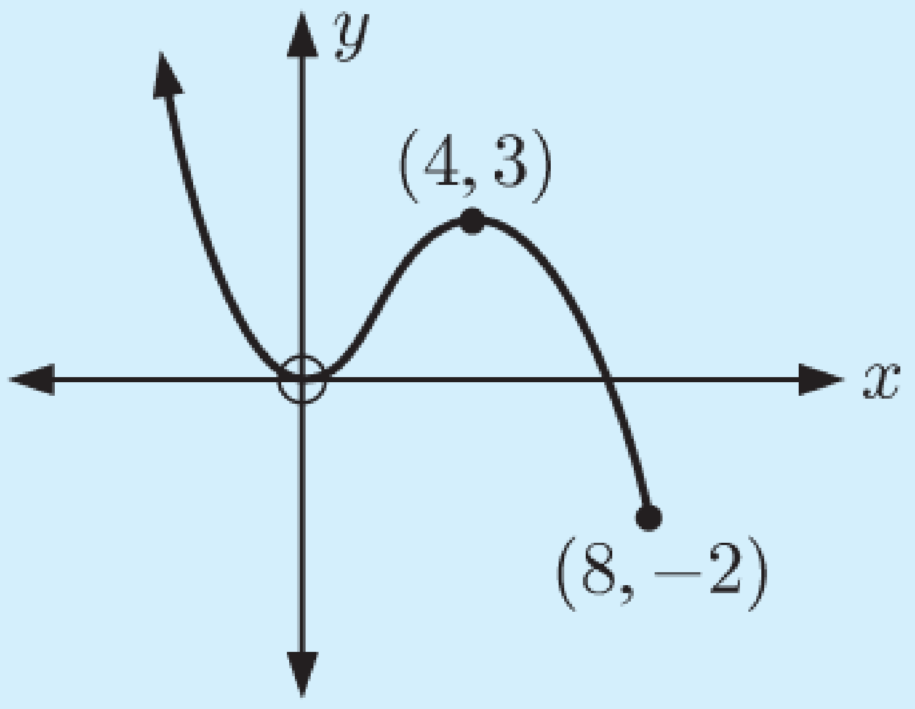
$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\frac{1}{x}$	$x \neq 0$
$\frac{1}{\sqrt{x}}$	$x > 0$

Example 4

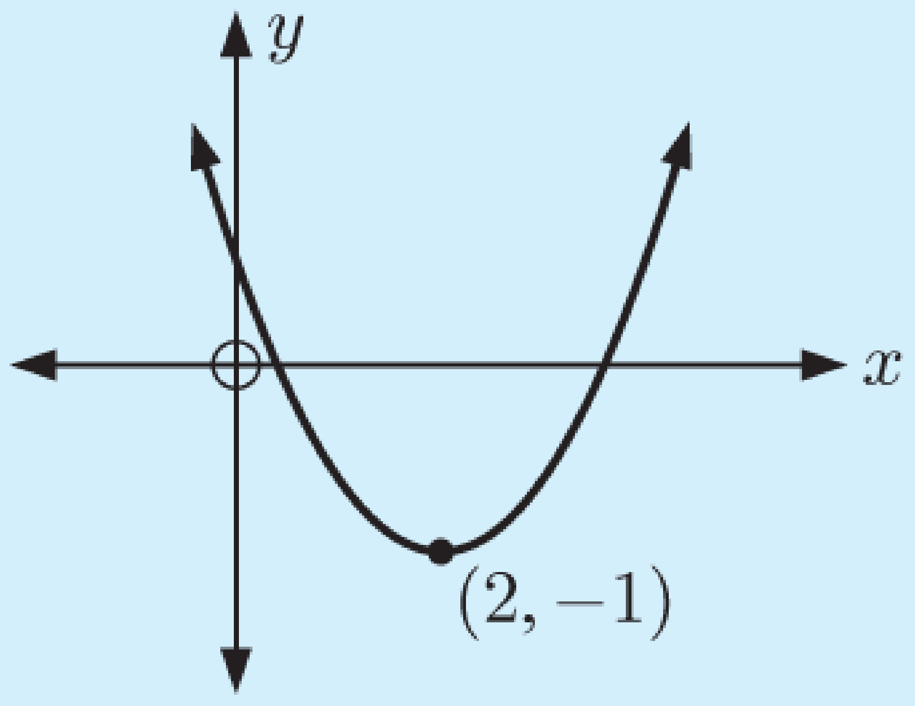
Self Tutor

For each of the following graphs, state the domain and range:

a



b



a Domain is $\{x \mid x \leqslant 8\}$.
Range is $\{y \mid y \geqslant -2\}$.

b Domain is $\{x \in \mathbb{R}\}$.
Range is $\{y \mid y \geqslant -1\}$.

EXERCISE 3C

1 A driver who exceeds the speed limit receives demerit points as shown in the table.

a Draw a graph to display this information.

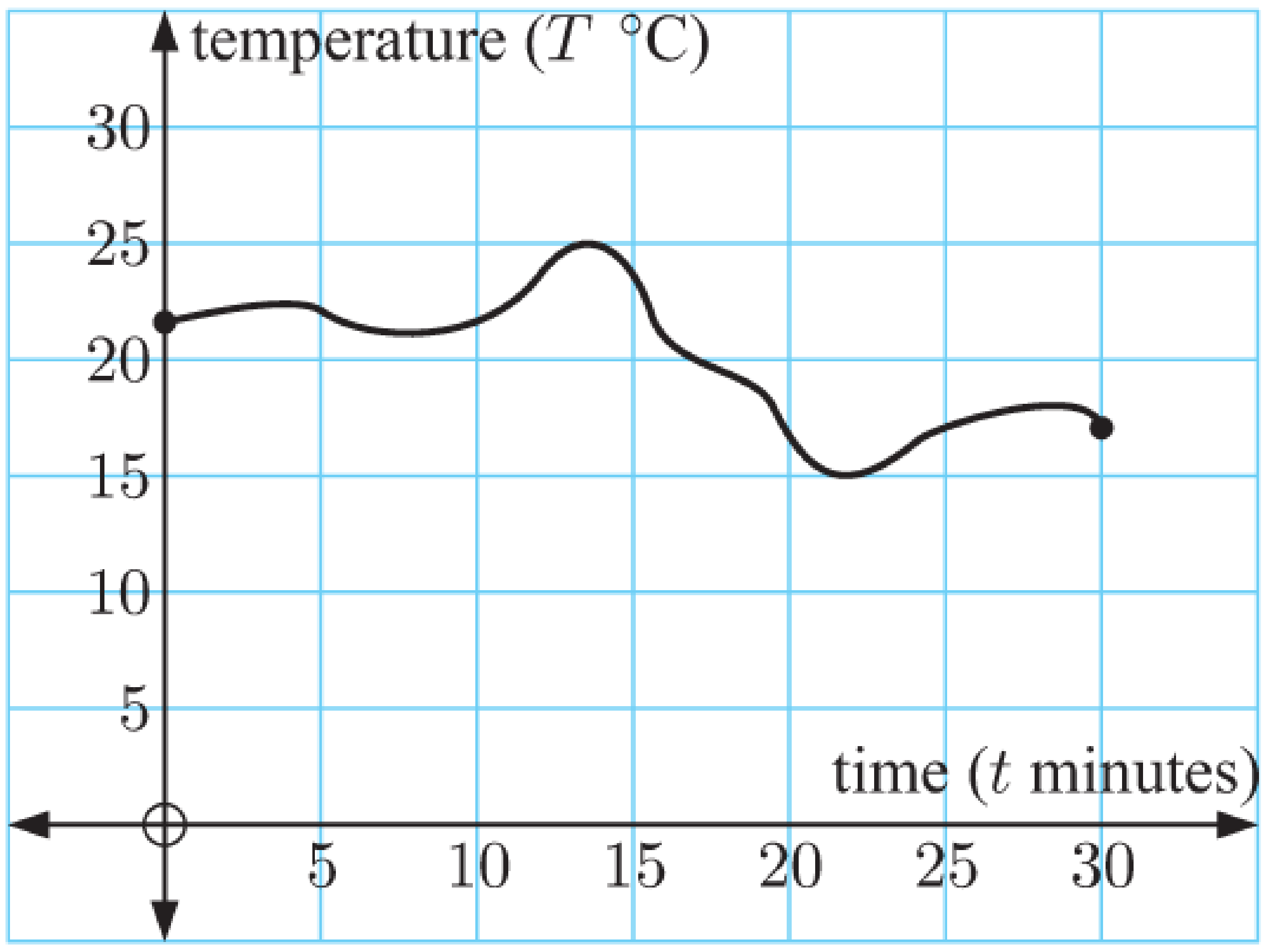
b Find the domain and range of the relation.

Amount over speed limit (x km h^{-1})	*Demerit points* (y)
$0 < x < 10$	2
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	5
$30 \leqslant x < 45$	7
$x \geqslant 45$	9

2 This graph shows the temperature in Barcelona over a 30 minute period as the wind shifts.

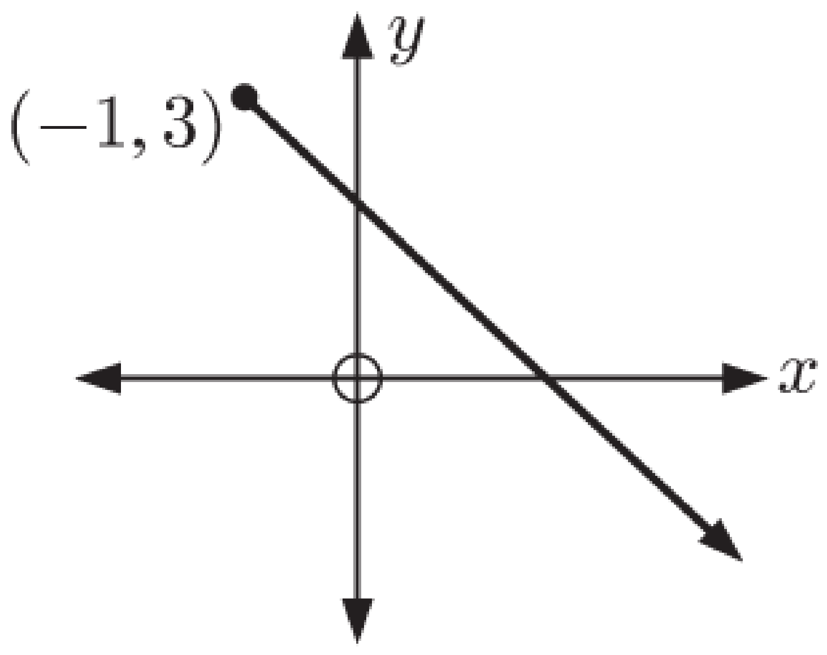
a Explain why a temperature graph like this must be a function.

b Find the domain and range of the function.

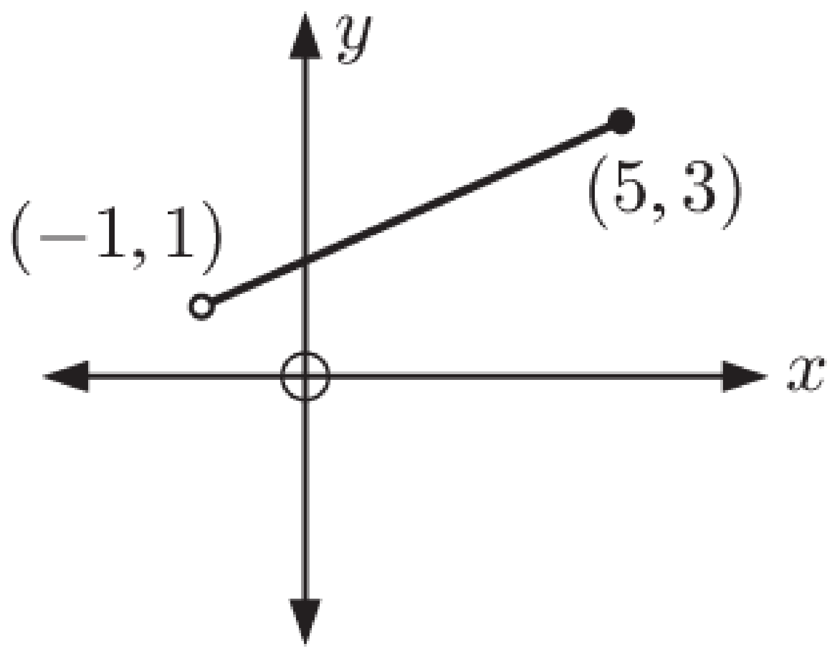


3 For each of the following graphs, find the domain and range:

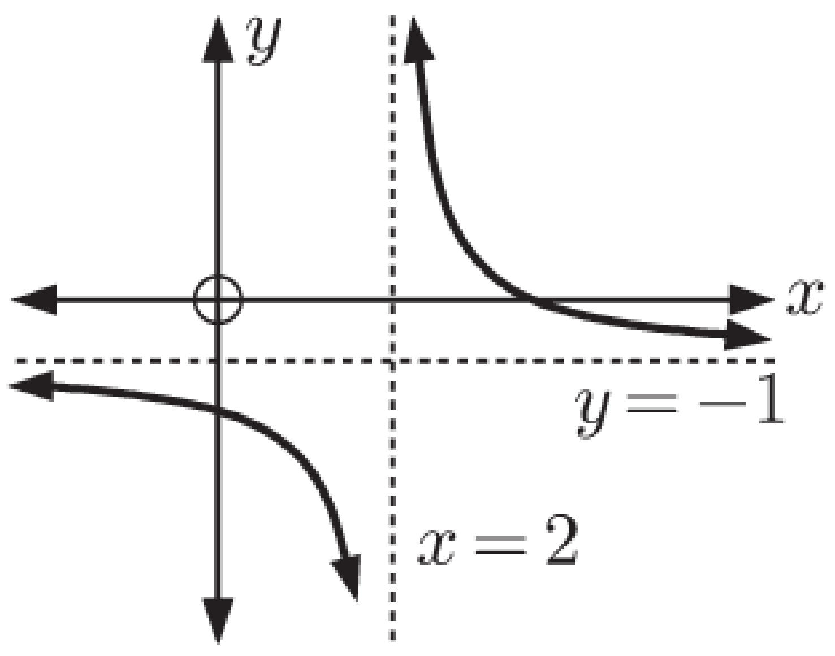
a



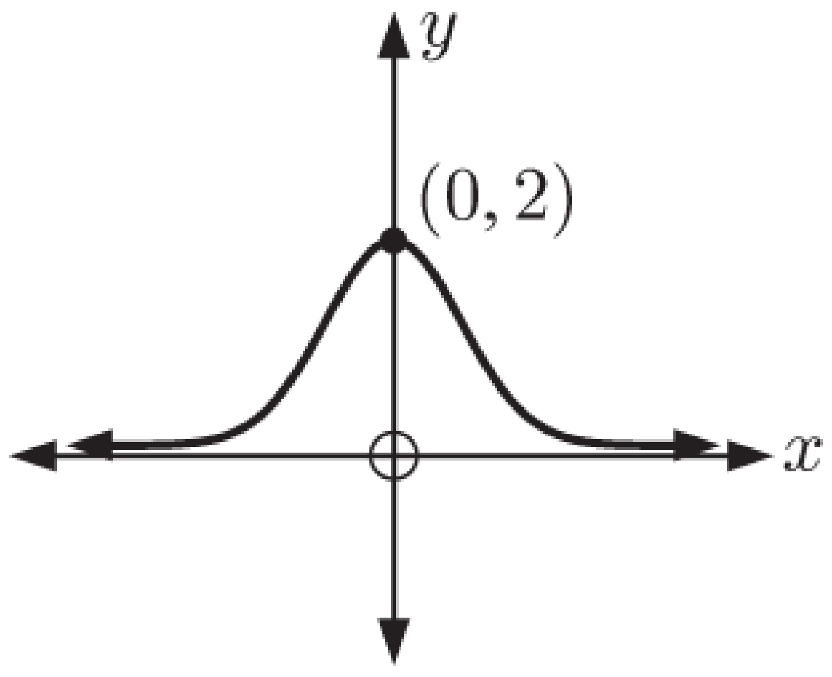
b



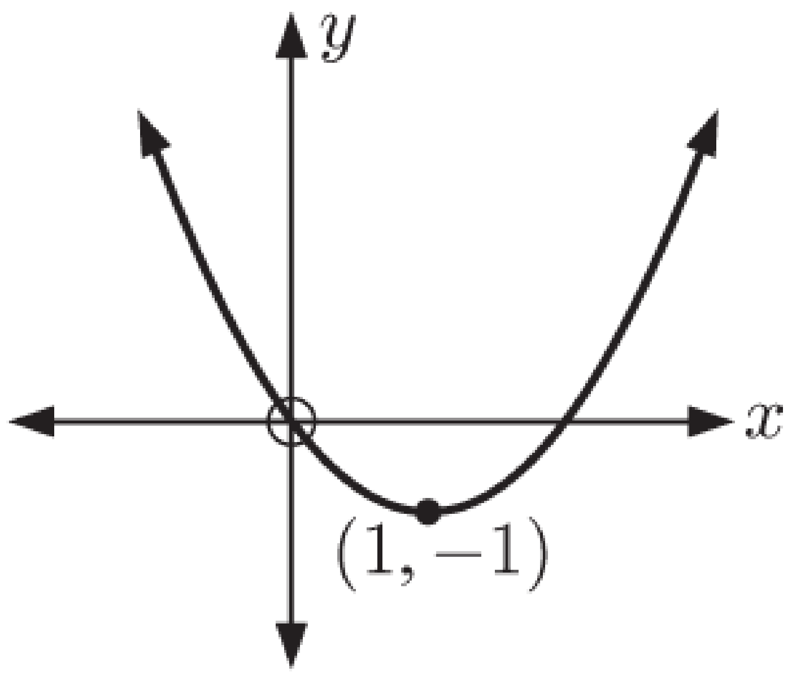
c



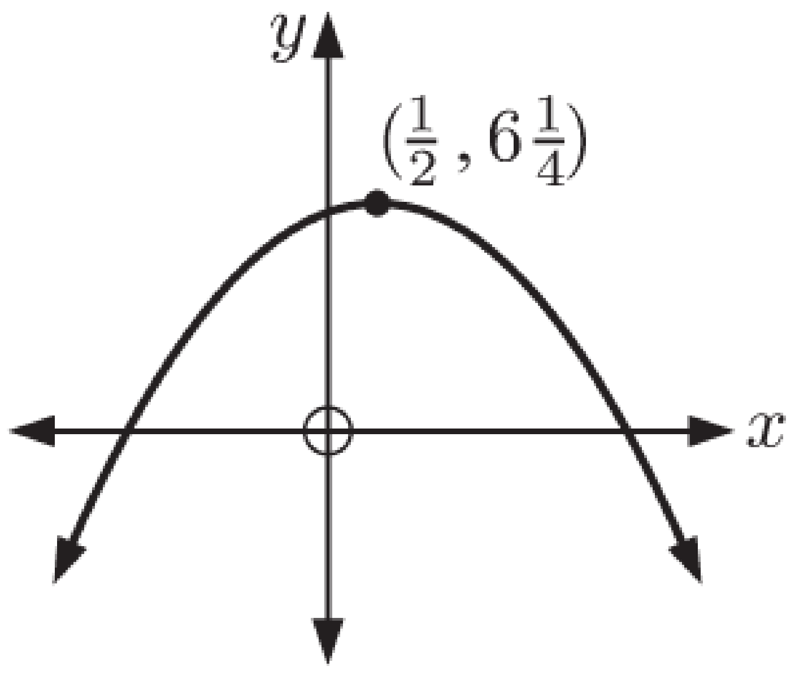
d



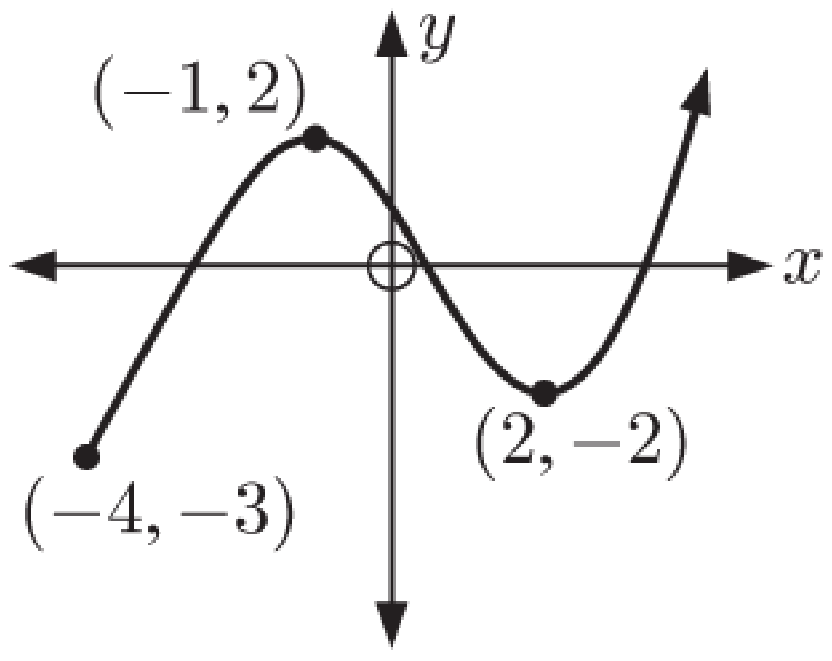
e



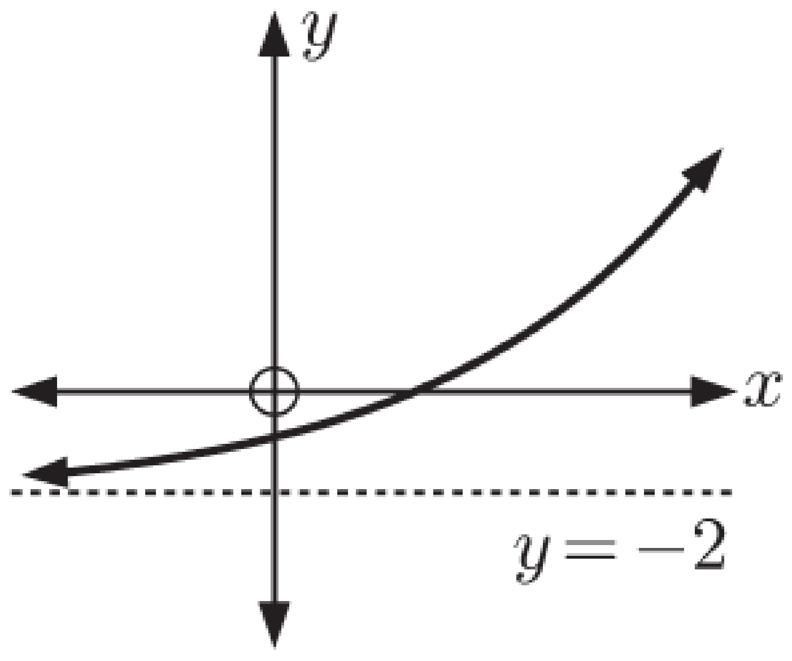
f



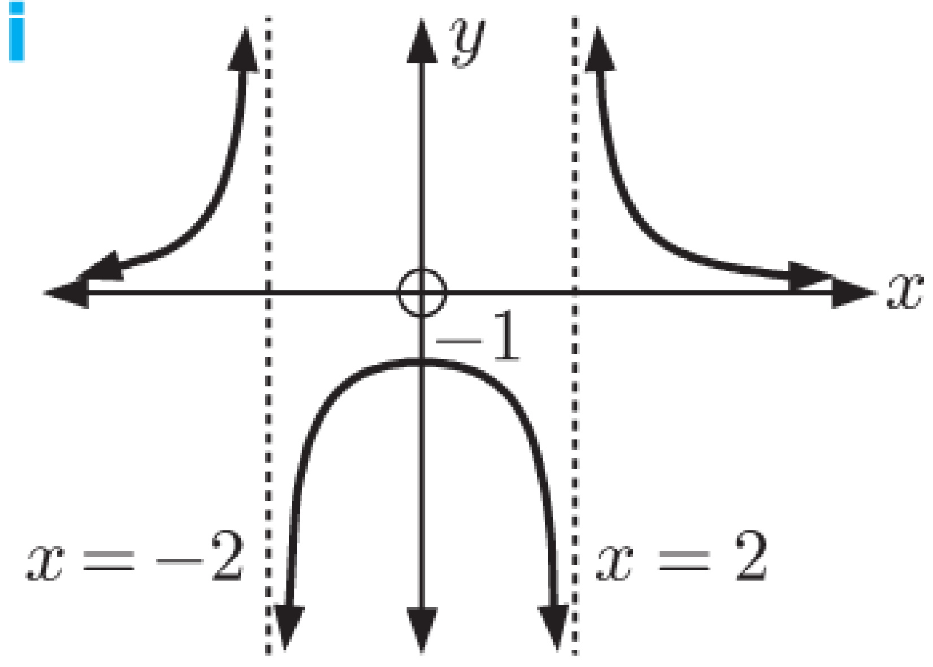
g



h



i



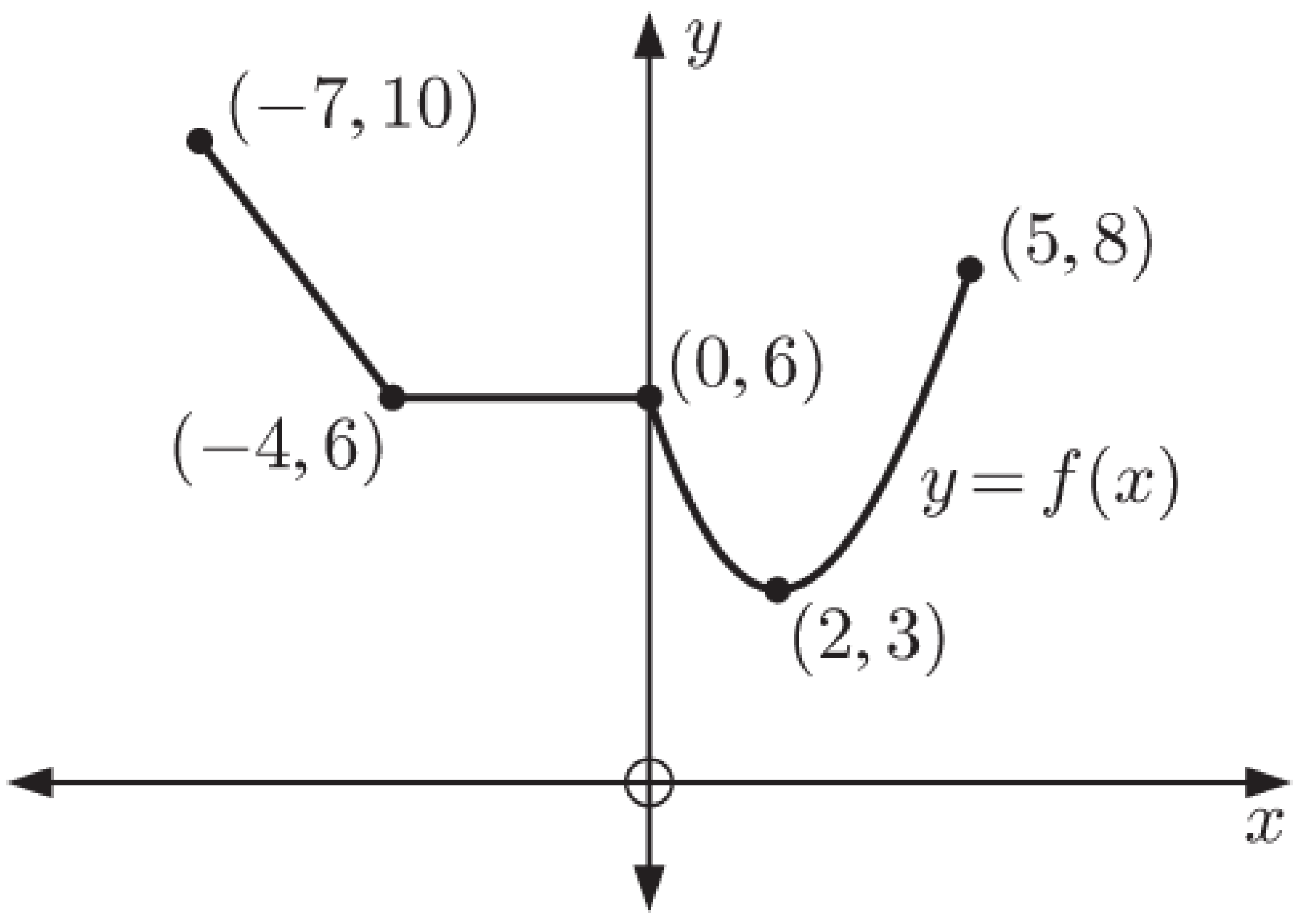
4 Consider the graph of $y = f(x)$ alongside.
Decide whether each statement is true or false:

a -5 is in the domain of f.

b 2 is in the range of f.

c 9 is in the range of f.

d $\sqrt{2}$ is in the domain of f.



Example 5

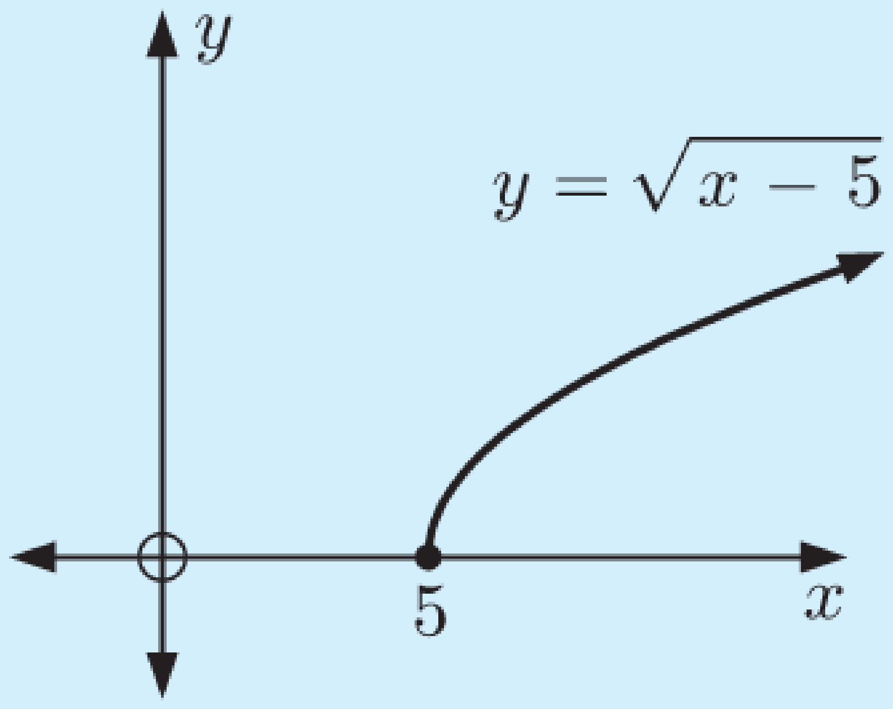
Self Tutor

State the domain and range of each of the following functions:

a $f(x) = \sqrt{x-5}$ **b** $f(x) = \dfrac{1}{x-5}$ **c** $f(x) = \dfrac{1}{\sqrt{x-5}}$

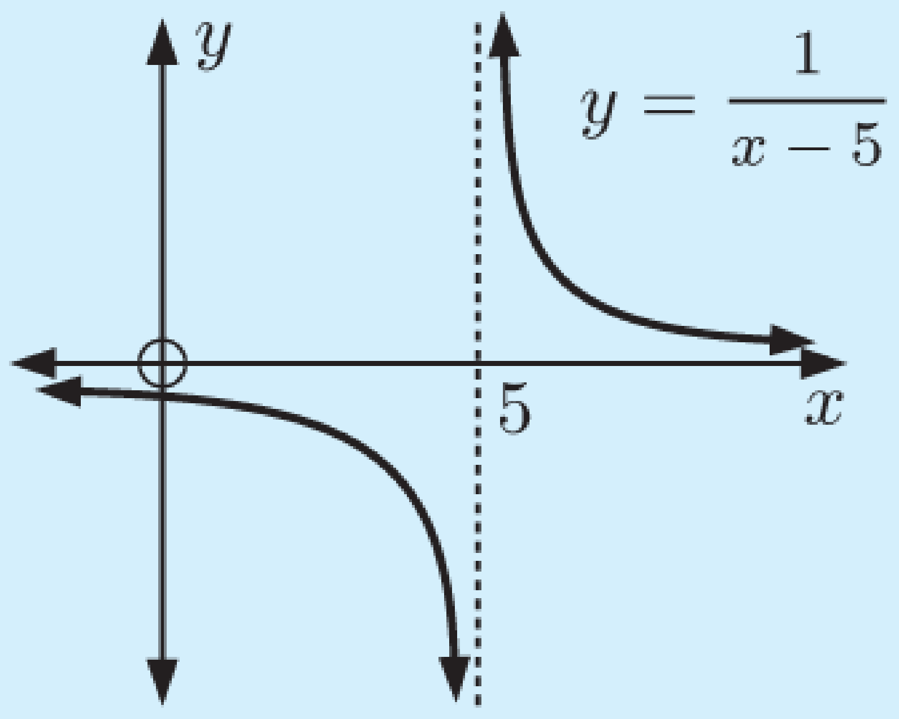
a $\sqrt{x-5}$ is defined when $x - 5 \geqslant 0$
$\therefore \ x \geqslant 5$

$\therefore$ the domain is $\{x \mid x \geqslant 5\}$.
A square root cannot be negative.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



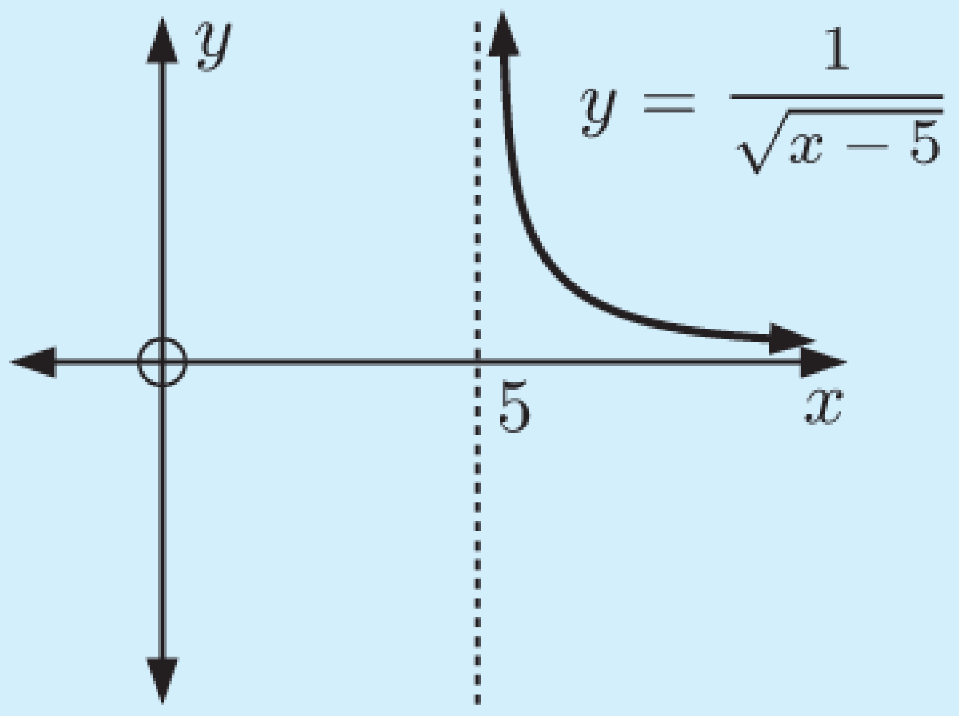
b $\dfrac{1}{x-5}$ is defined when $x - 5 \neq 0$
$\therefore \ x \neq 5$

$\therefore$ the domain is $\{x \mid x \neq 5\}$.
No matter how large or small x is,
$y = f(x)$ is never zero.
$\therefore$ the range is $\{y \mid y \neq 0\}$.



c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x - 5 > 0$
$\therefore \ x > 5$

$\therefore$ the domain is $\{x \mid x > 5\}$.
$y = f(x)$ is always positive and never zero.
$\therefore$ the range is $\{y \mid y > 0\}$.



5 Consider the function $f(x) = \sqrt{x}$.

a State the domain of the function.

b Copy and complete this table of values:

x	0	1	4	9	16
$f(x)$					

c Hence sketch the graph of the function.

d Find the range of the function.

DOMAIN AND RANGE

6 State the domain and range of each function:

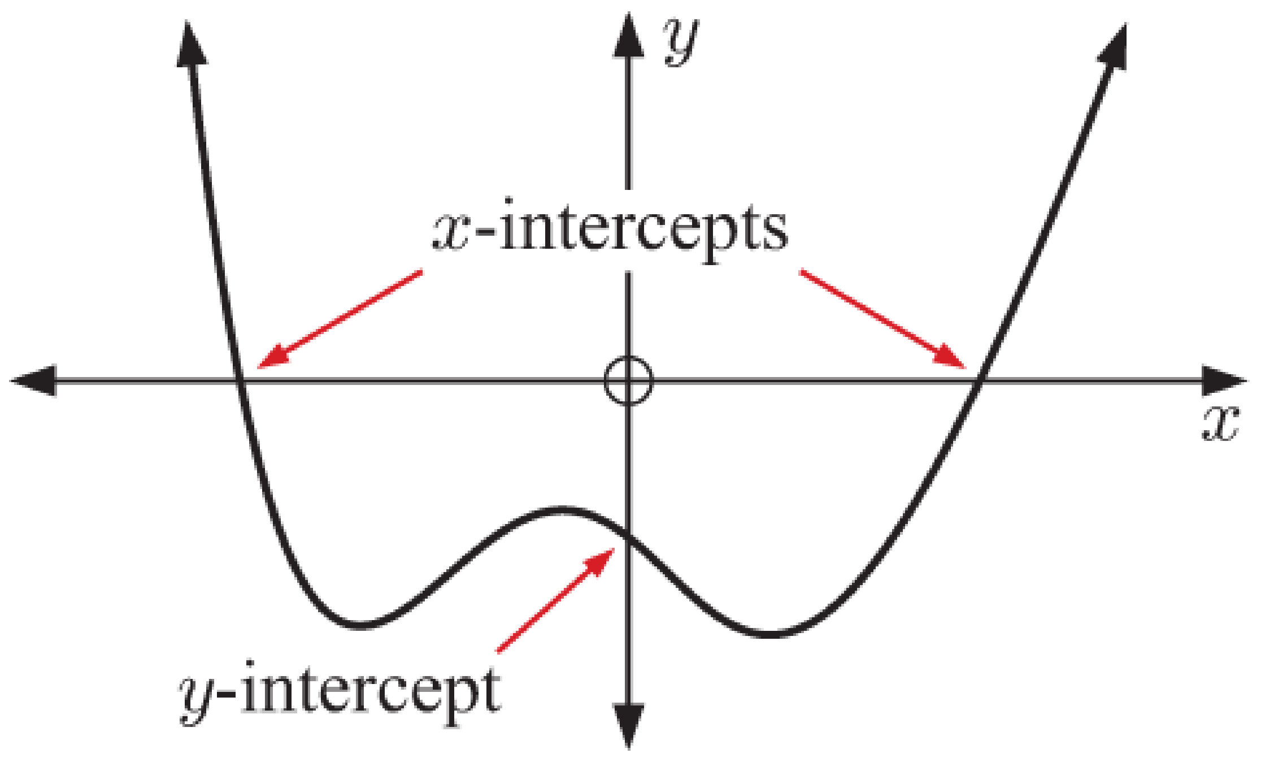
a $f(x)=\sqrt{x+6}$
b $f(x)=\dfrac{1}{x^2}$
c $f(x)=\dfrac{1}{x+1}$
d $f(x)=-\dfrac{1}{\sqrt{x}}$
e $f(x)=\dfrac{1}{3-x}$
f $f(x)=\sqrt{4-x}$

D GRAPHS OF FUNCTIONS

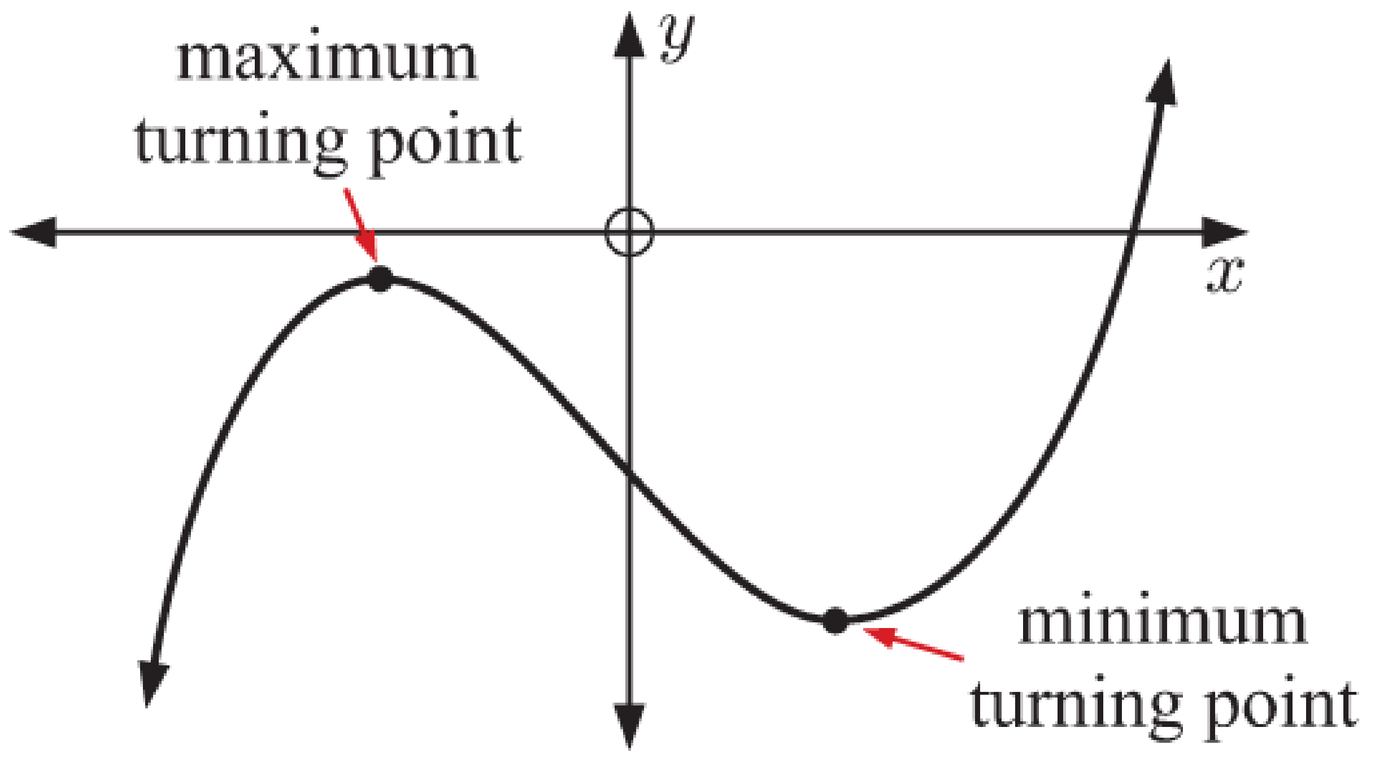
Many real world situations are modelled by mathematical functions which are difficult to analyse using algebra. However, we can use technology to help us graph and investigate the key features of an unfamiliar function.

The key features of a graph include:

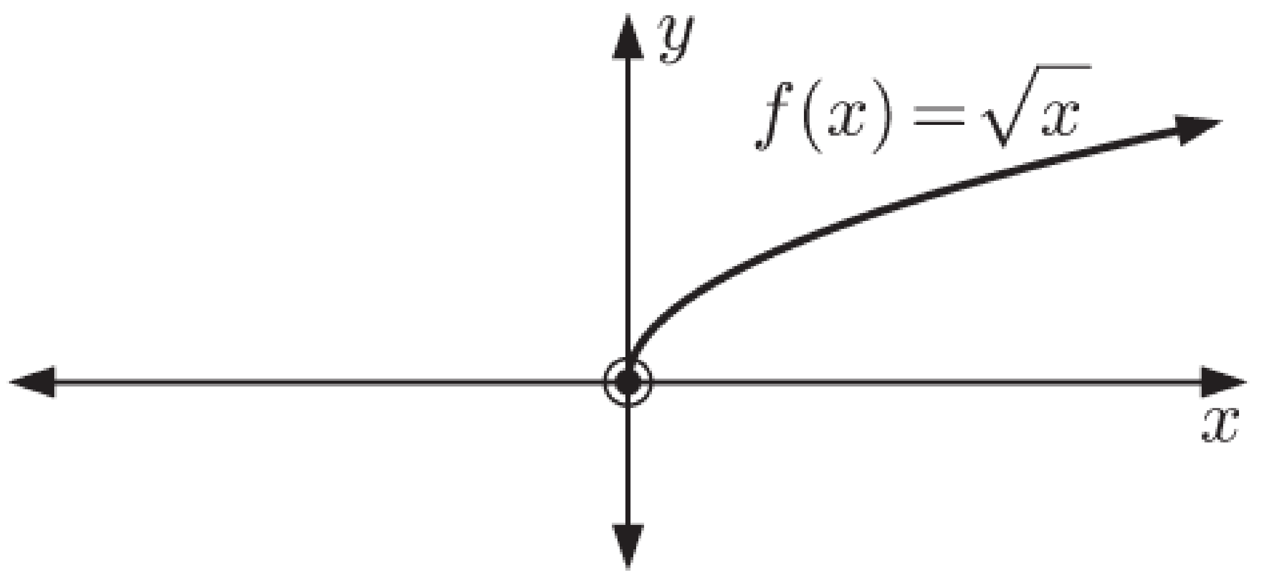
- **Axes intercepts** where the graph cuts the x and y-axes.



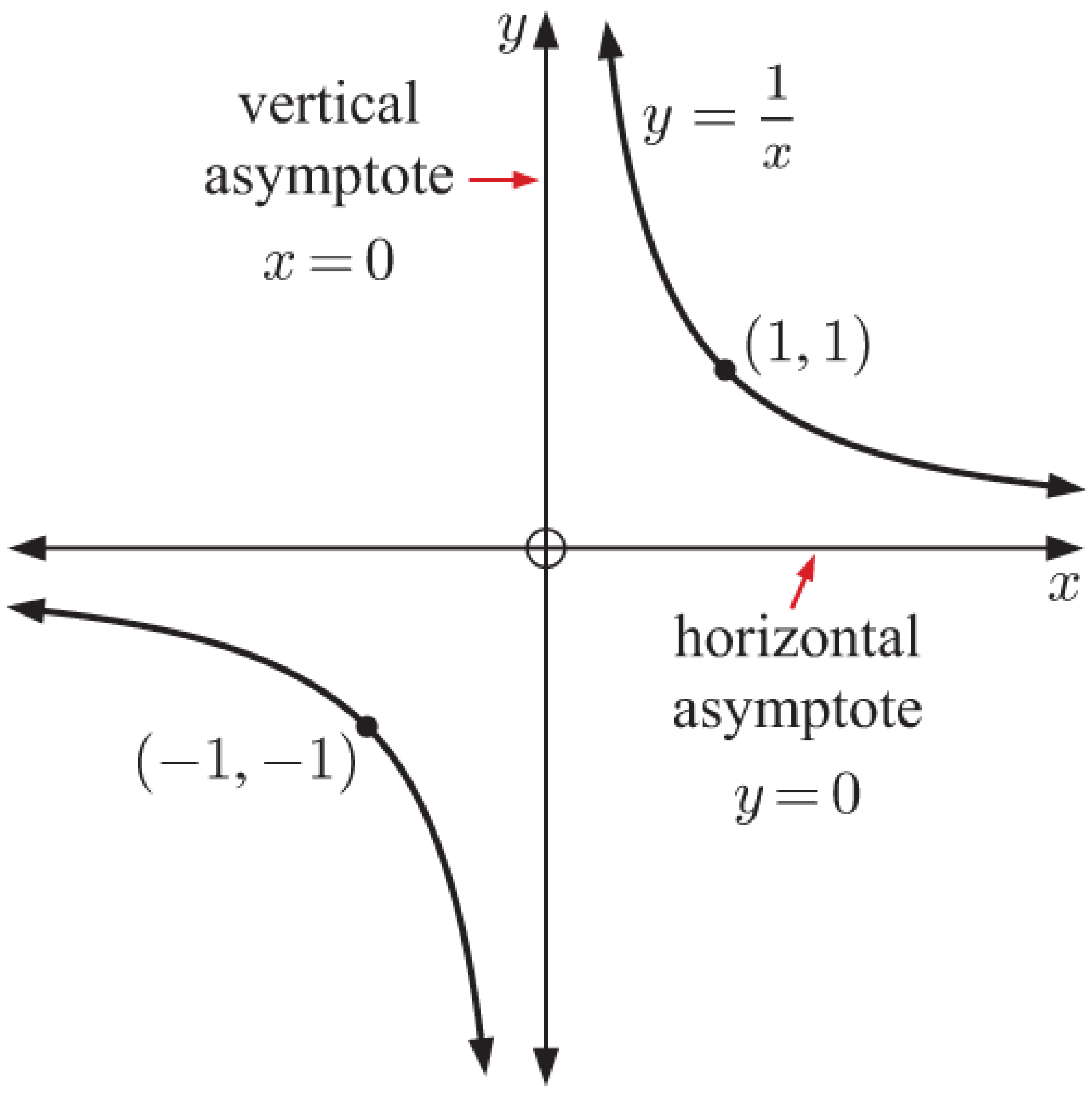
- **Minimum turning points** and **maximum turning points**.



- **Endpoints** of the graph and whether they are included in the domain.
  For example, $f(x)=\sqrt{x}$ does not exist for $x<0$.
  Its endpoint $(0, 0)$ is included in the graph, so it is marked with a filled-in dot.



- **Asymptotes**, which are lines that the graph gets closer and closer to, but never reaches.
  For example, in the graph of $y=\dfrac{1}{x}$, notice that:
  - $y=\dfrac{1}{x}$ is undefined when $x=0$
  - The graph gets closer and closer to the vertical line $x=0$, but never reaches it. We say that $x=0$ is a **vertical asymptote**.
  - The graph gets closer and closer to the horizontal line $y=0$, but never reaches it. We say that $y=0$ is a **horizontal asymptote**.



If there is no domain or range specified, start with a large viewing window. This will help to make sure you do not miss any features of the function.

Example 6 **Self Tutor**

Consider the function $y = \frac{2^x}{x} - 2$. Use technology to help answer the following:

a Find the axes intercepts.

b Find any turning points of the function.

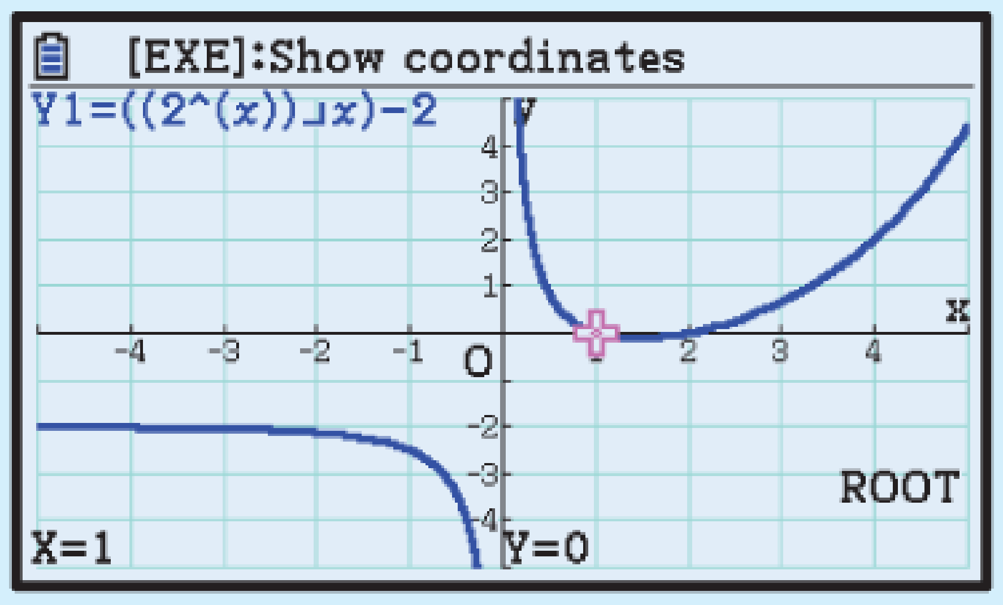
c Find any asymptotes of the function.

d State the domain and range of the function.

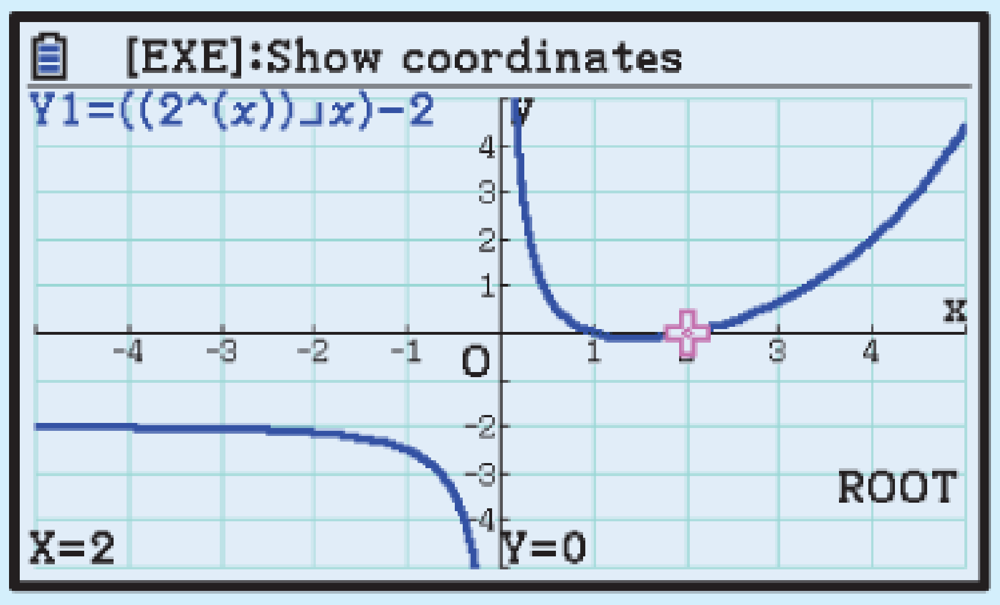
e Sketch the function, showing its key features.

GRAPHICS CALCULATOR INSTRUCTIONS

a When $x = 0$, $y = \frac{2^0}{0} - 2$ is undefined. There is no y-intercept.



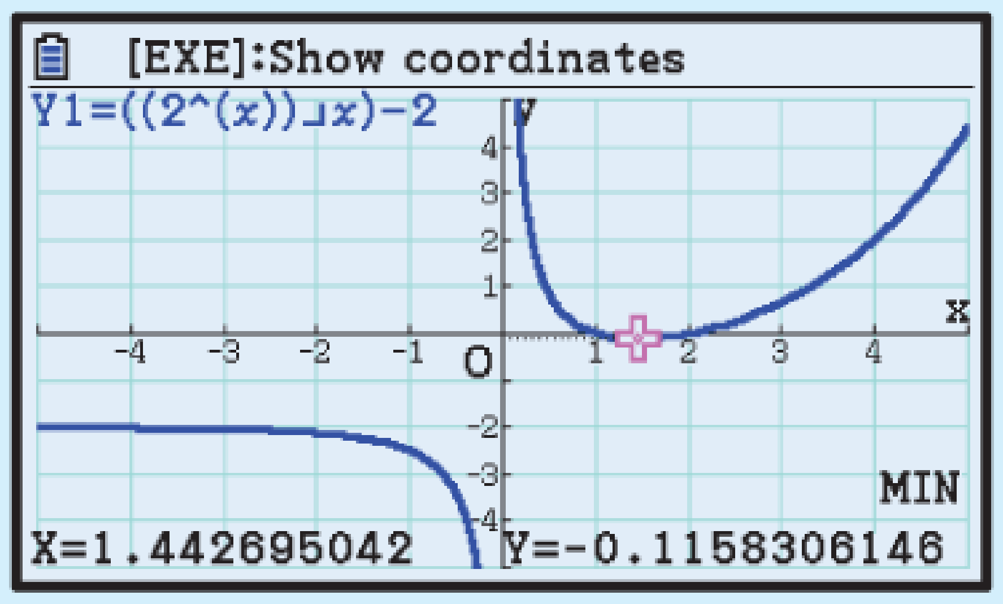
[EXE]:Show coordinates
Y1=((2^(x))⌟x)-2
ROOT
X=1 Y=0



[EXE]:Show coordinates
Y1=((2^(x))⌟x)-2
ROOT
X=2 Y=0

Using technology, the x-intercepts are 1 and 2.

b



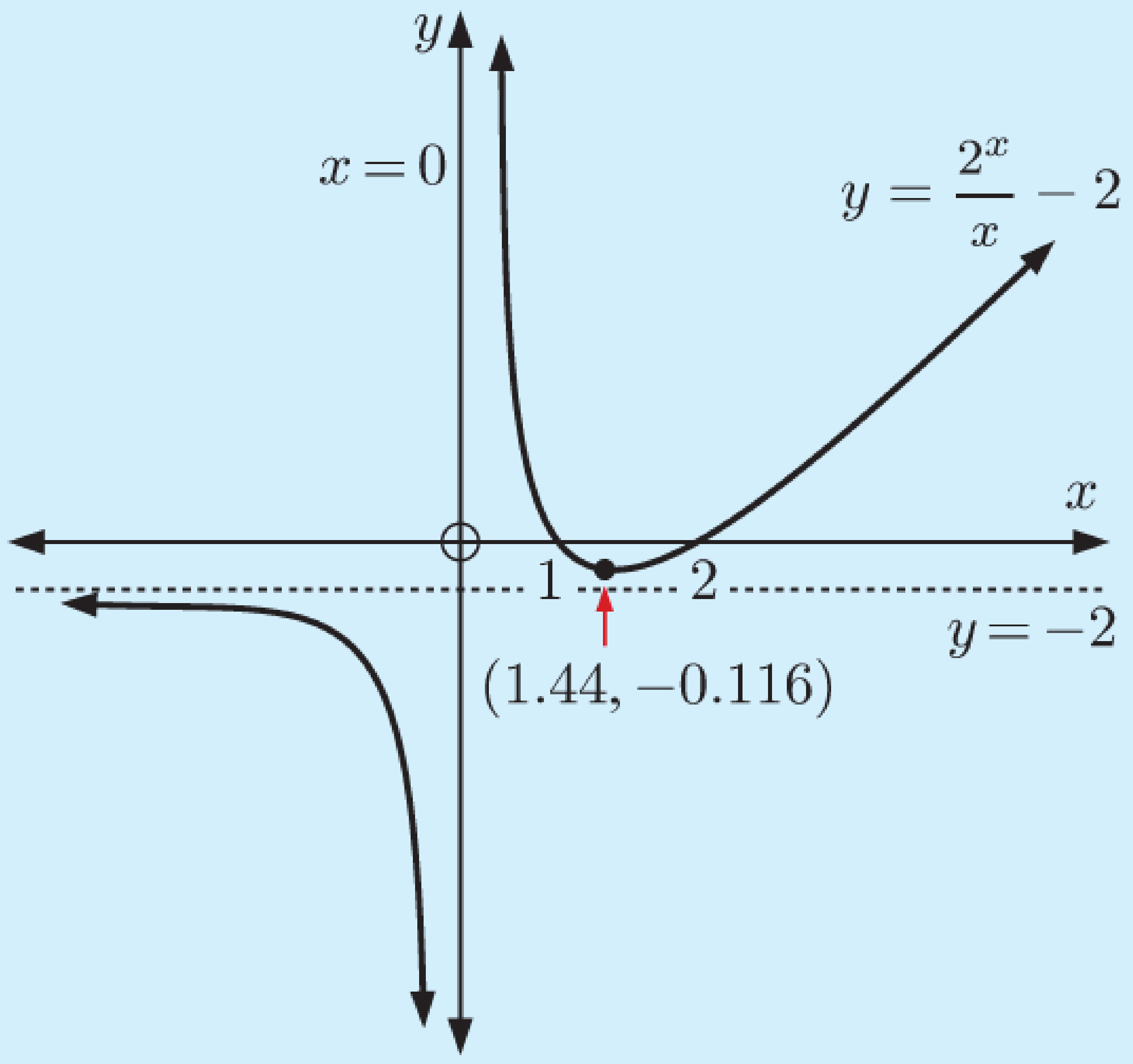
[EXE]:Show coordinates
Y1=((2^(x))⌟x)-2
MIN
X=1.442695042 Y=-0.1158306146

Using technology, there is a local minimum at $(1.44, -0.116)$.

c The graph appears to have a vertical asymptote at $x = 0$.
This is confirmed by the fact that y is undefined when $x = 0$.
As $x \to -\infty$, the graph gets closer to the line $y = -2$. So, $y = -2$ is a horizontal asymptote.

d The domain is $\{x \mid x \neq 0\}$.
The range is
$\{y \mid y < -2 \text{ or } y \geqslant -0.116\}$.

e

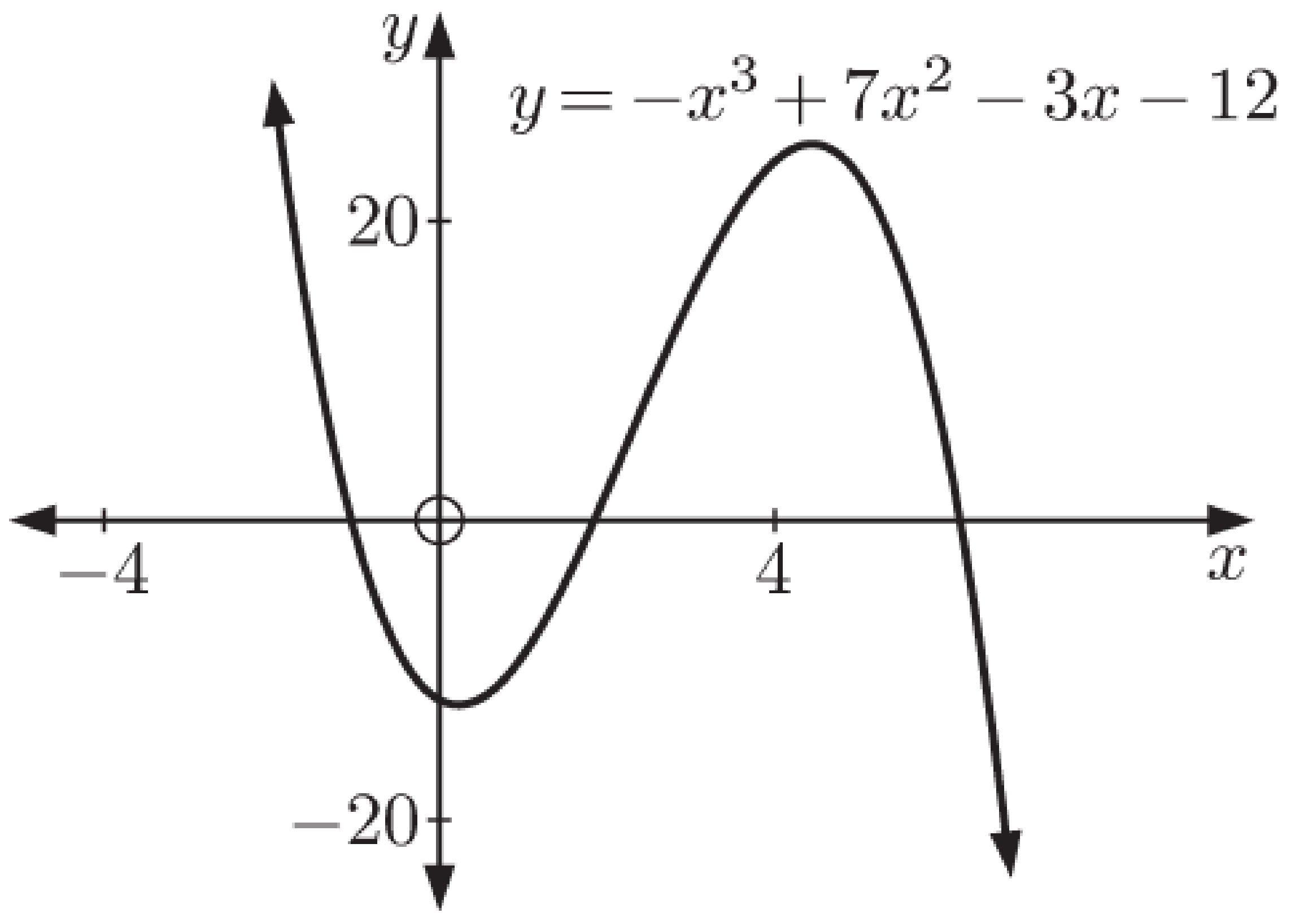


If you are asked to **sketch** a function, it should show the graph's general shape and its key features.

If you are asked to **draw** a function, it should be done more carefully to scale.

EXERCISE 3D

1 The graph of $y = -x^3 + 7x^2 - 3x - 12$ is shown alongside.



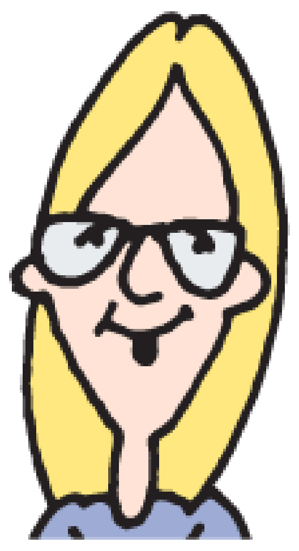
- **a** Find the y-intercept.
- **b** Use technology to find:
  - **i** the x-intercepts
  - **ii** the coordinates and nature of the turning points.

2 Consider the function $y = x^4 - 4x^3 + x^2 + 3x - 5$.

- **a** Find the axes intercepts.
- **b** Find the coordinates and nature of any turning points.
- **c** Discuss the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.
- **d** Sketch the function, showing the key features you have found.
- **e** State the range of the function.

3 Consider the function $y = \dfrac{6}{x-3} - 2$.

- **a** Find the axes intercepts.
- **b** Find the asymptotes of the function.
- **c** Sketch the function, including the features you have found.
- **d** State the domain and range of the function.



If x appears in the denominator of a fraction, there will be a vertical asymptote for any value of x which makes the denominator zero.

4 For each of the functions given, use technology to answer the following:

- **i** Find the axes intercepts.
- **ii** Find the coordinates and nature of any turning points.
- **iii** Find any asymptotes.
- **iv** State the domain and range of the function.
- **v** Sketch the function, showing its key features.

GRAPHING PACKAGE

a $y = \frac{1}{2}x(x-4)(x+3)$

b $y = x^3 - 2x^2 + 6$

c $y = \sqrt{x^2+4}$

d $y = \sqrt{x^2-4}$

e $y = \sqrt{9-x^2}$

f $y = \dfrac{3}{x-1} + 2$

g $y = \dfrac{x-6}{x+3}$

h $y = \dfrac{2x+3}{1-x}$

i $y = \dfrac{3x-9}{x^2-x-2}$

j $y = 3^x$

k $y = \frac{4}{5}x^4 + 5x^3 + 5x^2 + 2, \quad -5 \leqslant x \leqslant 1$

l $y = x\,2^{-x}$

m $y = \dfrac{1-x^2}{(x+2)^2}, \quad -5 \leqslant x \leqslant 5$

n $y = \dfrac{1}{1+2^{-x}}$

o $y = \dfrac{x^2}{2^x} - 1$

To solve $f(x) = g(x)$, find the x-coordinates of the intersection points of the graphs.

5 **a** Sketch the graphs of $f(x) = x^2 - \frac{4}{x}$ and $g(x) = -x^2 + 11$ for $-6 \leqslant x \leqslant 6$.

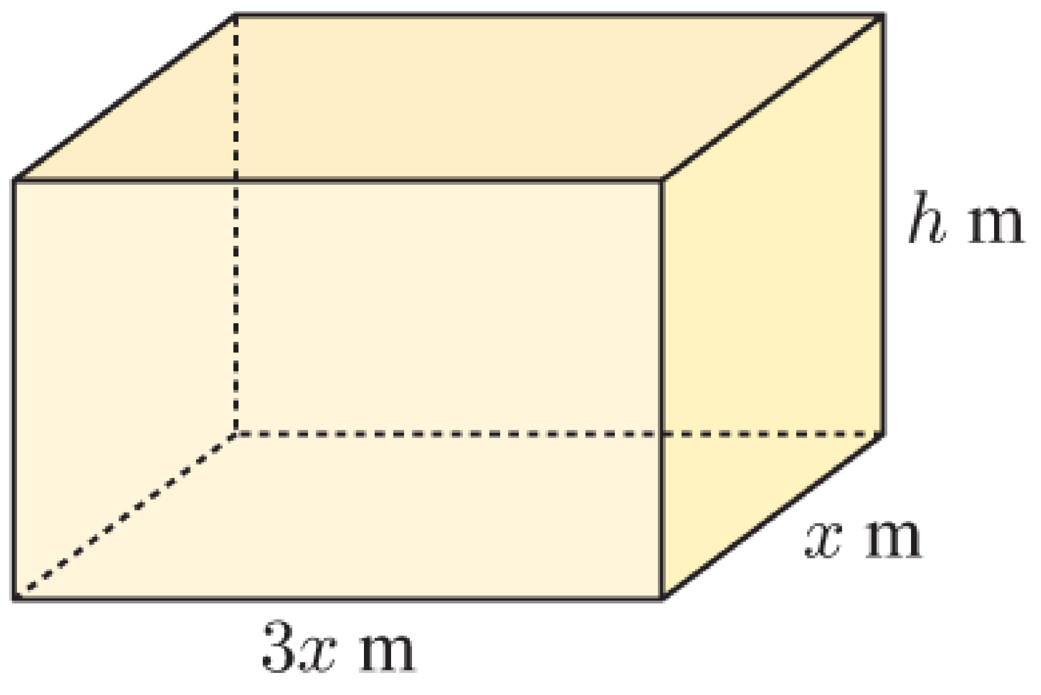
b One of the solutions to $f(x) = g(x)$ is $x \approx 2.51$. Use technology to determine the two negative solutions.

6 **a** Sketch the graphs of $f(x) = 3^{-x^2}$ and $g(x) = x^2$ on the same set of axes.

b Discuss the features of the graph of $y = 3^{-x^2}$.

c Solve $x^2 = 3^{-x^2}$.

7 A closed box is to be constructed with length three times its width. The volume of the box must be 6 m^3.



a If the box is x metres wide and h metres high, show that $h = \frac{2}{x^2}$.

b Hence show that the surface area of the box is given by $A(x) = 6x^2 + \frac{16}{x}$ m^2.

c Find the vertical asymptote of $y = A(x)$.

d Draw the graph of $y = A(x)$.

e Use your graph to estimate the surface area of a box with width 2 metres. Check your answer using technology.

f Find the coordinates of the turning point of $y = A(x)$. Explain the significance of this point.

8 When an anaesthetic is administered, its effect is modelled by

$$E(t) = 640t \times 4^{-t} \text{ units,}$$

where $t \geqslant 0$ is the time in hours after the injection of the drug.

a Find $E(1)$ and $E(4)$. Interpret your answers.

b Sketch the graph of $E(t)$ for $t \geqslant 0$.

c Discuss the behaviour of the function as t increases.

d Find the turning point of $E(t)$. Explain the significance of this point.

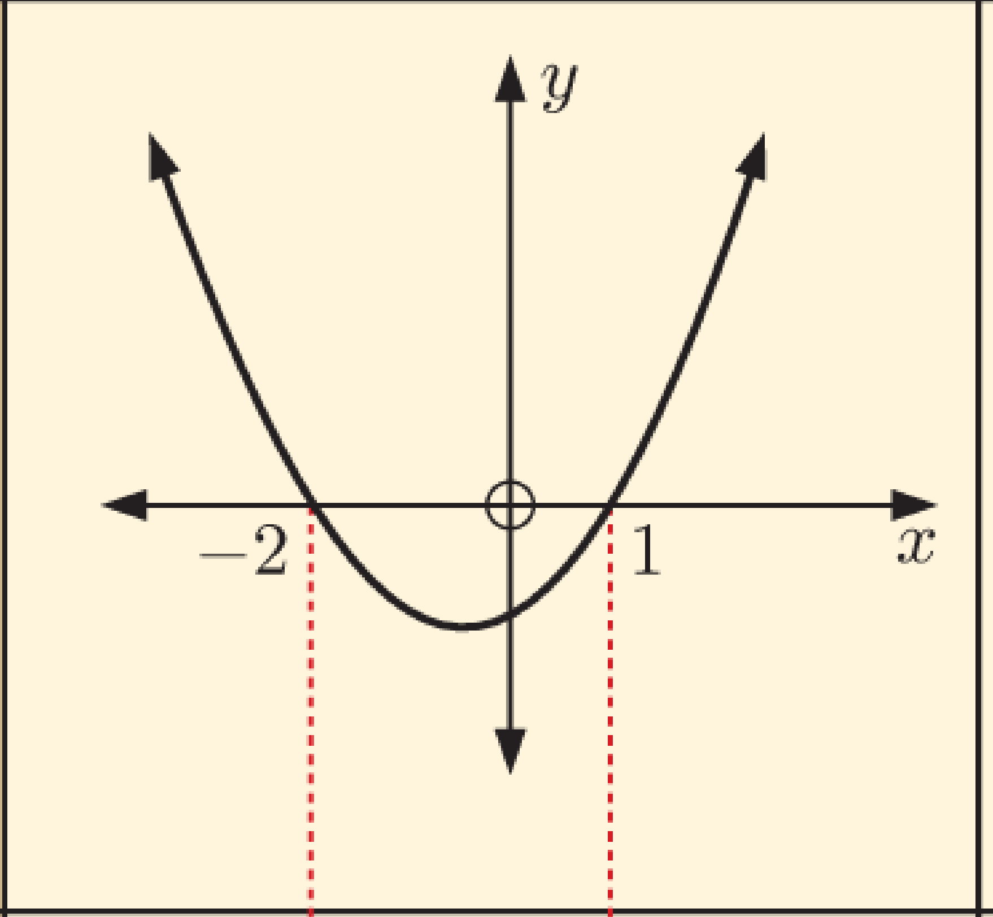
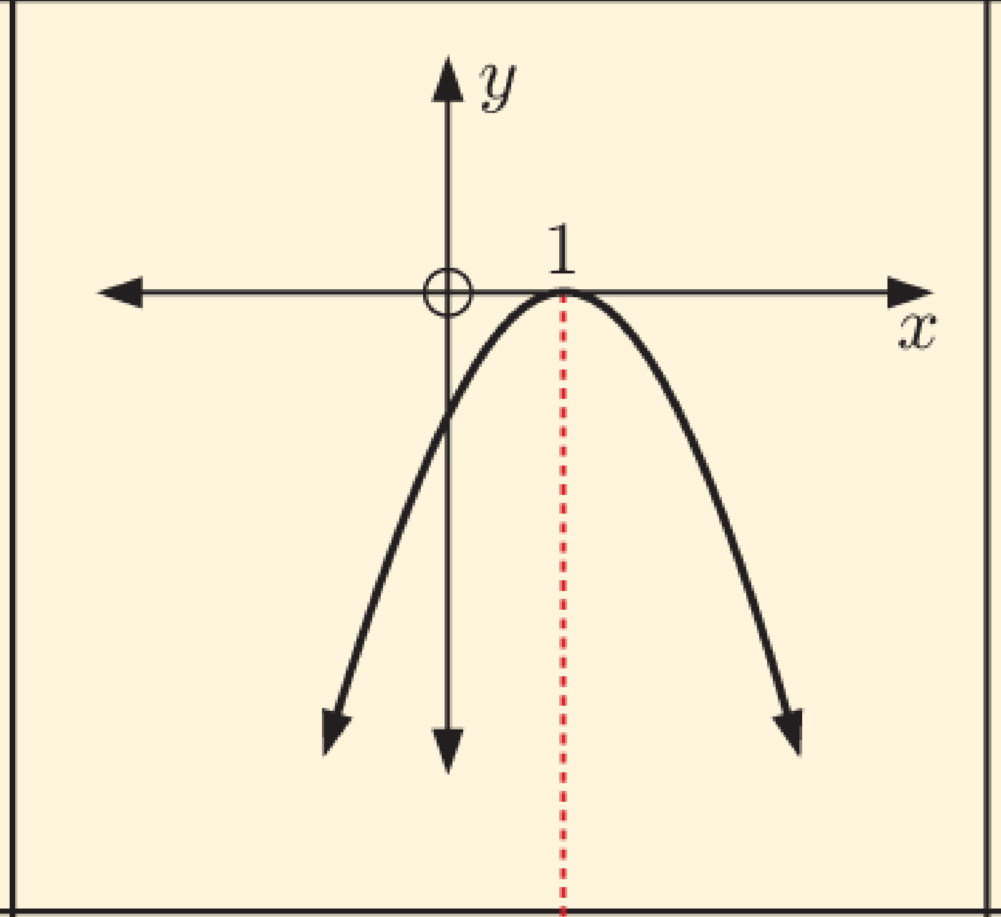
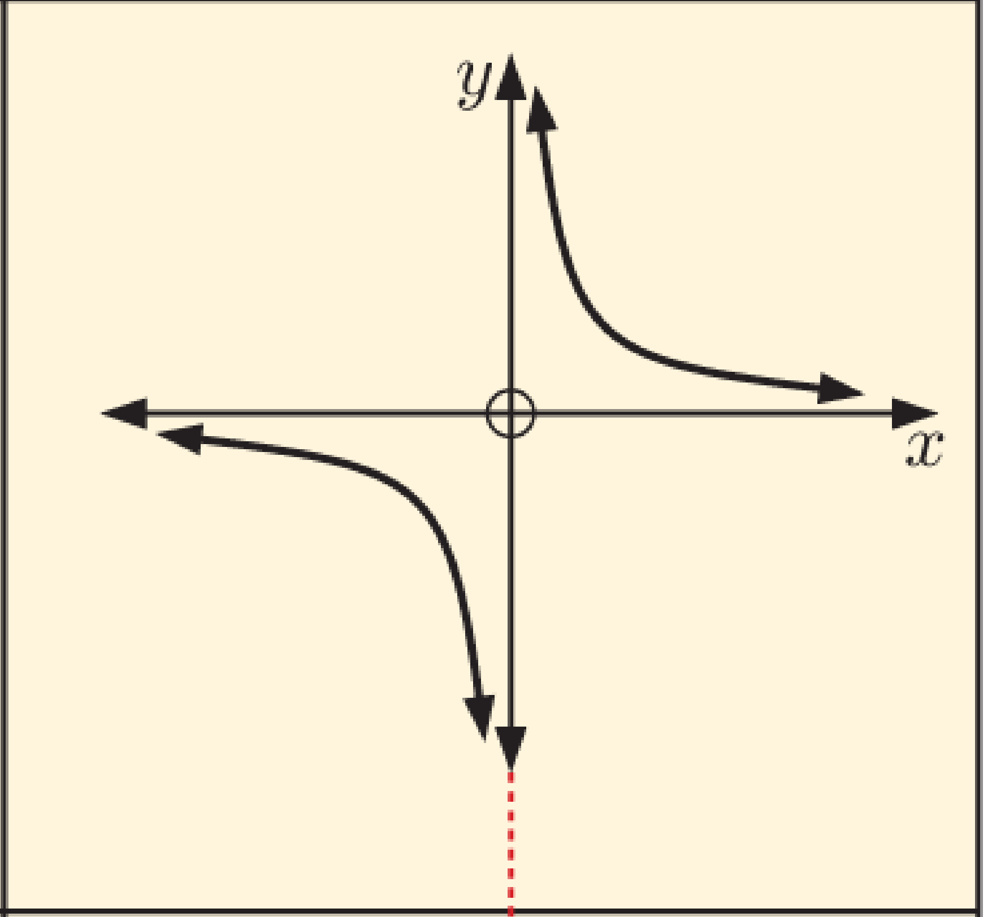
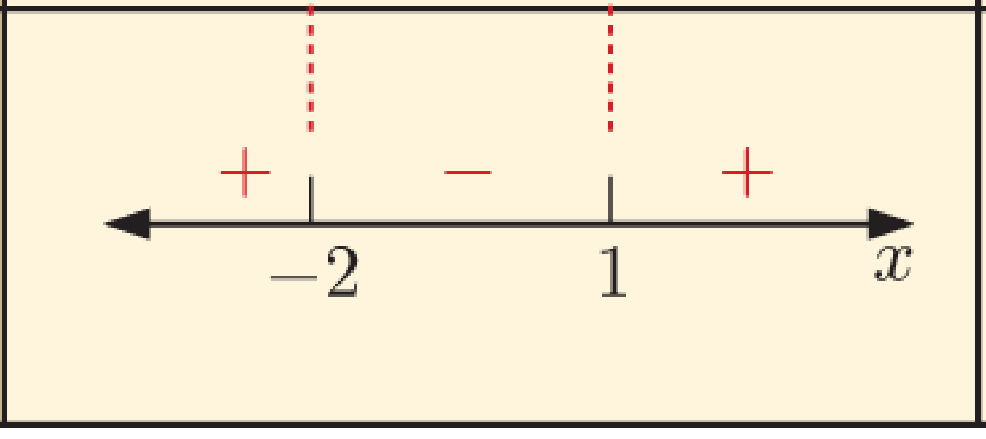
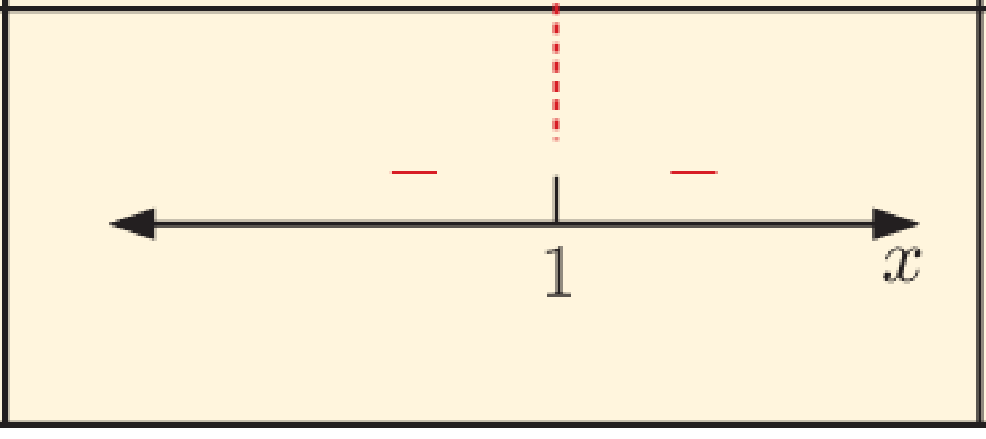
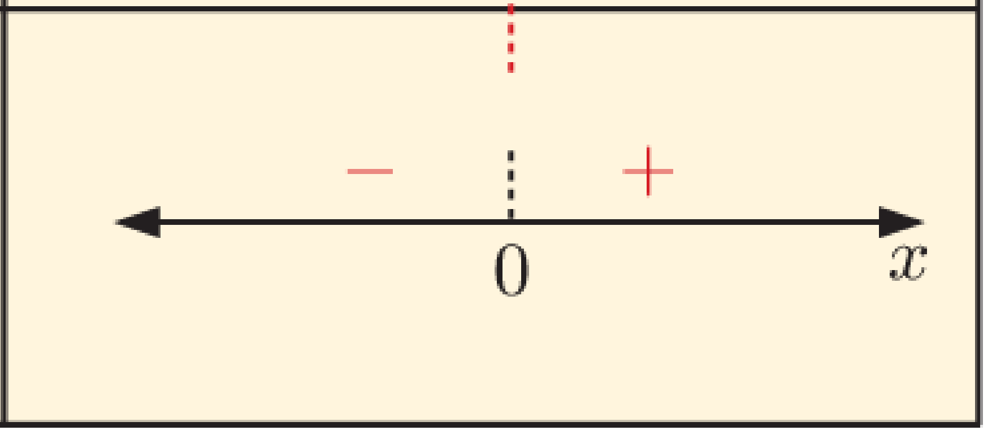
E SIGN DIAGRAMS

Sometimes we do not wish to draw a time-consuming graph of a function, but wish only to know when the function is positive, negative, zero, or undefined. A **sign diagram** allows us to do this.

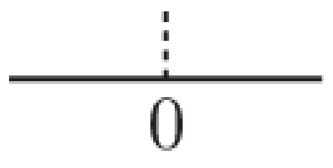
For the function $f(x)$, the sign diagram consists of:

- a **horizontal line** which represents the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating where the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of the graph of $y = f(x)$, and the **roots** of the equation $f(x) = 0$
- values of x where the graph is undefined.

Consider the three functions below:

Function	$y=(x+2)(x-1)$	$y=-2(x-1)^2$	$y=\dfrac{4}{x}$
Graph			
Sign diagram			

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates that the graph **cuts** the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates that the graph **touches** the x-axis.
-  indicates that a function is **undefined** at $x=0$.

DEMO

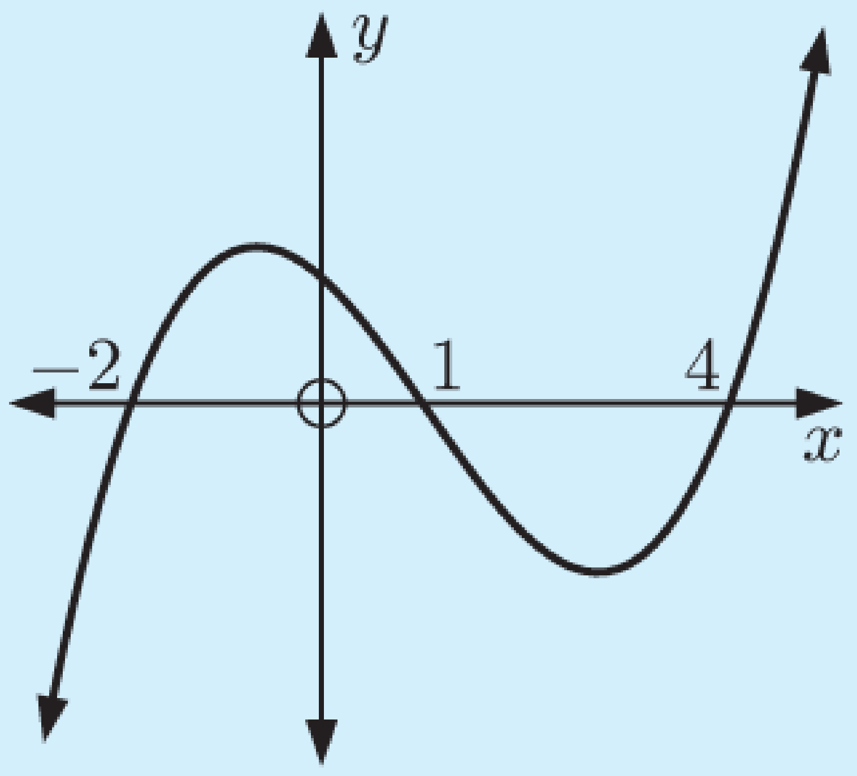
In general:

- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

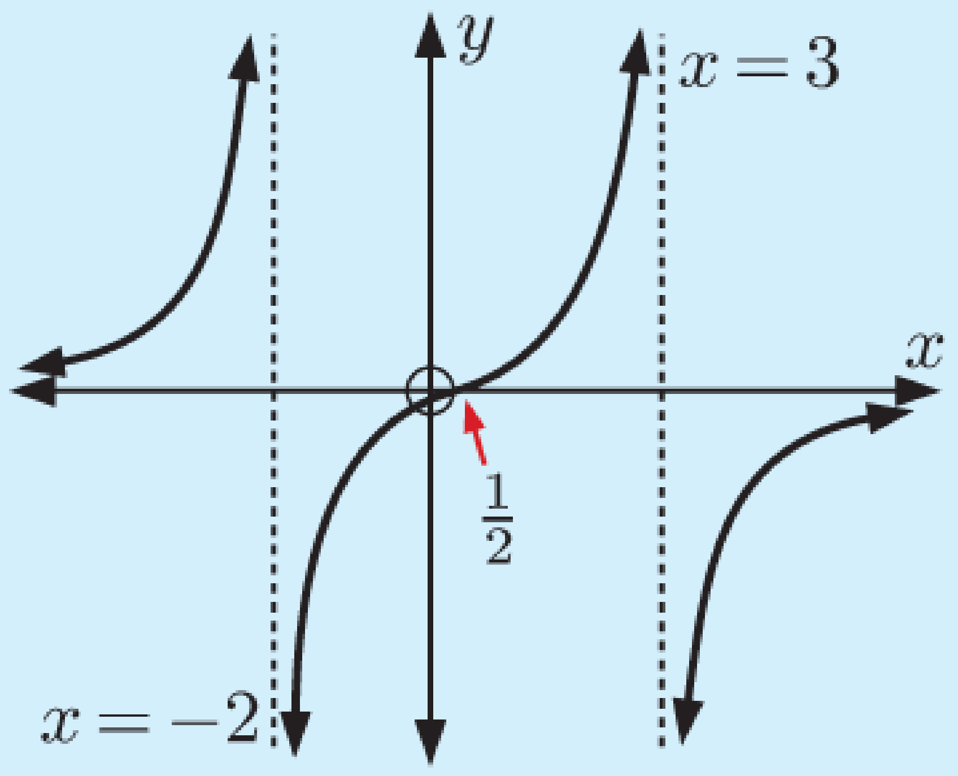
Example 7 **Self Tutor**

Draw a sign diagram for:

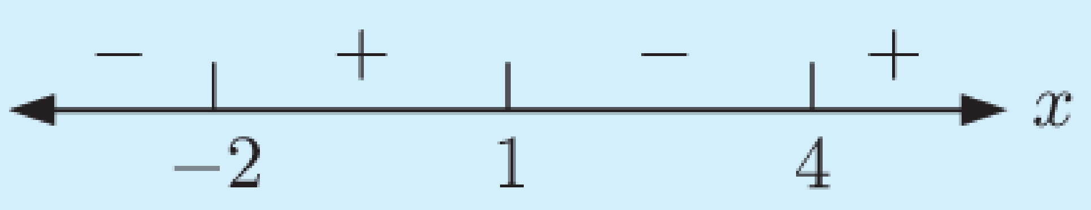
a



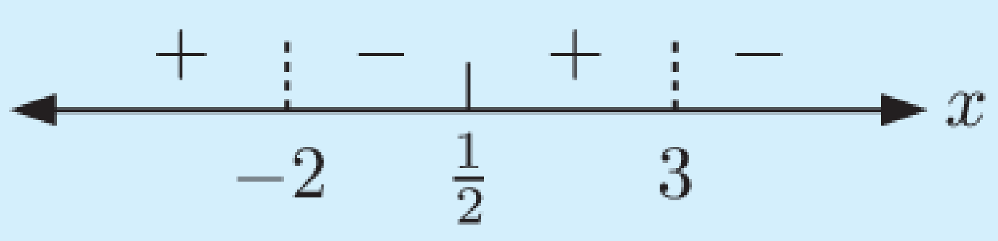
b



a



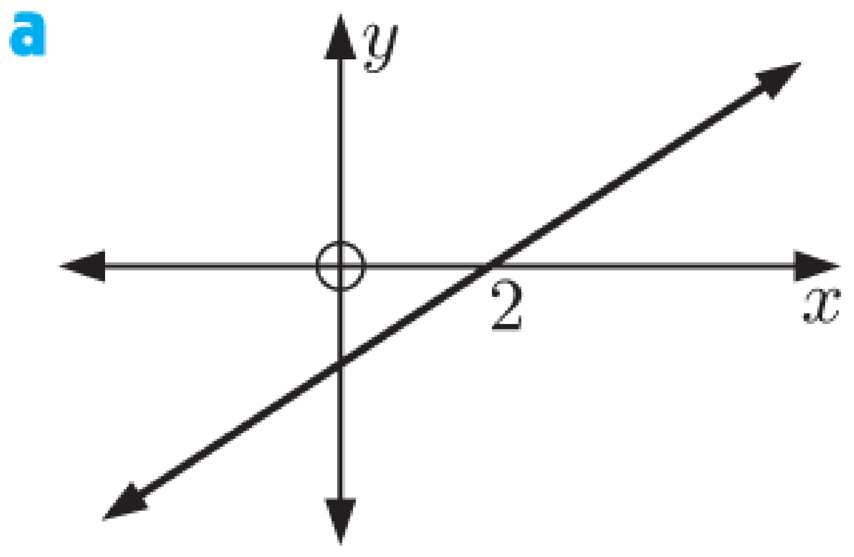
b



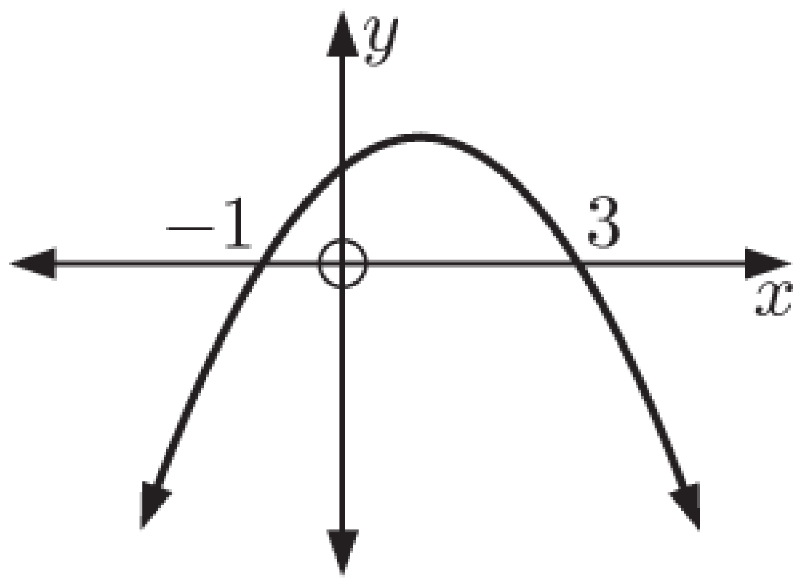
EXERCISE 3E

1 Draw a sign diagram for each graph:

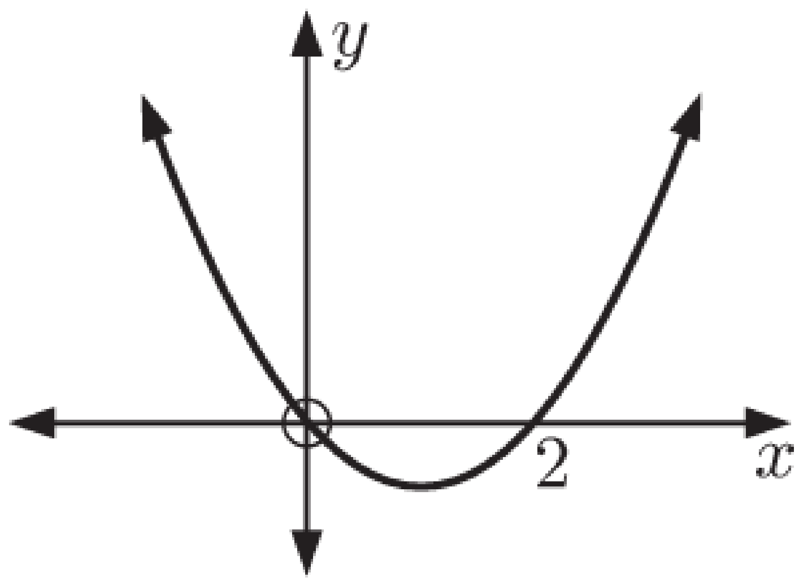
a



b



c



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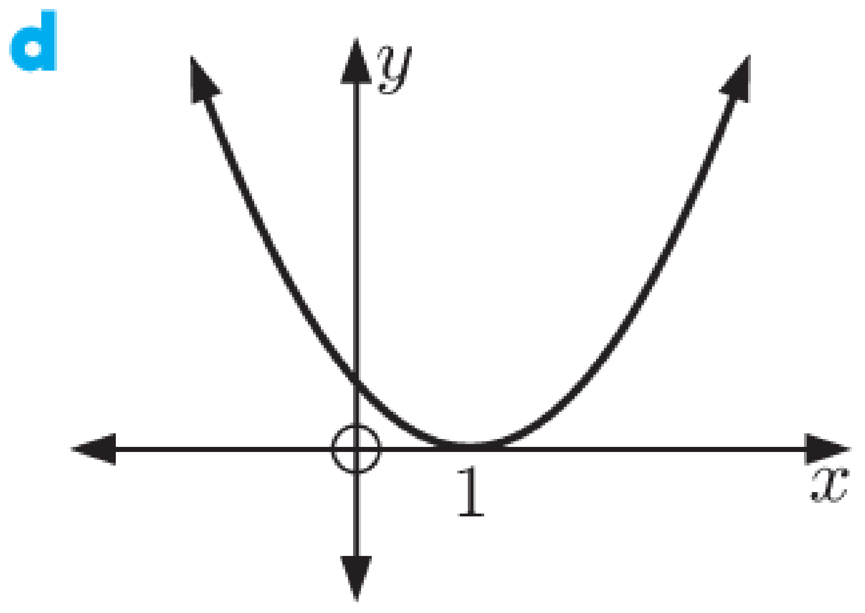

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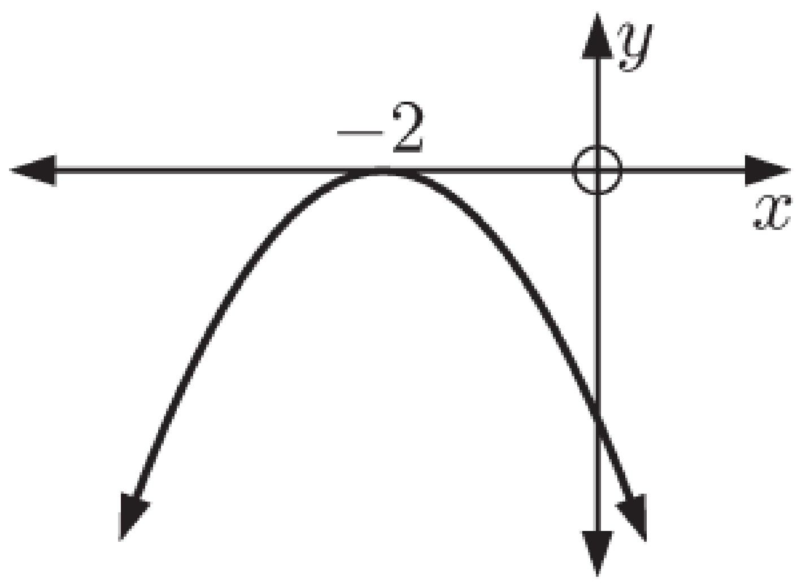

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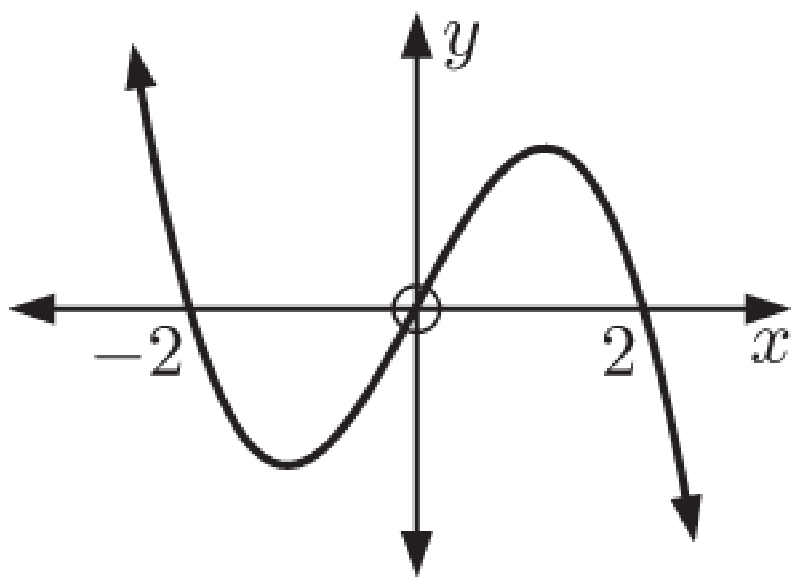

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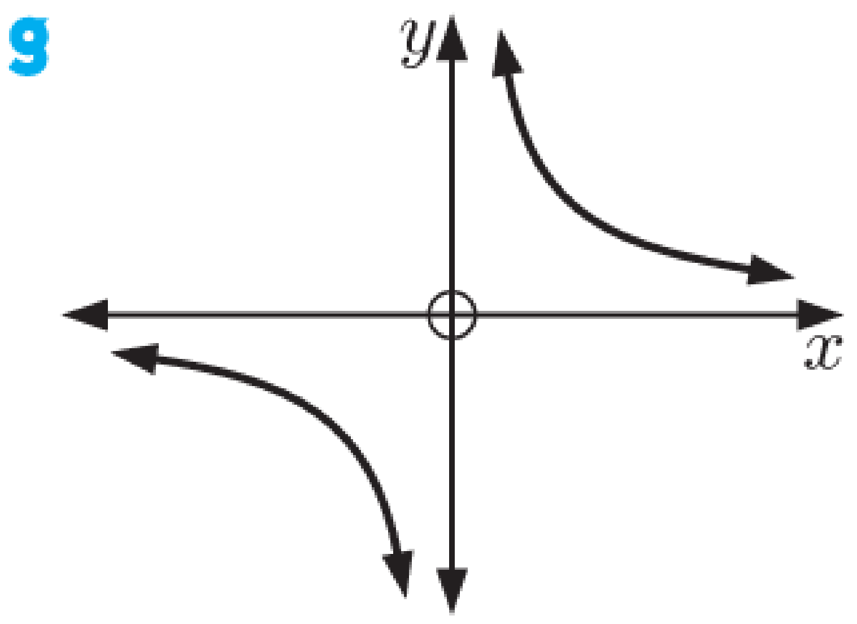

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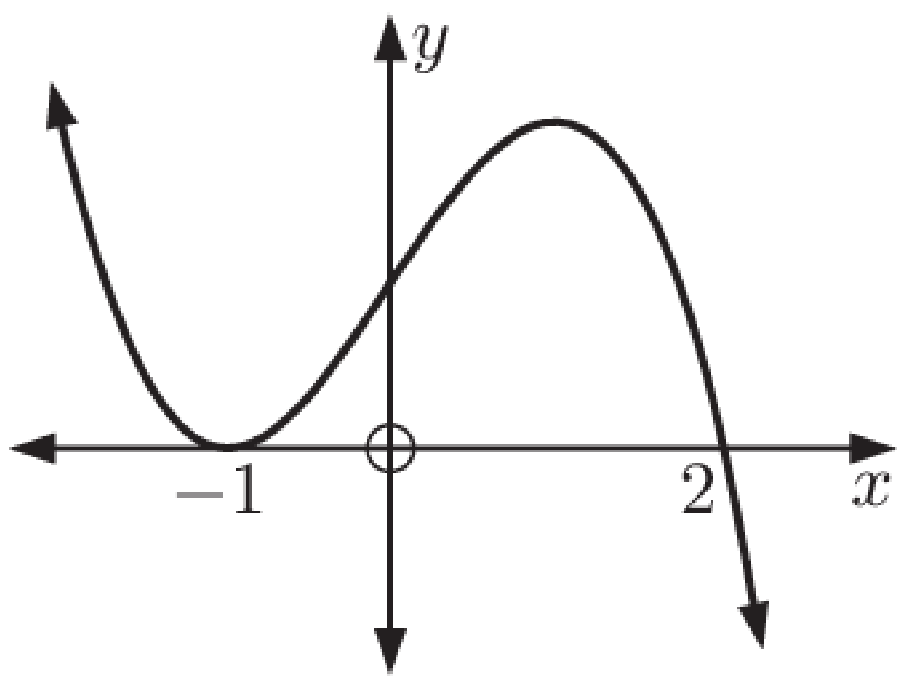

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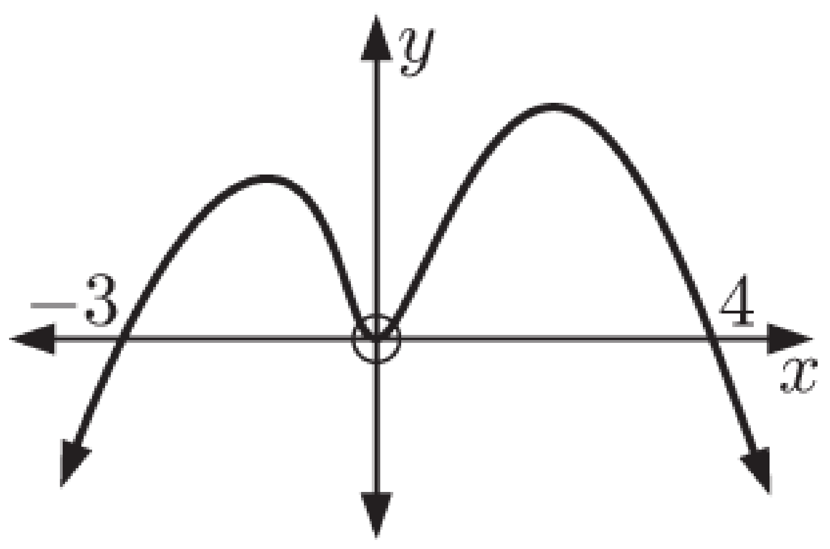

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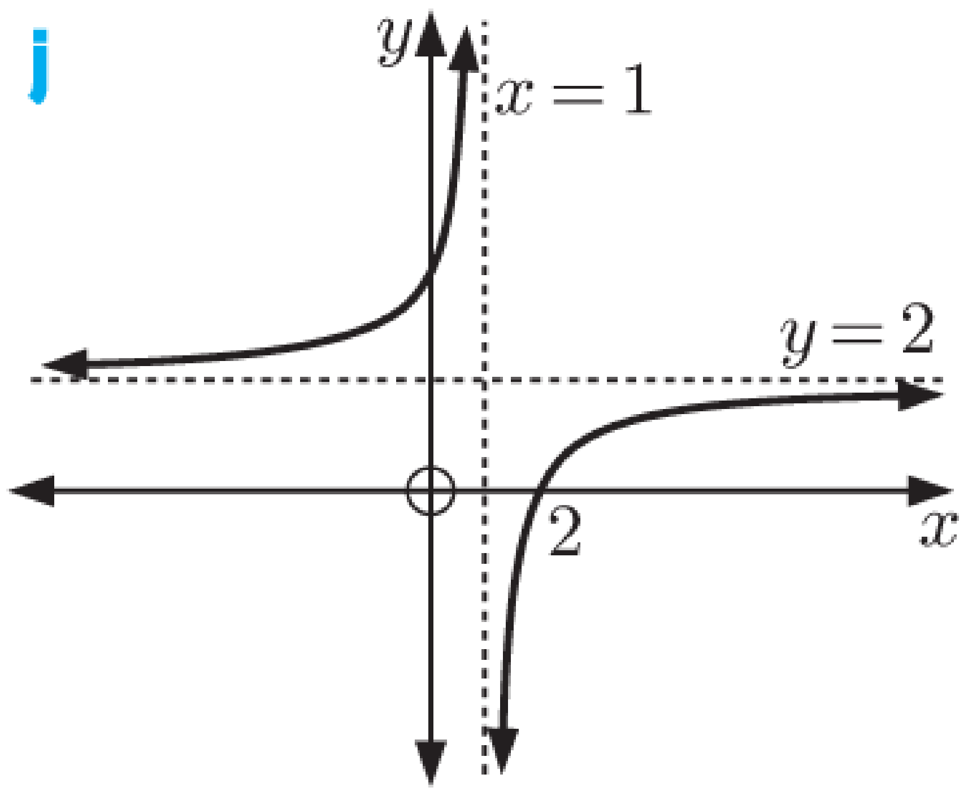

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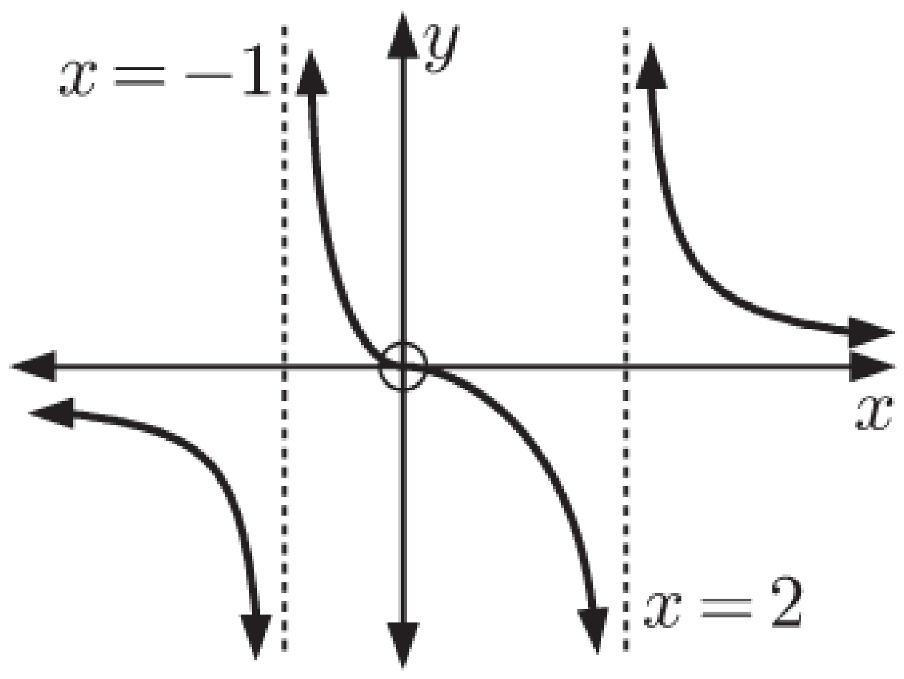

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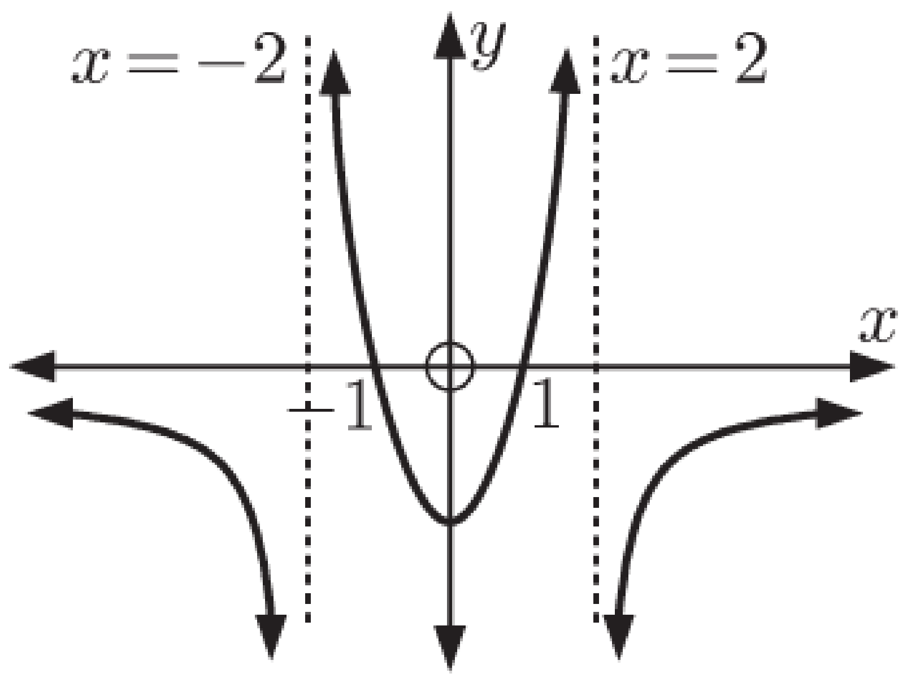


Example 8

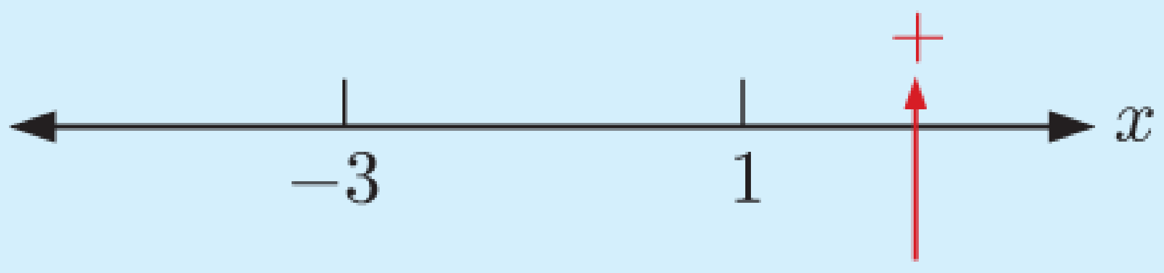
Self Tutor

Draw a sign diagram for:

a $(x+3)(x-1)$

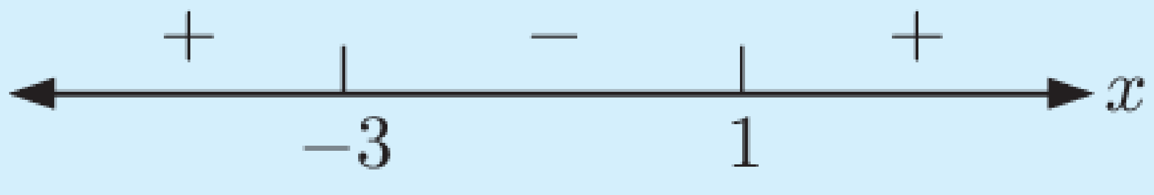
b $-4(x-3)^2$

a $(x+3)(x-1)$ has zeros -3 and 1.

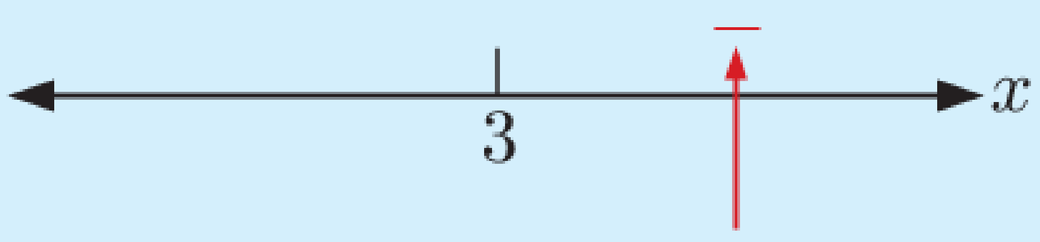


We substitute any number > 1.
When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

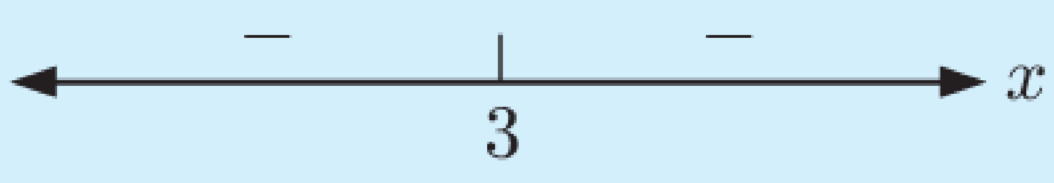


b $-4(x-3)^2$ has zero 3.



We substitute any number > 3.
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



2 Draw a sign diagram for:

a $(x+4)(x-2)$ **b** $(x+1)(x-5)$ **c** $x(x-3)$

d $x(x+2)$ **e** $(2x+1)(x-4)$ **f** $-(x+1)(x-3)$

g $-(3x-2)(x+1)$ **h** $(2x-1)(3-x)$ **i** $(5-x)(1-2x)$

3 Draw a sign diagram for:

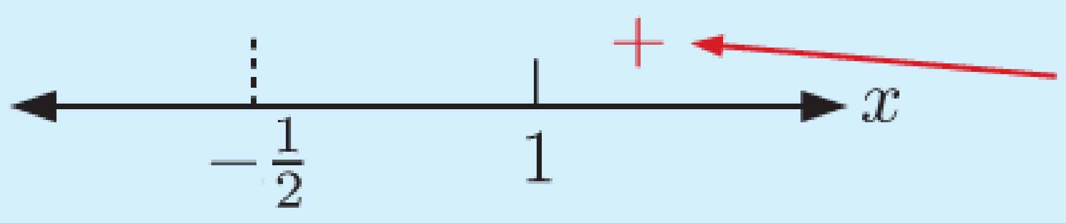
a $(x+2)^2$ **b** $(x-3)^2$ **c** $-(x-4)^2$

d $2(x+1)^2$ **e** $-3(x+4)^2$ **f** $-\frac{1}{2}(2x+5)^2$

Example 9 **Self Tutor**

Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x=1$ and undefined when $x=-\frac{1}{2}$.



When $x=10$, $\dfrac{x-1}{2x+1}=\dfrac{9}{21}>0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.

4 Draw a sign diagram for:

a $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{x+1}{x+5}$

d $\dfrac{x-2}{2x+1}$ **e** $\dfrac{2x+3}{4-x}$ **f** $\dfrac{4x-1}{2-x}$

g $\dfrac{3x}{x-2}$ **h** $\dfrac{(x-1)^2}{x}$ **i** $\dfrac{4x}{(x+1)^2}$

j $\dfrac{(x+2)(x-1)}{3-x}$ **k** $\dfrac{3-x}{(2x+3)(x-2)}$ **l** $\dfrac{(x+5)(x-1)}{(x+2)^2}$

F TRANSFORMATIONS OF GRAPHS

Different parts of a function's equation affect the properties of its graph. If we understand what part of the function controls a particular feature, we can learn how to **transform** the function. This is useful because we can often draw the graph of one function from the graph of a related function.

In this Section we will consider graphs of the form:

- $y=f(x)+d$
- $y=pf(x)$, $p>0$ and $y=f(qx)$, $q>0$
- $y=-f(x)$ and $y=f(-x)$.

INVESTIGATION TRANSFORMATIONS OF GRAPHS

In this Investigation we will explore transformations we can apply to $y = f(x)$ to obtain the graphs of related functions.

PART 1: GRAPHS OF THE FORM $y = f(x) + d$

What to do:

GRAPHING PACKAGE

1 Let $f(x) = 3x$.

a Write down: **i** $f(x) + 2$ **ii** $f(x) - 3$ **iii** $f(x) + 6$

b Graph $y = f(x)$ and the other three functions on the same set of axes. Record your observations.

2 Repeat **1** for the function $f(x) = x^3$.

3 Describe the transformation which maps $y = f(x)$ onto $y = f(x) + d$.

PART 2: GRAPHS OF THE FORM $y = pf(x)$, $p > 0$ AND $y = f(qx)$, $q > 0$

What to do:

GRAPHING PACKAGE

1 Let $f(x) = x + 2$.

a Find, in simplest form: **i** $3f(x)$ **ii** $\frac{1}{2}f(x)$ **iii** $5f(x)$

b Graph all four functions on the same set of axes.

c Copy and complete:

For the transformation $y = pf(x)$, $p > 0$, each point becomes ...... times its previous distance from the x-axis.

d Does any point on the graph *not* move under this transformation? Explain your answer.

2 Let $f(x) = x + 2$.

a Find, in simplest form: **i** $f(2x)$ **ii** $f(\frac{1}{3}x)$ **iii** $f(4x)$

b Graph all four functions on the same set of axes.

c Copy and complete:

For the transformation $y = f(qx)$, $q > 0$, each point becomes ...... times its previous distance from the y-axis.

d Does any point on the graph *not* move under this transformation? Explain your answer.

PART 3: GRAPHS OF THE FORM $y = -f(x)$ AND $y = f(-x)$

What to do:

GRAPHING PACKAGE

1 Let $f(x) = 2x + 3$.

a Find, in simplest form: **i** $-f(x)$ **ii** $f(-x)$

b Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.

2 Let $f(x) = x^3 + 1$.

a Find, in simplest form: **i** $-f(x)$ **ii** $f(-x)$

b Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.

3 What transformation moves:

a $y = f(x)$ to $y = -f(x)$ **b** $y = f(x)$ to $y = f(-x)$?

From the **Investigation** you should have discovered that:

- For $y = f(x) + d$, the effect of d is to **translate** the graph **vertically** through d units.
  - If $d > 0$ it moves **upwards**.
  - If $d < 0$ it moves **downwards**.
- $y = pf(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with **scale factor** p.
  - Each point becomes p times its previous distance from the x-axis.
  - If $p > 1$, points move further away from the x-axis.
  - If $0 < p < 1$, points move closer to the x-axis.
  - Points on the x-axis do not move.
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with **scale factor** $\frac{1}{q}$.
  - Each point becomes $\frac{1}{q}$ times its previous distance from the y-axis.
  - If $q > 1$, points move closer to the y-axis.
  - If $0 < q < 1$, points move further away from the y-axis.
  - Points on the y-axis do not move.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the **x-axis**.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the **y-axis**.

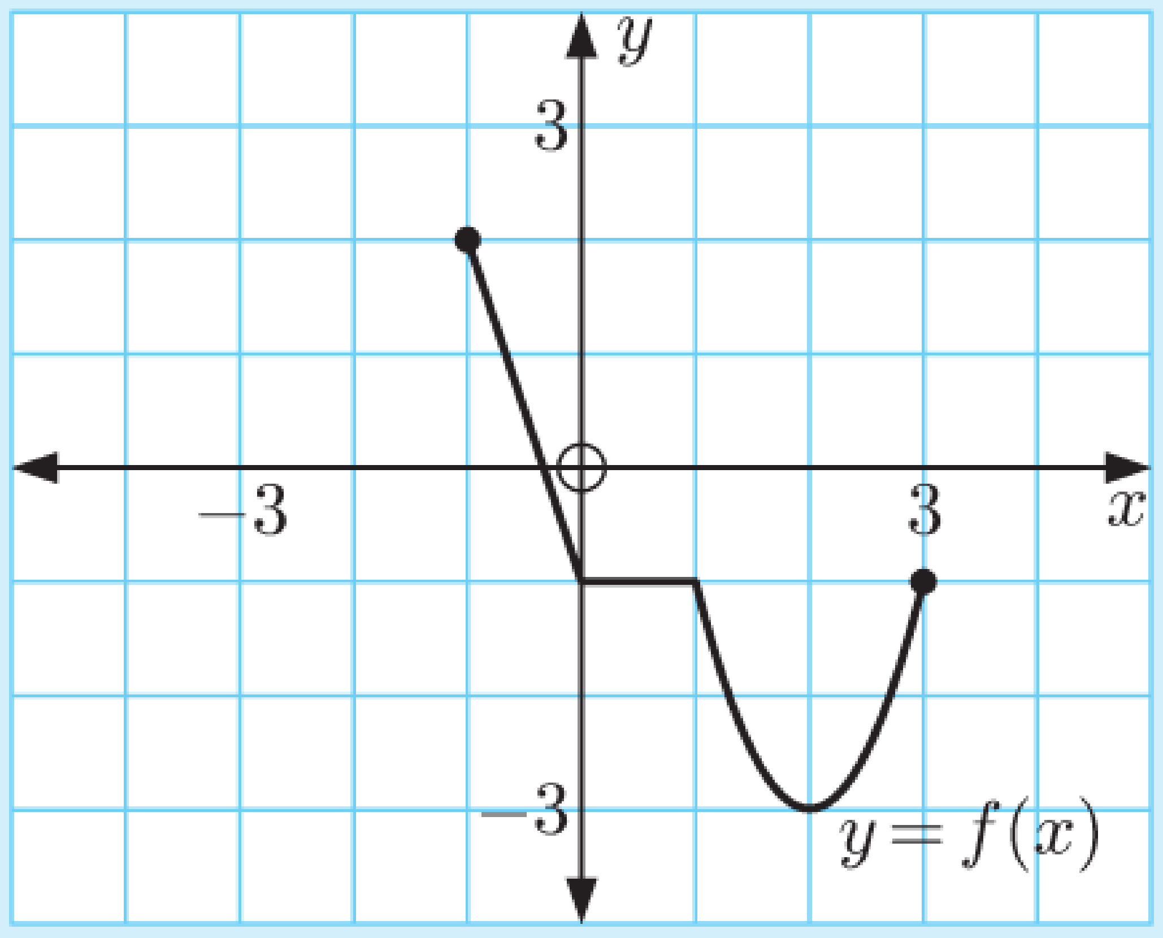
The word *dilation* is a common alternative to the word *stretch*.

Example 10 **Self Tutor**

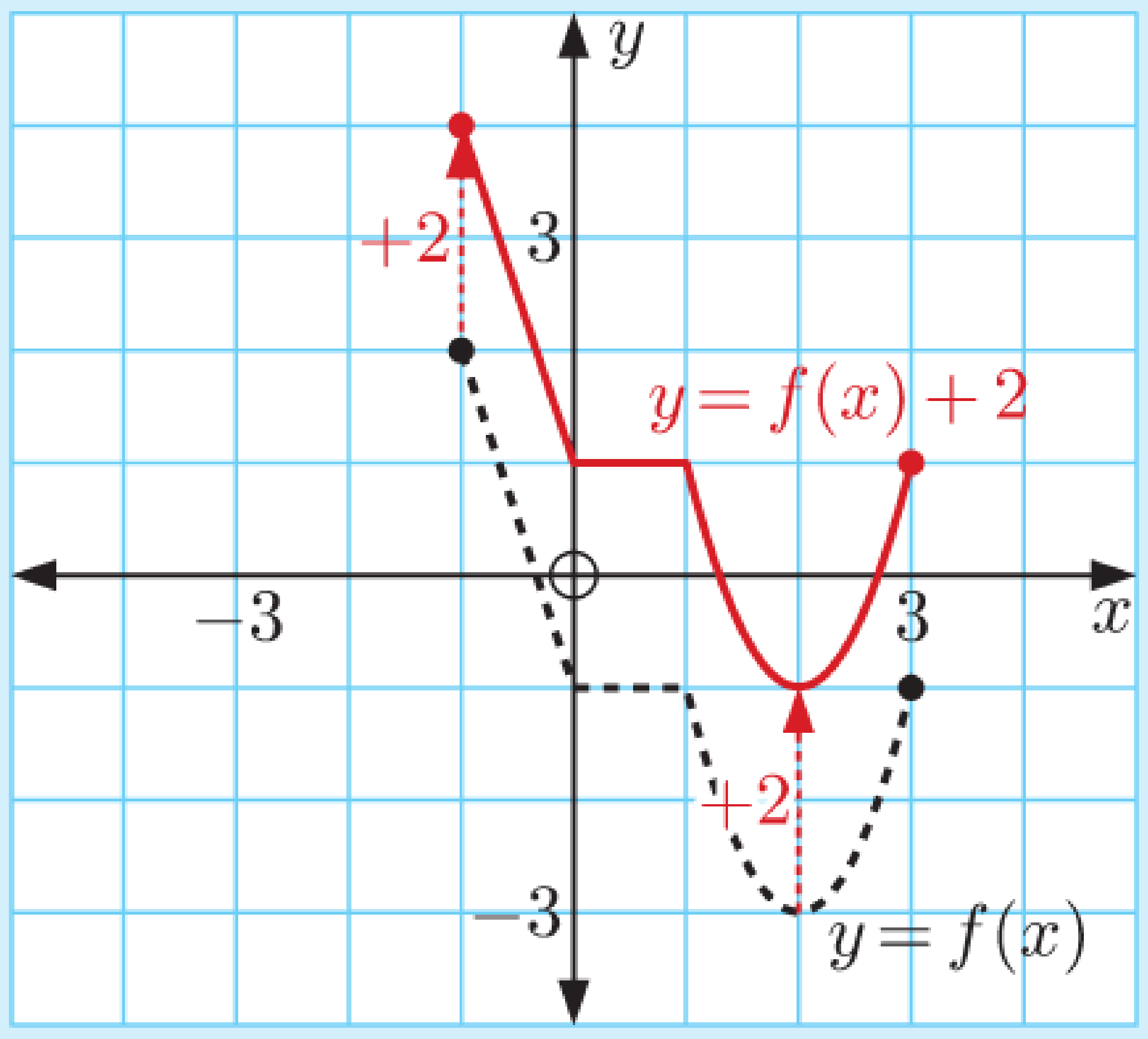
Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

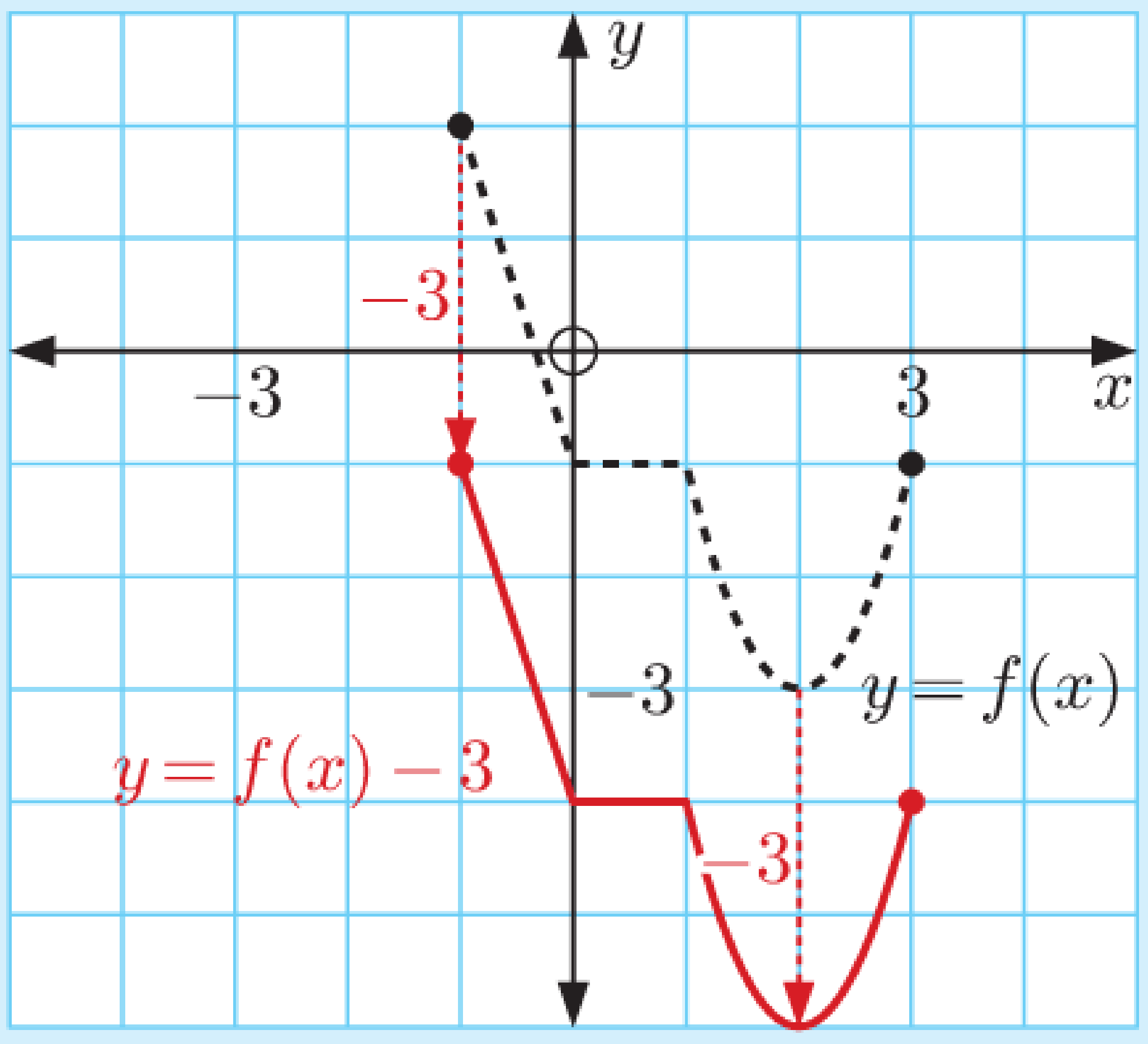
a $y = f(x) + 2$ **b** $y = f(x) - 3$



a The graph of $y = f(x) + 2$ is found by translating $y = f(x)$ 2 units upwards.



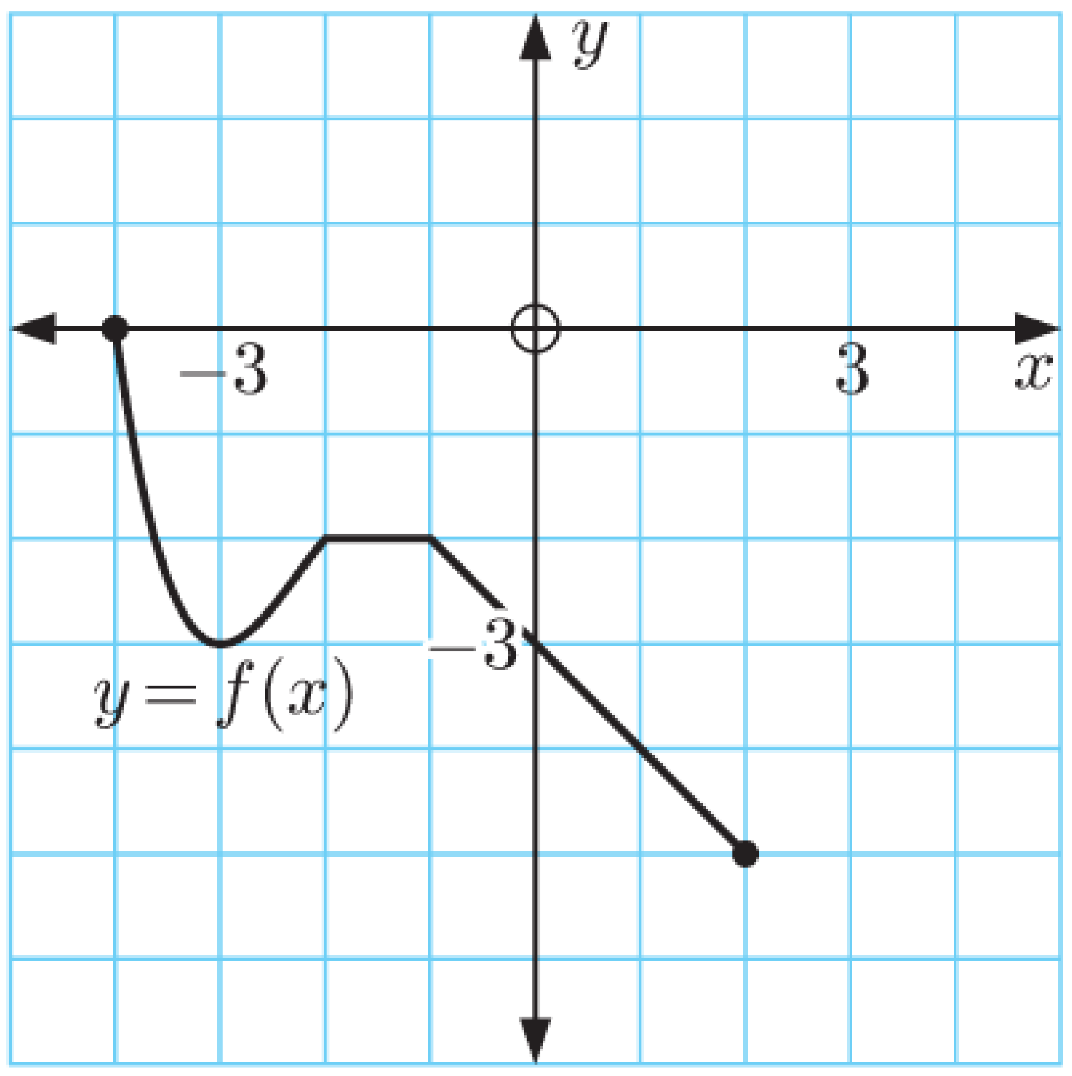
b The graph of $y = f(x) - 3$ is found by translating $y = f(x)$ 3 units downwards.



EXERCISE 3F

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

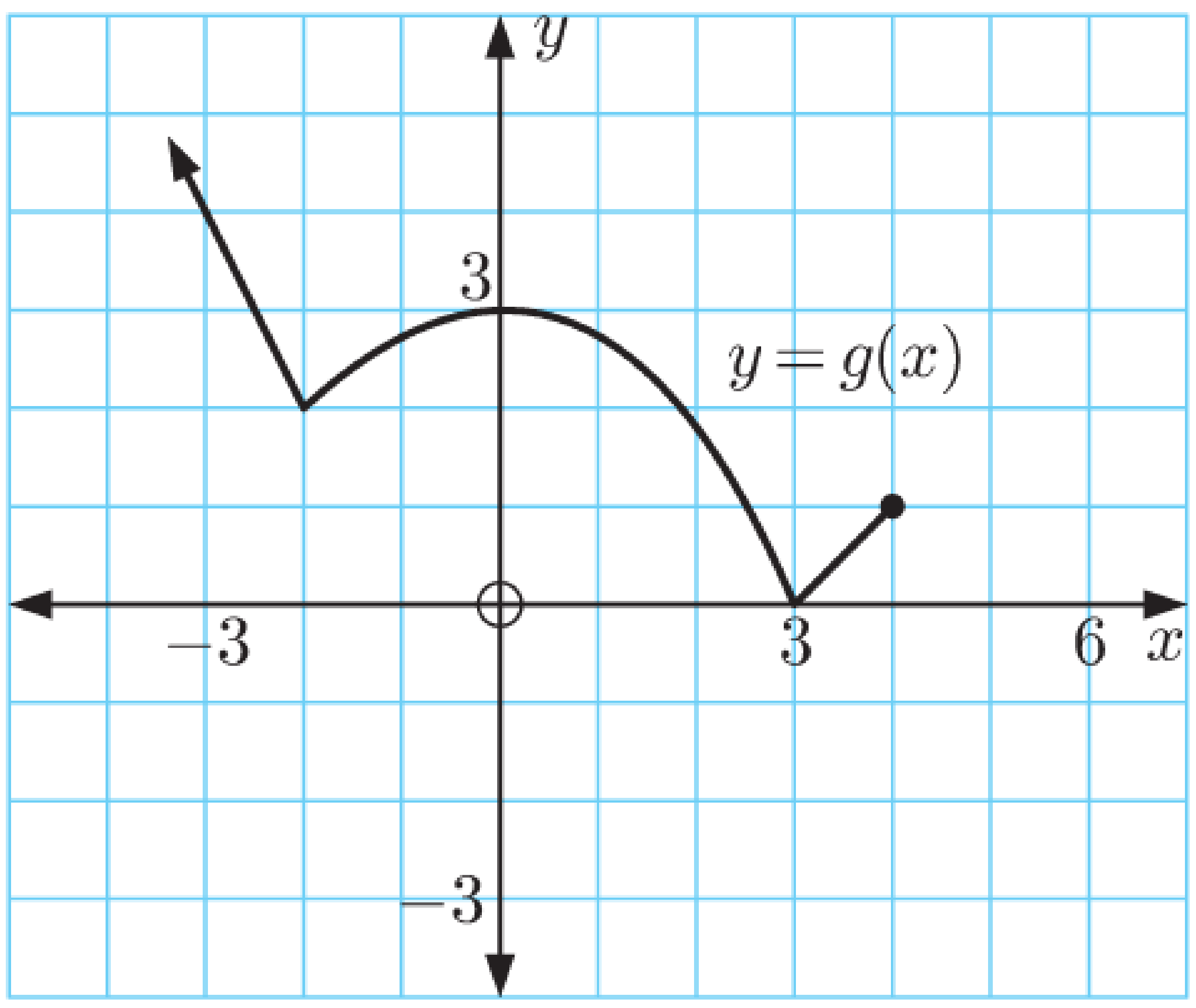
a $y = f(x) + 5$ **b** $y = f(x) - 2$



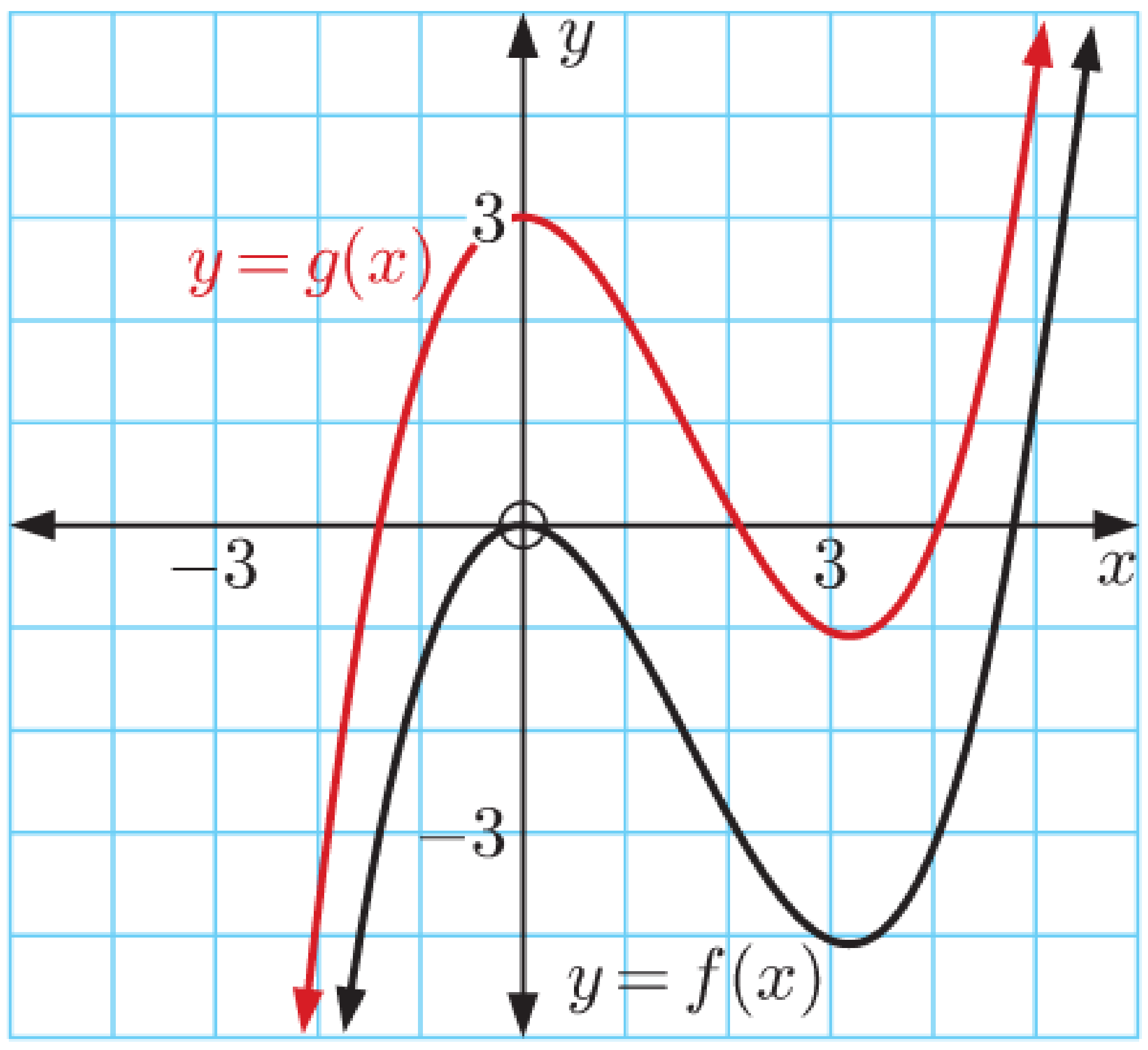
PRINTABLE GRIDS

2 Consider the graph of $y = g(x)$ alongside.
On separate axes, draw the graphs of:

a $y = g(x) + 1$ **b** $y = g(x) - 3$



3 Write $g(x)$ in terms of $f(x)$:



4 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 2x + 3$ is translated 4 units downwards

b $f(x) = -x^2 + 5x - 7$ is translated 3 units upwards

c $f(x) = x^3 + 2 - \frac{1}{x}$ is translated 6 units downwards.

5 **a** Sketch $f(x) = \frac{1}{x}$, using technology if necessary.

b Suppose $f(x)$ is translated k units upwards to produce $g(x)$.

i Write the function $g(x)$.

ii State the equation of the horizontal asymptote of $g(x)$.

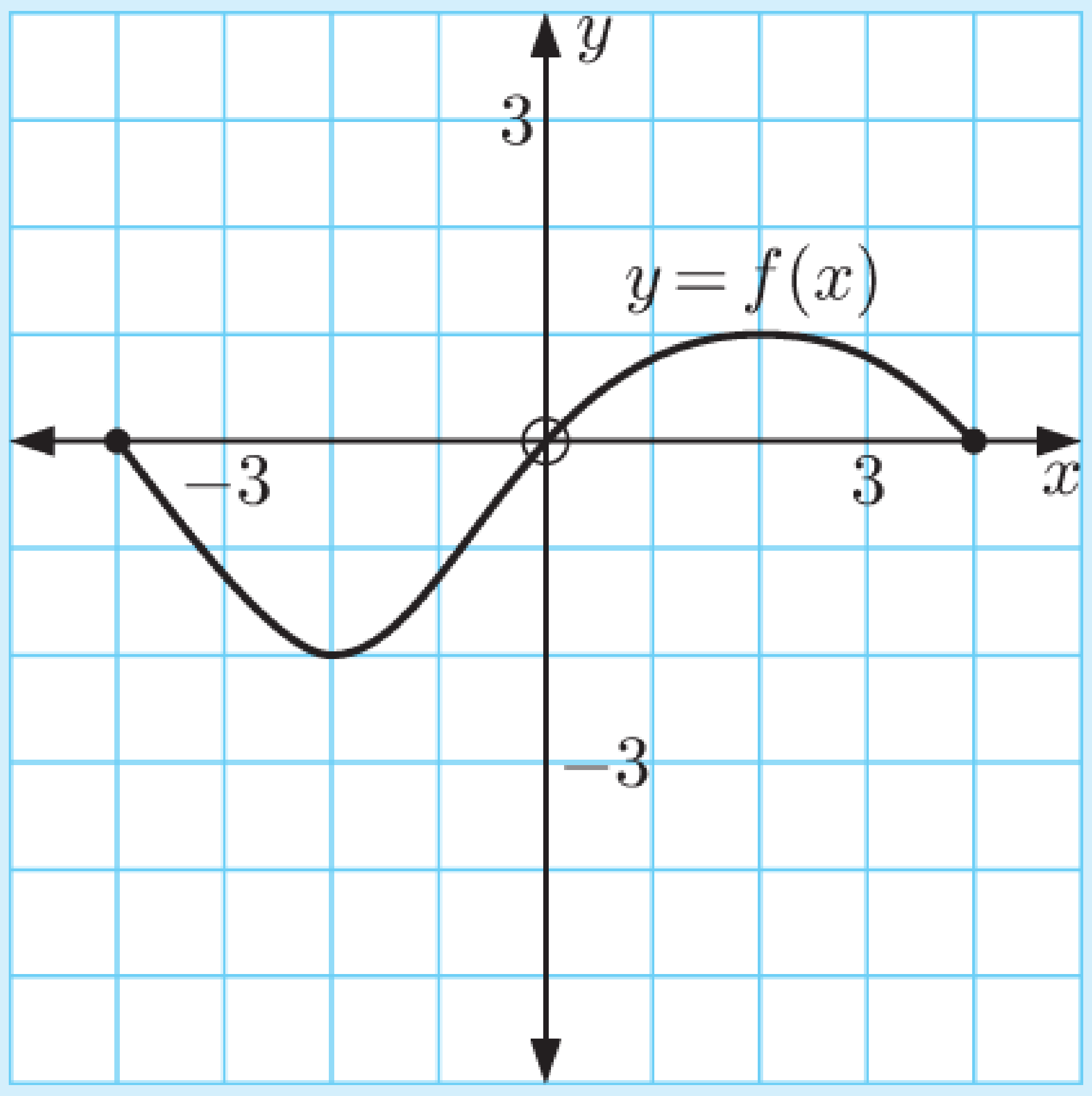
Example 11

Self Tutor

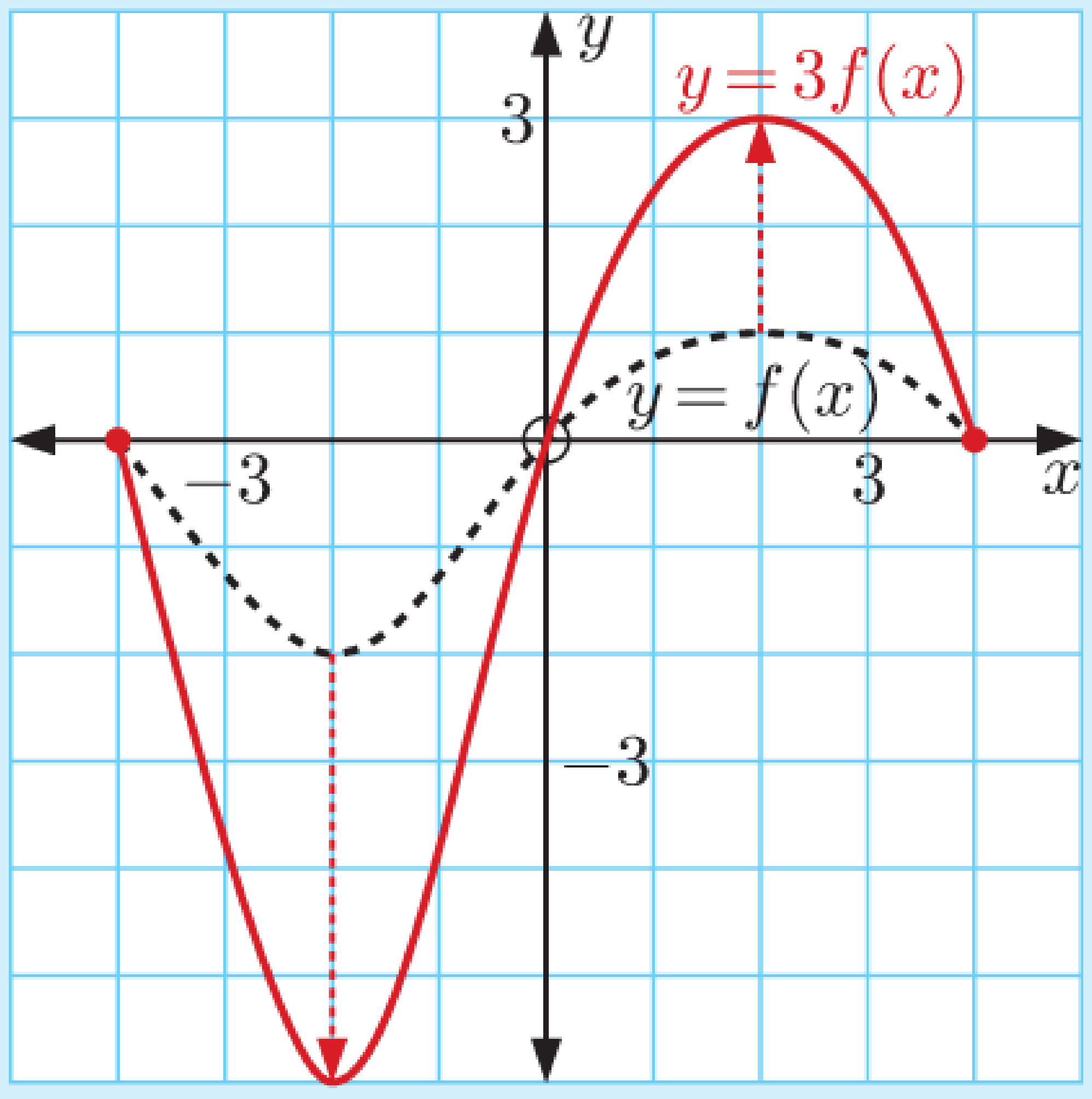
Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

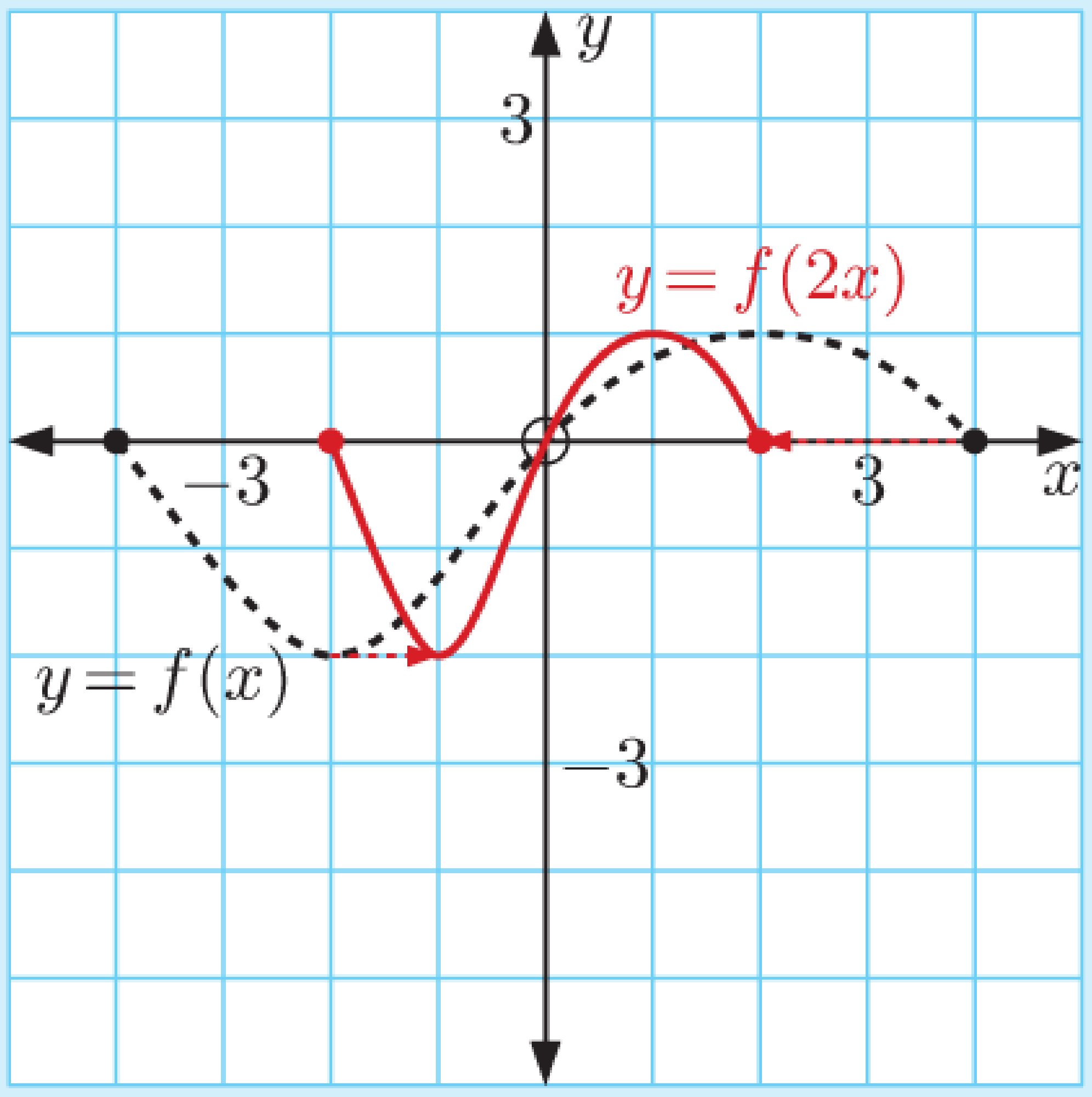
a $y = 3f(x)$ **b** $y = f(2x)$



a The graph of $y = 3f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.

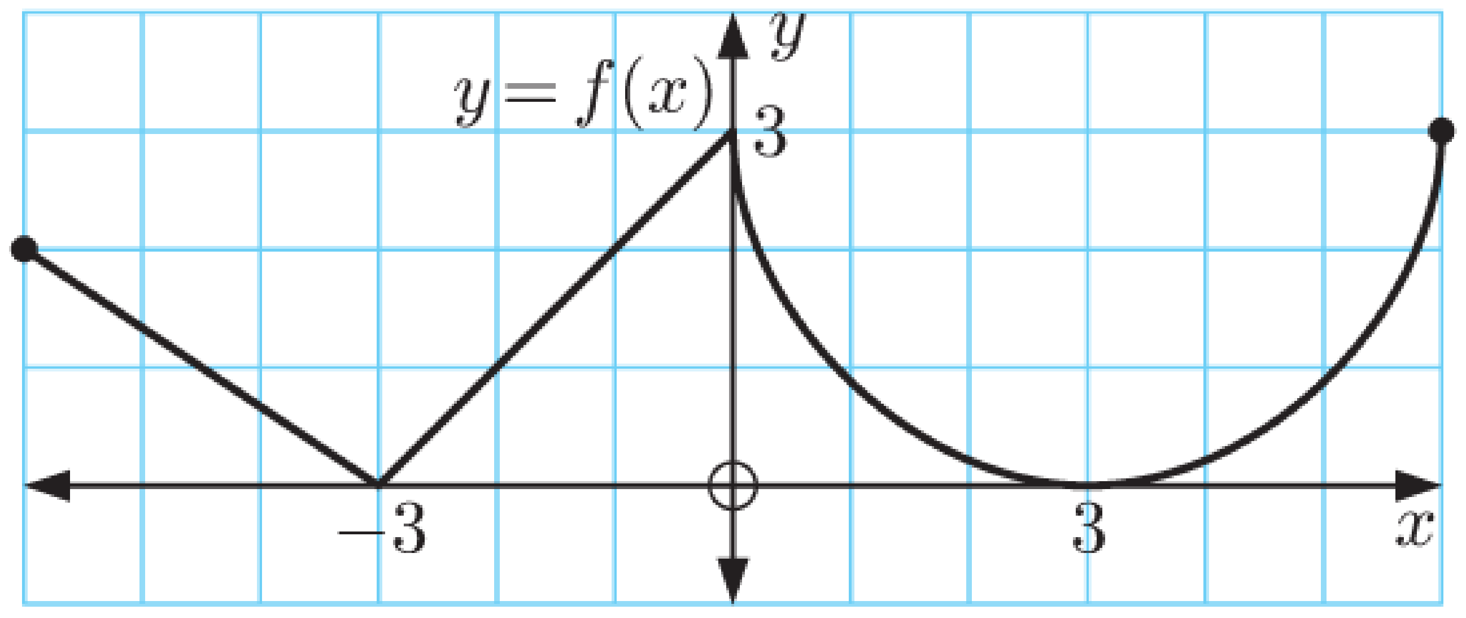


b The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.



6 Consider the graph of $y = f(x)$ alongside. On separate axes, draw the graphs of:

a $y = 2f(x)$ **b** $y = f(3x)$



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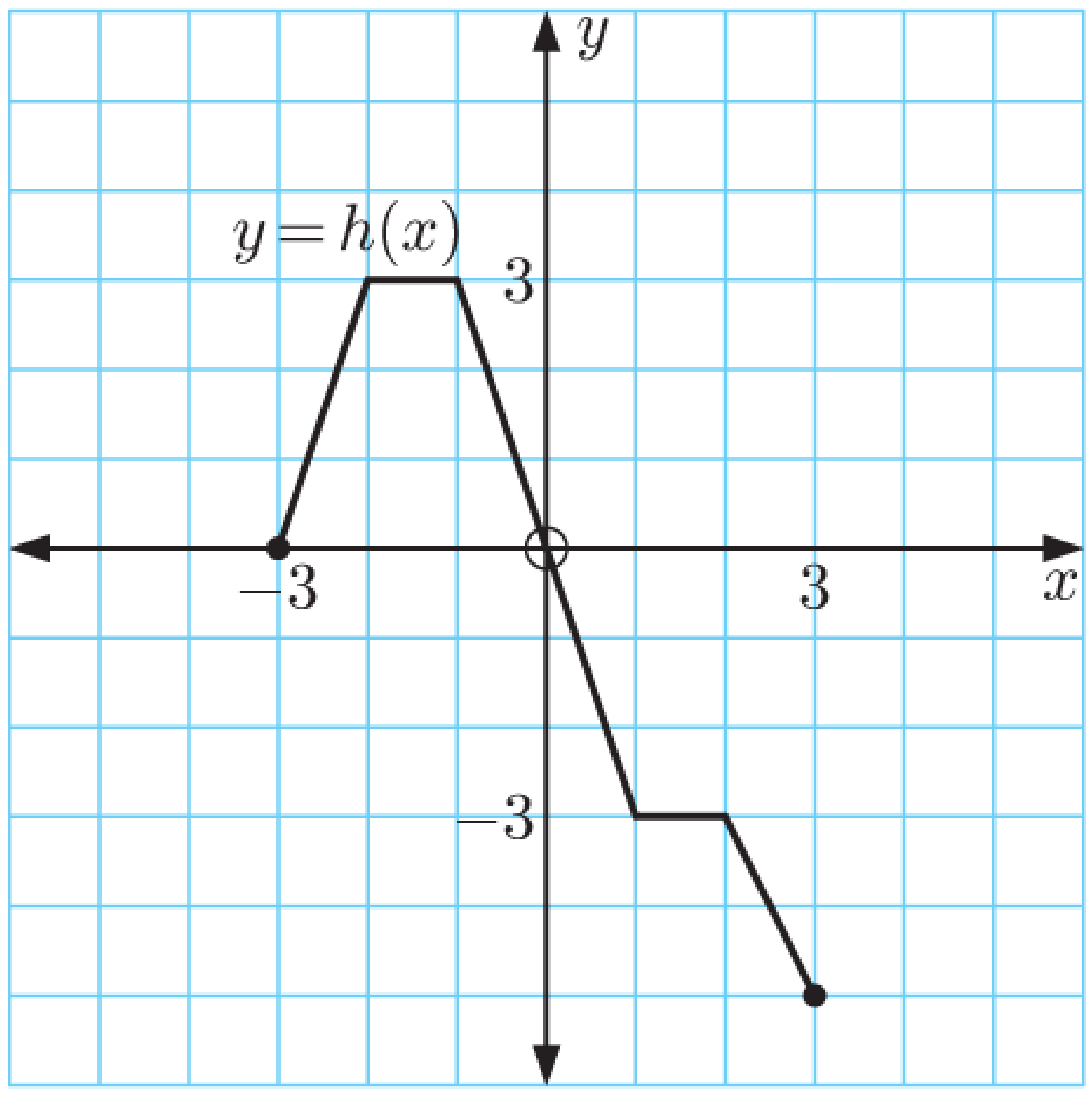


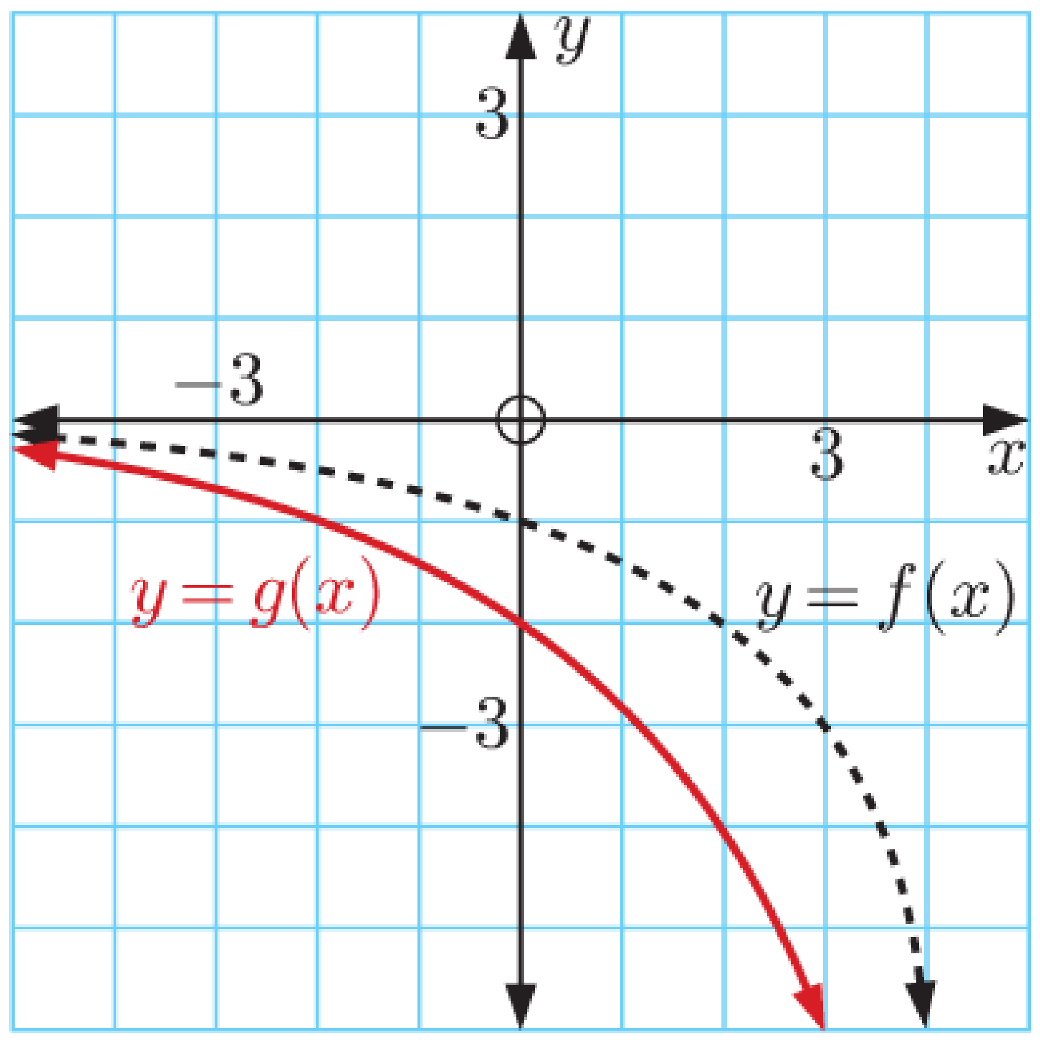
Consider the graph of $y = h(x)$ alongside. On separate axes, draw the graphs of:

a $y = \frac{1}{3}h(x)$ **b** $y = h\left(\frac{x}{2}\right)$

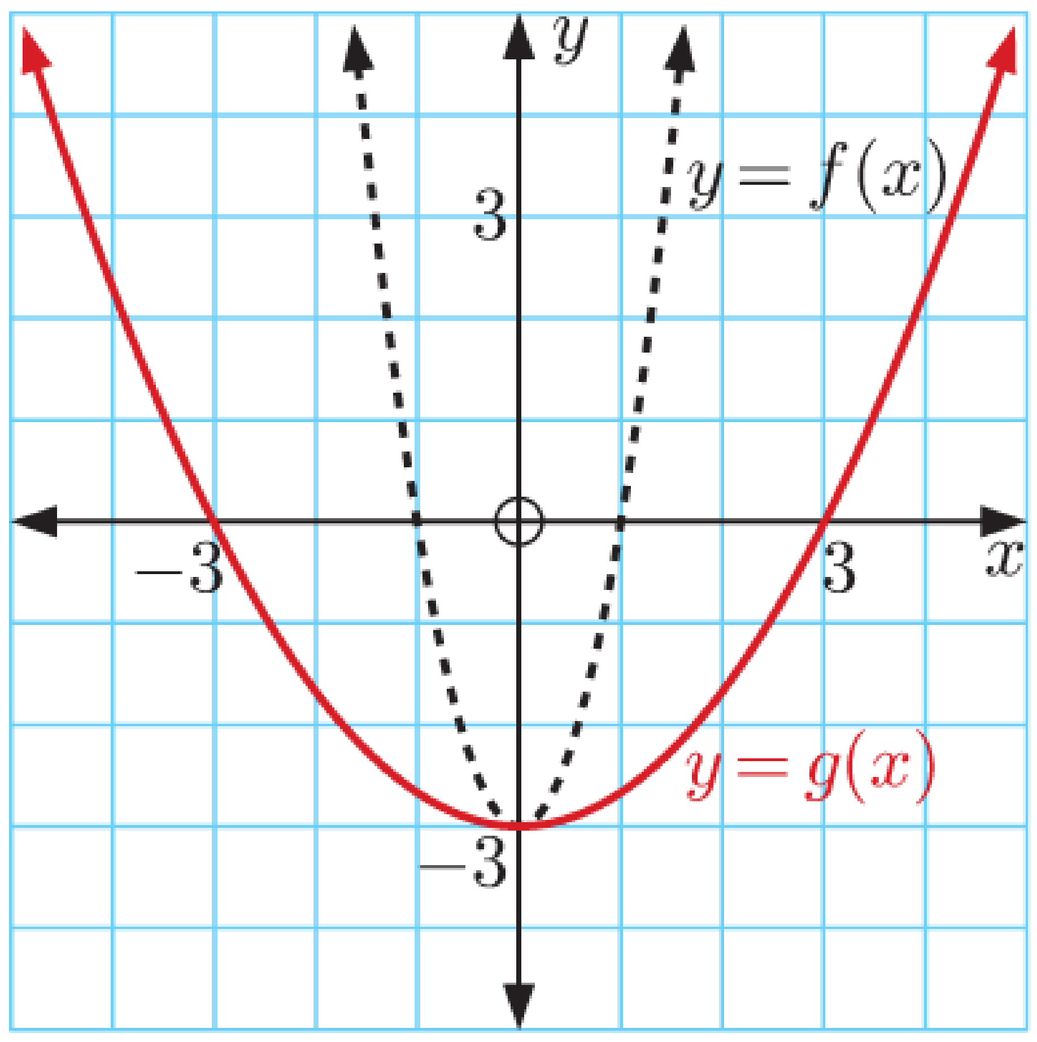
If scale factor > 1, the graph is *elongated*.
If $0 <$ scale factor < 1, the graph is *compressed*.

8 Write $g(x)$ in terms of $f(x)$:

a



b



9 A linear function with gradient m is vertically stretched with scale factor c. Find the gradient of the resulting line.

10 Find the equation of the resulting graph $g(x)$ when:

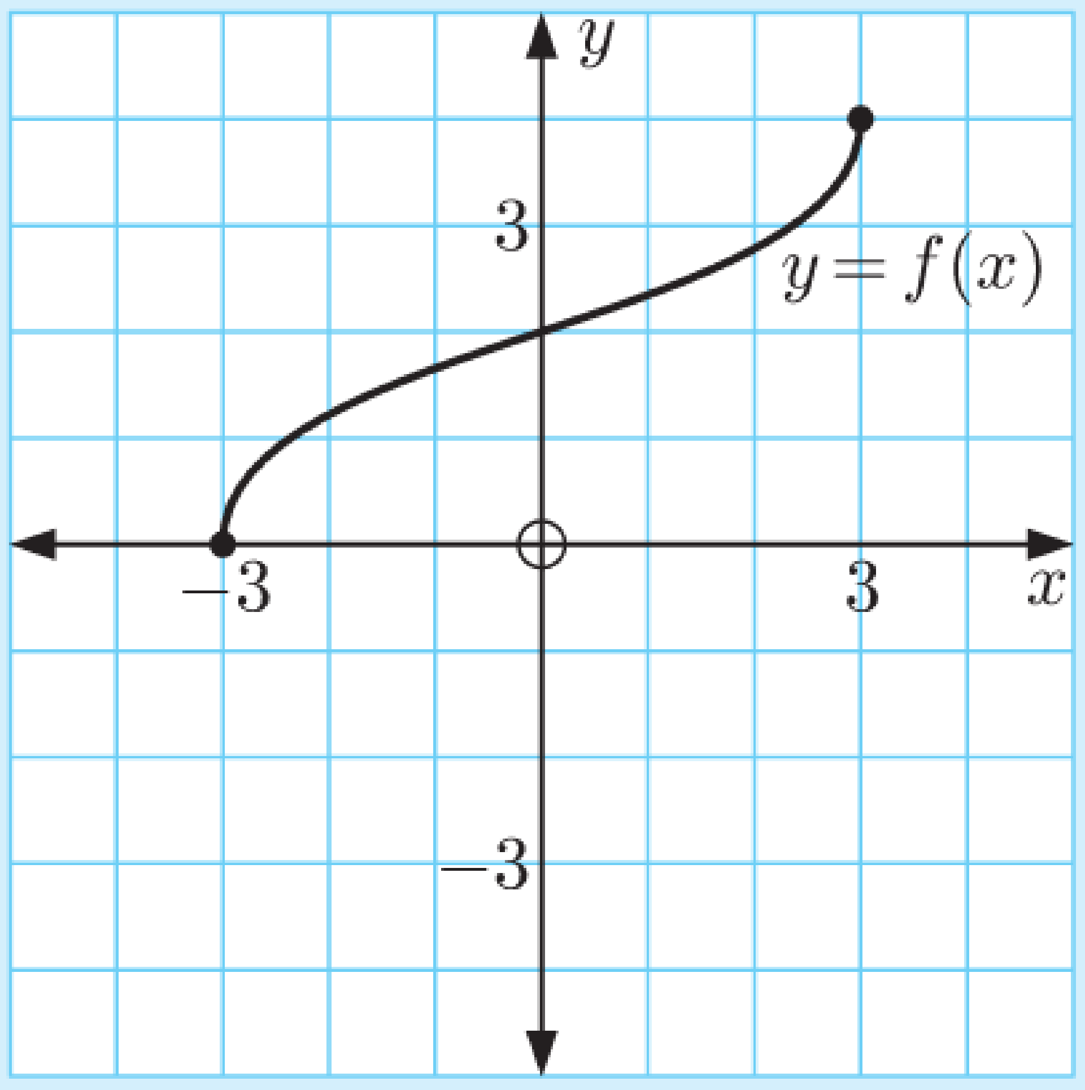
- **a** $f(x)=x^2+2$ is vertically stretched with scale factor 2
- **b** $f(x)=5-3x$ is horizontally stretched with scale factor 3
- **c** $f(x)=x^3+8x^2-2$ is vertically dilated with scale factor $\frac{1}{4}$
- **d** $f(x)=2x^2+x-3$ is horizontally dilated with scale factor $\frac{1}{2}$.

Example 12 **Self Tutor**

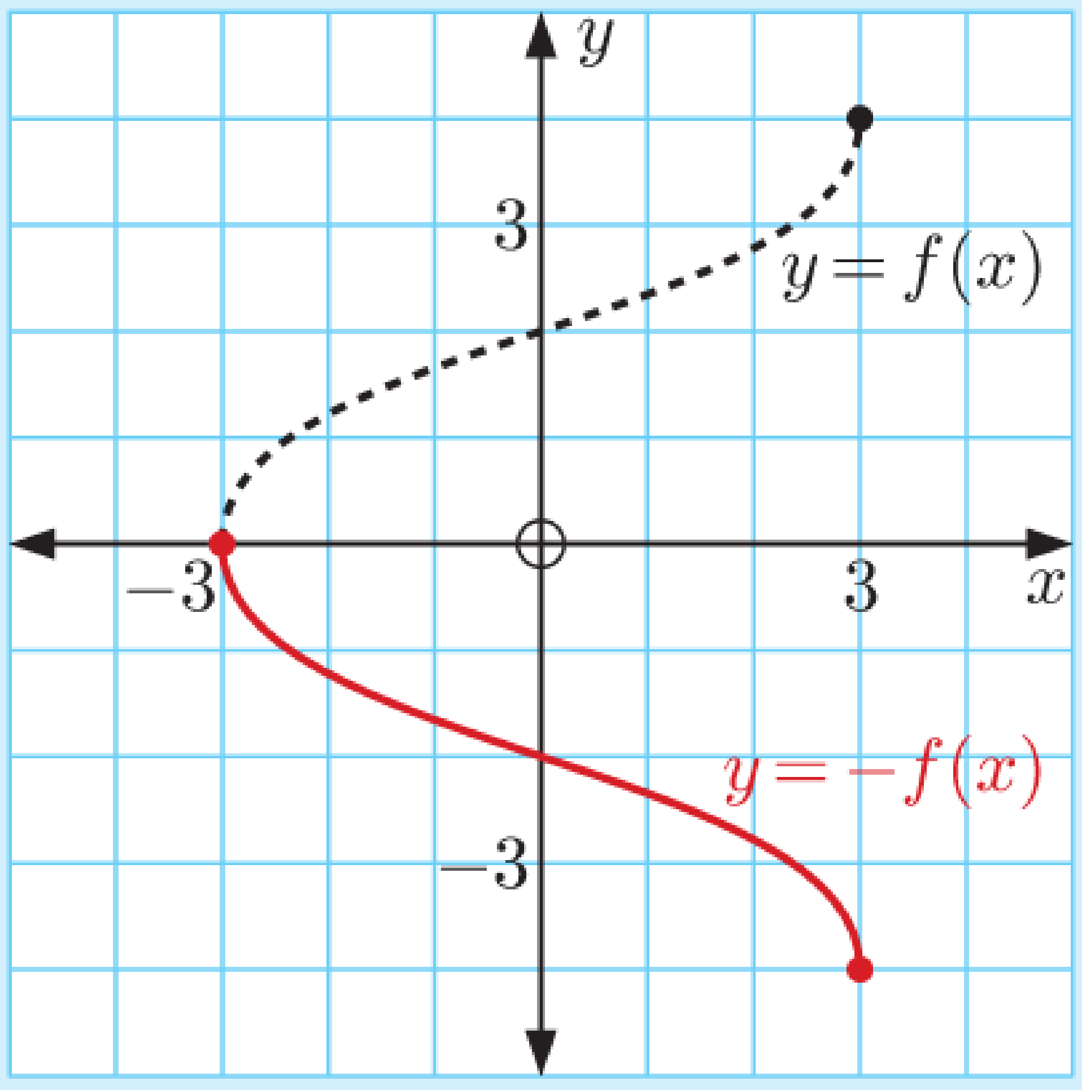
Consider the graph of $y=f(x)$ alongside.

On separate axes, draw the graphs of:

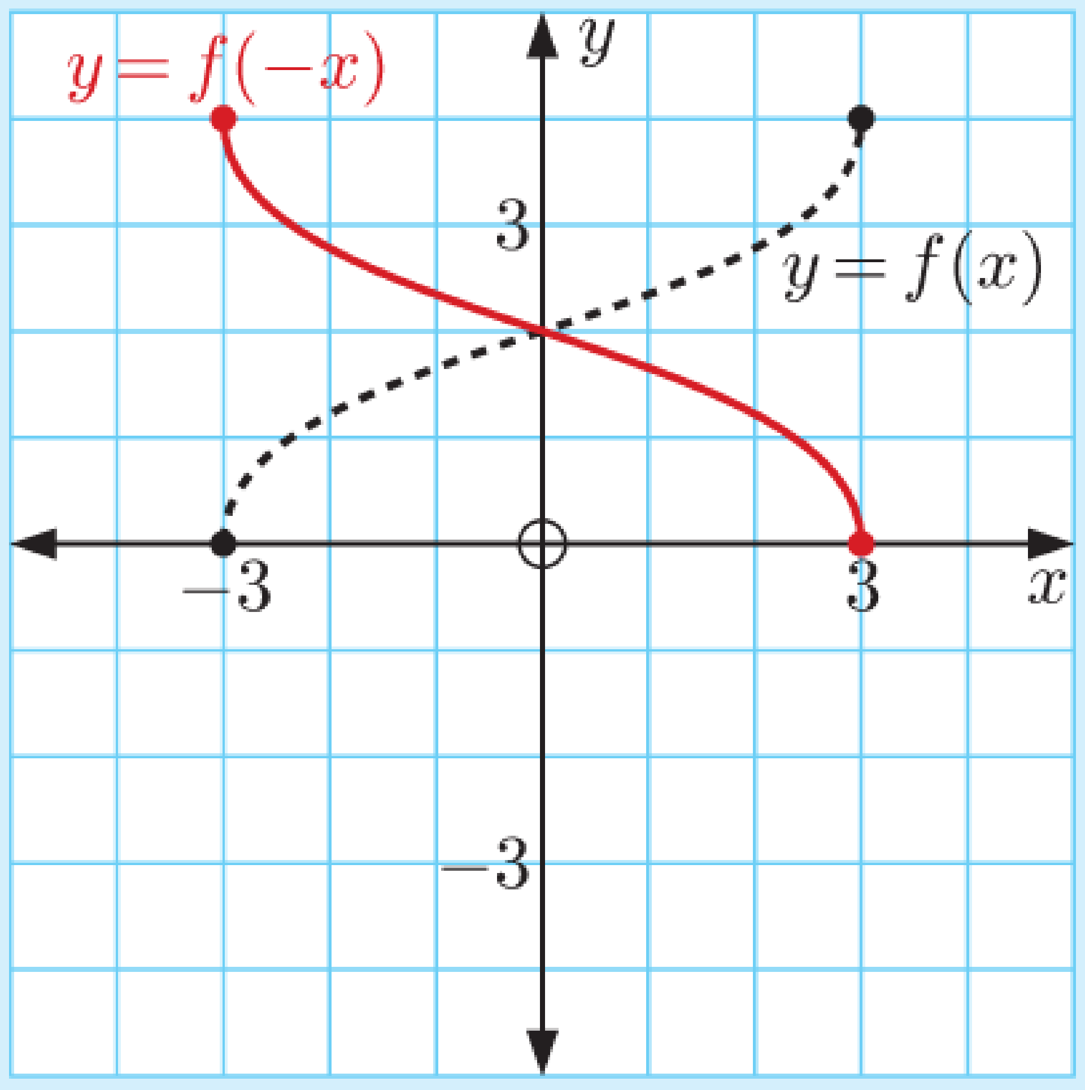
a $y=-f(x)$ **b** $y=f(-x)$



a The graph of $y=-f(x)$ is found by reflecting $y=f(x)$ in the x-axis.

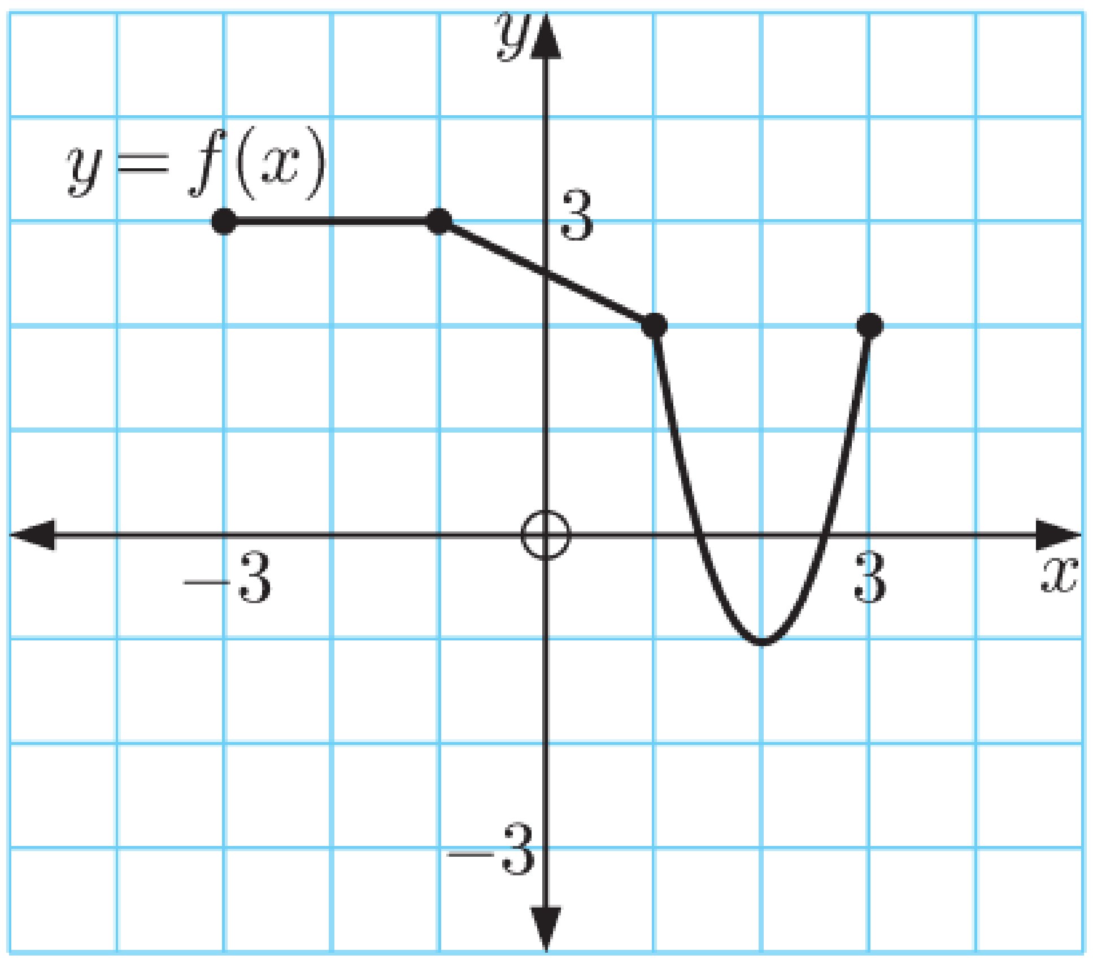


b The graph of $y=f(-x)$ is found by reflecting $y=f(x)$ in the y-axis.



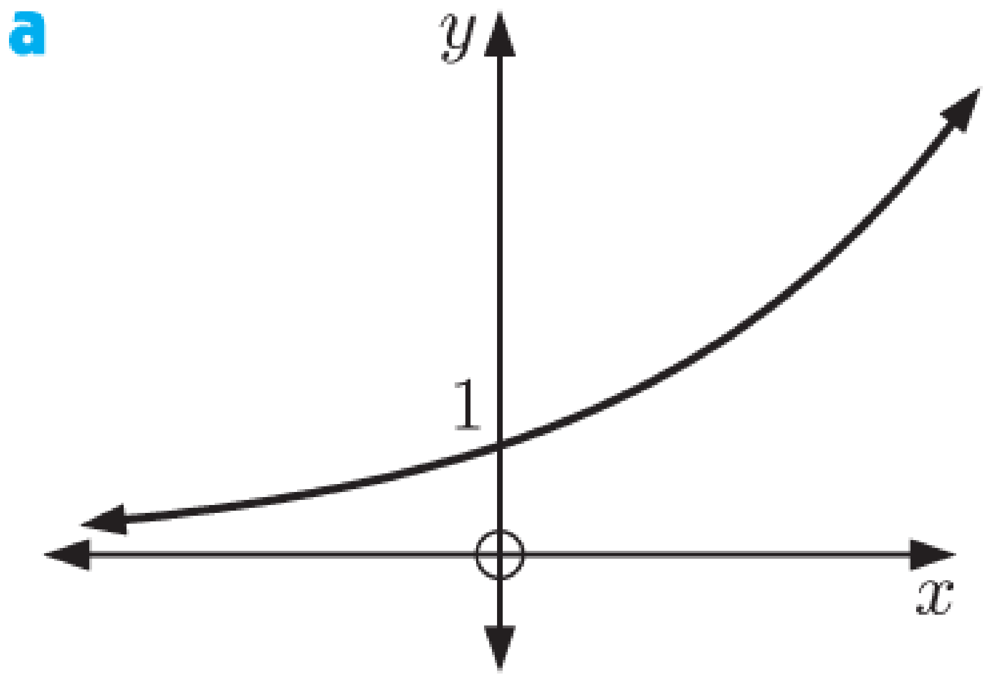
11 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = -f(x)$ **b** $y = f(-x)$

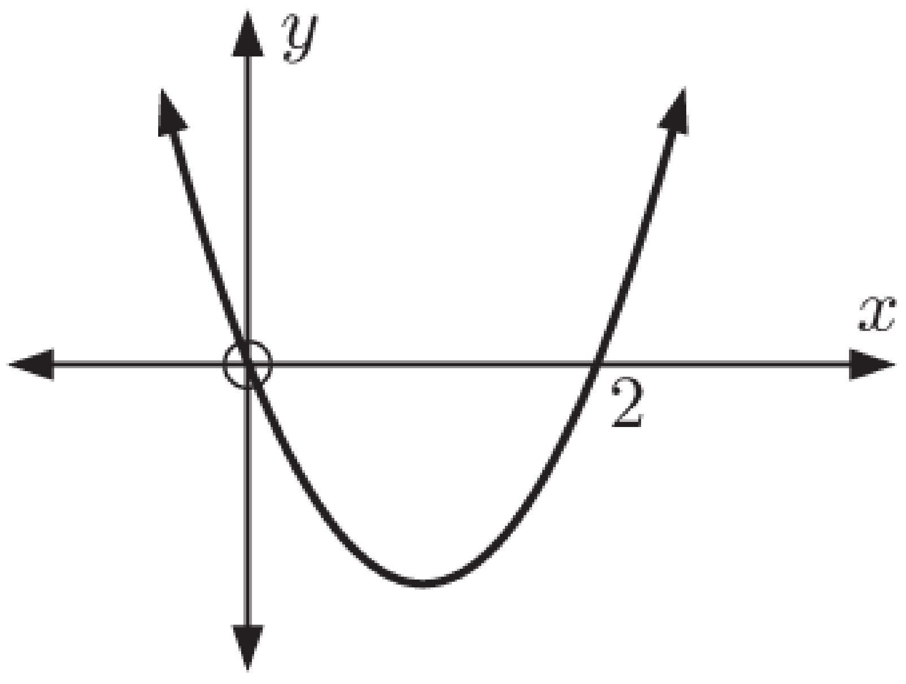


12 Copy the following graphs for $y = f(x)$ and sketch the graphs of $y = -f(x)$ on the same axes.

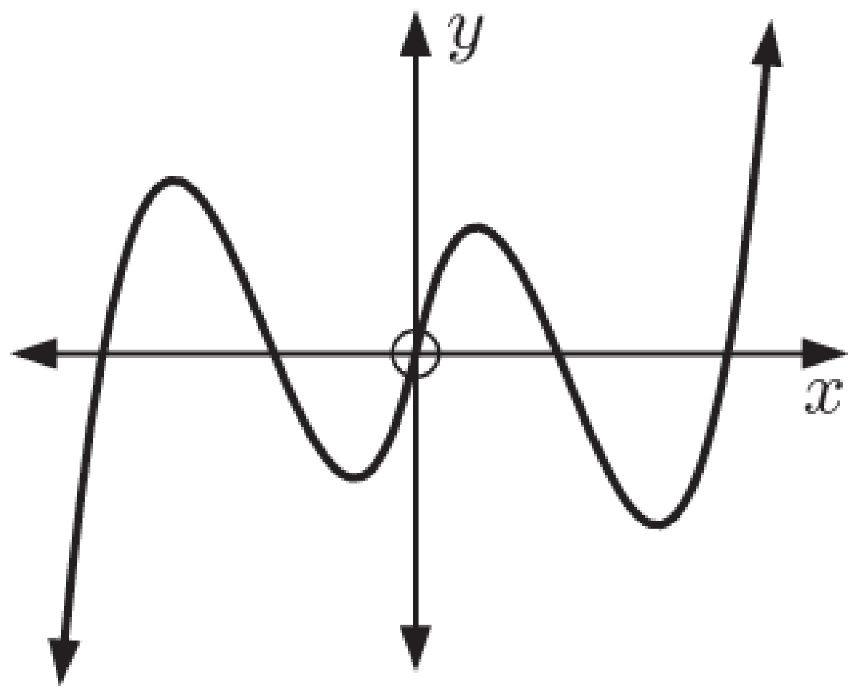
a



b

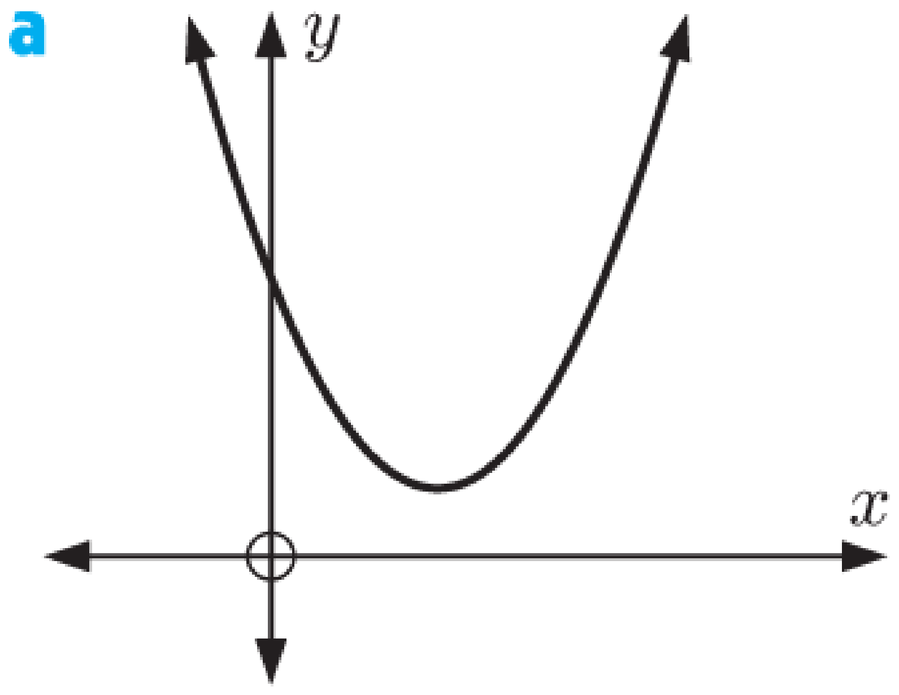


c

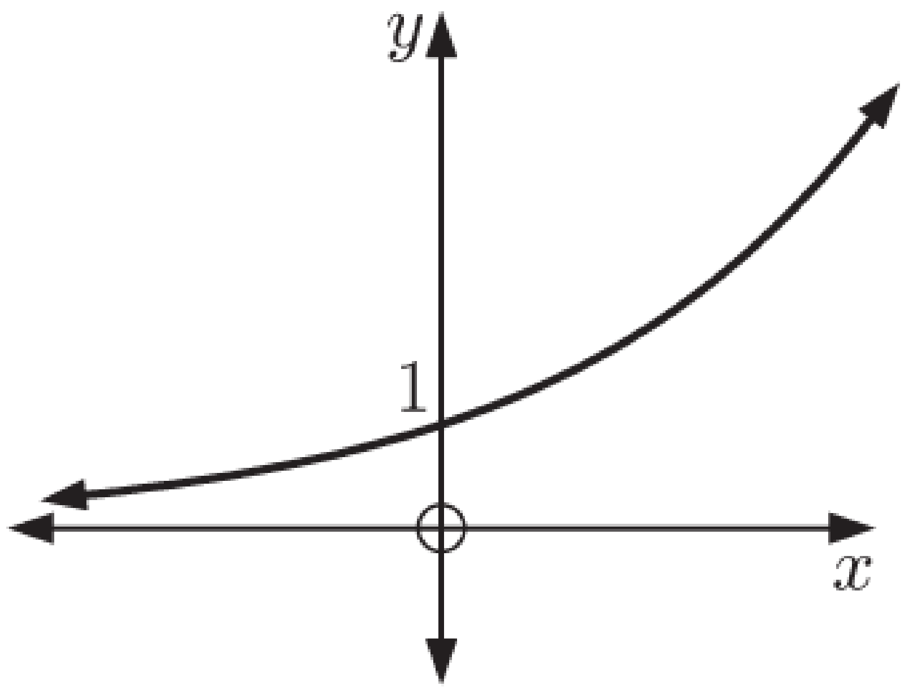


13 Copy the following graphs of $y = f(x)$ and sketch the graphs of $y = f(-x)$ on the same axes.

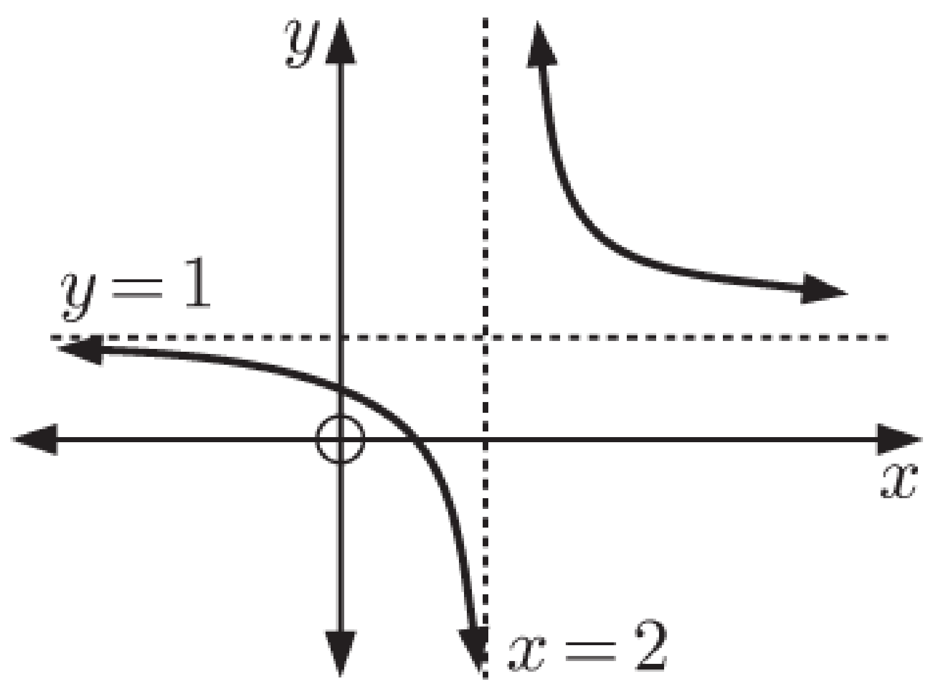
a



b



c



14 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 5x + 7$ is reflected in the x-axis

b $f(x) = 2^x$ is reflected in the y-axis

c $f(x) = 2x^2 + 1$ is reflected in the x-axis

d $f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7$ is reflected in the y-axis.

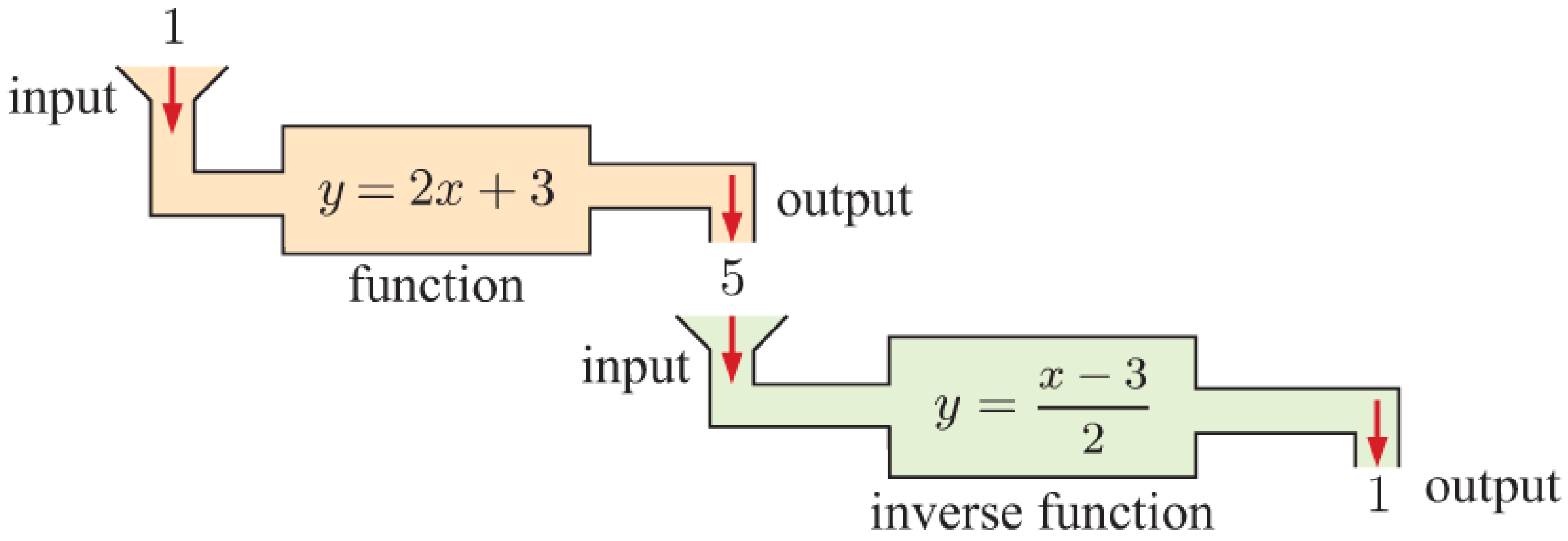
G INVERSE FUNCTIONS

The operations of + and −, × and ÷, are **inverse operations** as one "undoes" what the other does.

The function $y = 2x + 3$ can be "undone" by its *inverse* function $y = \frac{x-3}{2}$.

We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

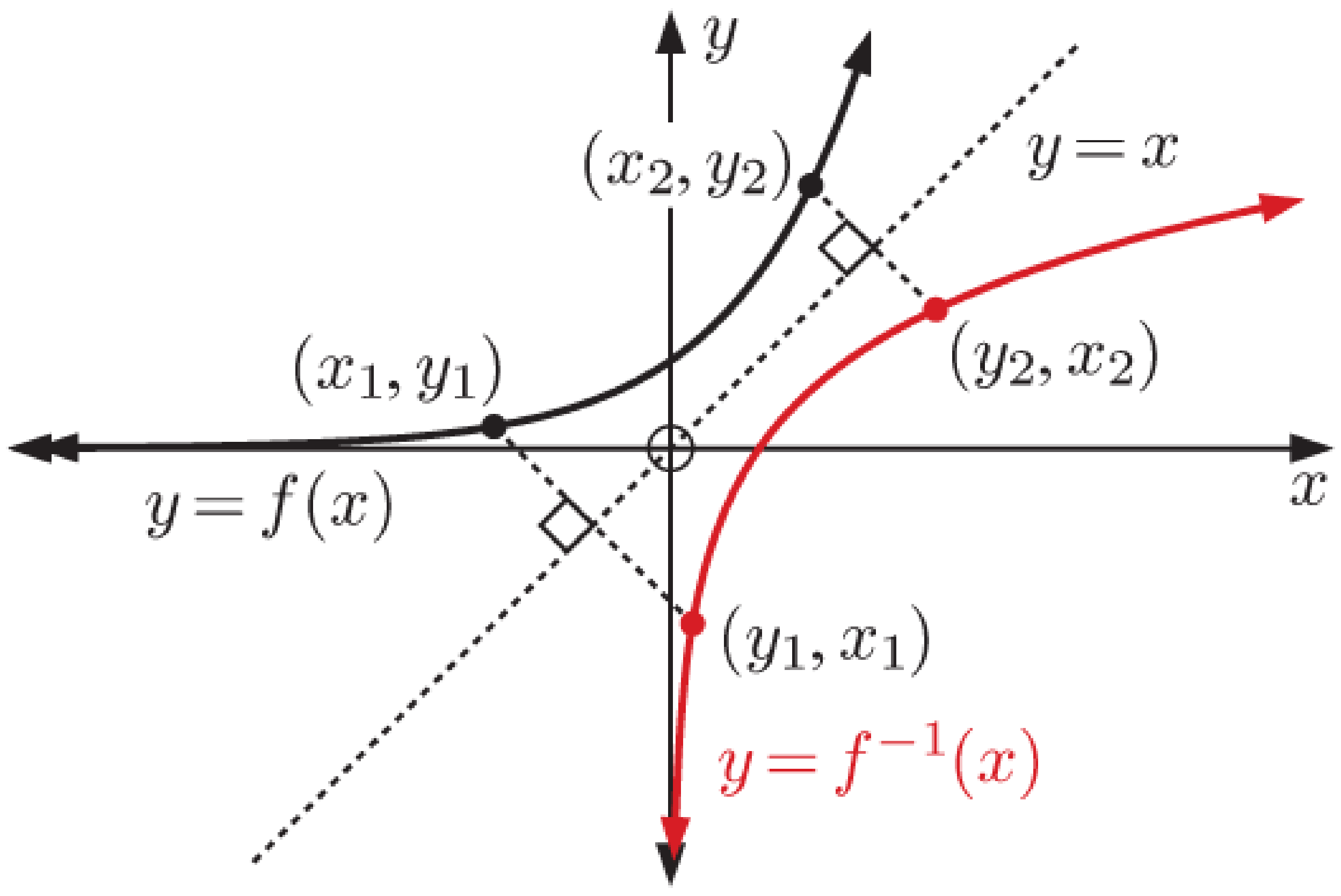
No matter what value of x enters the first machine, it is returned as the output from the second machine.



If a function $f(x)$ maps x to y, then its inverse function, denoted $f^{-1}(x)$, maps y back to x.

So, if the point (x, y) lies on the graph of $y = f(x)$, the point (y, x) must lie on the graph of $y = f^{-1}(x)$.

Geometrically, this is achieved by *reflecting* the graph of $y = f(x)$ in the line $y = x$.



A function $y = f(x)$ *may or may not* have an inverse function. To understand which functions do have inverses, we need some more terminology.

ONE-TO-ONE FUNCTION

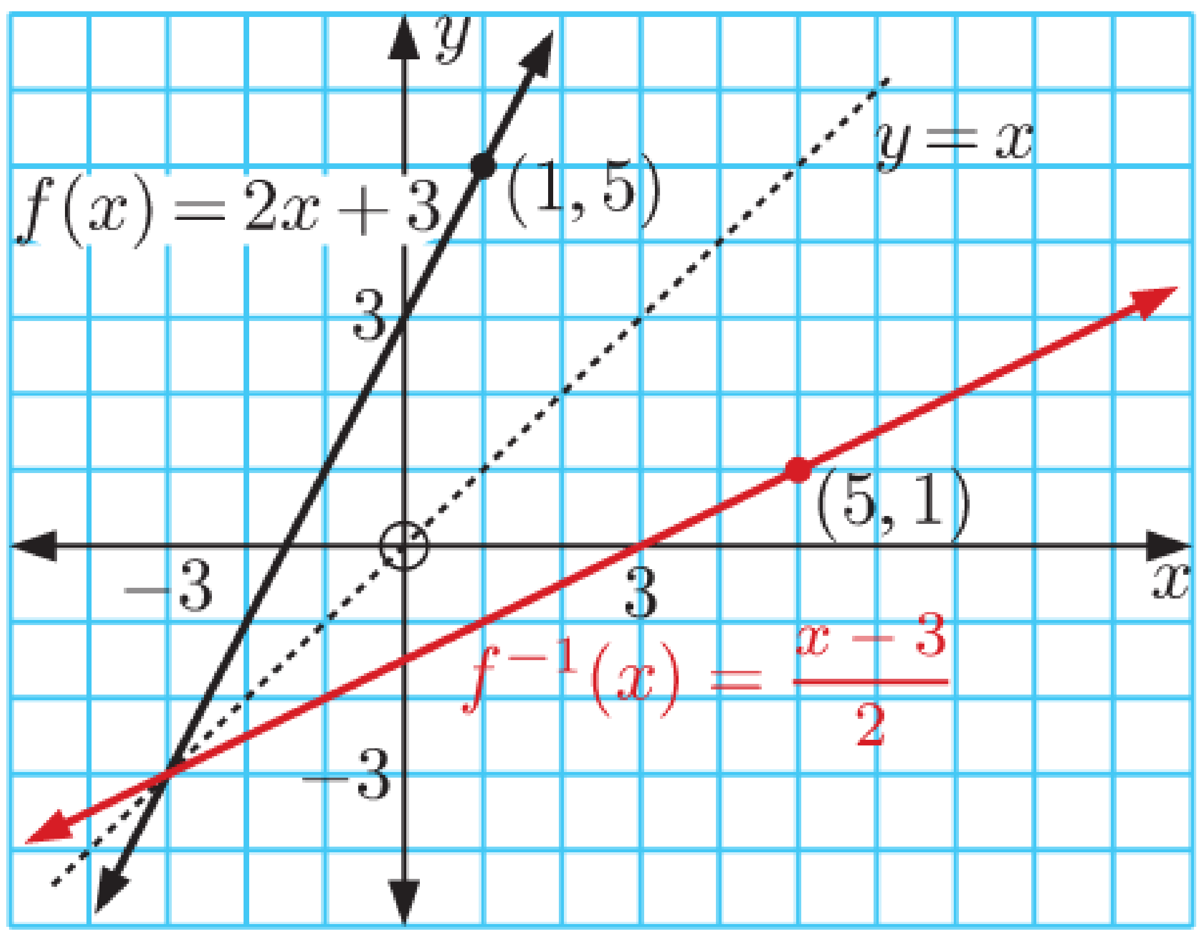
A function is **one-to-one** if, for each value of y, there is only one value of x.

One-to-one functions satisfy the **horizontal line test**. This means that no horizontal line can meet the graph more than once.

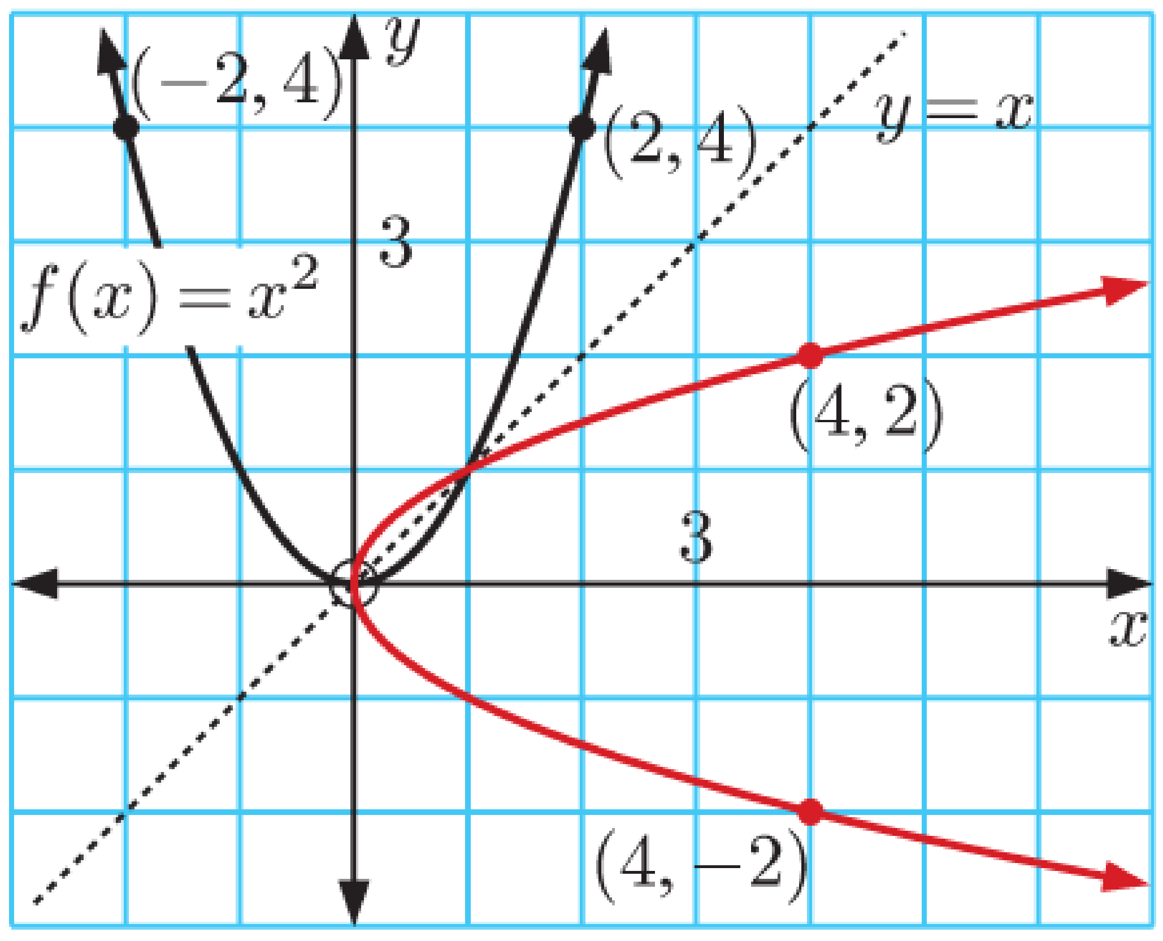
If the function $f(x)$ is **one-to-one**, it will have an inverse function which we denote $f^{-1}(x)$.

For example:

- The function $f(x) = 2x+3$ is one-to-one, as it passes the horizontal line test.
  When its graph is reflected in the line $y = x$, the result is the inverse function $f^{-1}(x) = \frac{x-3}{2}$.

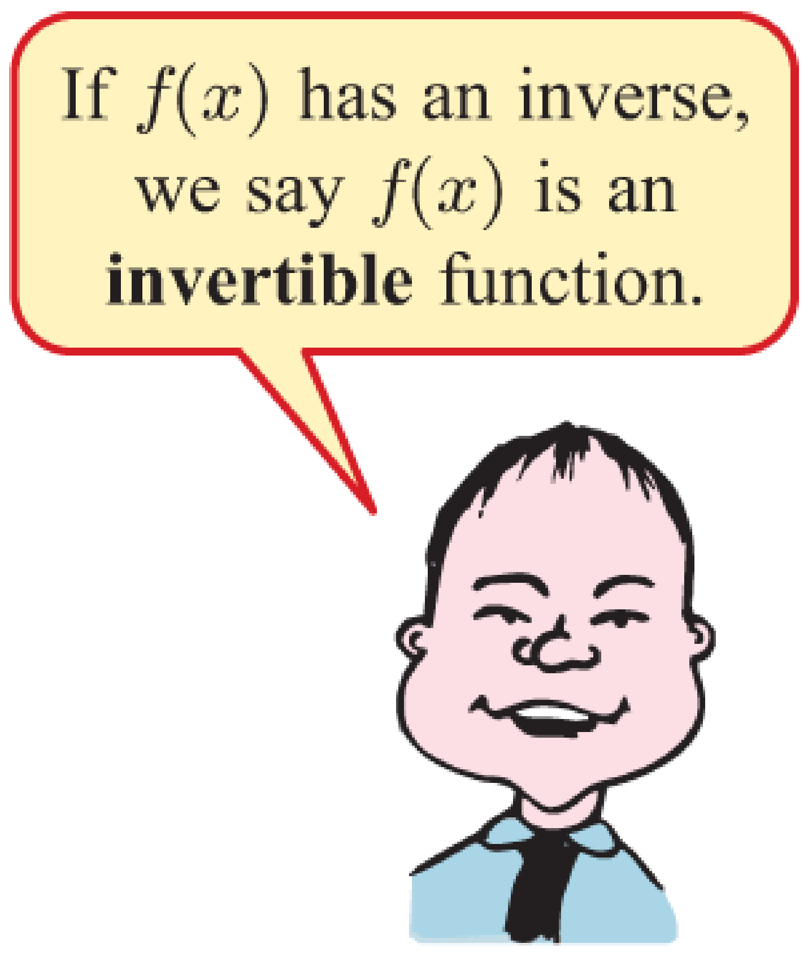


- The function $f(x) = x^2$ is not one-to-one, as it fails the horizontal line test.
  When its graph is reflected in the line $y = x$, the resulting graph is not a function.
  So, $f(x) = x^2$ does not have an inverse function.



If $f(x)$ has an inverse function $f^{-1}(x)$, then:

- The domain of f^{-1} is equal to the range of f.
- The range of f^{-1} is equal to the domain of f.


If $f(x)$ has an inverse, we say $f(x)$ is an **invertible** function.

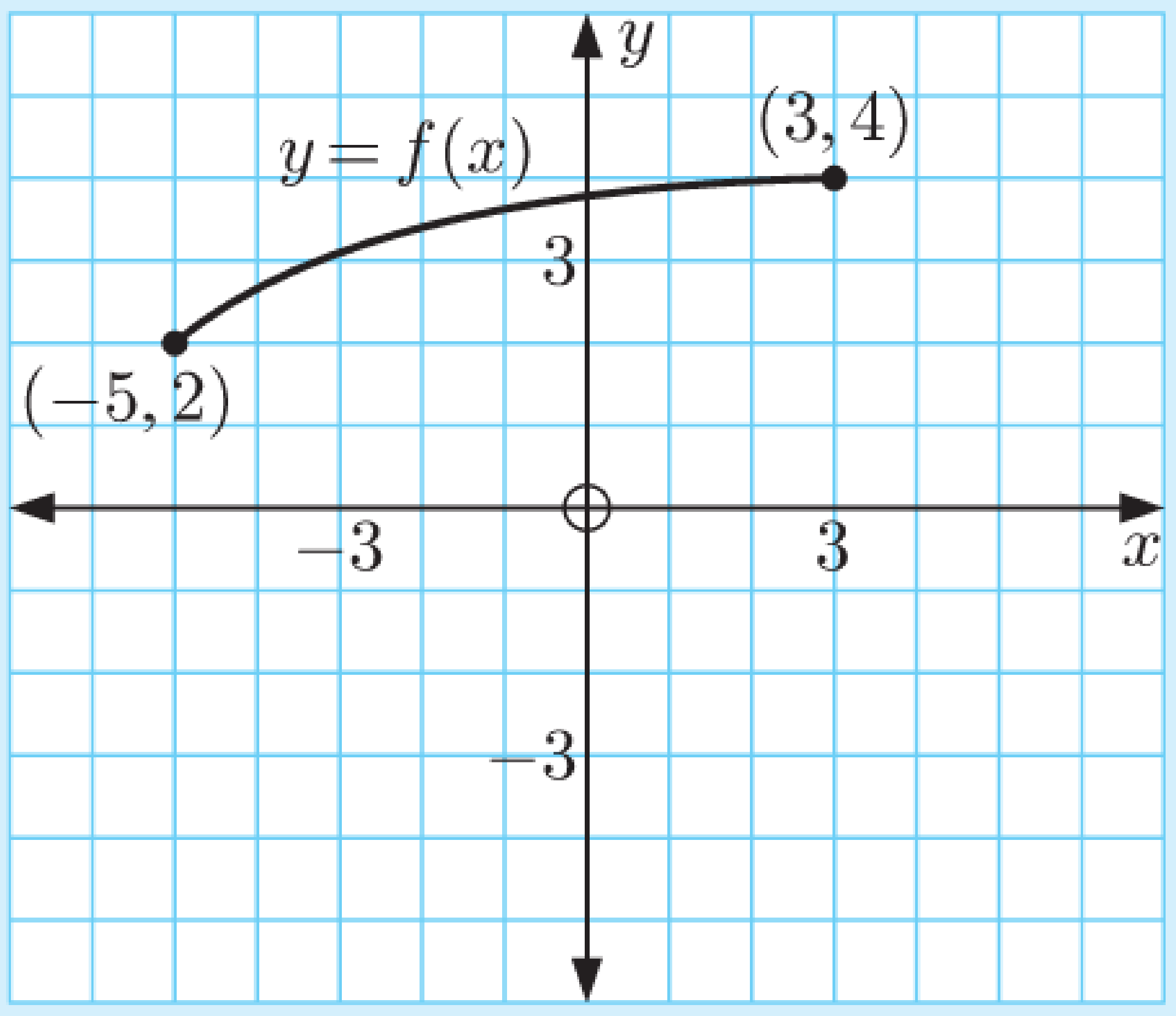
Example 13 **Self Tutor**

Consider the graph of $y = f(x)$ shown.

a Explain why $f(x)$ is an invertible function.

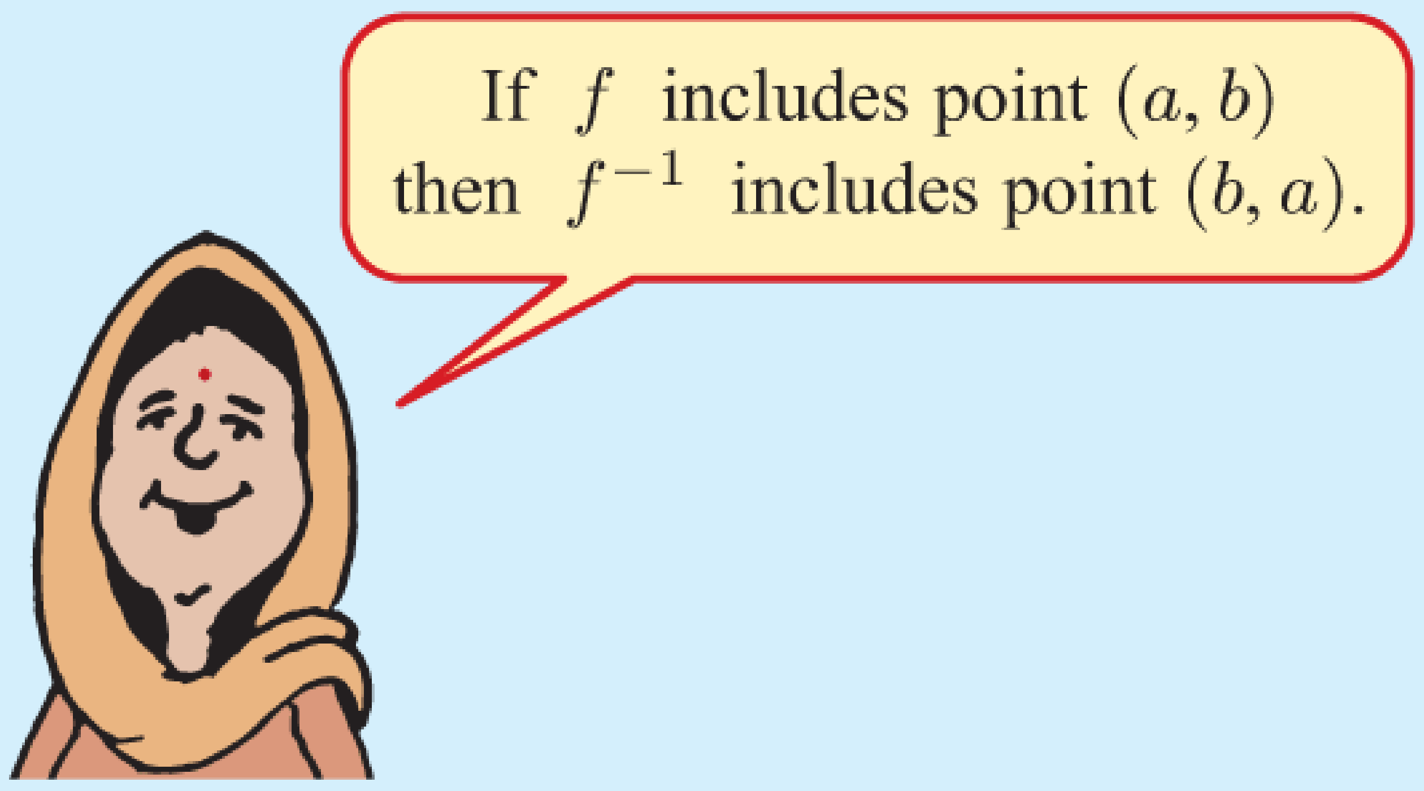
b Sketch the graph of the inverse function $y = f^{-1}(x)$.

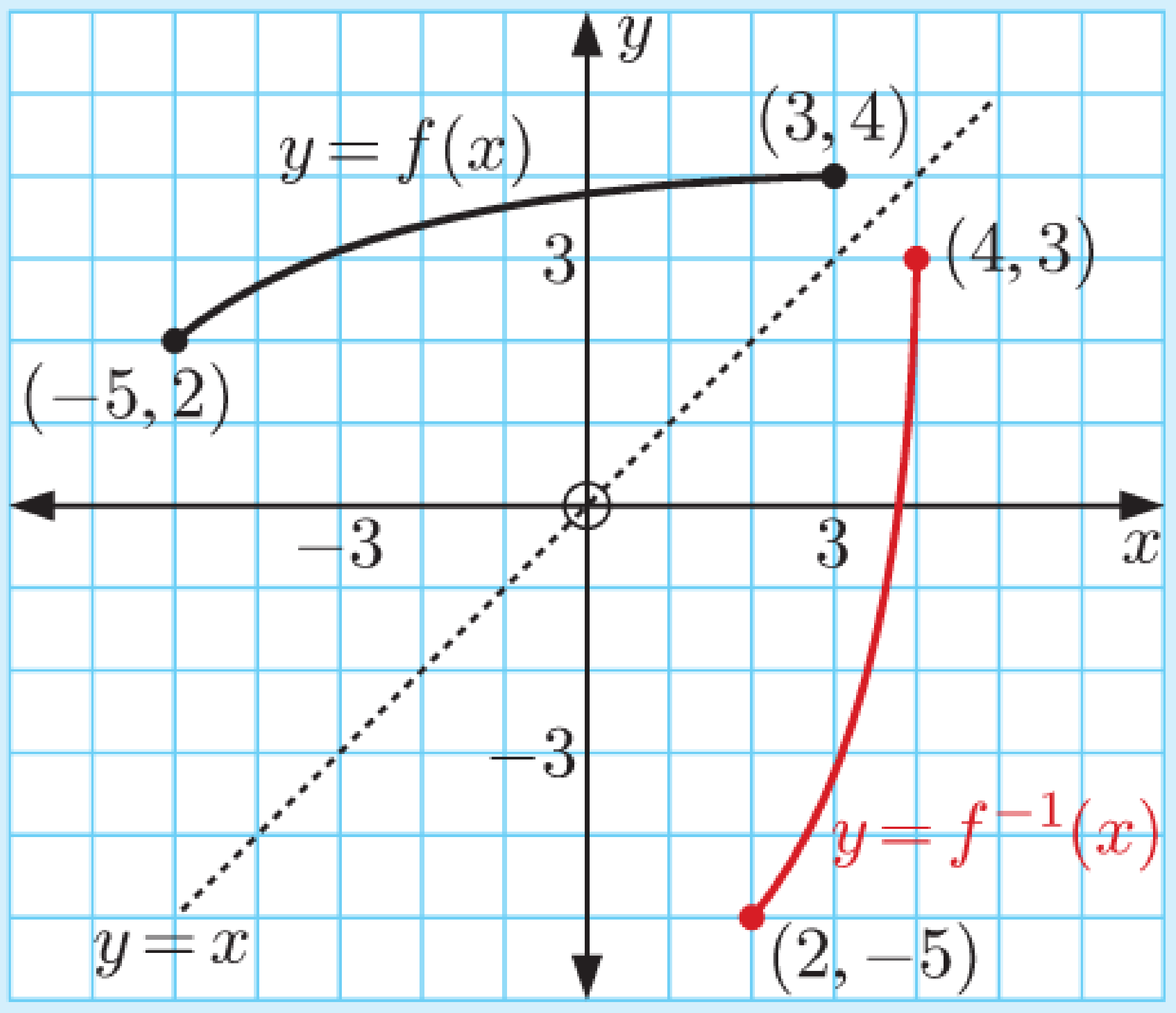
c State the domain and range of both $y = f(x)$ and $y = f^{-1}(x)$.



a $f(x)$ is a function because any *vertical* line cuts the graph at most once.
$f(x)$ is invertible because any *horizontal* line crosses the graph at most once.

b $y = f(x)$ passes through $(-5, 2)$ and $(3, 4)$.
$\therefore$ $y = f^{-1}(x)$ passes through $(2, -5)$ and $(4, 3)$.


If f includes point (a, b) then f^{-1} includes point (b, a).



c $y = f(x)$ has domain $\{x \mid -5 \leqslant x \leqslant 3\}$ and range $\{y \mid 2 \leqslant y \leqslant 4\}$.
$y = f^{-1}(x)$ has domain $\{x \mid 2 \leqslant x \leqslant 4\}$ and range $\{y \mid -5 \leqslant y \leqslant 3\}$.

EXERCISE 3G

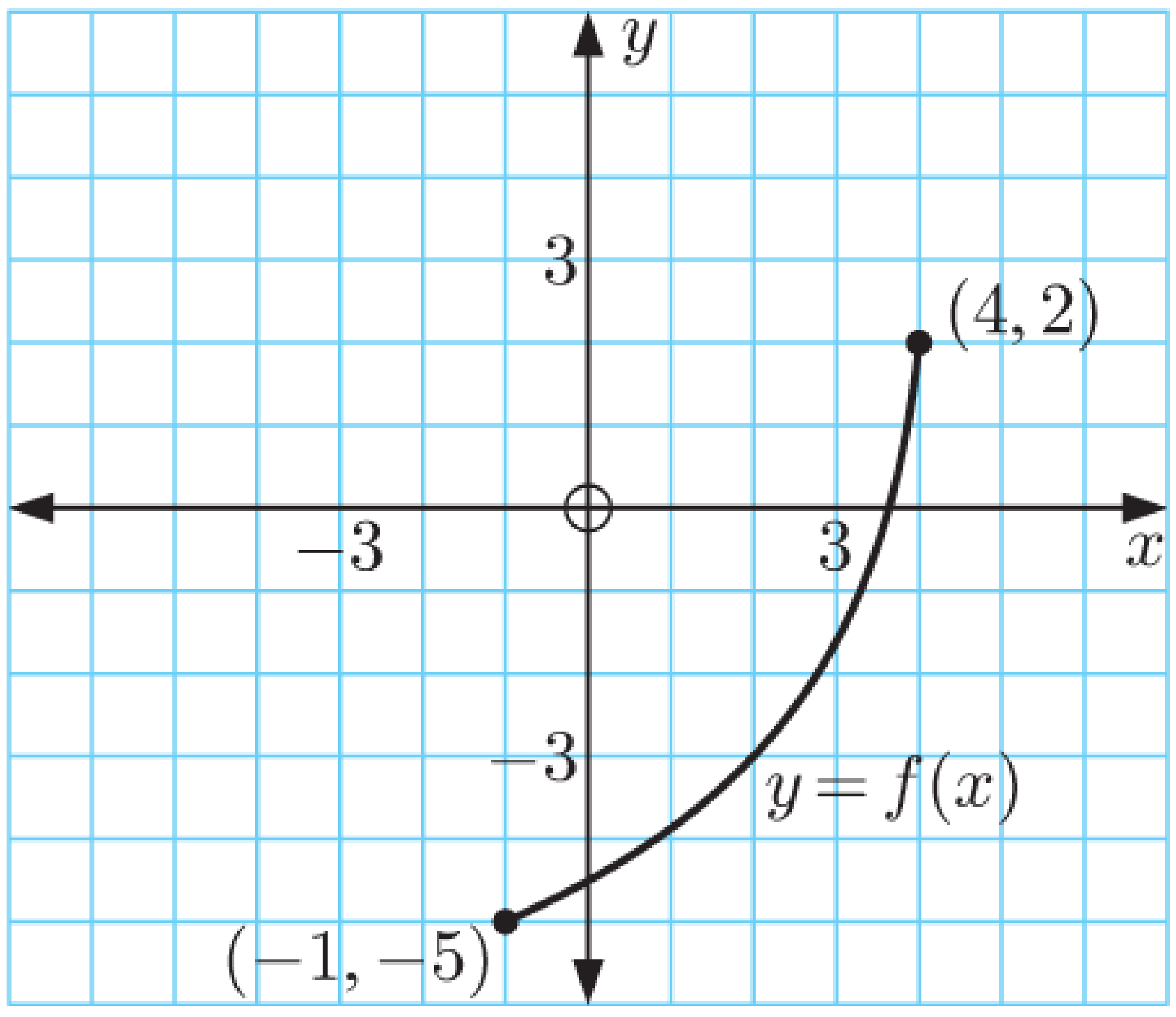
1 An invertible function f passes through the points $(-3, 7)$, $(0, 4)$, and $(2, -6)$. State three points which lie on the inverse function f^{-1}.

2 The graph of $y = f(x)$ is shown alongside.

a Explain why an inverse function $y = f^{-1}(x)$ exists.

b Copy the graph, and sketch the inverse function $y = f^{-1}(x)$.

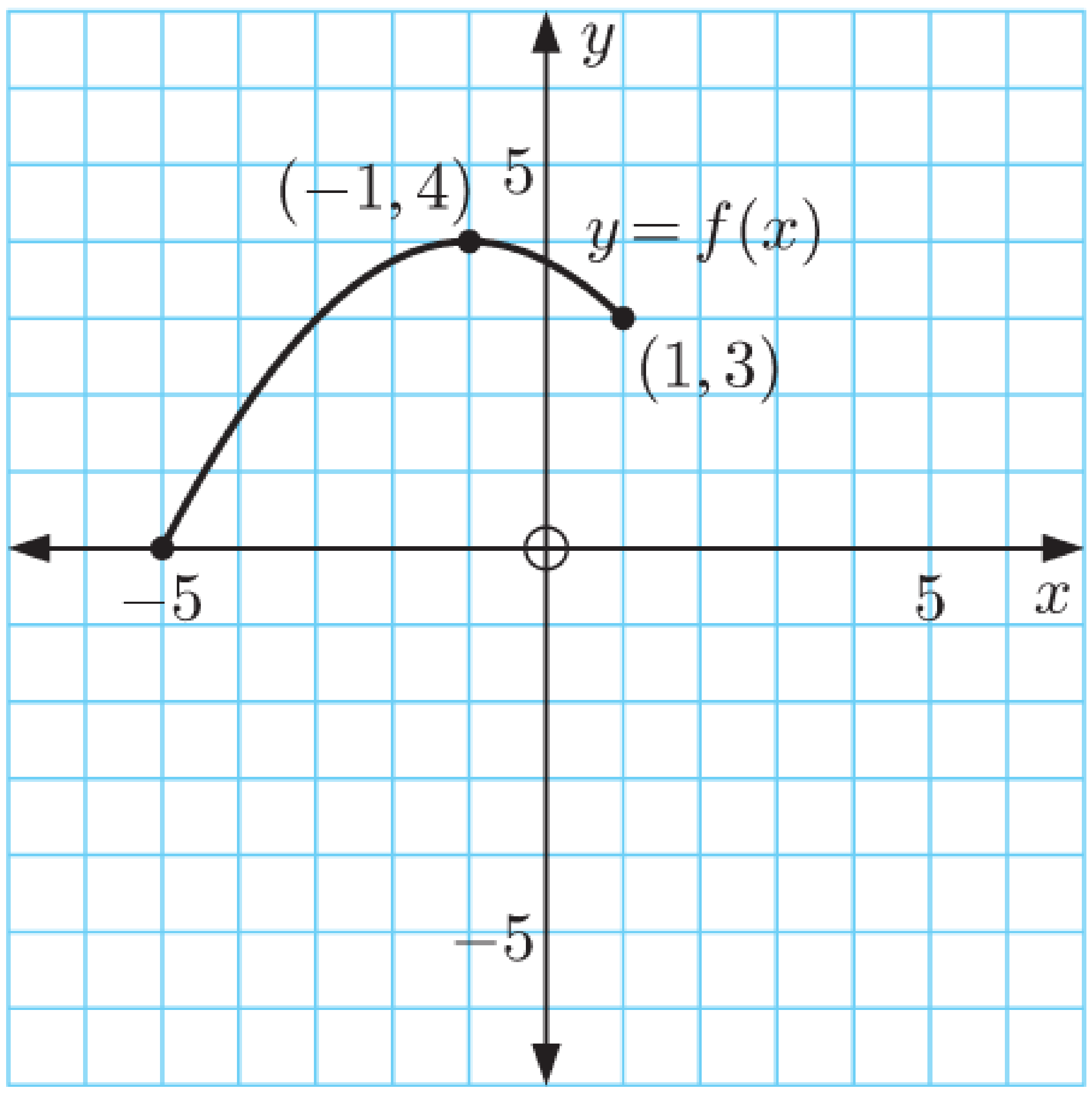
PRINTABLE GRAPHS



3 Consider the function $y = f(x)$ shown.

a Copy the graph, and sketch the reflection of $y = f(x)$ in the line $y = x$.

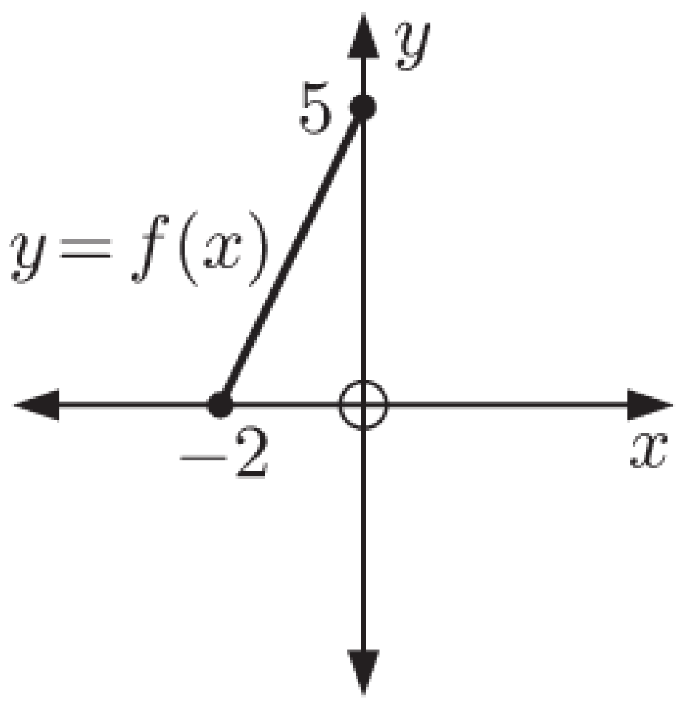
b Is the graph you have sketched the inverse function of $y = f(x)$? Explain your answer.



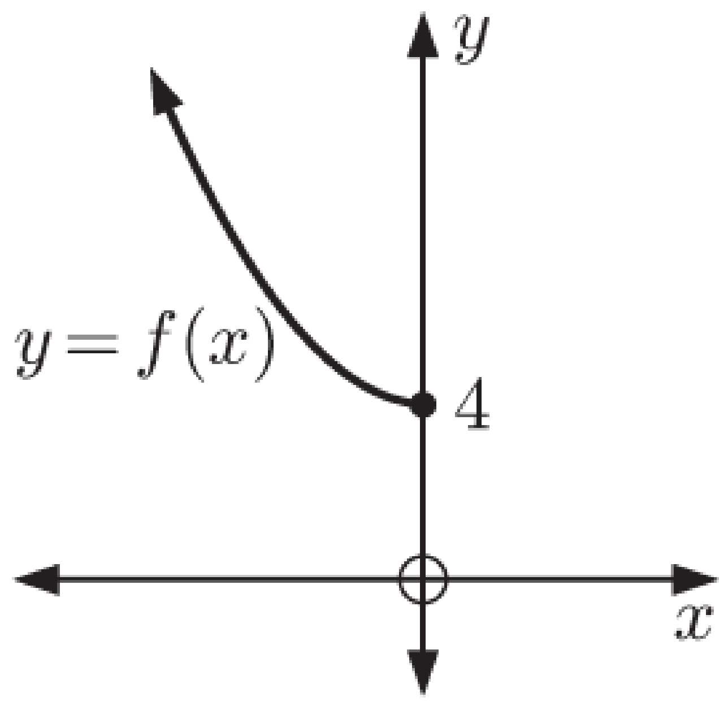
4 The invertible function $f(x)$ has domain $\{x \mid -2 \leqslant x < 3\}$. Find the range of its inverse $f^{-1}(x)$.

5 Copy the graphs of the following functions and draw the graphs of $y = x$ and $y = f^{-1}(x)$ on the same set of axes. In each case, state the domain and range of both f and f^{-1}.

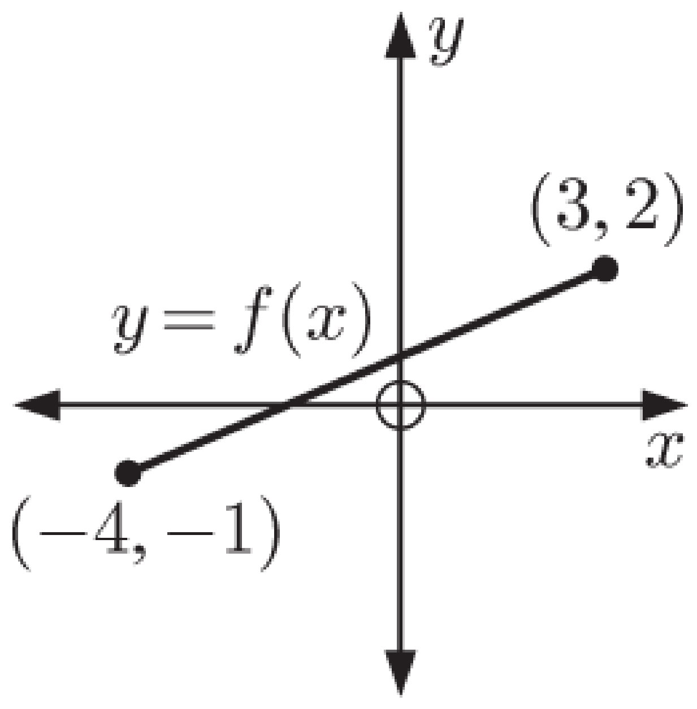
a



b



c



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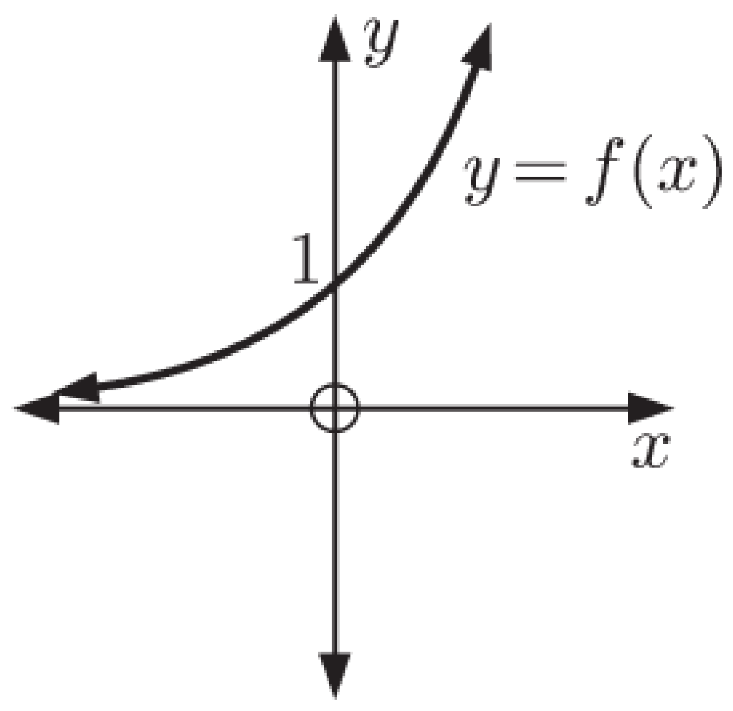

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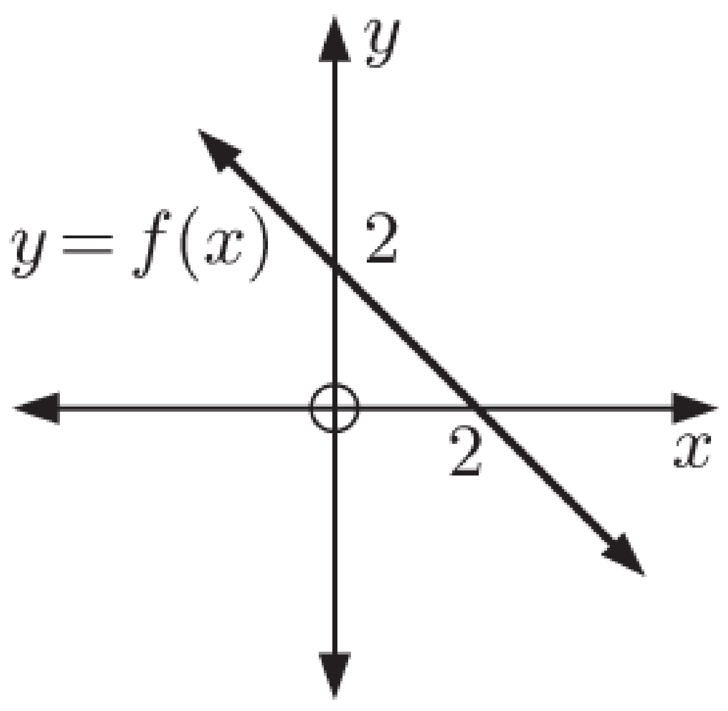
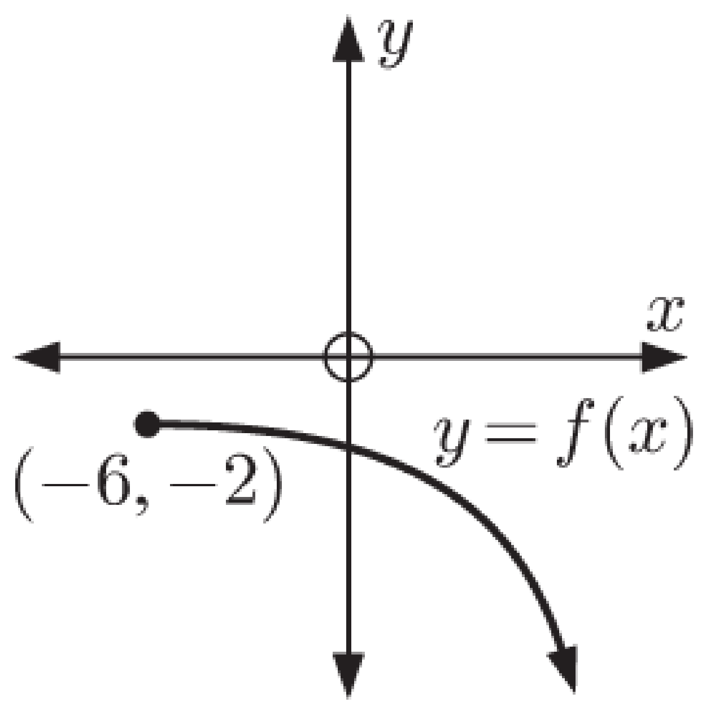


f



6 An invertible function $g(x)$ has x-intercept 5 and y-intercept -3. State the axes intercepts of its inverse function $g^{-1}(x)$.

7 A function $f(x)$ has x-intercepts -3 and 2, and y-intercept 4. Explain why $f(x)$ is not invertible.

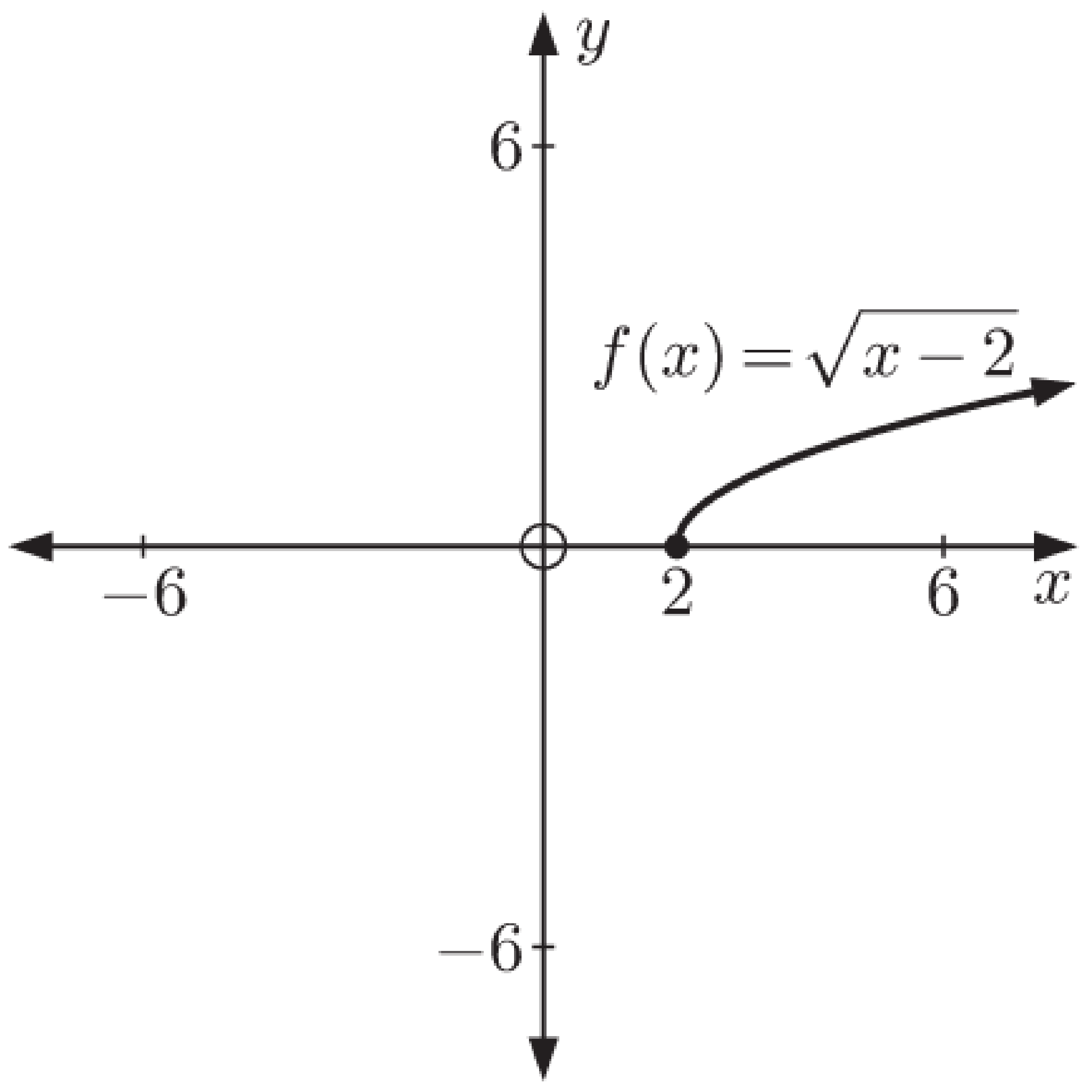
8 The graph of $f(x) = \sqrt{x-2}$ is shown alongside.

a Find the domain and range of $f(x)$.

b Copy the graph, and sketch the inverse function $f^{-1}(x)$.

c State the domain and range of $f^{-1}(x)$.

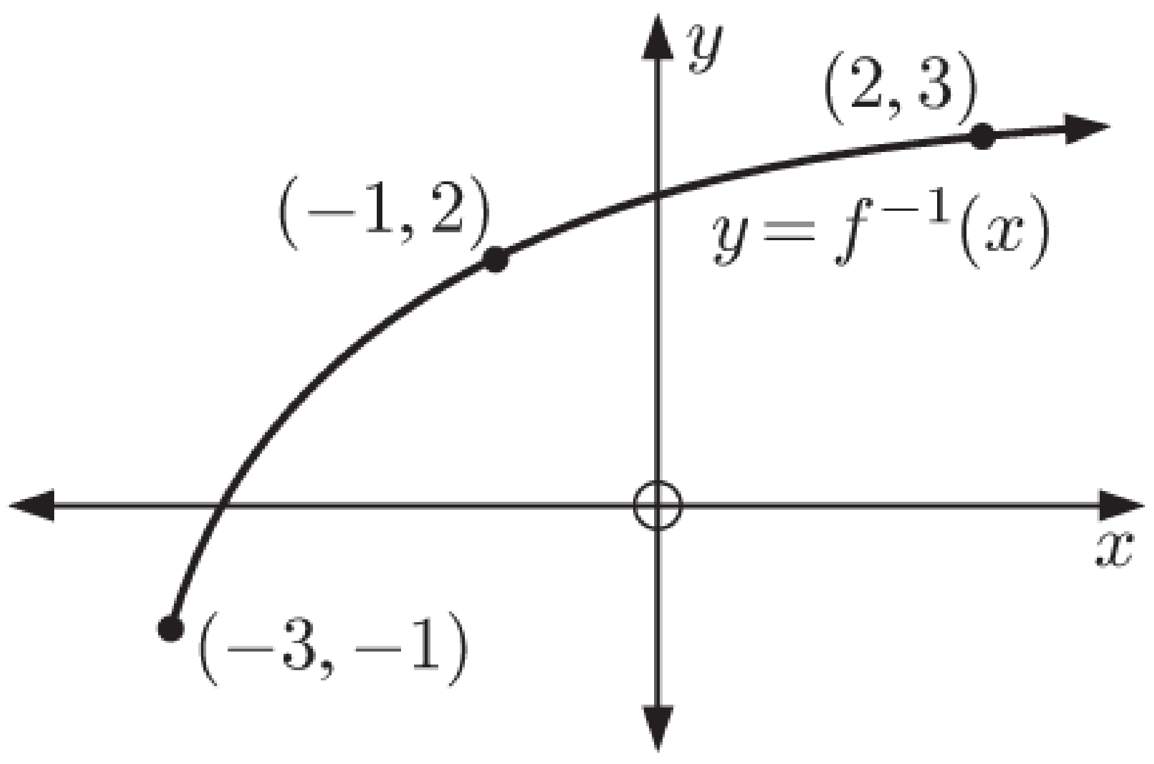
d Solve $f^{-1}(x) = 3$.



9 The inverse function $f^{-1}(x)$ of an invertible function $f(x)$ is graphed alongside.

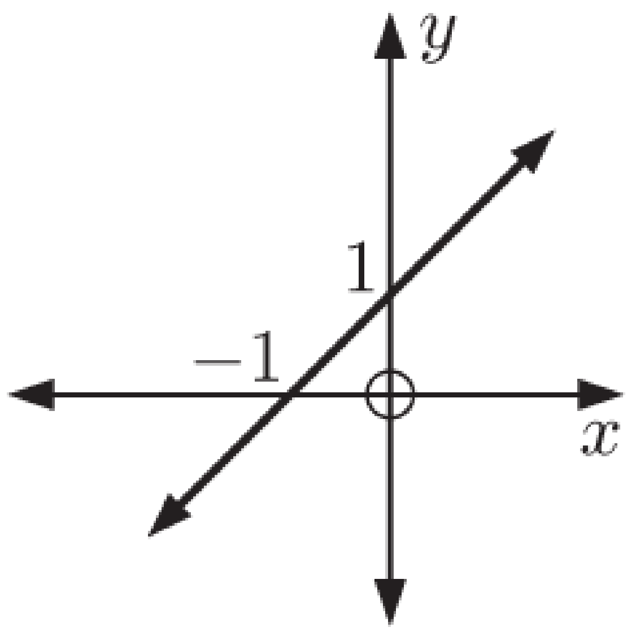
a State the domain and range of $f(x)$.

b Solve $f(x) = 2$.

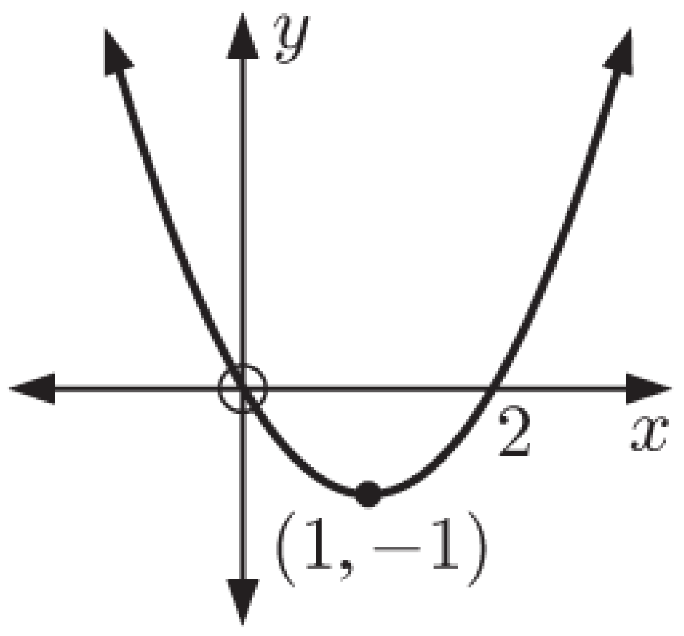


10 Which of the following functions have an inverse function?

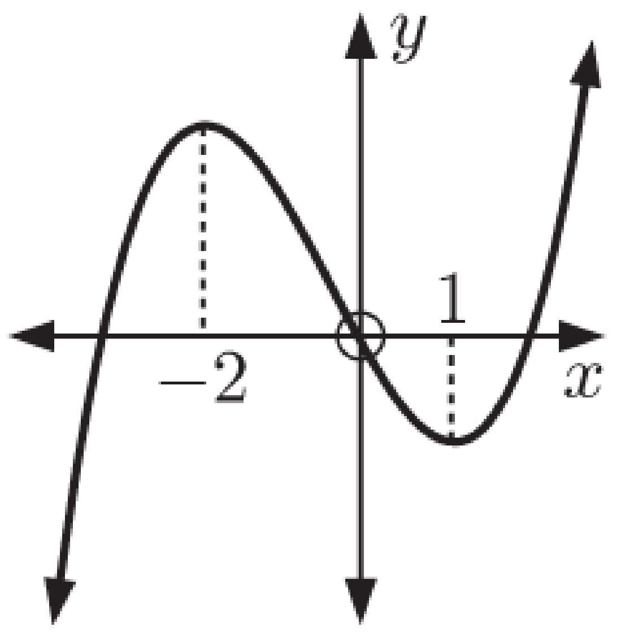
a



b



c



11 Consider the function $f(x) = x^2 - 2x + 5$.

a Copy and complete this table of values:

x	-2	-1	0	1	2
$f(x)$					

b Hence explain why $f(x)$ is not invertible.

12 Consider the function $f(x) = 3x + 6$.

a Find the axes intercepts of the function.

b Draw the graph of $f(x)$ and its inverse function $f^{-1}(x)$.

c Determine the equation of $f^{-1}(x)$.

13 Given the linear function $f(x) = mx + c$, $m \neq 0$, find the equation of the inverse function $f^{-1}(x)$.

REVIEW SET 3A

1 Determine whether the set of ordered pairs is a function. Explain your answer.

a $\{(1, 5), (4, 2), (2, 5), (6, -1)\}$

b $\{(-4, 0), (3, 2), (0, -2), (3, 5)\}$

2 If $f(x) = 2x - x^2$, find:

a $f(2)$

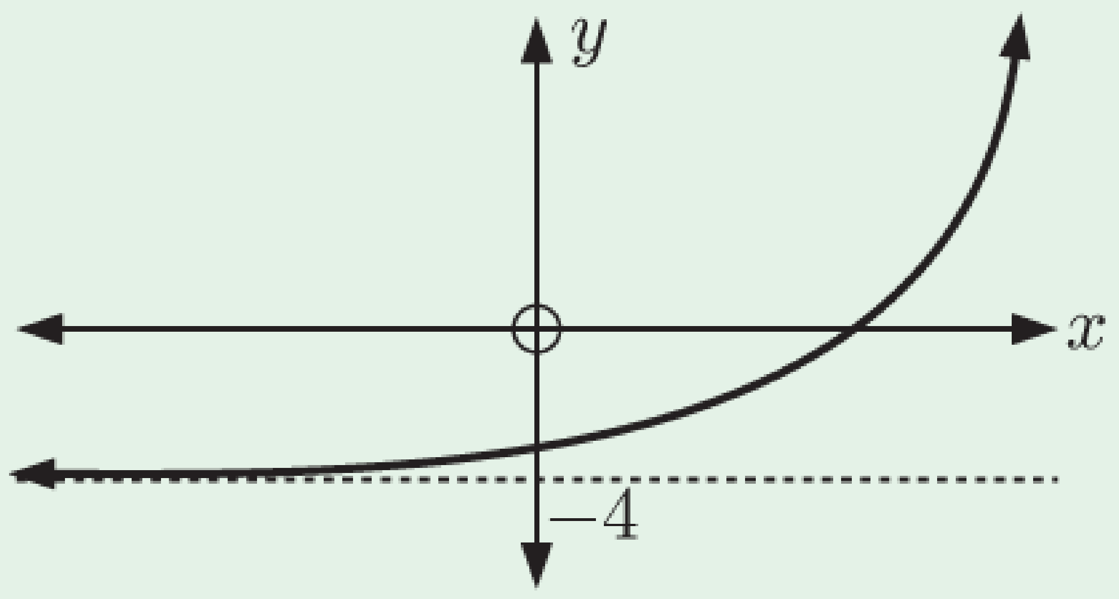
b $f(-3)$

c $f(-\frac{1}{2})$

3 For each graph, state:

i the domain **ii** the range **iii** whether the graph shows a function.

a



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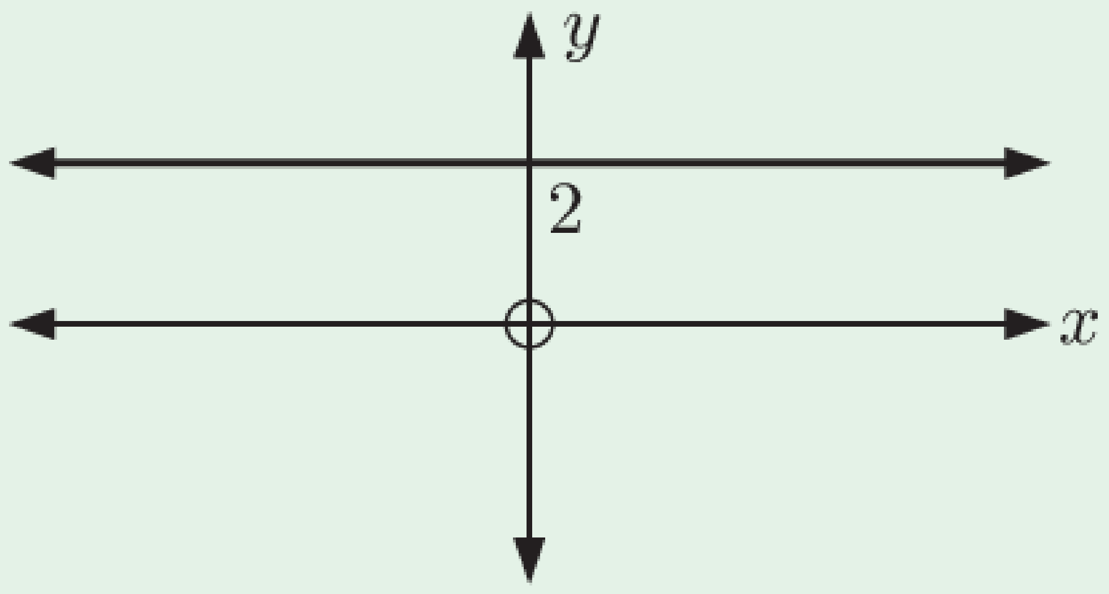
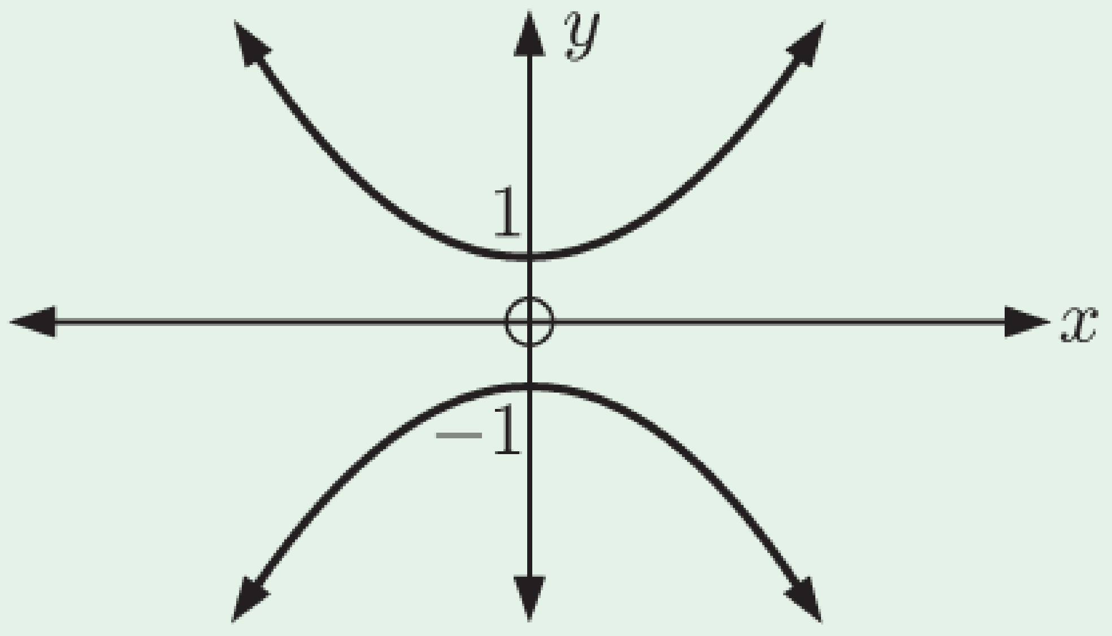


c



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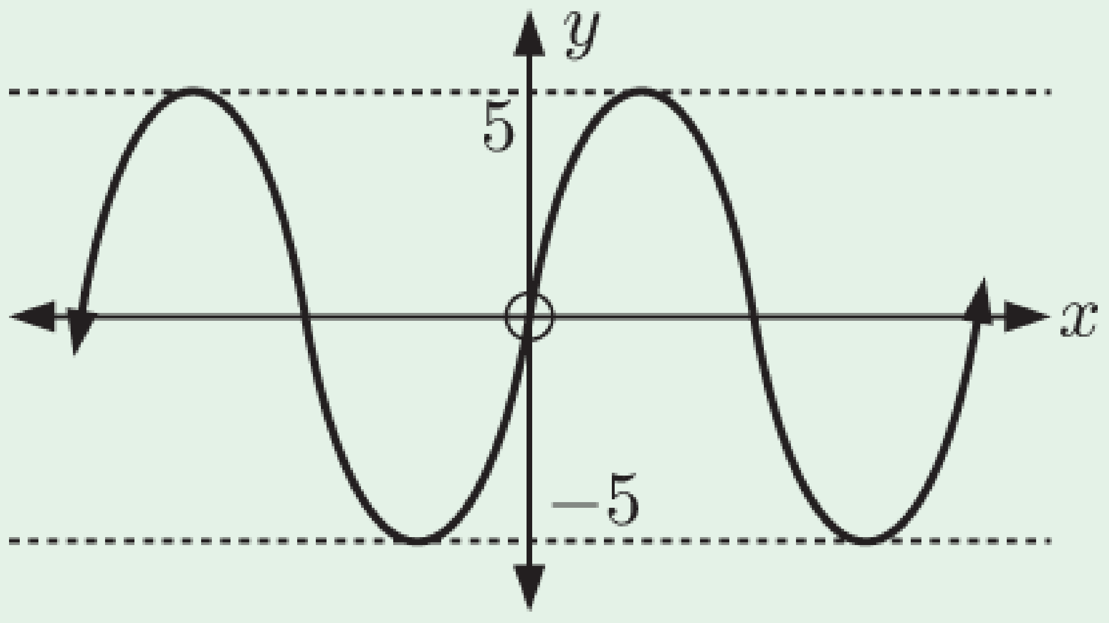
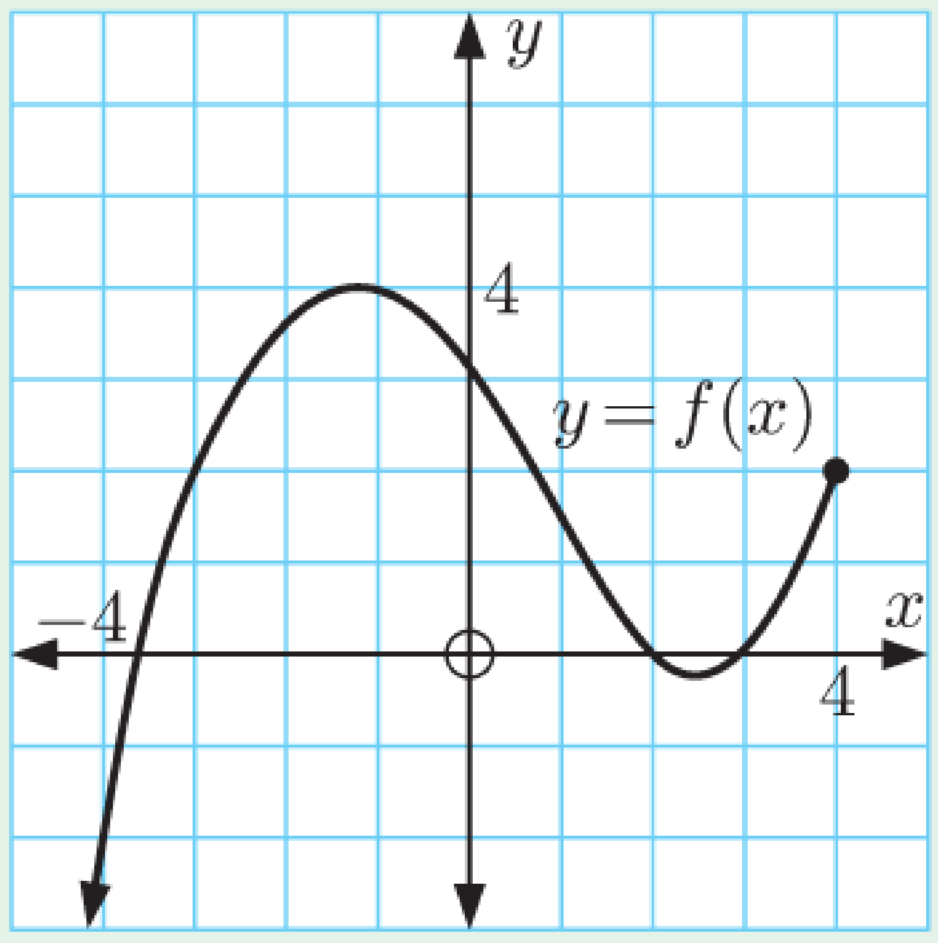


4 The graph of $y = f(x)$ is shown alongside.

a Find:

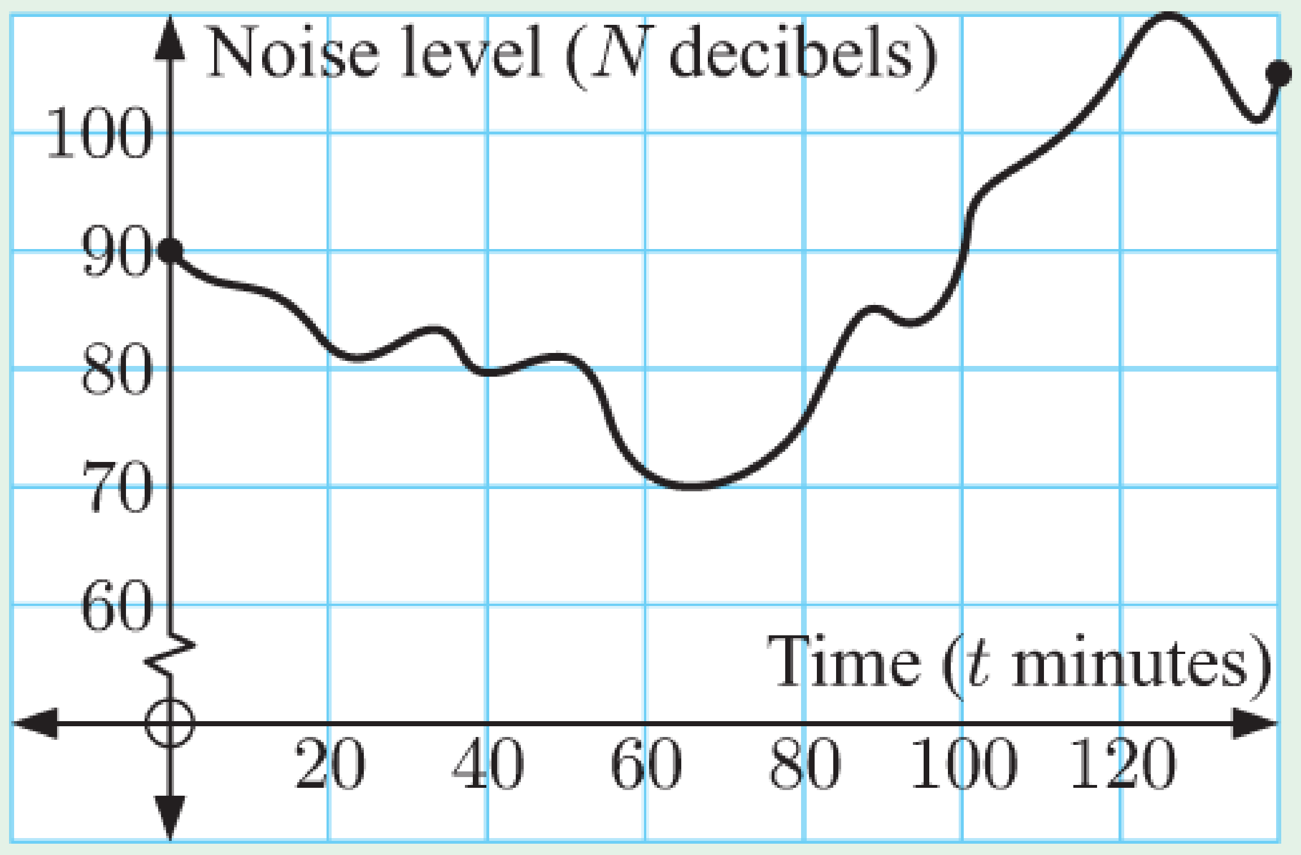
i $f(-3)$ **ii** $f(2)$

b Find the value of x such that $f(x) = -2$.



5 Find a and b given $f(x) = ax + b$, $f(1) = 7$, and $f(3) = -5$.

6 This graph shows the noise level at a stadium during a football match.
Find the domain and range of the function.



7 Consider $f(x) = \frac{-2}{x^2}$.

a For what value of x is $f(x)$ undefined, or not a real number?

b Sketch the function using technology.

c State the domain and range of the function.

8 Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

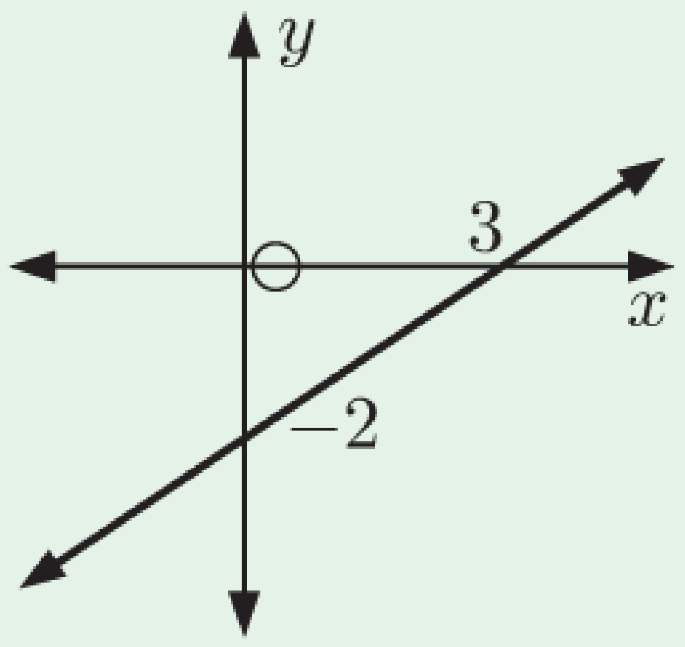
a Show that $f(-3) = g(-\frac{4}{3})$. **b** Find x such that $g(x) = f(5)$.

9 If $f(x) = x^2 - 2x$, find in simplest form:

a $f(x) + 5$ **b** $3f(x)$ **c** $f(2x)$ **d** $f(-x)$

10 Draw a sign diagram for each graph:

a



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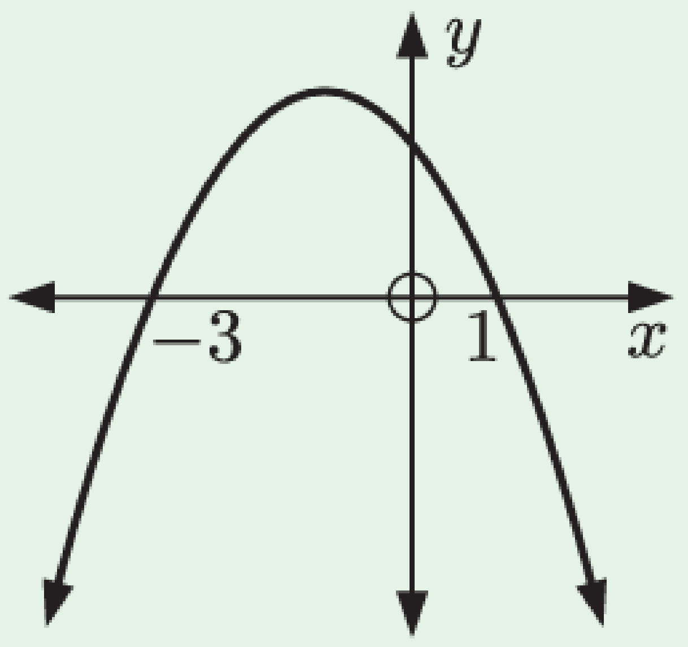
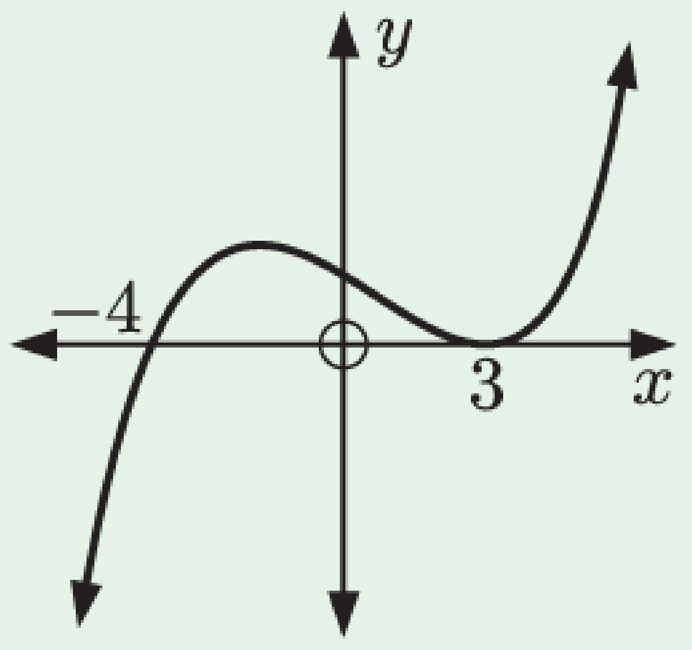


c



11 Draw a sign diagram for:

a $(x-5)(x+2)$

b $-(x+3)^2$

c $\dfrac{-11}{(x+1)(x+8)}$

12 For each of the following functions:

- **i** Determine the axes intercepts.
- **ii** Find the coordinates and nature of any turning points.
- **iii** Find the equation(s) of any asymptotes.
- **iv** Sketch $y = f(x)$, showing its key features.

a $f(x) = x^3 - 4x^2 + 3$

b $f(x) = 1 + \dfrac{8}{x-2}$

13 The height of a seedling t days after planting is given by $H(t) = \dfrac{10}{1 + 4 \times (1.5)^{-1.2t}}$ cm for $t \geqslant 0$.

a Use technology to help sketch the graph of $H(t)$ for $0 \leqslant t \leqslant 15$.

b How high was the seedling when it was planted?

c How high was the seedling after 1 week?

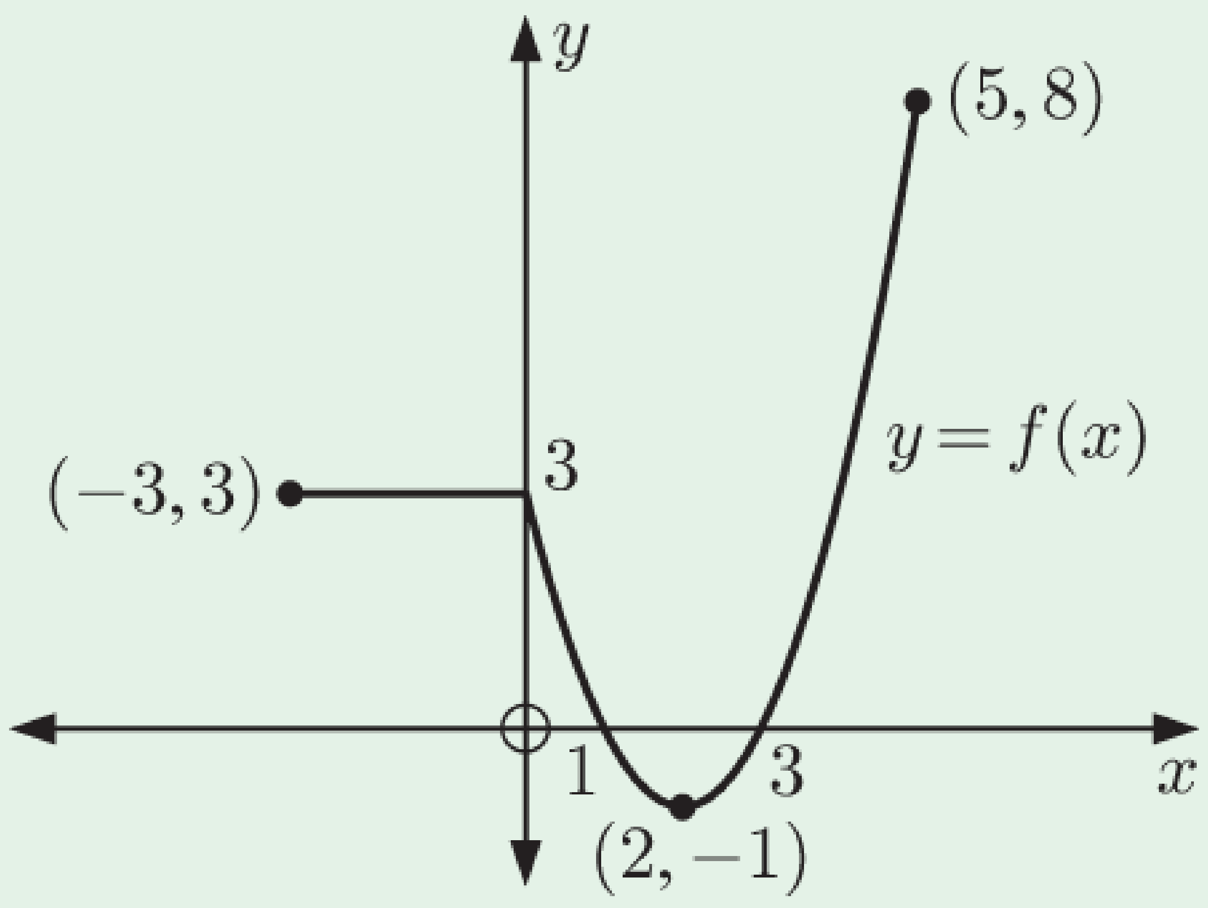
d How long did it take for the seedling to reach a height of 5 cm?

e Is there a limit to how high the seedling can grow? If so, what is it?

14 Given the graph of $y = f(x)$ shown, sketch the graph of:

a $y = -f(x)$

b $y = f(x) + 2$



15 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 4x - 7$ is translated 3 units downwards

b $f(x) = x^2 + 6$ is vertically stretched with scale factor 5

c $f(x) = 2x^2 - x + 4$ is horizontally stretched with scale factor 3

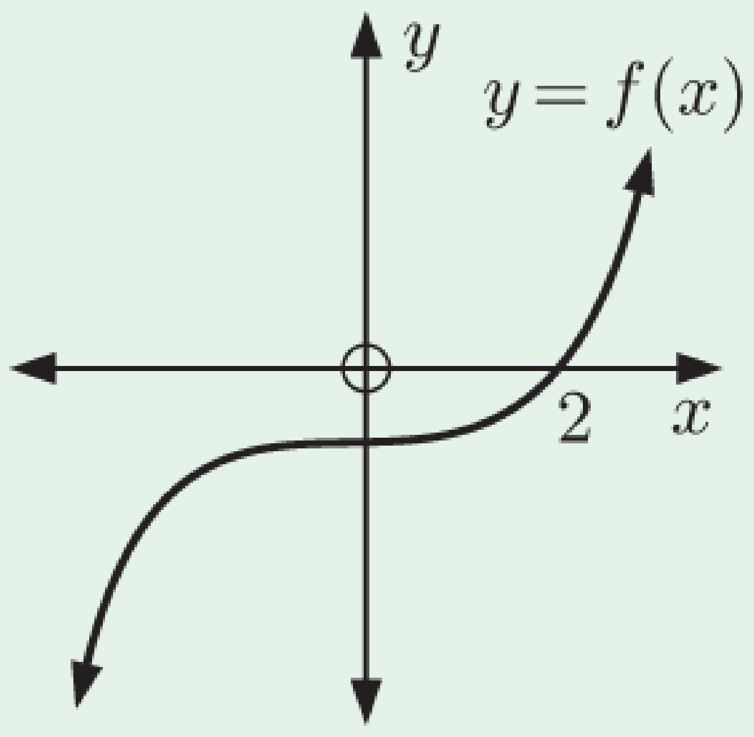
d $f(x) = x^3$ is reflected in the y-axis.

16 The function $f(x)$ has domain $\{x \mid -2 \leqslant x \leqslant 3\}$ and range $\{y \mid -1 \leqslant y \leqslant 7\}$. Find the domain and range of $g(x) = f(x) - 4$. Explain your answers.

17 An invertible function f passes through the points $(-1,\ 4)$ and $(6,\ -2)$. State two points which lie on the inverse function f^{-1}.

18 Decide whether each function has an inverse function:

a



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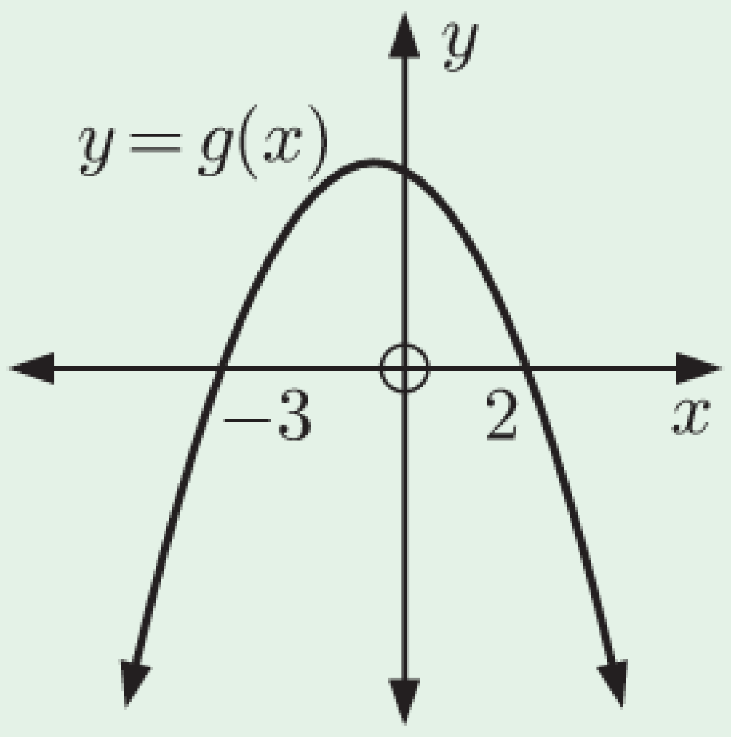
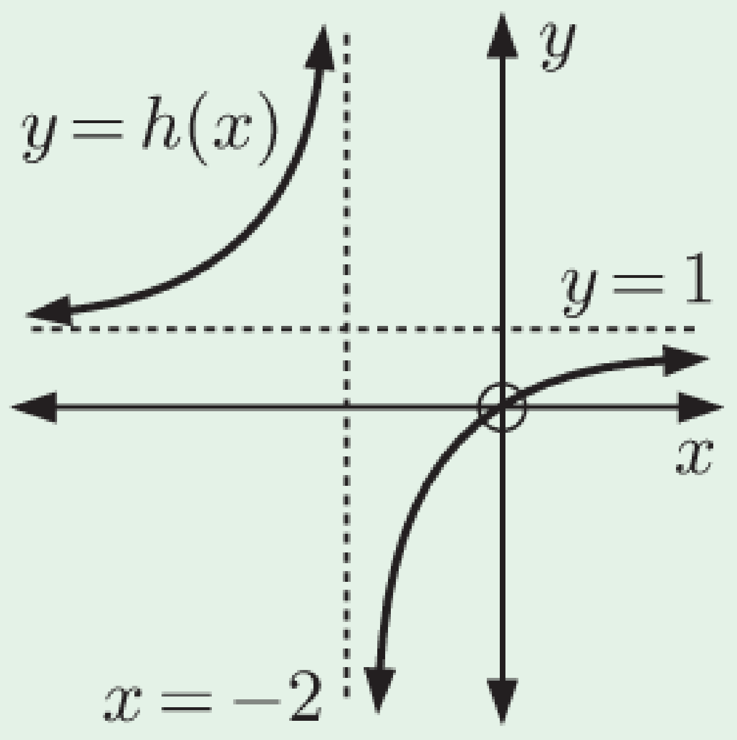
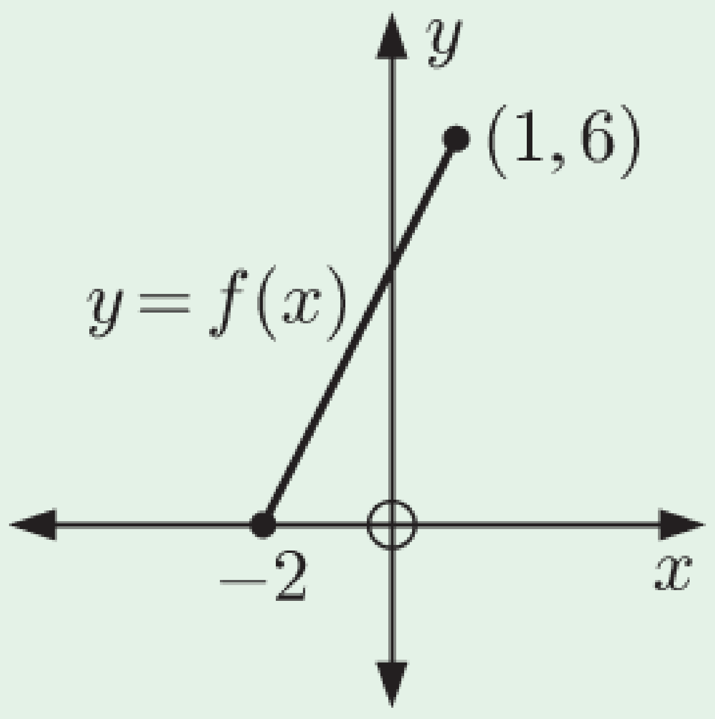


c



19 Copy each graph and draw $y = f^{-1}(x)$ on the same set of axes. In each case, state the domain and range of both f and f^{-1}.

a



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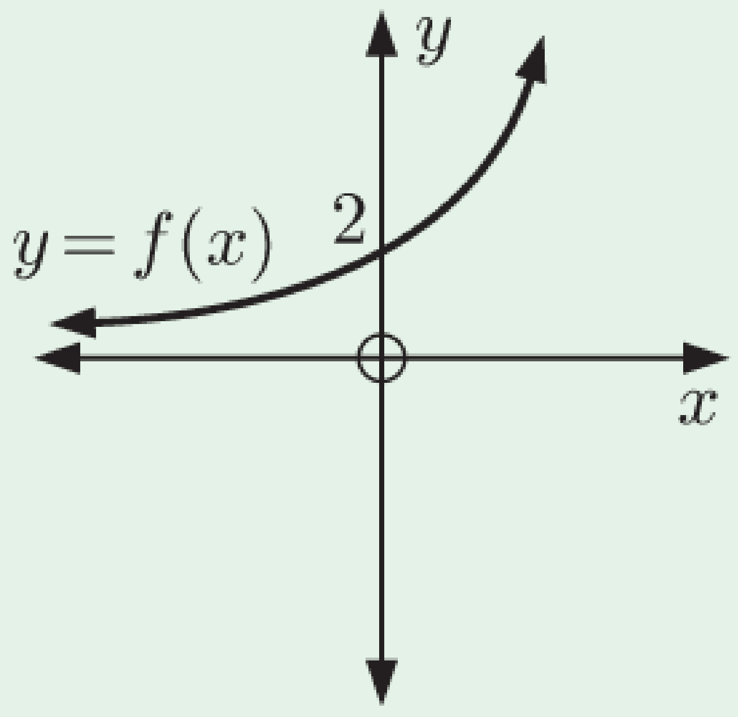
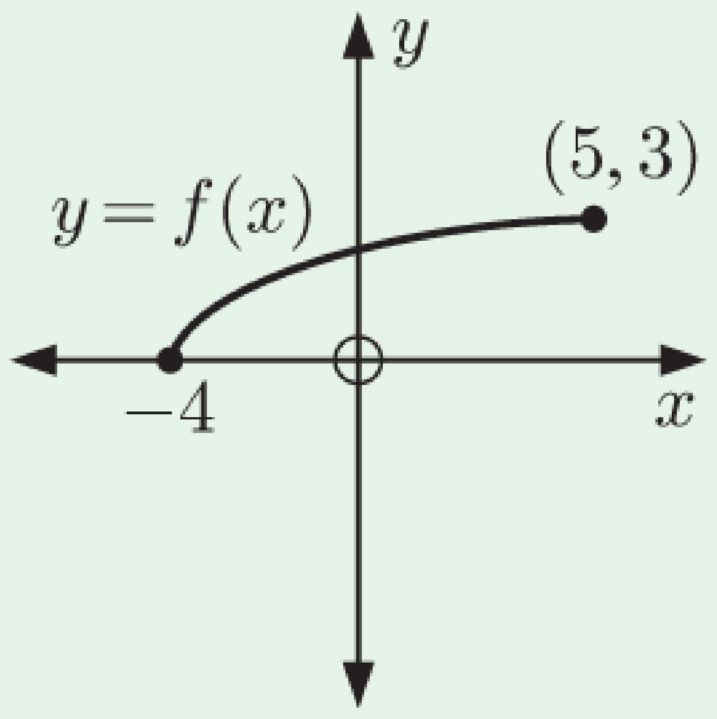


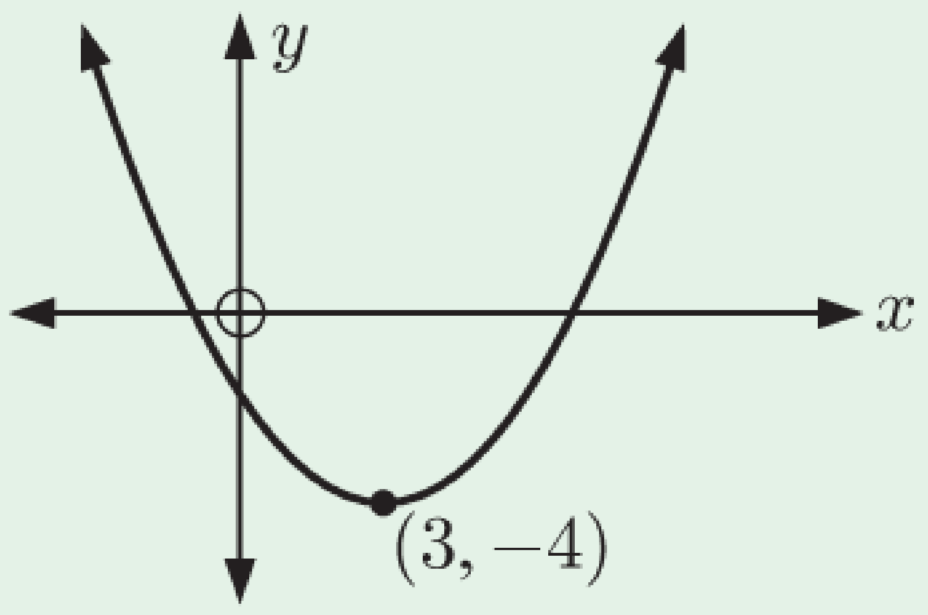
c



REVIEW SET 3B

1 State the domain and range of each function:

a



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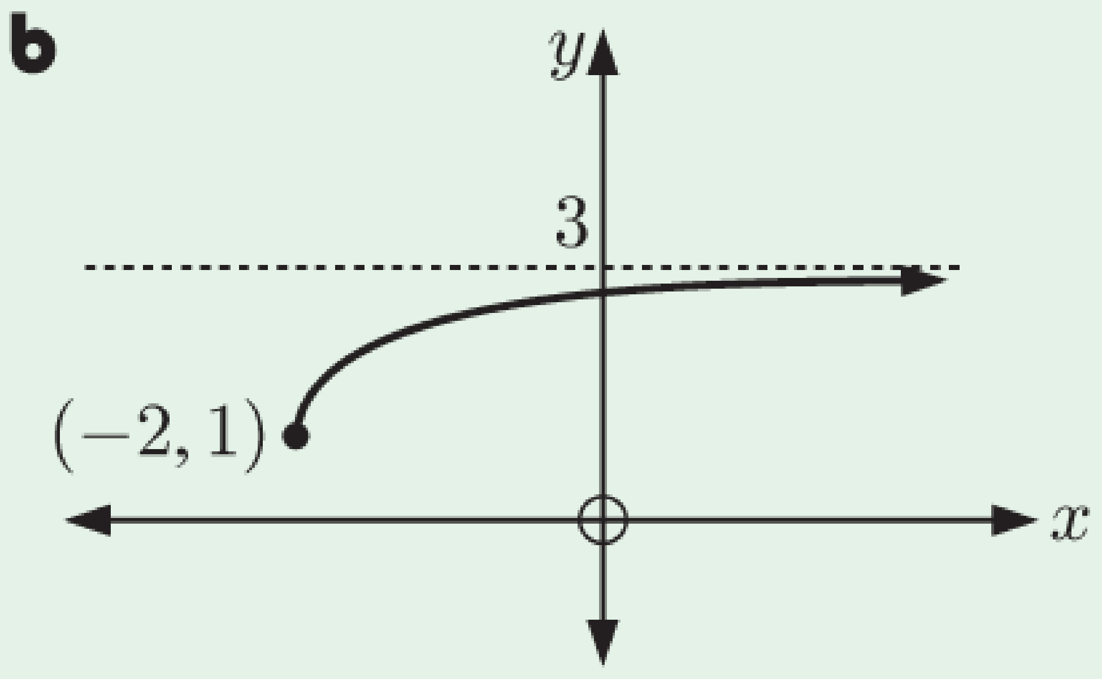

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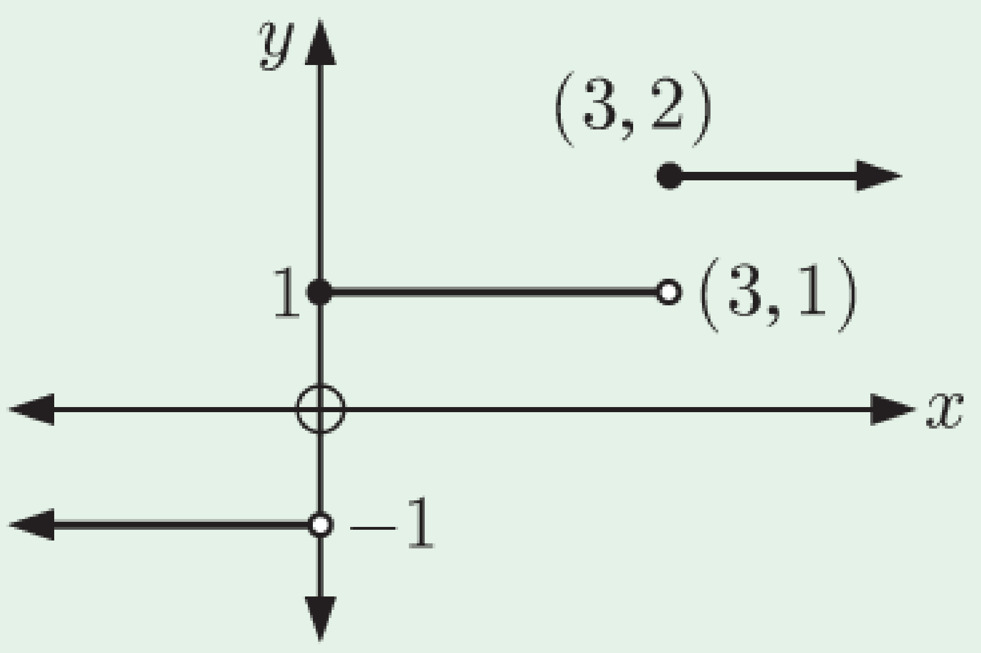


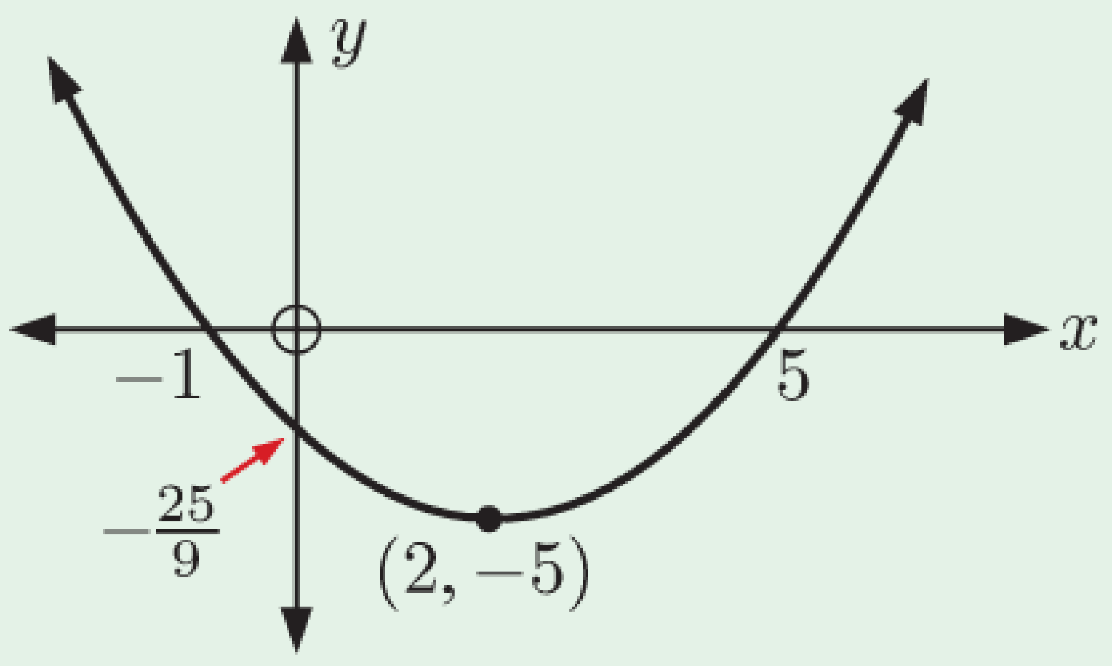
2 If $g(x) = x^2 - 3x$, find in simplest form:

a $g(2)$ **b** $g(x+1)$ **c** $g(4x)$

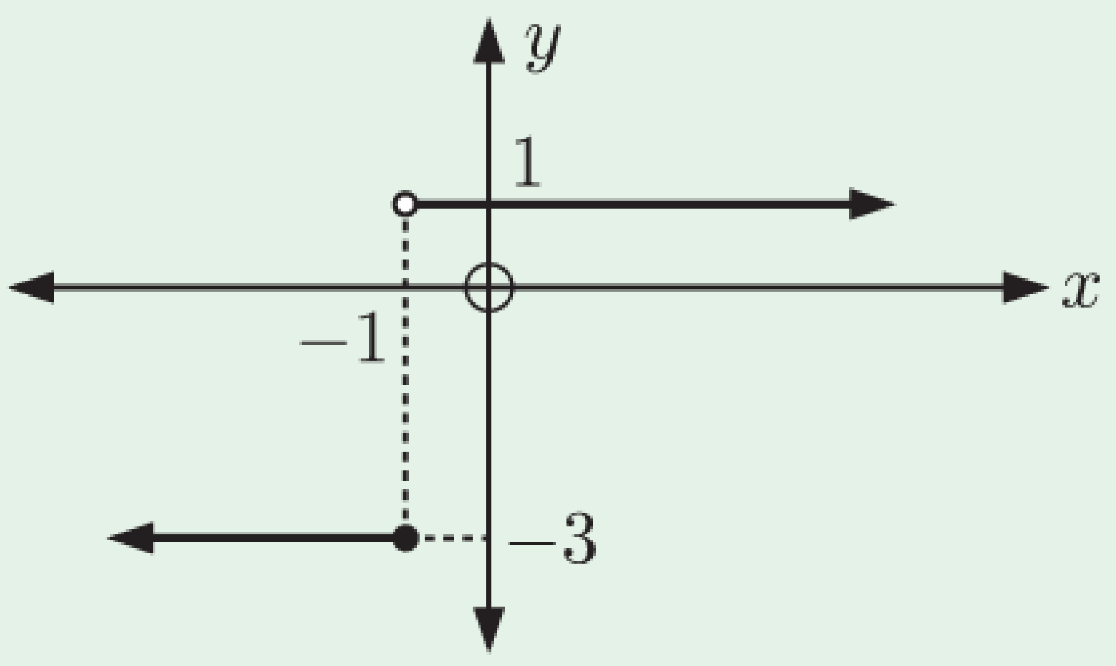
3 For each of the following graphs, determine:

i the domain and range **ii** the x and y-intercepts **iii** whether it is a function.

a



b



4 Use algebraic methods to determine whether these relations are functions:

a $x + 2y = 10$ **b** $x + y^2 = 10$

5 Suppose $f(x) = \dfrac{3x-1}{x+2}$.

- **a** Evaluate: **i** $f(-1)$ **ii** $f(0)$ **iii** $f(5)$
- **b** Find a value of x such that $f(x)$ does not exist.
- **c** Find $f(x-1)$ in simplest form.
- **d** Find x if $f(x) = 4$.

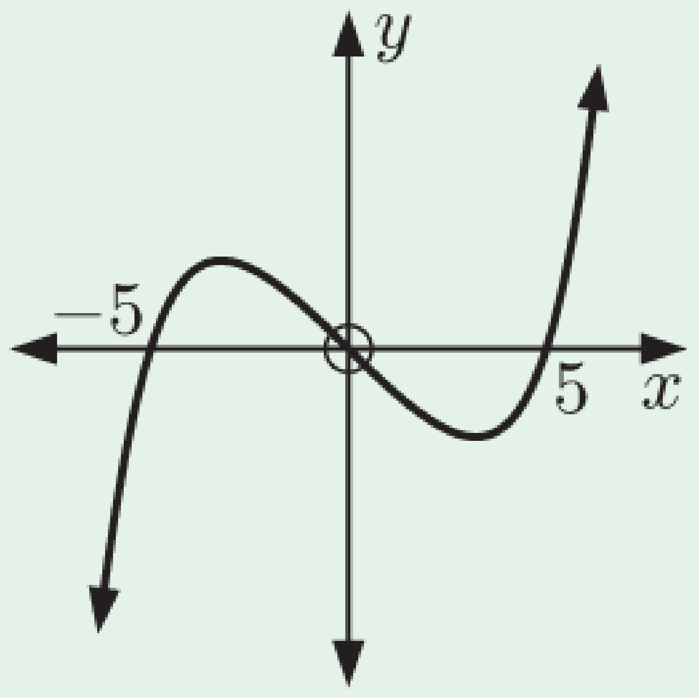
6 Given $f(x) = x^2 + 3$, find:

- **a** $f(-3)$
- **b** x such that $f(x) = 4$.

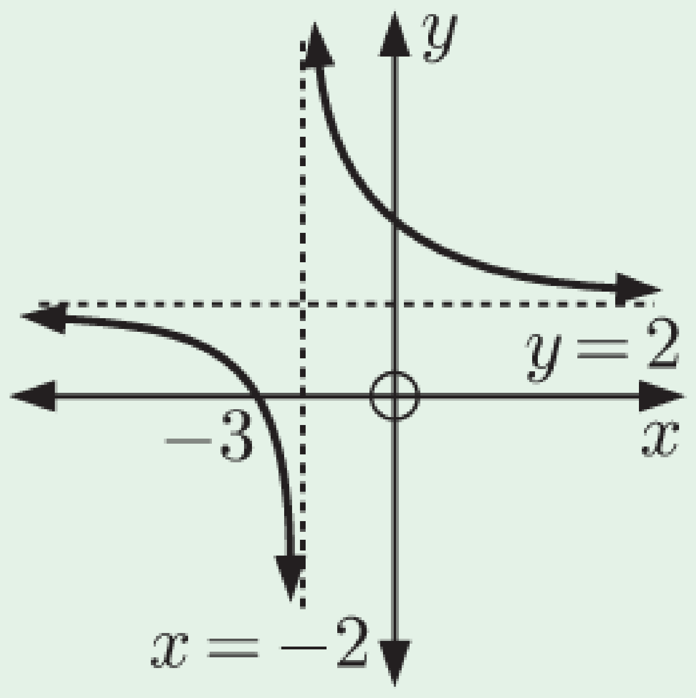
7
- **a** Use technology to help sketch the graph of the relation $y = \sqrt{9-x}$.
- **b** Determine whether the relation is a function.
- **c** Find the domain and range of the relation.

8 Draw a sign diagram for each graph:

a



b



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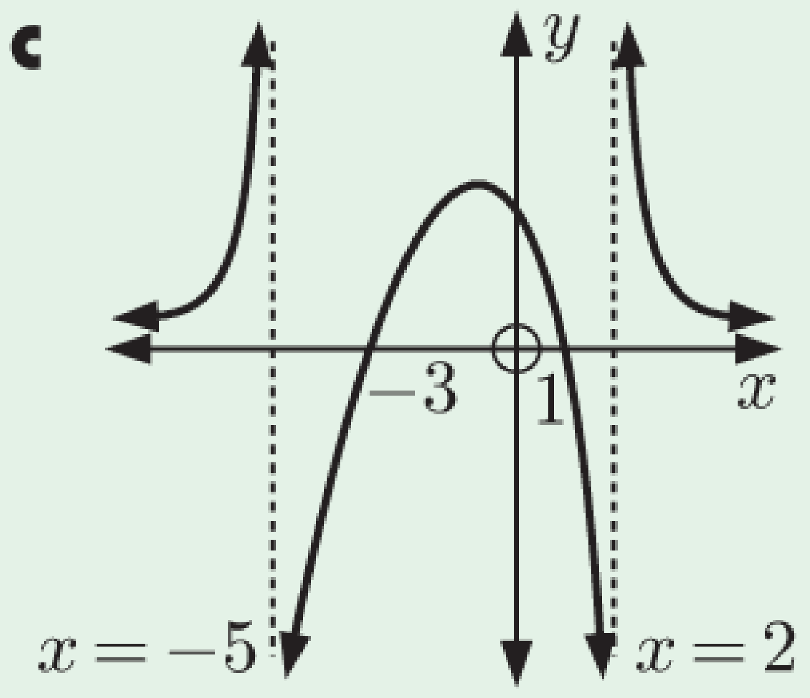


9 Draw a sign diagram for:

- **a** $(3x+2)(4-x)$
- **b** $\dfrac{x-3}{(x+2)^2}$

10 For each of the functions given, use technology to answer the following:

- **i** Find the axes intercepts.
- **ii** Find the coordinates and nature of any turning points of the function.
- **iii** State the domain and range of the function.
- **iv** Sketch the function, showing its key features.

- **a** $y = x^3 - 7x^2 + 4x + 12$
- **b** $y = \sqrt{x^2 - 4}$

11 The number of people infected with a virus t days after an outbreak is given by $N(t) = 100t \times (1.3)^{-t}$ for $t \geqslant 0$.

- **a** Sketch the graph of $N(t)$ for $0 \leqslant t \leqslant 20$.
- **b** How many people were infected after 1 day?
- **c** At what time was the outbreak greatest? How many people were infected at this time?
- **d** How long did it take before the outbreak was contained to 15 people?

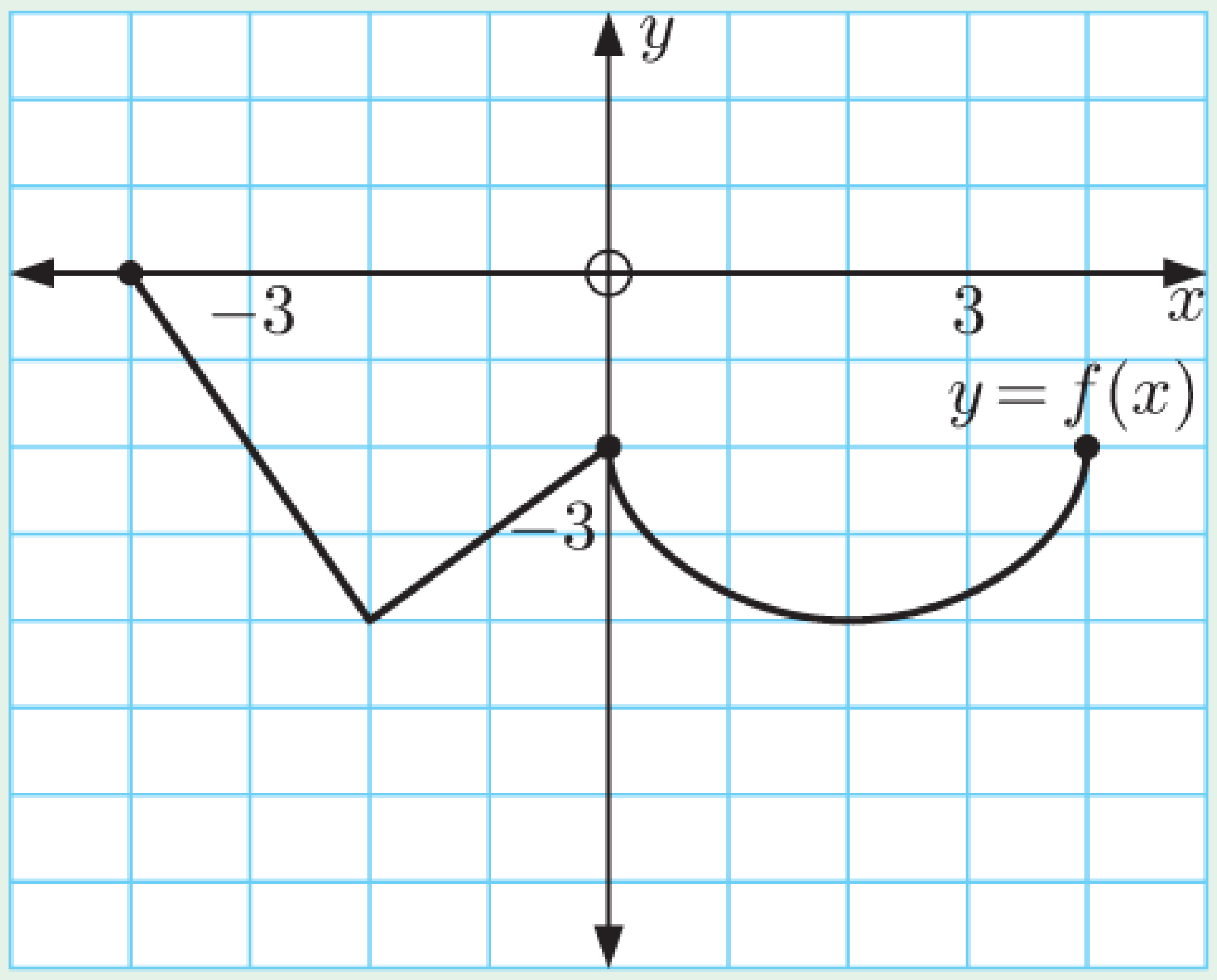
12 Find the equation of the resulting graph $g(x)$ when:

- **a** $f(x) = x^2 - 3x$ is reflected in the x-axis
- **b** $f(x) = 14 - x$ is translated 2 units upwards
- **c** $f(x) = \frac{1}{3}x + 2$ is horizontally dilated with scale factor 4.

13 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = f(2x)$ **b** $y = 2f(x)$ **c** $y = f(-x)$

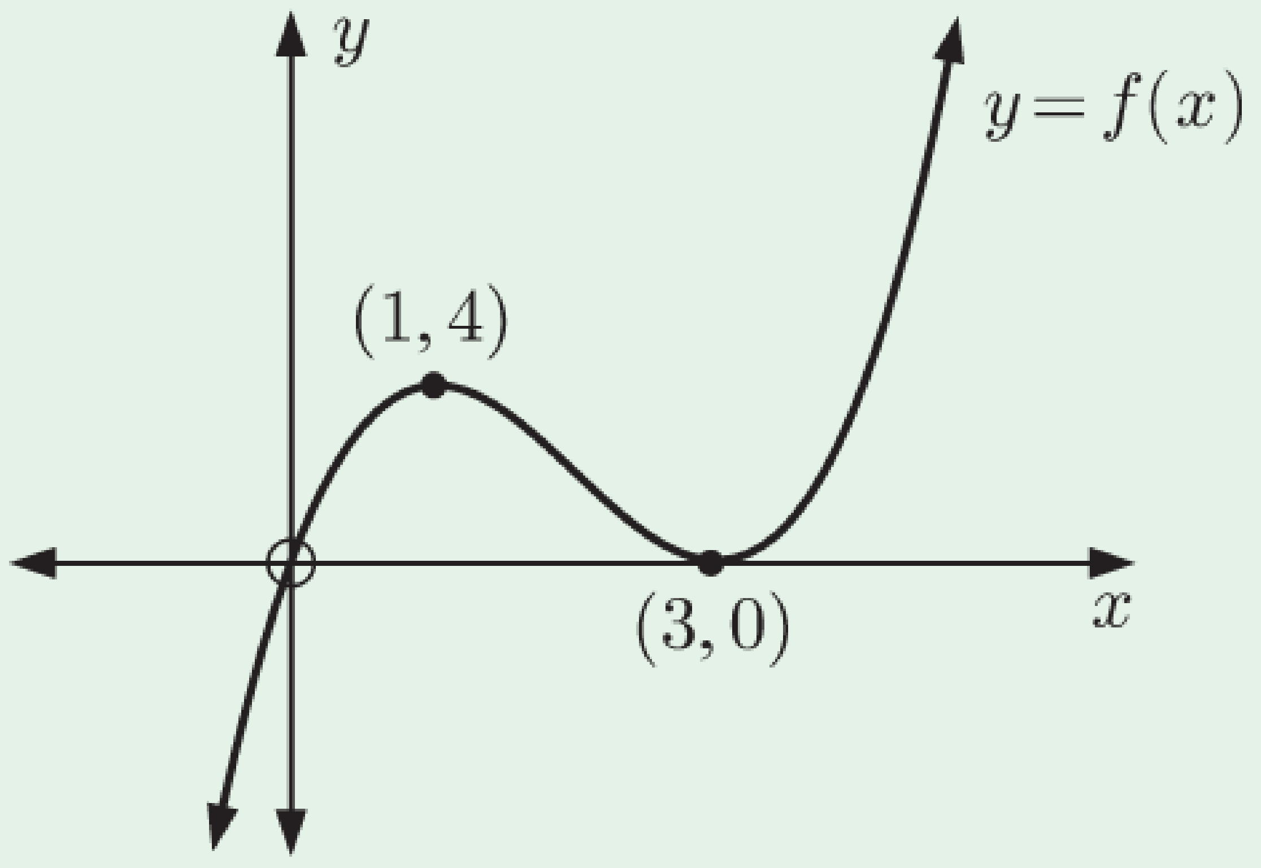
PRINTABLE GRIDS



14 The graph of a cubic function $y = f(x)$ is shown alongside.

a Sketch the graph of $g(x) = -f(x)$.

b State the coordinates of the turning points of $y = g(x)$.



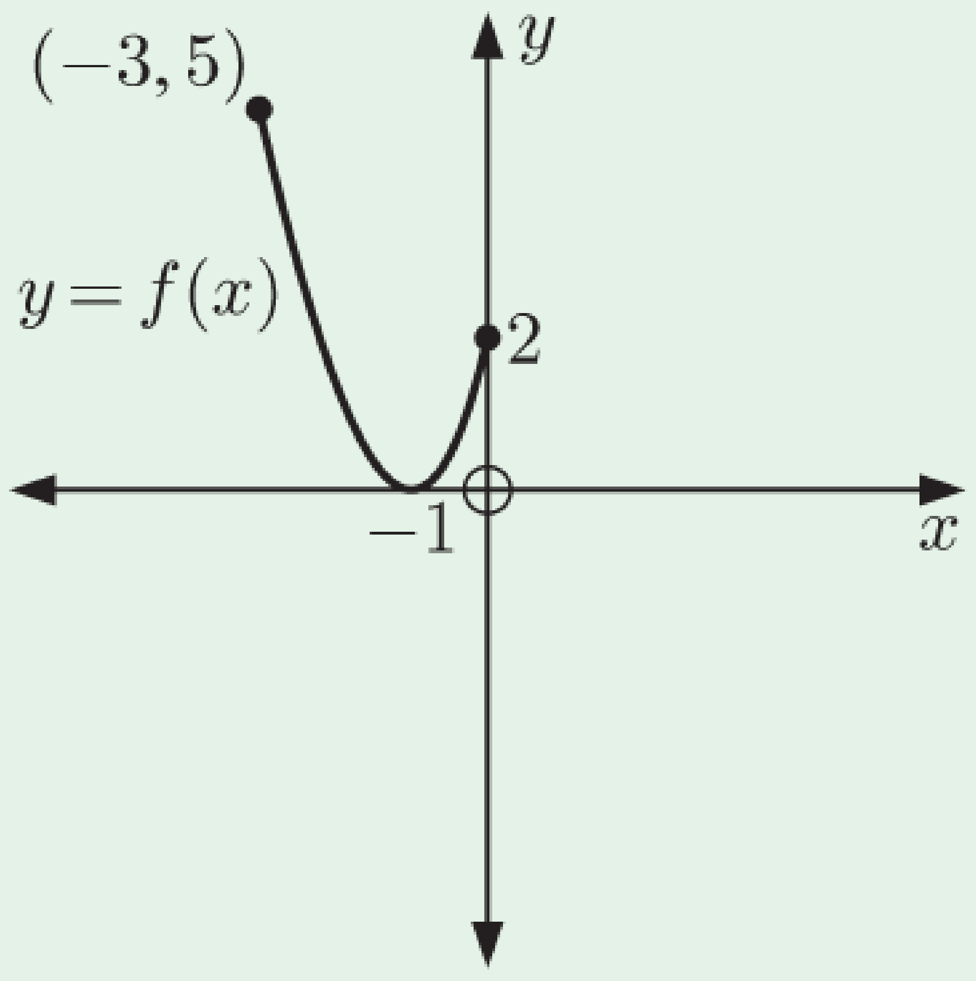
15 An invertible function $f(x)$ has x-intercept 4 and y-intercept -1. Find the axes intercepts of:

a $y = f(-x)$ **b** $y = f^{-1}(x)$

16 Consider the graph of $y = f(x)$ alongside.

a Copy the graph, and sketch its reflection in the line $y = x$.

b Is the graph you have sketched the inverse function of $y = f(x)$? Explain your answer.



17 Consider the function $f(x) = 8 - 2x$.

a Find the axes intercepts of the function and draw its graph.

b On the same set of axes, draw the inverse function $f^{-1}(x)$.

c Determine the equation of $f^{-1}(x)$.

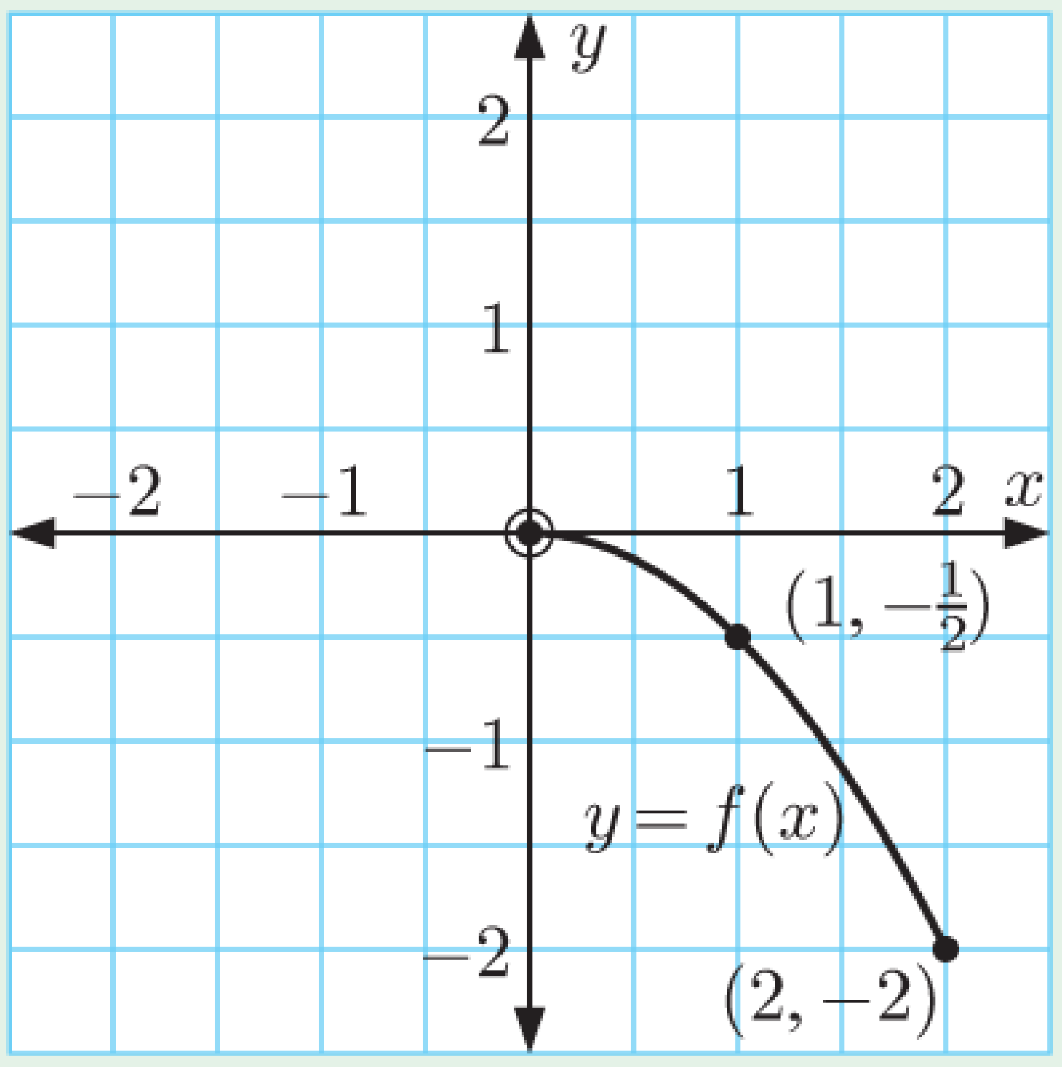
18 The graph of the function $f(x) = -\frac{1}{2}x^2$, $0 \leqslant x \leqslant 2$ is shown alongside.

a Sketch the graph of $y = f^{-1}(x)$.

b State the range of f^{-1}.

c Solve:

i $f(x) = -\frac{3}{2}$ **ii** $f^{-1}(x) = 1$



Chapter 4

Modelling

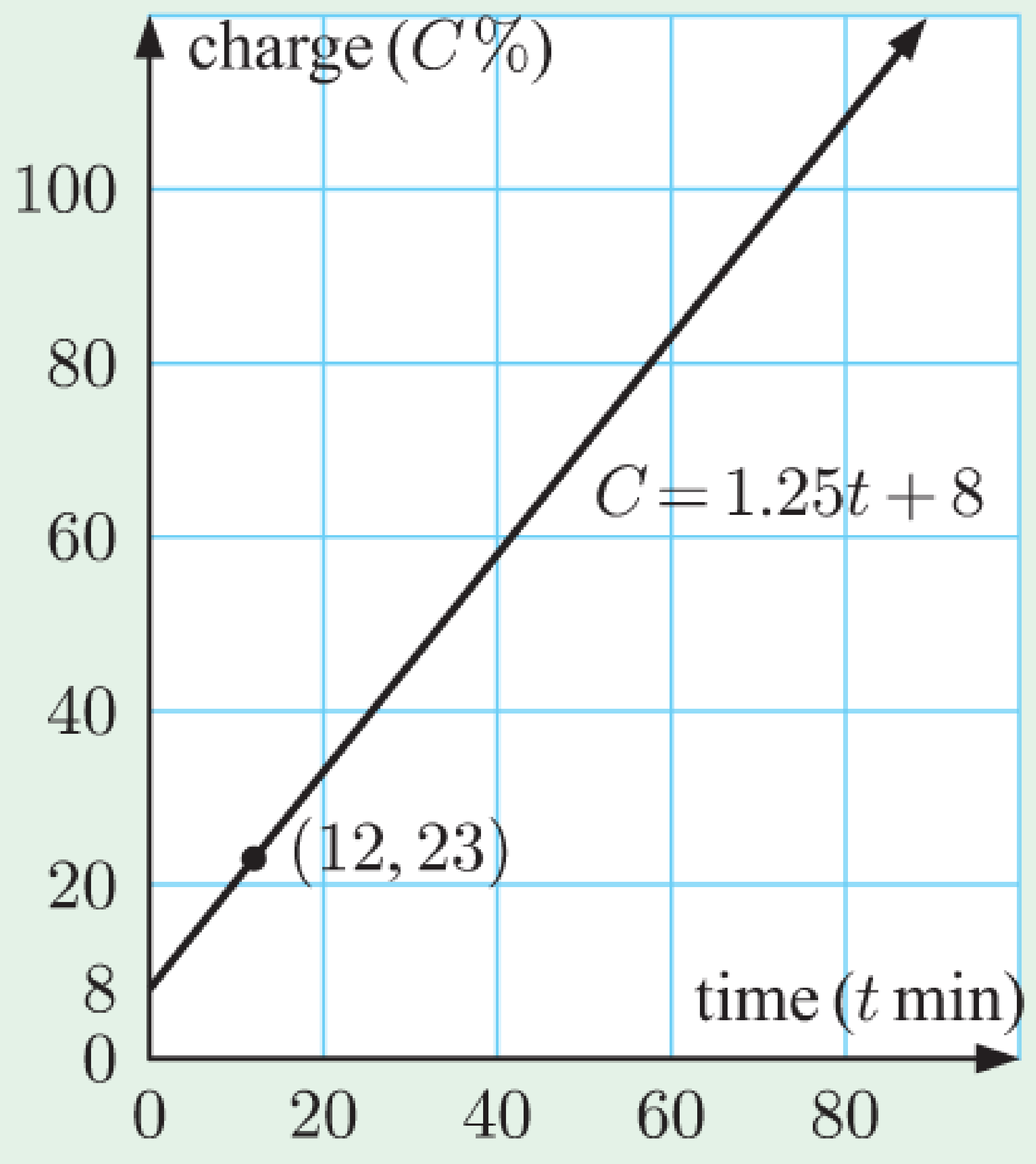
Contents:

A The modelling cycle
B Linear models
C Piecewise linear models
D Systems of equations

OPENING PROBLEM

Briony's laptop had 8% charge left when she started to recharge it. 12 minutes later, the laptop had 23% charge.

Briony would like to know when the laptop will be fully charged. She constructs the model alongside to describe the laptop's charge after t minutes.



Things to think about:

- **a** Can you explain how Briony constructed her model? What assumptions did Briony make? Do you think the assumptions are reasonable?
- **b** What range of values can each variable take? Has Briony accounted for this correctly?
- **c** According to Briony's model, how long will it take for the laptop to be fully charged?
- **d** The laptop actually took 79 minutes to be fully charged.
  - **i** Why do you think the actual time differed from the time predicted by Briony's model?
  - **ii** Do you think Briony's model was accurate enough to be useful?

We can apply mathematics to real-world problems to help understand the world around us.

A **mathematical model** is a simplified description of a real system using mathematical concepts and language. The process of developing this model is called **mathematical modelling**.

A THE MODELLING CYCLE

Beyond basic problems of counting, most real-world problems are complicated. We generally cannot find an exact, perfect solution.

In mathematical modelling, we simplify the problem by making assumptions and removing less important details. This allows us to construct a mathematical description of the problem which is simple enough for us to work with, and which we can solve to approximate the real-world situation.

Mathematical models are developed using a **modelling cycle**:

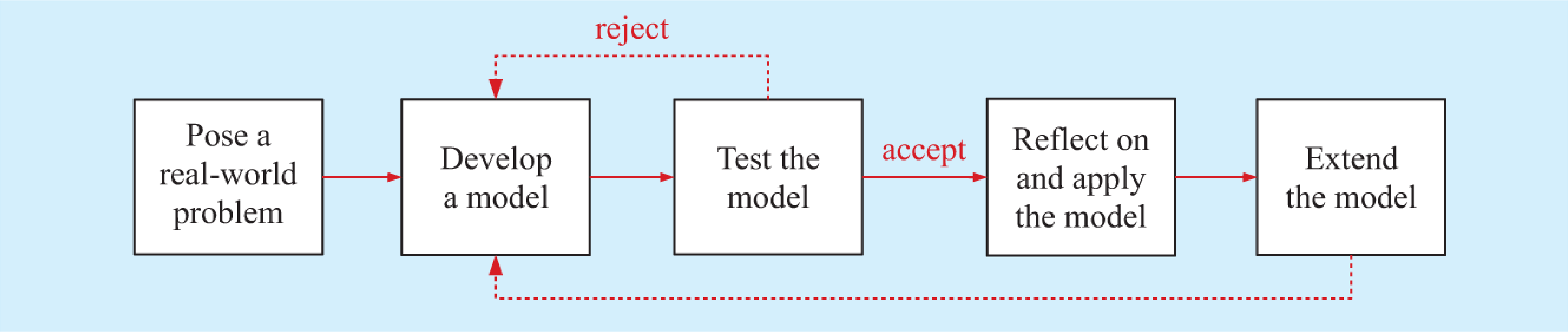
Step 1: **Pose** a real-world problem. Make **assumptions** which simplify the problem without missing key features.

Step 2: **Develop** a model which represents the problem with mathematics. This may involve a formula or an equation. The model should consider constraints such as the range of possible values each variable can take in the real world.

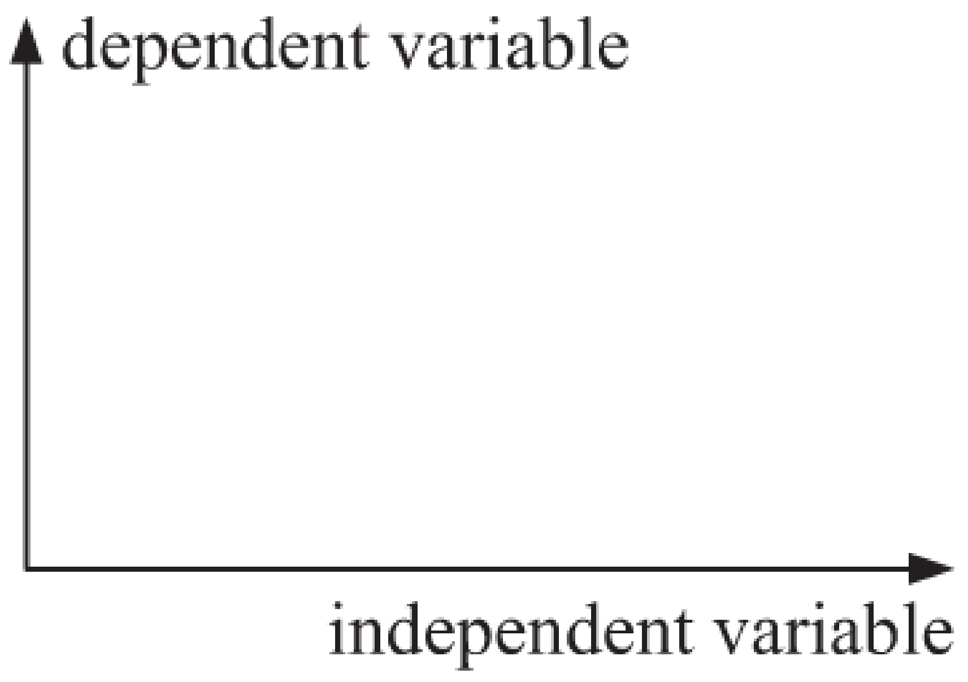
Step 3: **Test** the model by comparing its predictions with known data. If the model is unsatisfactory, return to *Step 2*.

Step 4: **Reflect** on your model and **apply** it to your original problem, interpreting the solution in its real-world context.

Step 5: If appropriate, **extend** your model to make it more general or accurate as needed.

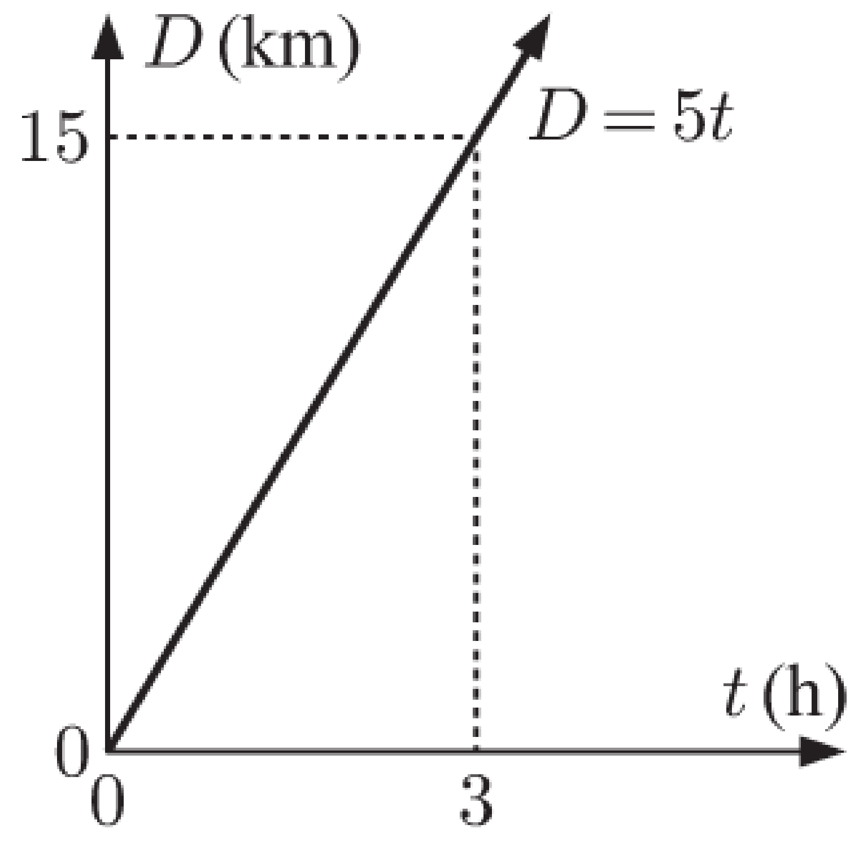


In general, when we consider the relationship between two variables, the value of one of the variables is *dependent* on the value of the other *independent* variable. We place the independent variable on the horizontal axis, and the dependent variable on the vertical axis.



For example, suppose we are interested in the distance a person could walk in a certain time. *Time* is the independent variable and *distance* is the dependent variable. In reality, the person will not walk at the same speed the whole time, and they will not walk at *exactly* 5 km h^{-1}. However, we might *assume* that the person walks at this typical speed. This assumption allows us to construct a simplified model for the distance (D km) travelled in t hours:

$$\begin{aligned}\text{speed} &= \frac{\text{distance}}{\text{time}}\\ \therefore\ \text{distance} &= \text{speed} \times \text{time}\\ &= 5\ \text{km h}^{-1} \times t\ \text{h}\\ \therefore\ D &= 5t\ \text{km}\end{aligned}$$



We can use the model to predict that, for example, in 3 hours the person will travel 15 km.

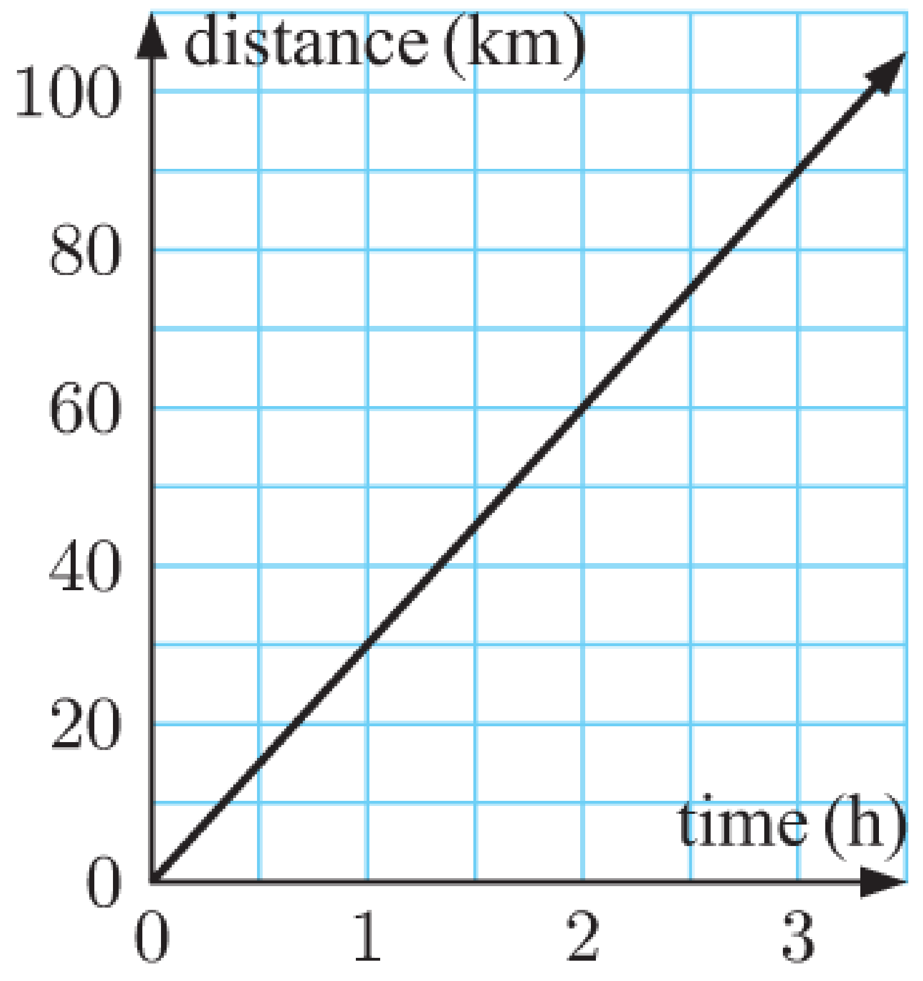
If this model is not sufficiently accurate, we might consider using different speeds for walking uphill, on flat ground, and downhill.

EXERCISE 4A

1 The graph alongside is a model for the distance travelled by a cyclist.

a What assumptions have been made in constructing the model? Do you think the assumptions are reasonable?

b Use the model to predict the distance travelled by the cyclist in 2 hours.



2 Answer the **Opening Problem** on page **80**.

3 Rick takes 15 seconds to run 100 metres.

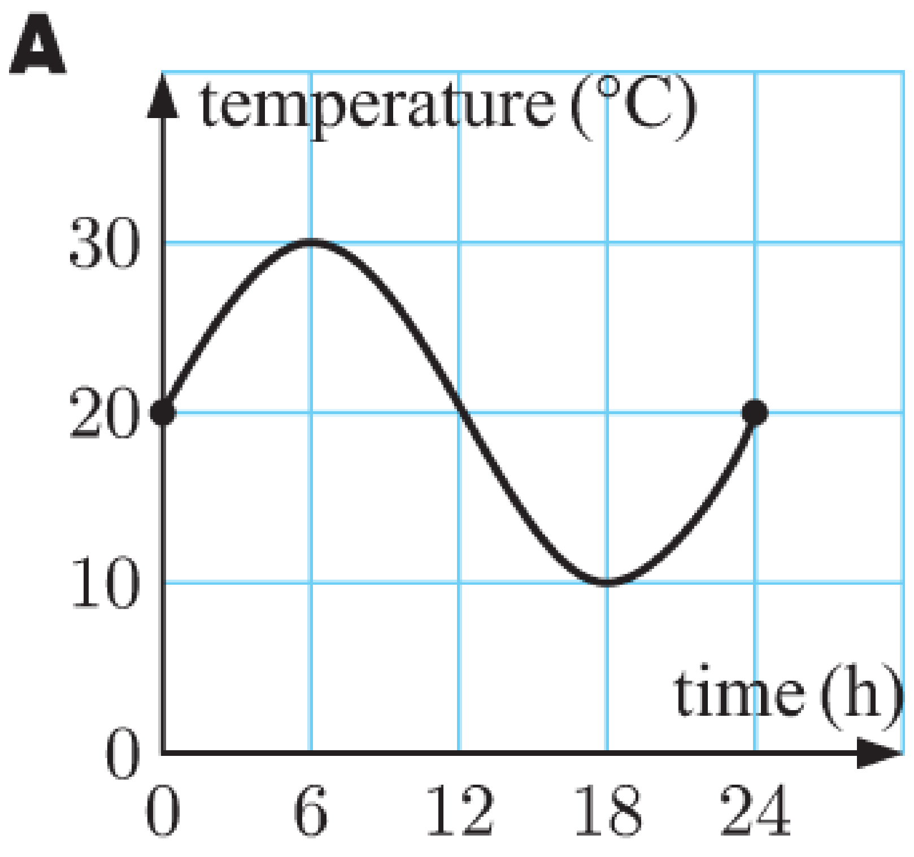
a Construct a model to describe how long Rick takes to run d metres.

b Hence predict how long Rick takes to run 500 metres.

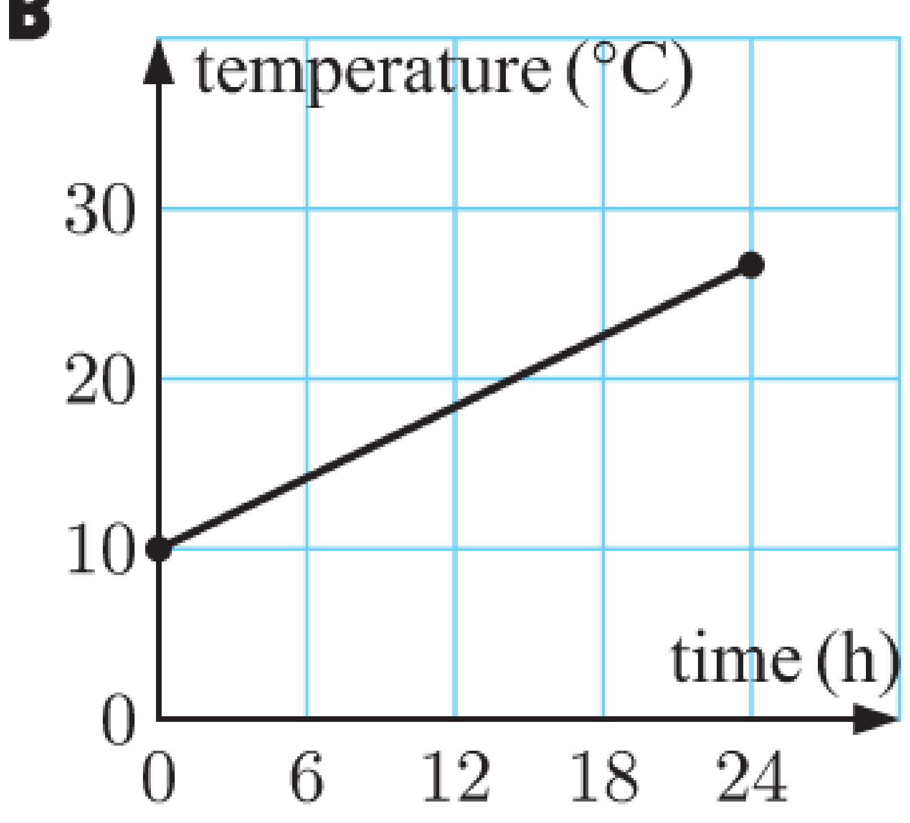
c Do you think the actual time Rick takes to run 500 metres will be longer or shorter than your prediction? Explain your answer.

4 **a** Which of these graphs is the most appropriate model for the temperature of a city on a particular day? Explain your reasoning. Time is measured in hours after midnight.

A



B



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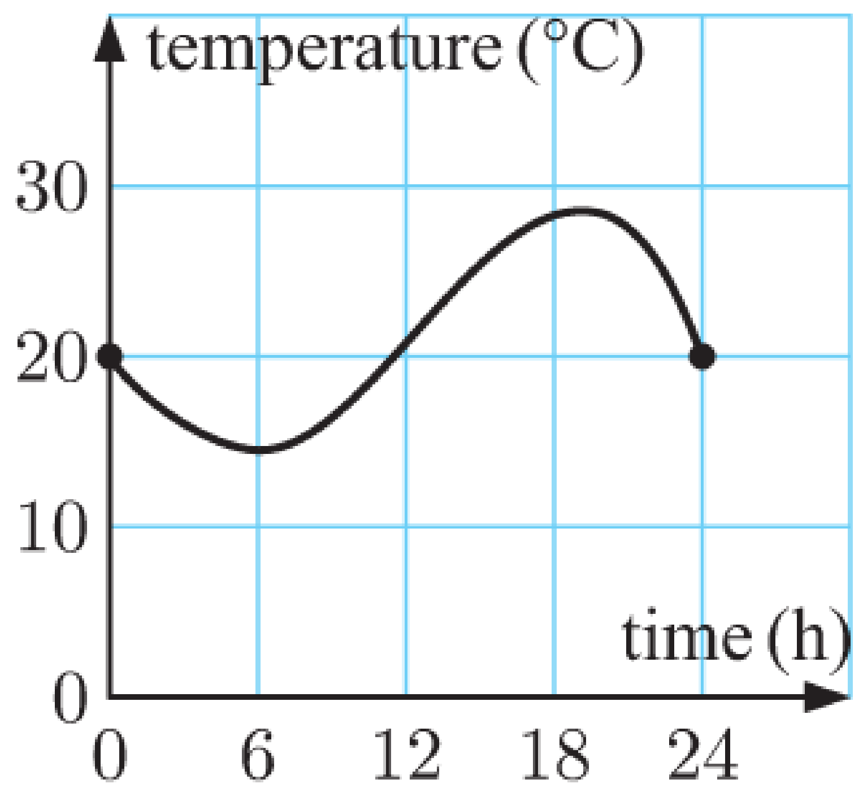
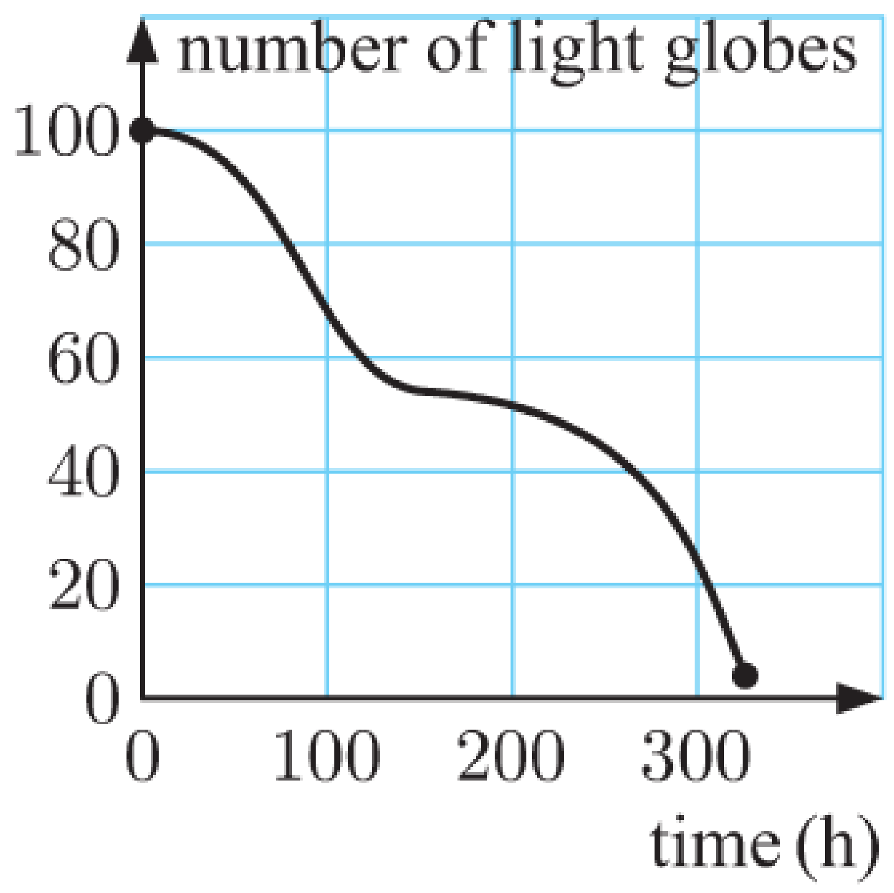


b Use the model you selected in **a** to predict the temperature of the city at 8 pm.

5 A particular type of light globe is claimed to last for 200 hours. The manufacturer simultaneously tests 100 of these globes.

a Which of these graphs do you think is the most appropriate model for the number of light globes still working at any given time? Explain your reasoning.

A



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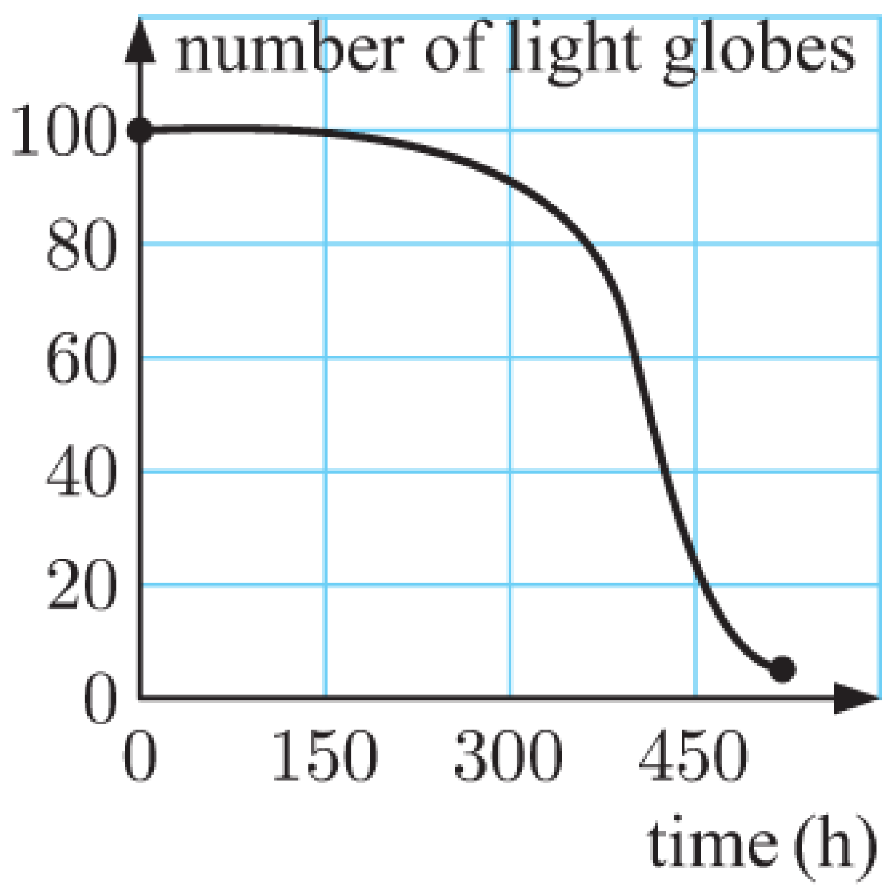

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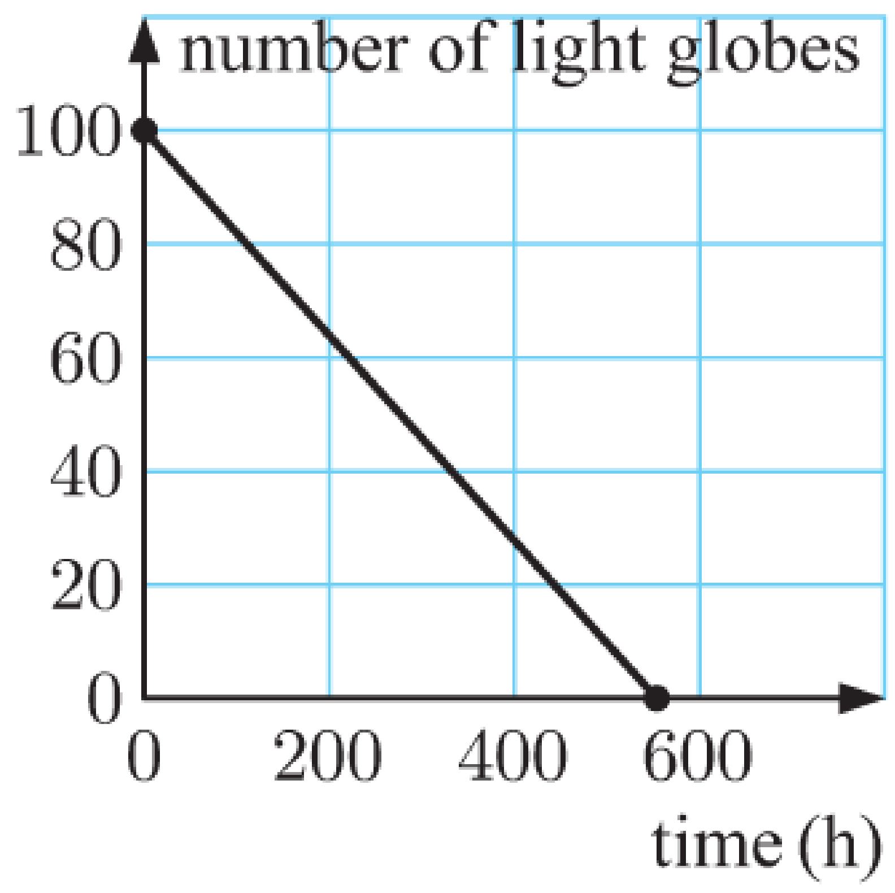
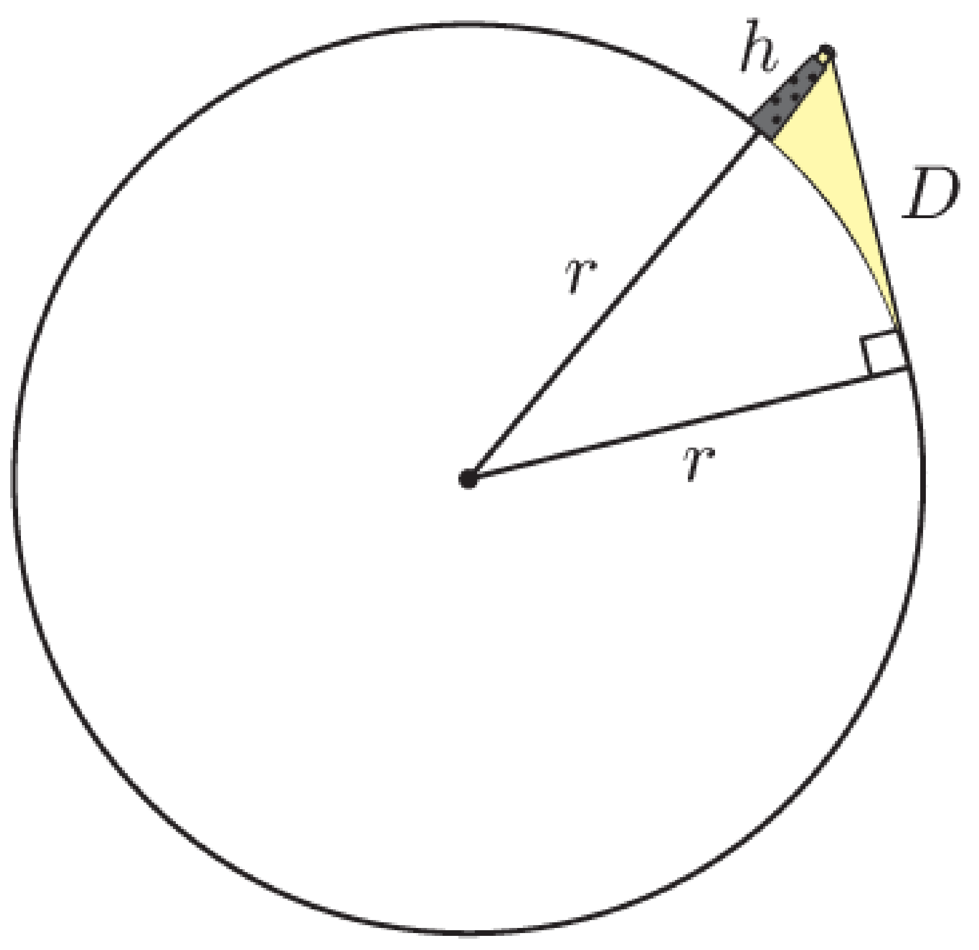


b Use the model you selected in **a** to predict the number of light globes still working after 250 hours.

6



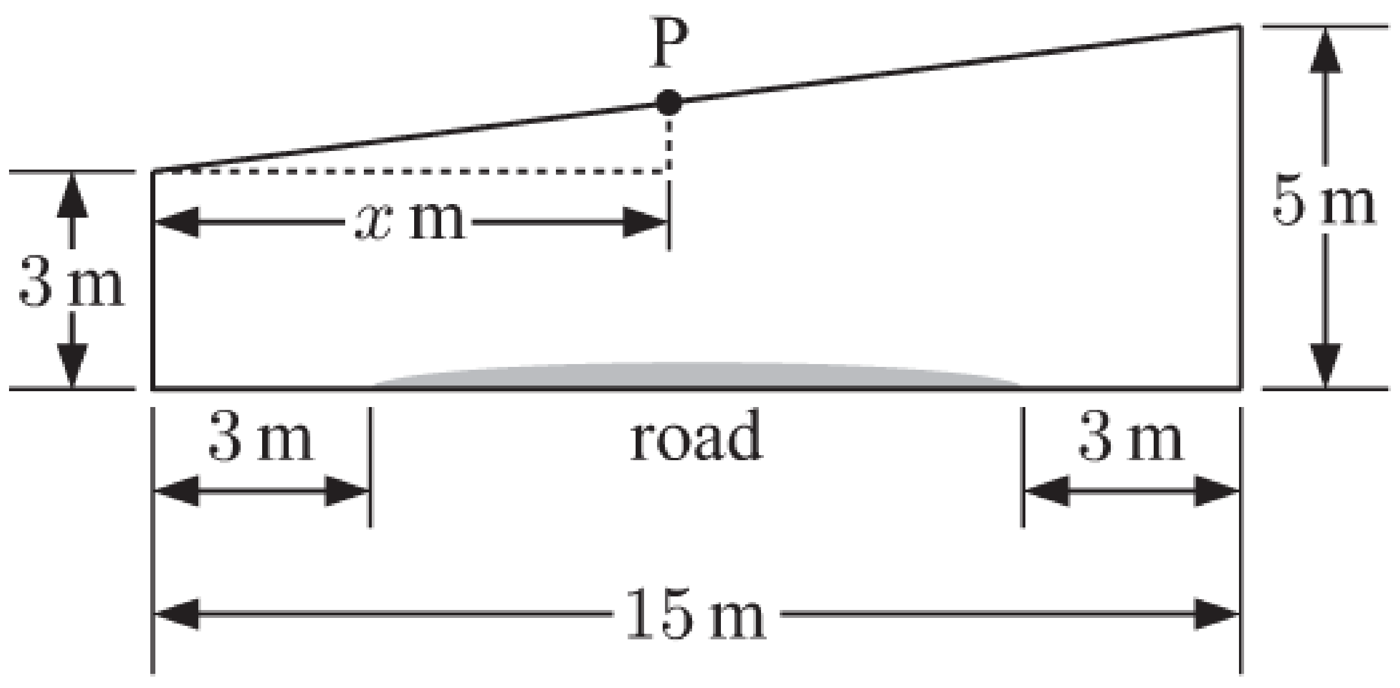
A lighthouse is 40 m tall. Darren wants to know how far the light at the top can be seen from.

He draws the diagram alongside to describe the situation, where r is the Earth's radius, h is the height of the lighthouse, and D is the required distance.

a What assumptions has Darren made in constructing this model? Do you think they are reasonable?

b Given that the Earth's radius is approximately 6370 km, use the model to find D.

7 A power line is connected between two poles as shown. Hashni needs to know the height of the power line above the road, as he needs to work out whether a truck can fit underneath.



a Use similar triangles to construct a model for the height of the power line above ground level at point P. Describe any assumptions you make when constructing your model.

b For what values of x is it reasonable to apply the model?

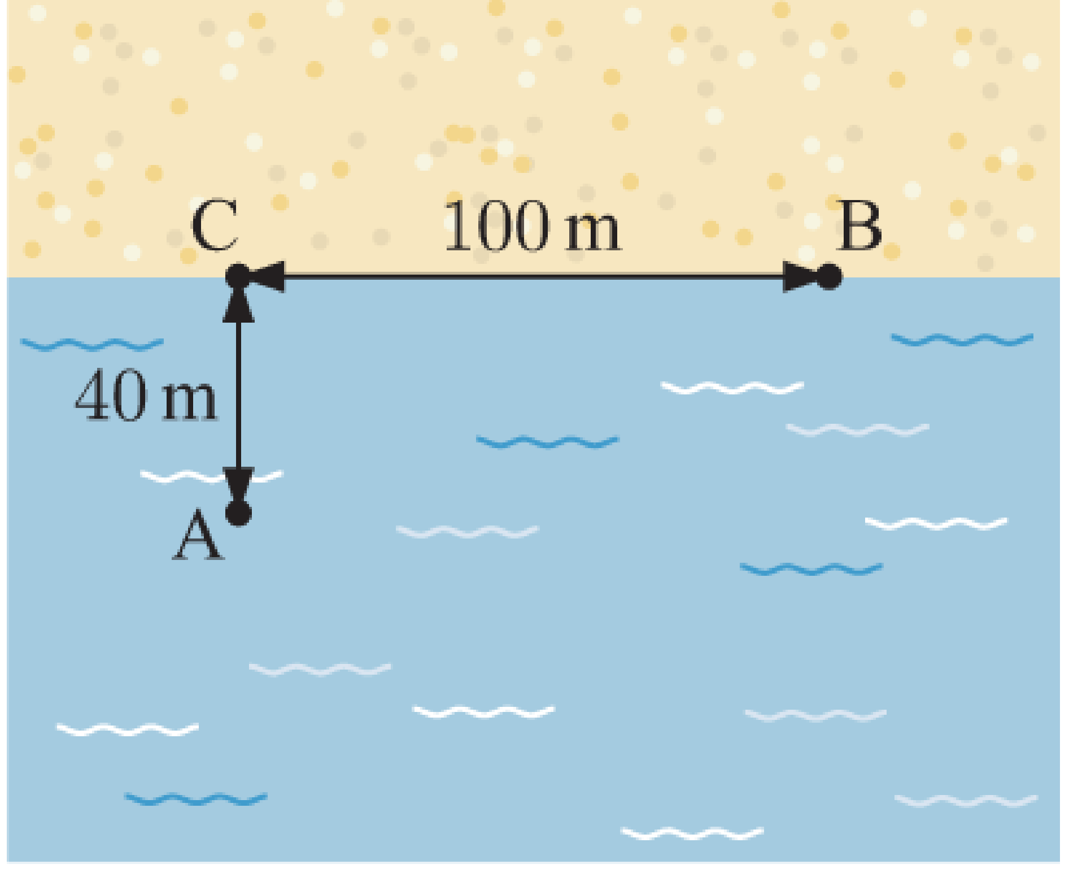
c Use your model to predict the height of the power line above ground level as it passes over the middle of the road.

d Does your answer to **c** seem reasonable? Explain your answer.

e Do you think your model would be useful for Hashni's purposes? Explain your answer.

8 Antonio is swimming at A, and his belongings are at B. Feeling cold, he wonders how he can return to his belongings quickest.

Antonio can swim at 1.5 m s^{-1} and can jog at 4 m s^{-1}.



a Find the time Antonio would take to return to his belongings by:

- **i** swimming directly to B
- **ii** swimming to C, then jogging to B.

State any additional assumptions you have made in your calculations.

b Which option appears to be quicker?

c Can you find another option that is even quicker?

Example 1 Self Tutor

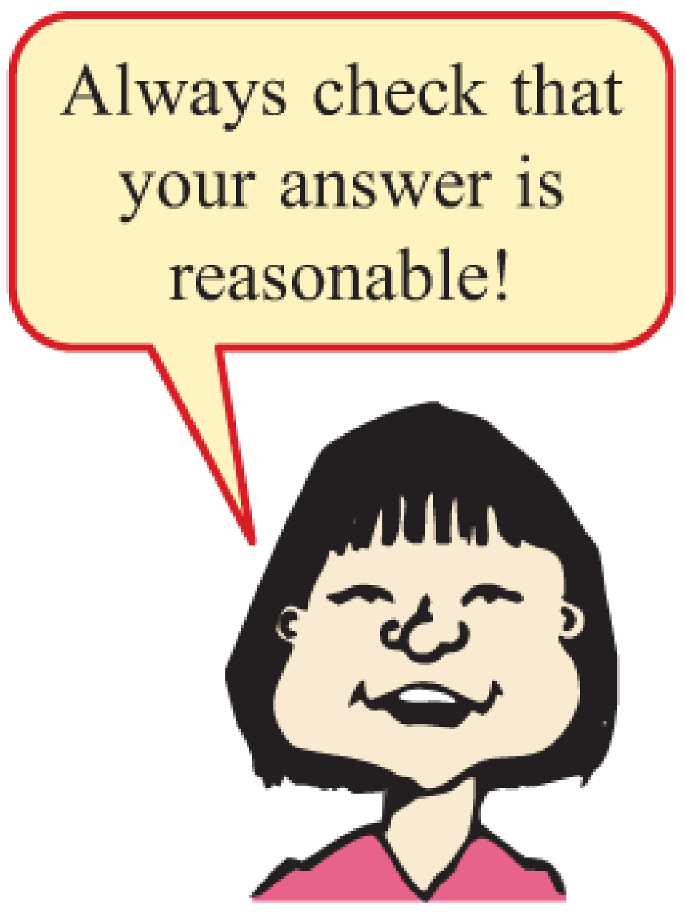
Henri takes 6 hours to clean the house. Marcia takes 5 hours to clean the house. How long would they take to clean the house working together?

Henri can clean $\frac{1}{6}$ of the house each hour, and Marcia can clean $\frac{1}{5}$ of the house each hour.

We assume Henri and Marcia can work without getting in each other's way when they work together. So, working together they will clean $\frac{1}{6}+\frac{1}{5}=\frac{11}{30}$ of the house each hour.

$\therefore$ it would take them $\frac{30}{11}\approx 2.73$ hours $\approx$ 2 hours 44 minutes to clean the house together.

This answer seems reasonable. With two people working we expect them to take about half the time, and 2 hours 44 minutes is between half of Henri's time and half of Marcia's time.


Always check that your answer is reasonable!

9 Kate takes 7 hours to dig a hole. Lenny takes 12 hours to dig a hole. How long would they take to dig a hole working together?

10 A tank holds 1200 litres of water. Pulling the plug would drain the tank in 15 minutes. A hole develops which would drain the tank in 25 minutes. How long would it take to drain the tank through the hole and the plug together?

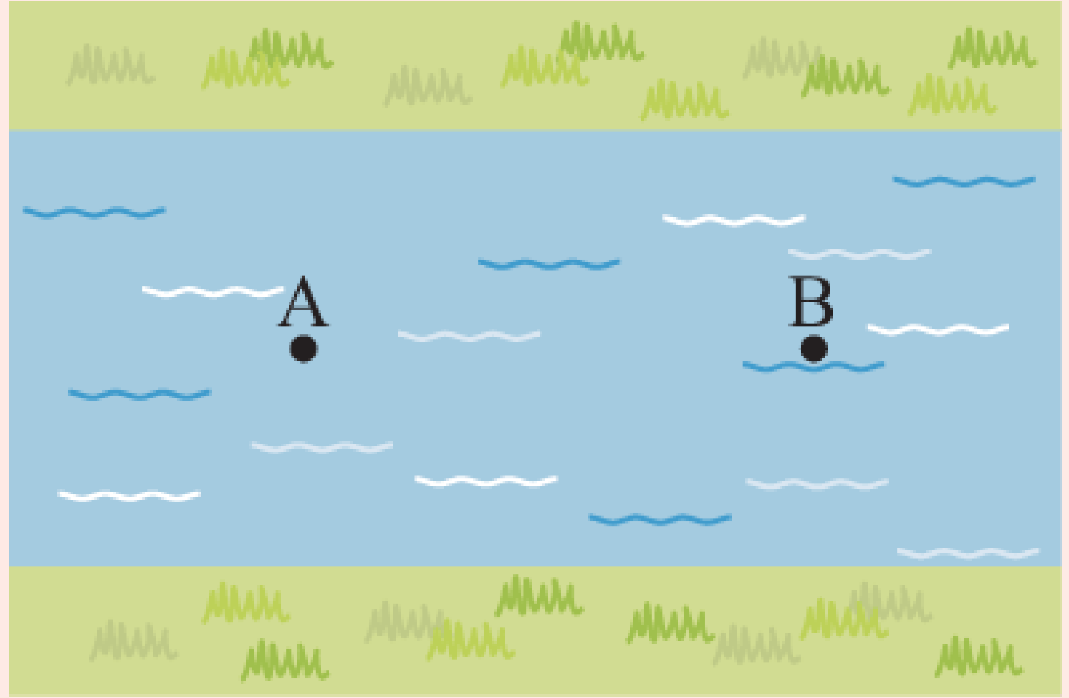
11 Aaron can make 4 widgets per hour, Bonnie can make 3 widgets per hour, and Calum can make 2 widgets per hour. How long would they take to make 135 widgets working together?

12 Angus can paint a room in 3 hours. Angus and Beitidh would take 2 hours to paint the same room together. How long would Beitidh take to paint the room on her own?

ACTIVITY 1 MATHEMATICAL MODELLING

What to do:

Working in small groups, try to solve the problems presented below. You will need to make some assumptions in order to simplify each problem to one that can be solved mathematically. Record your assumptions as well as your mathematical solution.



- How long would it take to count to 1 million out loud?
- Do people who take showers or baths use more water?
- A swimmer must swim up a river from A to B, and then back to A.
  Would a current in the river make the task take more time or less time?
- One tonne of sand is dumped into a pile on the ground. How high is the pile?

Compare your answers with those of other groups. If your answers differ significantly, compare the assumptions you have made.

INVESTIGATION CALENDARS

A **solar day** is the time it takes Earth to rotate about its axis. A **tropical year** is the time it takes the Earth to orbit around the Sun.

There are approximately 365.242 199 solar days in a tropical year. However, it is much more convenient to have a calendar with a whole number of days in the year. In this Investigation we will explore some models for achieving this.

What to do:

1 If we fixed a year as 365 days then quite quickly we would find the months drifting apart from the seasons. July would eventually move to winter in the Northern hemisphere!

  a Find the difference between 365 days and a tropical year.

  b How long would it take for the difference between 365 days and a tropical year to accumulate to one whole day?

  c How many years would it take for the difference to accumulate to six months?

2 To fix this problem, we have leap years in which an extra day is added to the end of February. The **Julian calendar**, devised by **Julius Caesar** in 46 BC, adds an extra day to every fourth year, giving the average length of a year as 365.25 days.

  a In days per year, how inaccurate is the Julian calendar?

  b How long would it take for this inaccuracy to accumulate to:

    i one whole day **ii** six months?

3 In the mid 1200s, **Roger Bacon** (1219 - 1292) noticed that the Julian calendar had already drifted by about nine days. Religious festivals such as Easter were being celebrated at the wrong time! However, it was only in 1582 that **Pope Gregory XIII** introduced the **Gregorian calendar** used in most of the world today.

In the Gregorian calendar, every 4th year is a leap year unless it is divisible by 100 and *not* by 400.

For example:

- 1800 was divisible by 100 but not divisible by 400, so it was *not* a leap year.
- 2000 was divisible by 100 and by 400, so 2000 *was* a leap year.

When the Gregorian calendar was first introduced, people who went to bed on October 4th 1582 woke up the next day on October 15th. People did not like having so many days "stolen" from them! Few European countries used the new system immediately. For example, the Kingdom of Great Britain waited until 1752, Russia until 1917, and China until 1949. The Eastern Orthodox Church has still not adopted the Gregorian calendar.

a In each period of 400 years, how many leap years does the Gregorian calendar add?

b Find the length of an average year in the Gregorian calendar.

c How long will it take for the inaccuracy in the Gregorian calendar to accumulate to one whole day?

THEORY OF KNOWLEDGE MOVEMENT OF THE PLANETS

People have always been fascinated by the Moon, the planets, and the stars. Many attempts have been made to accurately model their movement, given their importance in navigation and for calendars.

Early teaching in Europe put the Earth at the centre of the universe. However, **Nicolaus Copernicus** (1473 - 1543) created a new model of the solar system which placed the Sun at the centre. In this model, the orbits of the planets were circles centred at the Sun. The orbit of the Moon was also a circle, centred at the Earth. Putting the Sun at the centre made it much easier to describe the positions of the planets.

Nicolaus Copernicus

By carefully analysing astronomical observations, **Johannes Kepler** (1571 - 1630) realised that the positions of the planets did not exactly fit the theory of circular orbits. He proposed his *three laws of planetary motion*, the first two of which appeared in his book *Astronomia nova* in 1609:

- The planets move in *elliptical* orbits, with the Sun at one focus.
- The speed of the planet is proportional to the area swept out by the orbit.

In the Renaissance, clockwork mechanisms were developed to display the solar system. A mechanism like this is called an **orrery**.

In the late 17th century, **Isaac Newton** developed his law of gravitation and his laws of motion in which he suggested that forces result in *acceleration* and not velocity. His new theories now *predicted* the elliptical orbits suggested by Kepler.

However, Newton's elliptical orbits did not account for the gravitational pull of the planets on each other. A revised model was needed to more accurately account for the observed movement of the planets. Indeed, the existence of the planet Neptune was predicted by the mathematical model before it was directly observed. More recent mathematics has improved models of how bodies move under gravity, and the "three body problem" is the subject of ongoing mathematical research.

1. How did the different models for the solar system build on previous models?
2. Identify assumptions made in each model.
3. Can a mathematical model ever be "complete"?
4. The British statistician **George Box** stated that "All models are wrong, but some are useful".
   Discuss what he meant by this.

B LINEAR MODELS

Two variables are **linearly related** if the graph connecting them is a straight line.

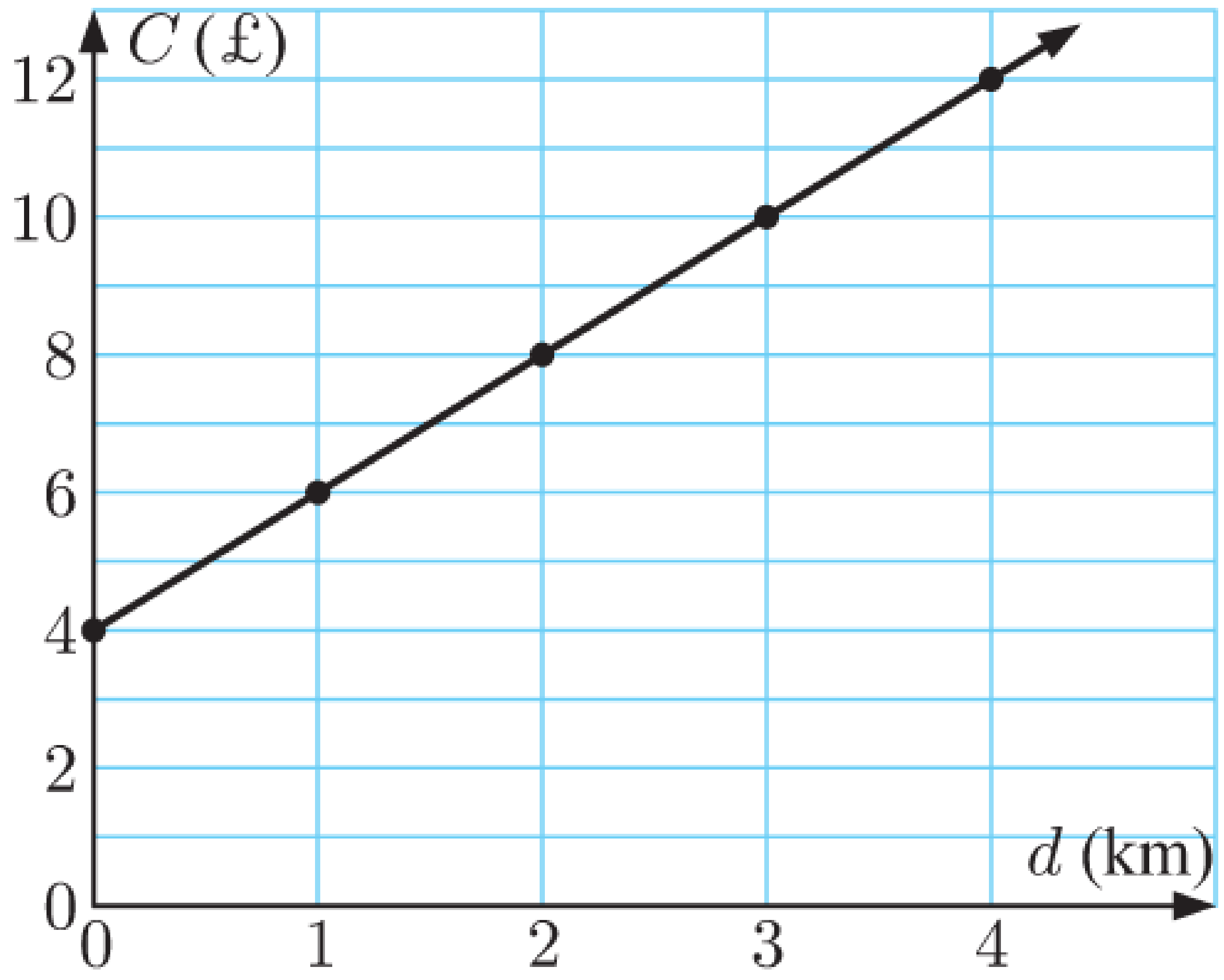
Ted the taxi driver charges passengers an initial fee of £4, and then £2 for each kilometre travelled.

To study the relationship between the *distance travelled* (d km) and the *cost* (C pounds), we construct a table of values and draw a graph. The *cost* is dependent on the *distance travelled*, so we place d on the horizontal axis, and C on the vertical axis.

Distance travelled (d km)	0	1	2	3	4
Cost (£C)	4	6	8	10	12

+2 +2 +2 +2

The graph of C against d is a straight line, so C and d are **linearly related**.



Notice in the graph that:

- The C-intercept of the graph is 4. This is the initial fee, in pounds, of the taxi ride.
- The gradient of the graph is 2. This is the cost, in pounds, of each additional kilometre travelled.
- The variables are related by the **linear model** $C = 2d + 4$.

The model can be used to predict the value of one variable given the value of the other variable. For example, for a 10 km journey, $d = 10$, and $C = 2(10) + 4 = 24$. The cost of the 10 km journey is £24.

EXACT AND APPROXIMATE MODELS

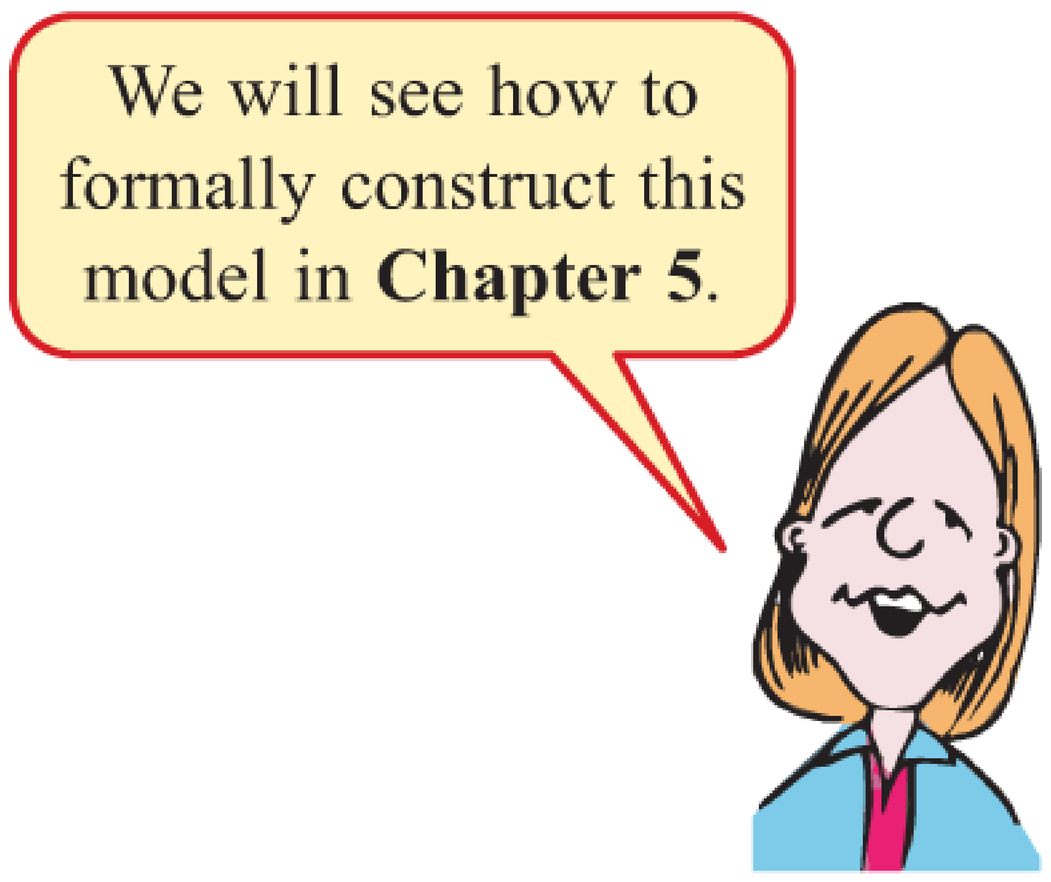
In the taxi situation above, the cost follows a specific rule, and the relationship between C and d is *exact*. We can use our linear model to find the exact cost C for any positive distance d.

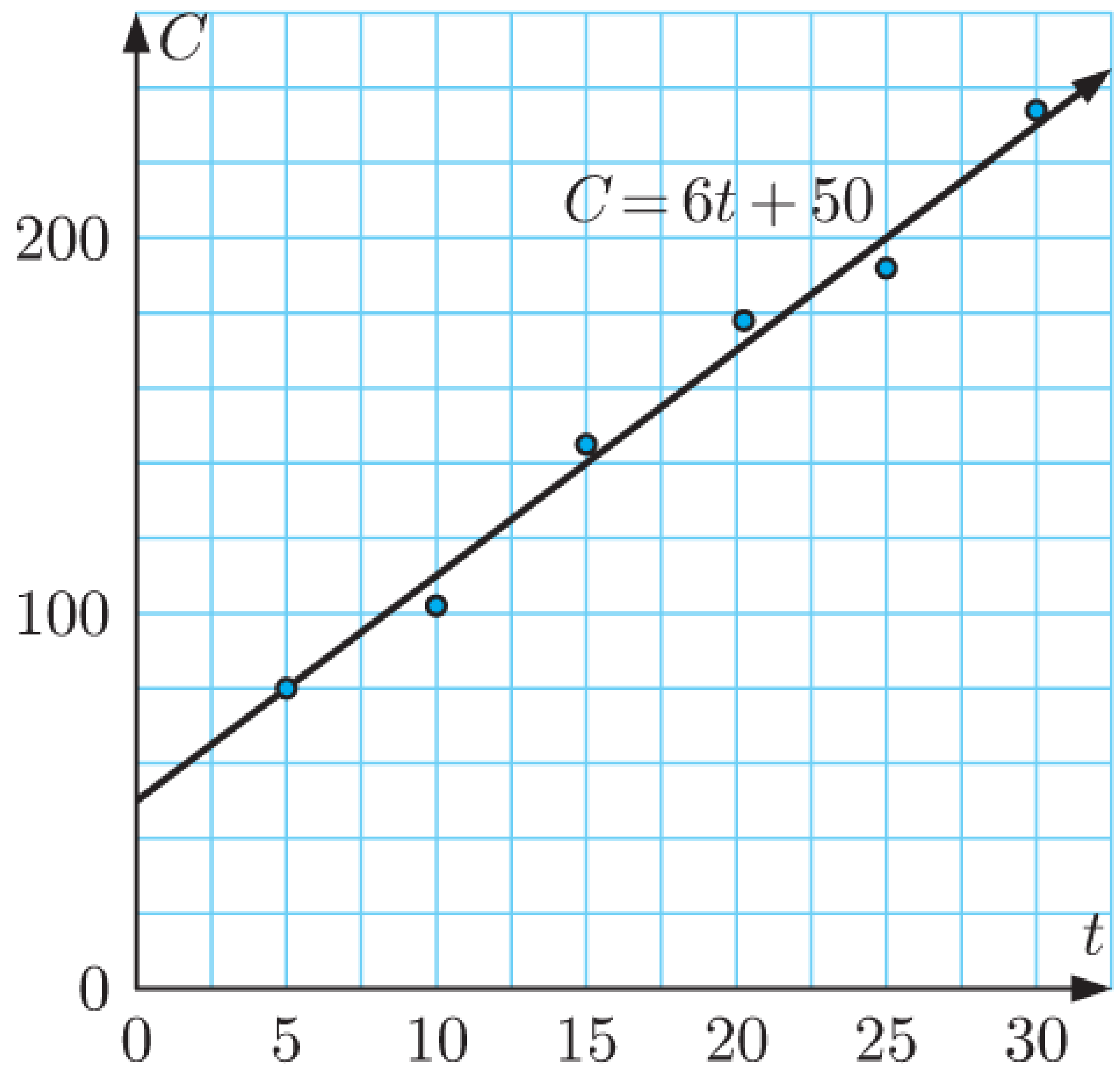
In many situations, however, the connection between the variables is not exact. When we plot a set of data points, we may observe a *linear trend*, which suggests a linear model will be a good *approximation* of the situation.

For example, the table below shows the number of customers at a restaurant every 5th day after it opens.

Time (t days)	5	10	15	20	25	30
Number of customers (C)	80	102	145	178	192	234

If we plot these points on a graph, we can see the variables are approximately linearly related. The line $C = 6t + 50$ is an *approximate* model for the data.


We will see how to formally construct this model in **Chapter 5**.



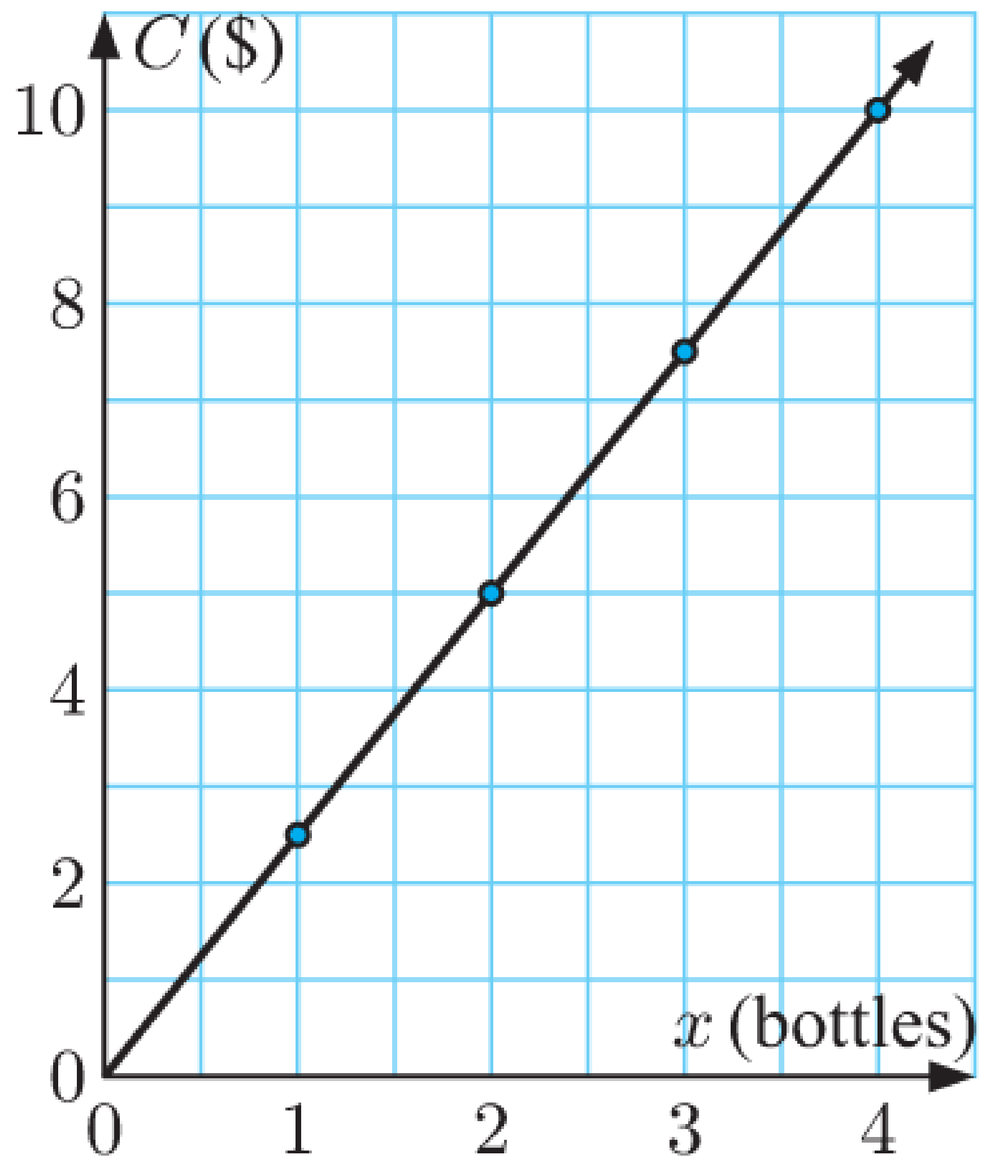
Care must be taken when using an approximate model to predict values beyond the range of the given data points. This process is called **extrapolation**.

For example, we could use the model to predict that, on the 100th day, the restaurant will receive $6(100) + 50 = 650$ customers. However, this assumes that the linear trend will continue far beyond the given data. This is unrealistic given the restaurant will have a maximum capacity.

EXERCISE 4B

1 This graph shows the cost $\$C$ of buying x bottles of juice.

- **a** Find the model connecting C and x.
- **b** Is the model exact or approximate? Explain your answer.
- **c** Can the model be used to find the exact cost of 12 bottles of juice? If so, find the cost.



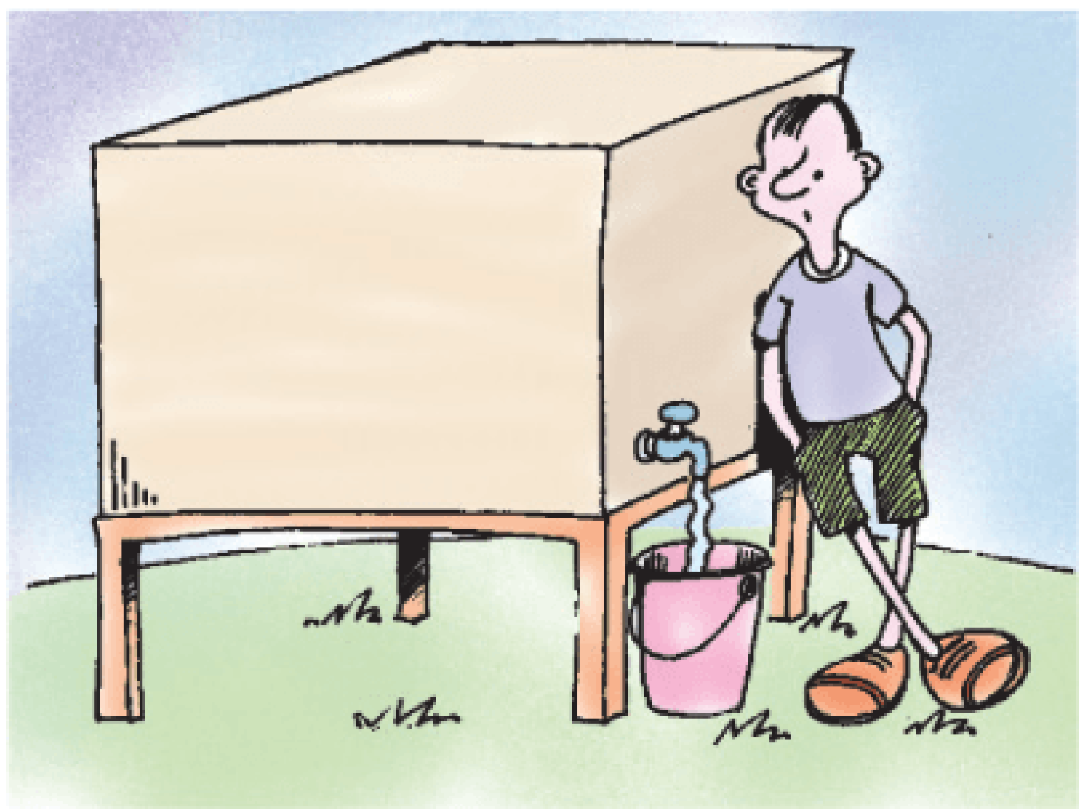
2 An electrician charges $60 callout plus $45 per hour he spends working on the job.

- **a** Copy and complete this table:

Time (t hours)	0	1	2	3	4	5
Cost ($\$C$)						

- **b** Draw the graph of C against t.
- **c** Find the linear model connecting C and t.
- **d** Use your model to determine the electrician's total cost for a job lasting $6\frac{1}{2}$ hours. Use your graph to check your answer.

3 A tank contains 265 litres of water. The tap is left on and 11 litres escape per minute.



a Construct a table of values for the volume V litres left in the tank after t minutes, for $t = 0$, 1, 2, 3, 4, and 5.

b Draw the graph of V against t.

c Find the linear model connecting V and t.

d Determine:

i the amount of water left in the tank after 15 minutes

ii the time taken for the tank to empty.

4 Xuanyu planted a 30 cm high bamboo plant in her garden bed. She found that with consistent weather it grew 10 cm each day.

a Copy and complete this table of values which gives the height H of the bamboo after t days.

t (days)	0	1	2	3	4	5	6
H (cm)							

b Draw the graph of H against t.

c Discuss whether it is reasonable to continue the line for $t < 0$ and for $t > 6$ days. Hence state the domain of the function $H(t)$.

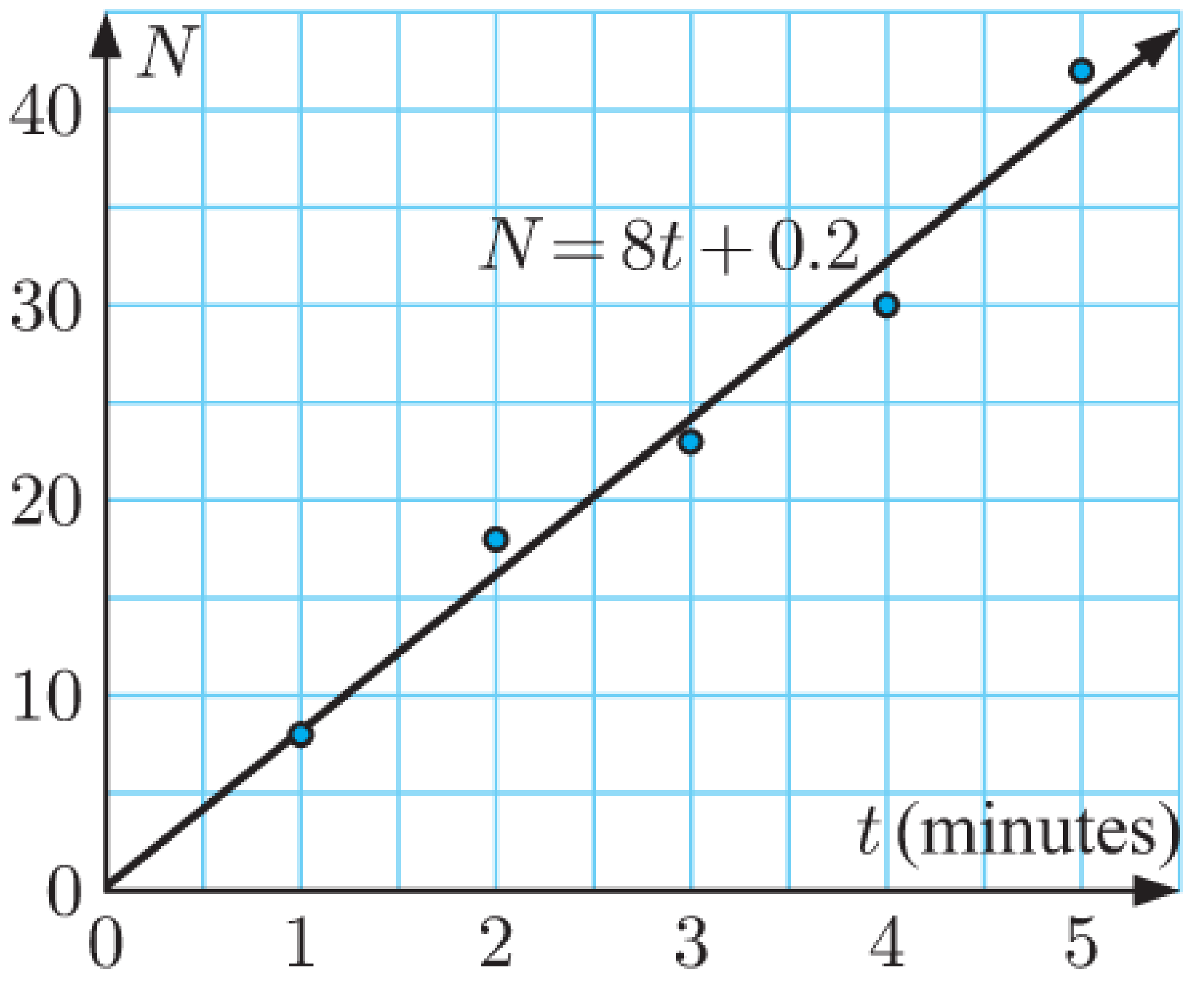
d Find the model connecting H and t.

e How long will it take for the bamboo to be 1 m high?

5 As punishment for misbehaving, Jack must pick up litter at lunchtime. This table shows how many pieces of litter he has picked up after t minutes.

Time (t minutes)	1	2	3	4	5
Number of pieces of litter (N)	8	18	23	30	42

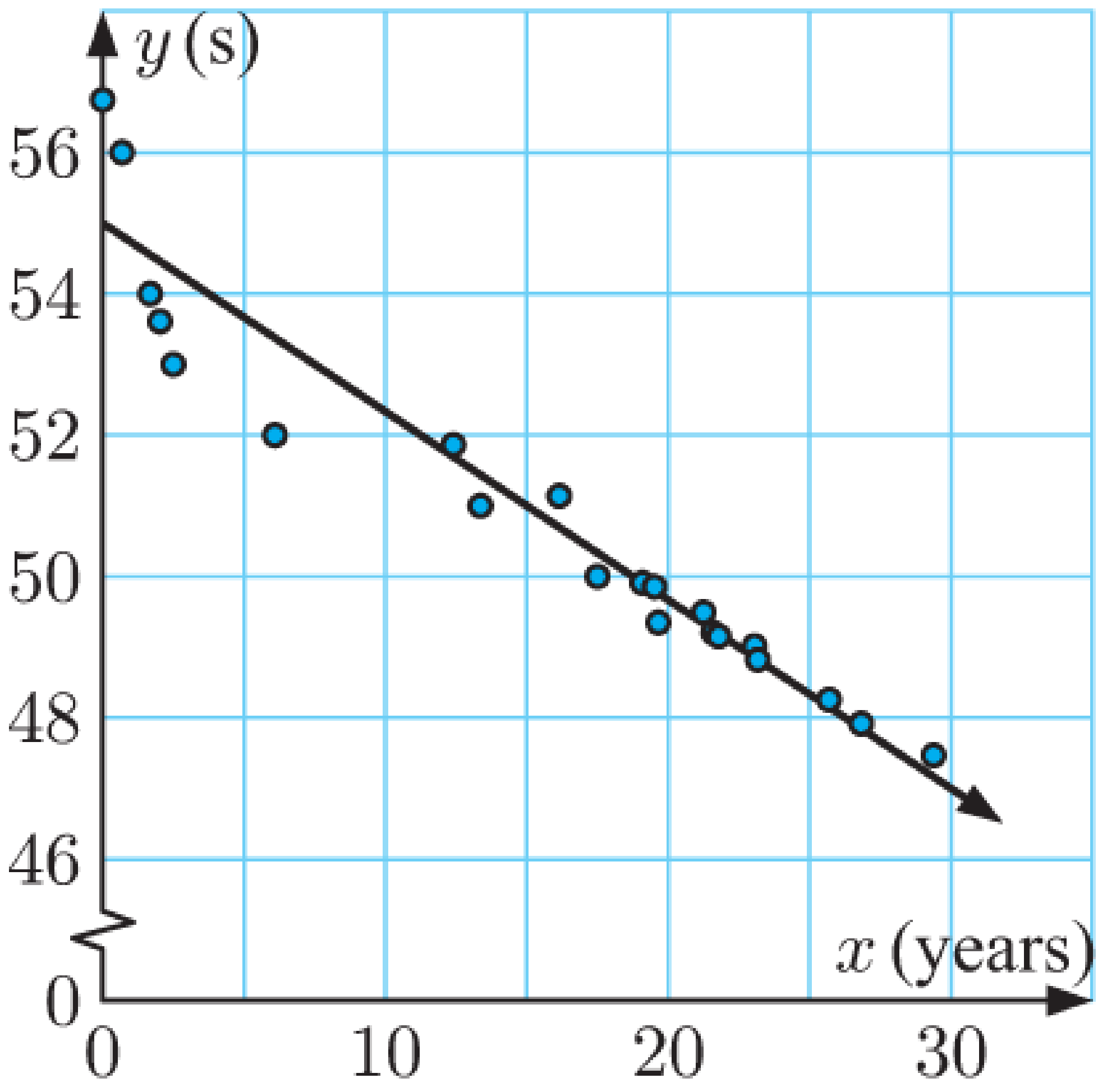
The graph alongside shows these data points, as well as the linear model $N = 8t + 0.2$.



a Is the linear model exact or approximate? Explain your answer.

b Use the model to predict how many pieces of litter Jack will pick up in 20 minutes. Discuss the accuracy of your prediction.

6 The graph alongside shows the world record times for the women's 400 m, between Marlene Mathews' record of 57.0 s on January 6, 1957 and Marita Koch's record of 47.60 s on October 6, 1985. The data is modelled with the straight line $y = 54.85 - 0.2660x$ where y is the time in seconds and x is the number of years since 1957.



a What does the model predict the time will be in 1957? What was the time, and why are these values different?

b Could the model be used to estimate world record times prior to 1957? Explain your answer.

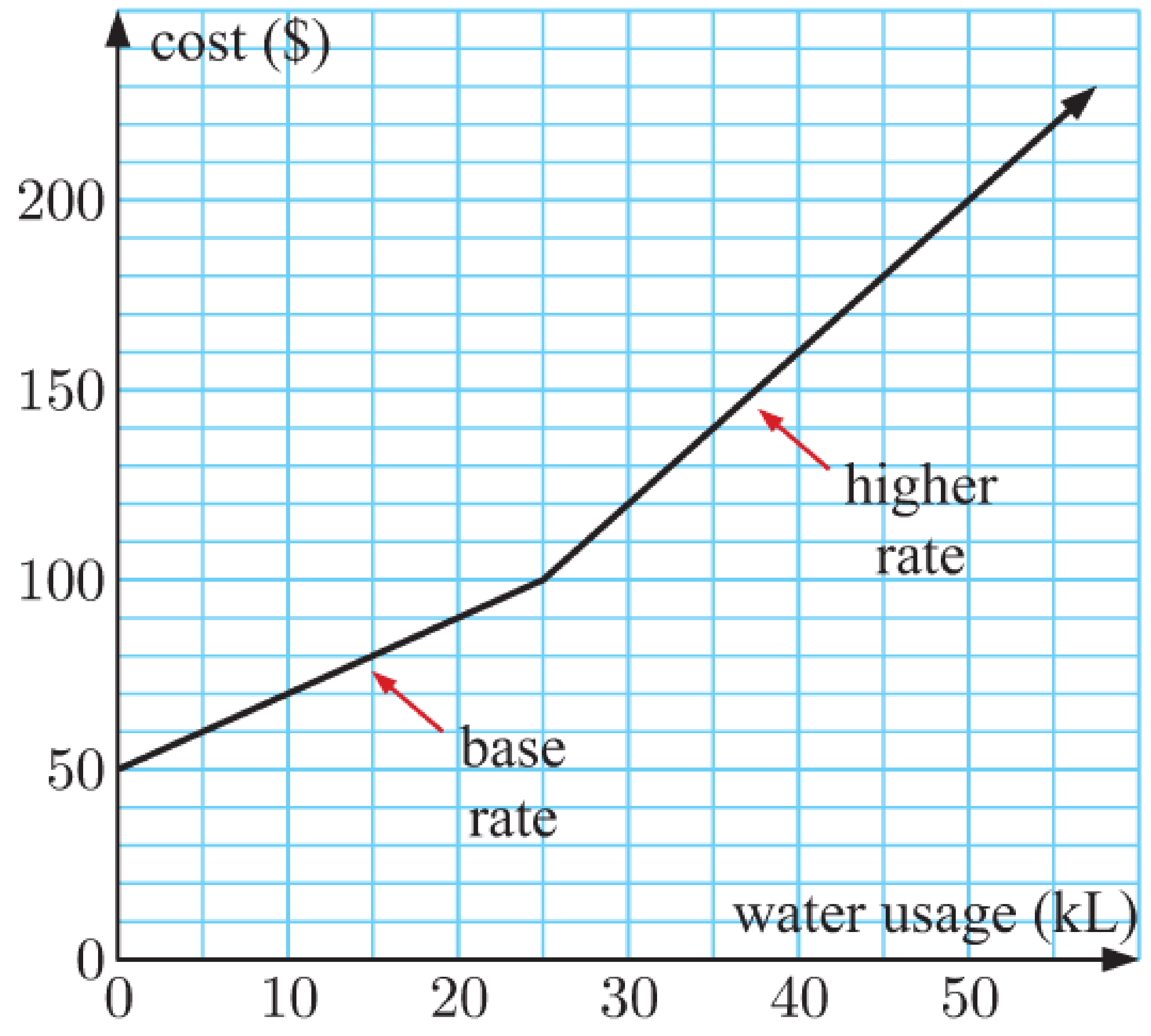
c Florence Griffith-Joyner holds the women's 100 m record of 10.49 s. In what year does this model predict the women's 400 m will be completed in $10.49 \times 4 = 41.96$ s?

d Marita Koch's time of 47.60 s is still the world record in 2018. Discuss the reliability of this model for predicting future world record times.

C PIECEWISE LINEAR MODELS

A **piecewise linear model** is a model made up of several straight line segments.

This piecewise linear graph shows the relationship between the *water used* by a household and the *cost* of the water. Water usage up to 25 kL is charged at a base rate, and any water usage above 25 kL is charged at a higher rate. We observe this in the graph because after 25 kL, the gradient of the graph suddenly increases.



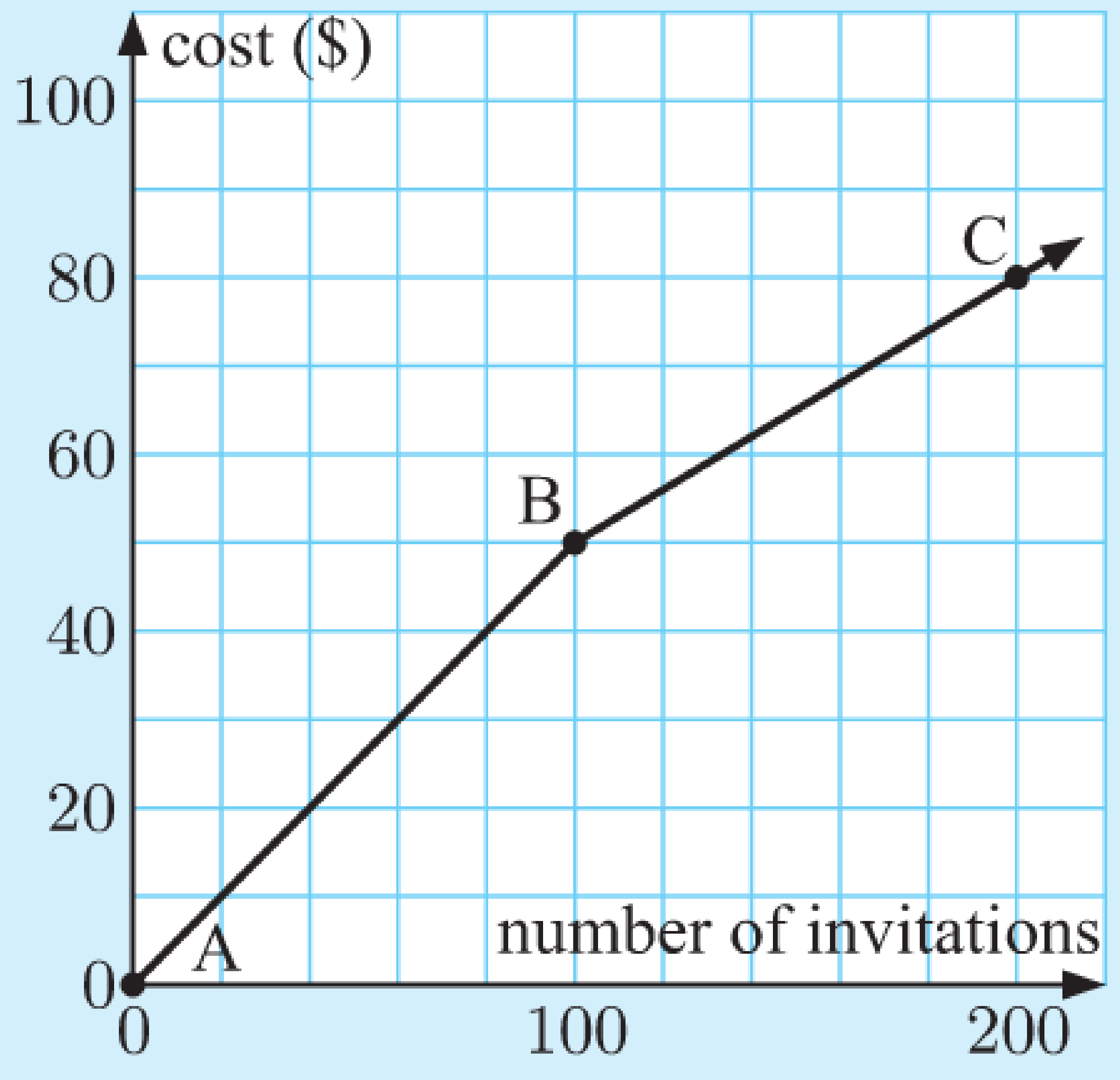
Example 2 **Self Tutor**

The *cost* of printing invitations depends on the *number of invitations* printed, as shown in this piecewise linear model.

a Find the gradient of: **i** [AB] **ii** [BC].
Interpret your answer.

b Find the cost of printing 60 invitations.

c How many invitations can be printed for $74?



a **i** A is $(0, 0)$ and B is $(100, 50)$

$\therefore$ gradient of [AB] $= \dfrac{50-0}{100-0} = 0.5$

The first 100 invitations cost 50 cents each to print.

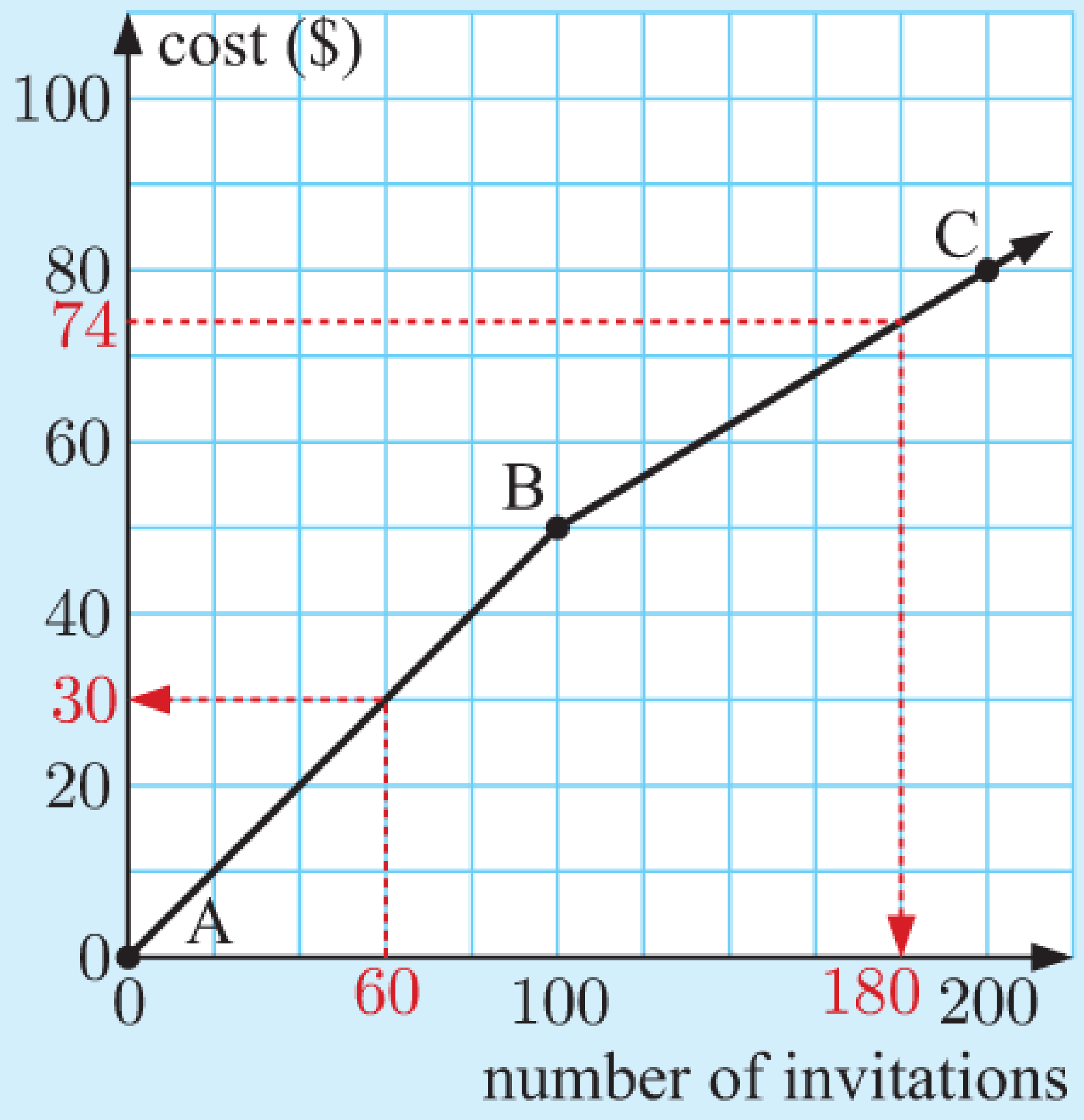
ii B is $(100, 50)$ and C is $(200, 80)$

$\therefore$ gradient of [BC] $= \dfrac{80-50}{200-100} = 0.3$

Each invitation after the first 100 costs 30 cents to print.

b It costs $30 to print 60 invitations.

c 180 invitations can be printed for $74.



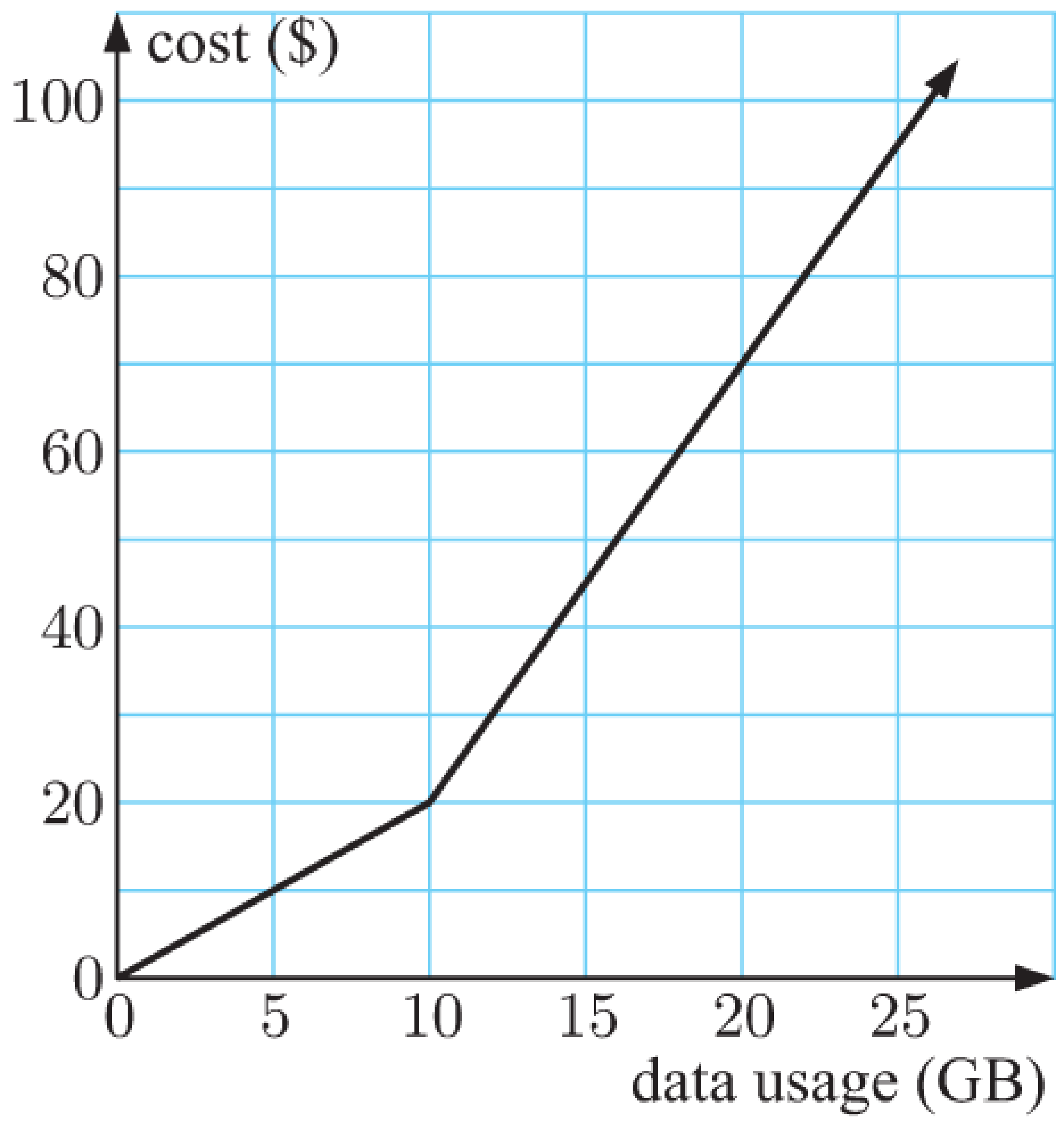
EXERCISE 4C

1 Consider the water costs graph on the previous page.

- **a** Find the gradient of each line segment. Interpret your answers.
- **b** Find the cost of using 40 kL of water.
- **c** Kelly's last water bill was \$80. How much water did she use?

2 Each month, Colin pays \$2 per gigabyte for mobile phone data up to his allowance, and then a higher rate for excess data.

- **a** What is Colin's data allowance?
- **b** At what rate is Colin charged for excess data?
- **c** How much does Colin need to pay for using 15 GB of data in a month?
- **d** What data usage would result in a bill of \$60?



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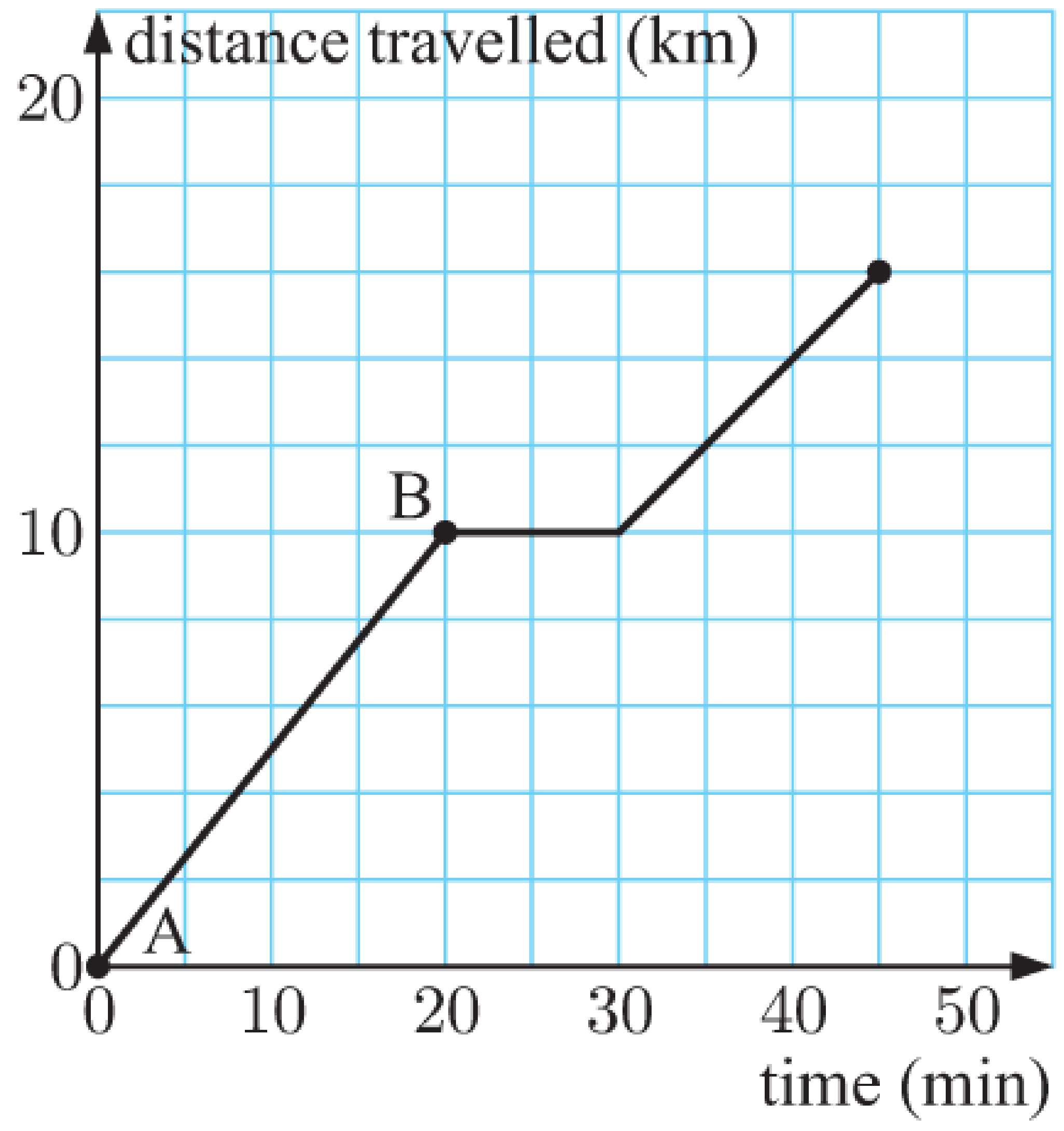
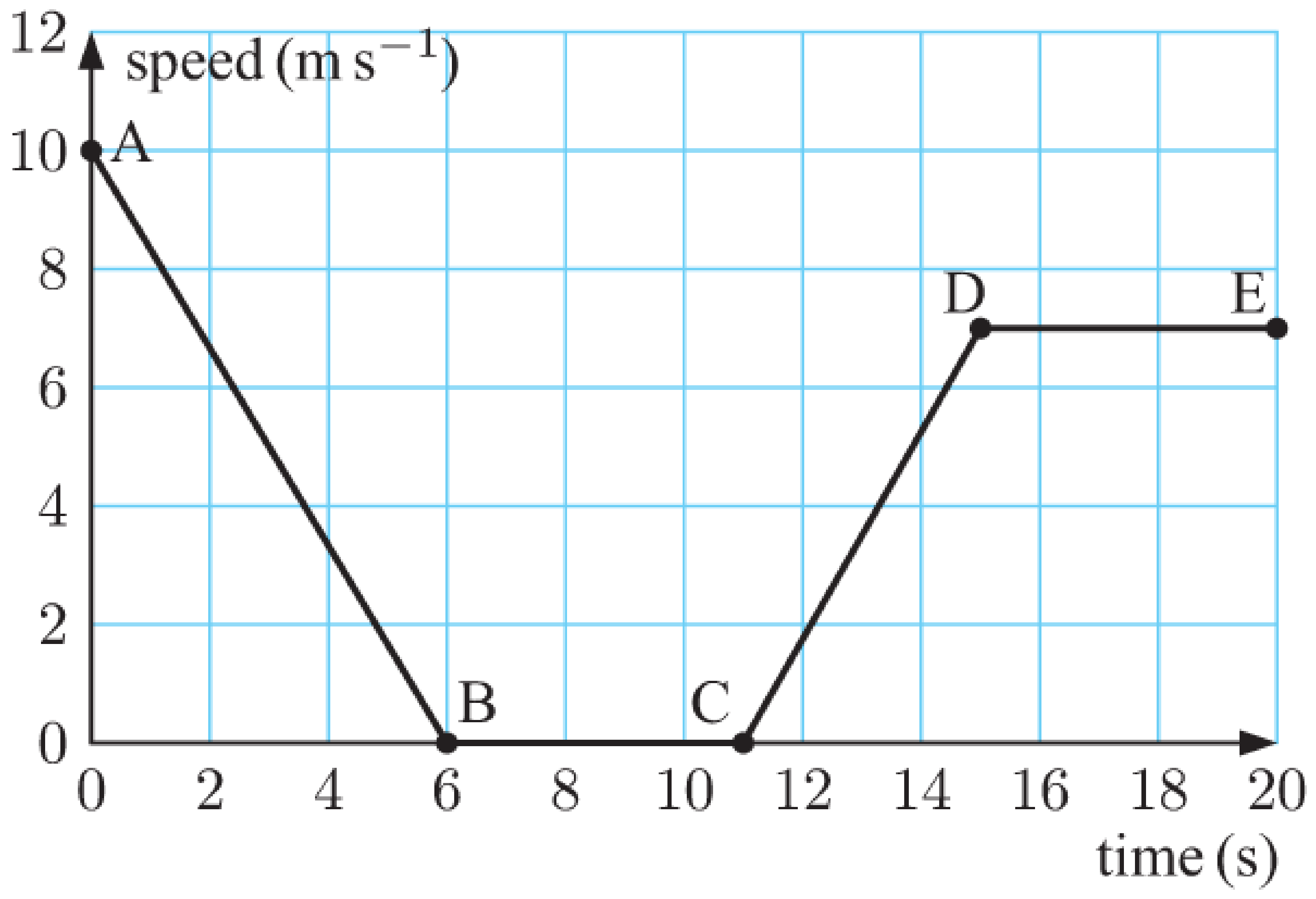


This graph shows the distance travelled by a cyclist on her way to work.

- **a** How long did it take the cyclist to ride to work?
- **b** How far did she travel in total?
- **c** Find the gradient of [AB]. Interpret your answer.
- **d** For how long did the cyclist stop to fix her tyre?
- **e** How far had the cyclist travelled after 35 minutes?

4 This graph models the speed of a car over a 20 second period.

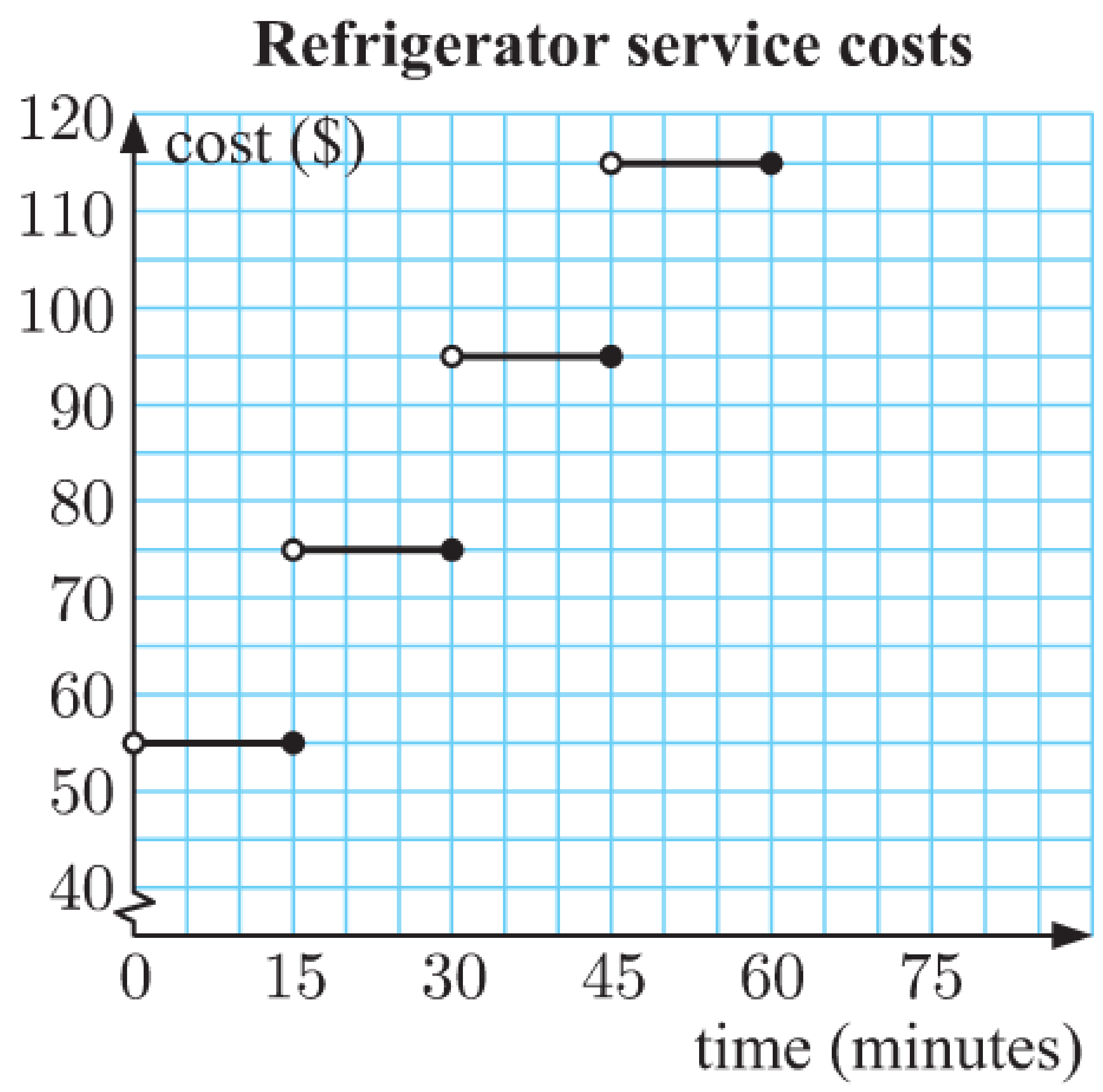
- **a** Describe the car's journey over this period.
- **b** What assumptions have been made when using this graph to model the car's speed? Are these assumptions reasonable?
- **c**
  - **i** Find the *average* speed of the car between A and B.
  - **ii** Hence find the distance travelled between A and B.
- **d** Find the total distance travelled by the car.



5 Jasper picks berries on a farm. He earns \$5 per kg for the first 12 kg of berries he picks each day, and \$8 for each additional kilogram.

- **a** Draw a graph to display the relationship between the *weight of berries* Jasper picks and his *daily wage*.
- **b** How much does Jasper earn for picking 9 kg of berries?
- **c** How many kilograms of berries must Jasper pick to earn \$100 in a day?

6



This graph shows the costs for refrigerator services.

- **a** Find the cost of a service which takes:
  - **i** 20 minutes
  - **ii** 45 minutes.
- **b** Find the maximum time for a service costing:
  - **i** \$55
  - **ii** \$115
- **c** Predict the cost of an 80 minute service, stating any assumptions you are making.

7 The cost of sending a parcel overseas depends on the weight of the parcel.

- **a** Draw a graph to display the information given in the table, for weights up to 5 kg.
- **b** Find the cost of sending a parcel weighing:
  - **i** 750 g
  - **ii** 1.6 kg
  - **iii** 3.4 kg
- **c** Heather has a 1.7 kg parcel and a 2.8 kg parcel to send to her mother. How much money will she save by sending the parcels together as a single package, rather than separately?

Weight	*Cost*
Up to 500 g	\$13
Over 500 g up to 1 kg	\$23
Over 1 kg up to 2 kg	\$32
Over 2 kg up to 3 kg	\$40
Extra kg or part thereof	\$5

8 A swimmer in a 200 m medley swims all 4 strokes (butterfly, backstroke, breaststroke, and freestyle) in one race. Laura's times for each 50 m leg are shown in the table.

- **a** Sketch a distance-time graph to model Laura's position during the race.
- **b** Describe any assumptions you made when constructing your graph.
- **c** Use your graph to estimate how far Laura had swum after 1 minute.

Stroke	*Time* (s)
Butterfly	28.20
Backstroke	29.28
Breaststroke	33.91
Freestyle	27.43

Example 3 **Self Tutor**

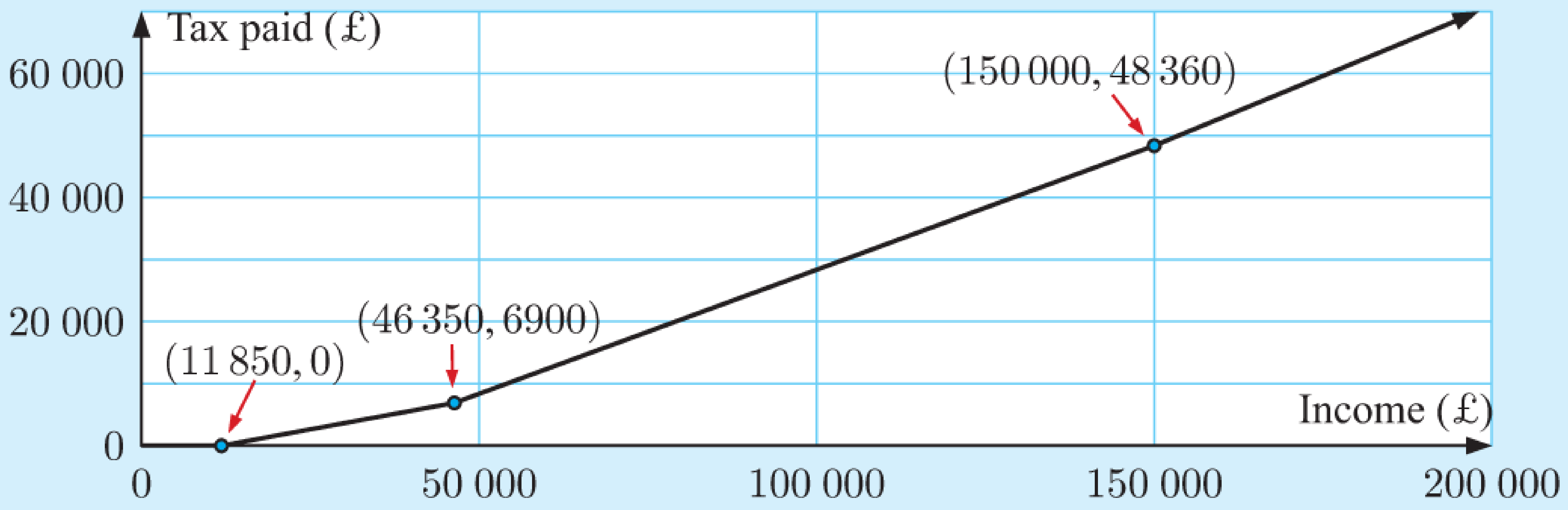
This table shows the income tax rates in the United Kingdom in 2018.

Taxable income (per year)	*Tax paid*
Up to £11 850	Nil
£11 851 - £46 350	£0.20 for each £1 above £11 850
£46 351 - £150 000	£6900 + £0.40 for each £1 above £46 350
Over £150 000	£48 360 + £0.45 for each £1 above £150 000

a Draw a graph of tax paid against income.

b Find the amount of tax paid by a person who earns £50 000 per year.

a



b £50 000 = £46 350 + £3650

$\therefore$ the tax paid = £6900 + £0.40 × 3650

= £8360

9 Use the income tax rates in **Example 3** to answer the following problems:

- **a** The minimum wage in the United Kingdom is £7.83 per hour. Find the annual tax paid by a person working on minimum wage for 36 hours per week, 50 weeks of the year.
- **b** Harold is a train driver earning £53 172 per year.
  - **i** How much income tax must Harold pay?
  - **ii** Find Harold's salary after income tax has been deducted.
- **c** On April 1, 2018 the United Kingdom Prime Minister's total salary was £153 907.
  - **i** How much income tax did the Prime Minister pay?
  - **ii** What *percentage* of the Prime Minister's income was paid as income tax?

10 For 2018 - 2019, the Australian government set the following income tax rates for residents:

Taxable income (per year)	*Tax paid*
Up to $18 200	Nil
$18 201 - $37 000	$0.19 for each $1 over $18 200
$37 001 - $90 000	$3572 plus $0.325 for each $1 over $37 000
$90 001 - $180 000	$20 797 plus $0.37 for each $1 over $90 000
$180 001 and over	$54 097 plus $0.45 for each $1 over $180 000

- **a** Draw a graph of tax paid against income for incomes between $0 and $200 000.

b On a separate set of axes, draw a graph showing the *rate* of tax against income for incomes between \$0 and \$200 000.

c Calculate the income tax for the following annual incomes:

i \$28 000 **ii** \$48 300 **iii** \$96 150

ACTIVITY 2 NON-LINEAR PIECEWISE MODELS

The models we have studied in the previous Section are all built out of straight line segments. In the world around us, there are many situations where piecewise models involve curves. It is useful to logically consider each process and how the curves in the model result from the physical situation.

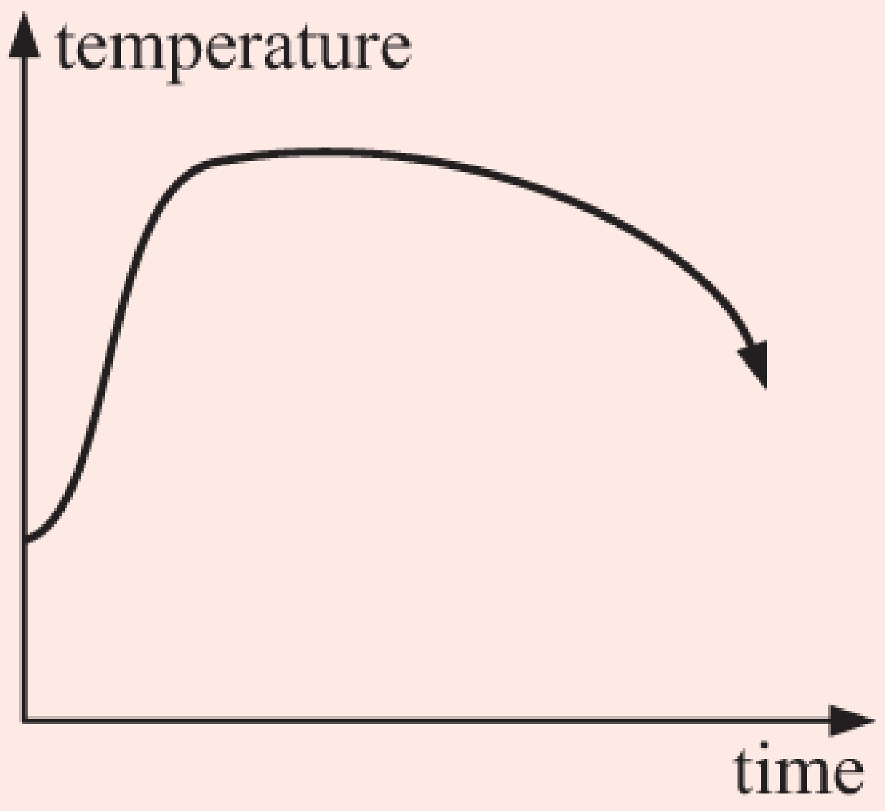
What to do:

1 When a kettle is turned on, the energy added to the water causes the temperature to rise. While the water is in liquid phase, the rise is fairly constant. As the water approaches boiling, large amounts of energy are needed to convert the liquid water into water vapour. Energy is also lost as bubbles are formed and burst.

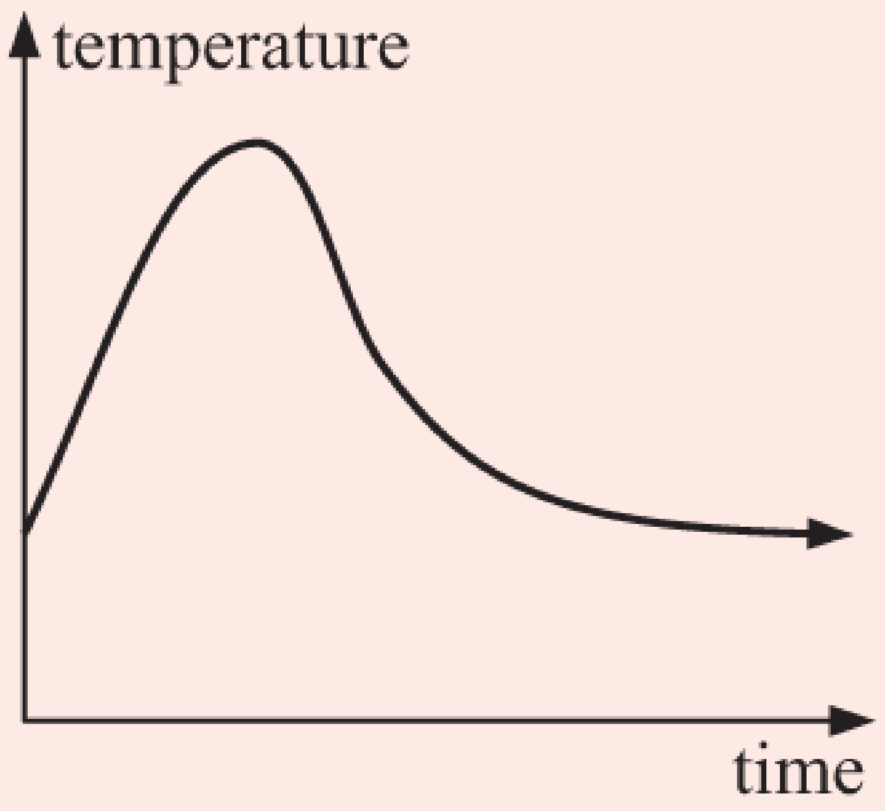
After the kettle is switched off, the water starts to cool back to the temperature of the room around it. The rate at which its temperature falls is determined by the difference in temperature with the room, so it initially falls rapidly, then slows down over time.

Which of the following piecewise graphs accurately describes the temperature of the water over time? Explain your answer by describing how each part of the process corresponds to the graph.

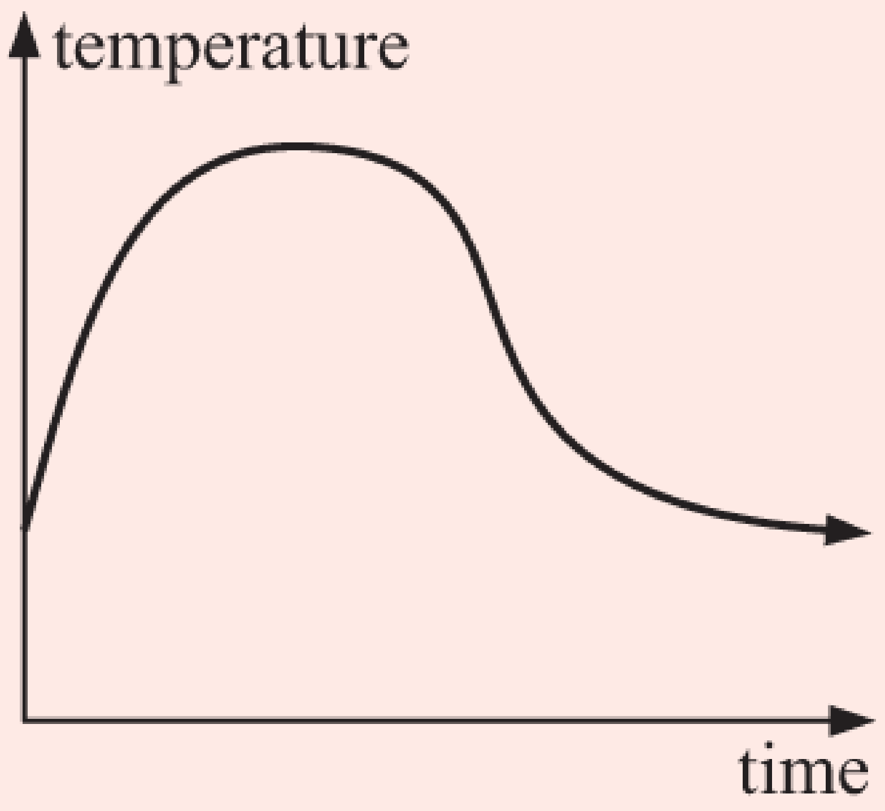
A



B



C



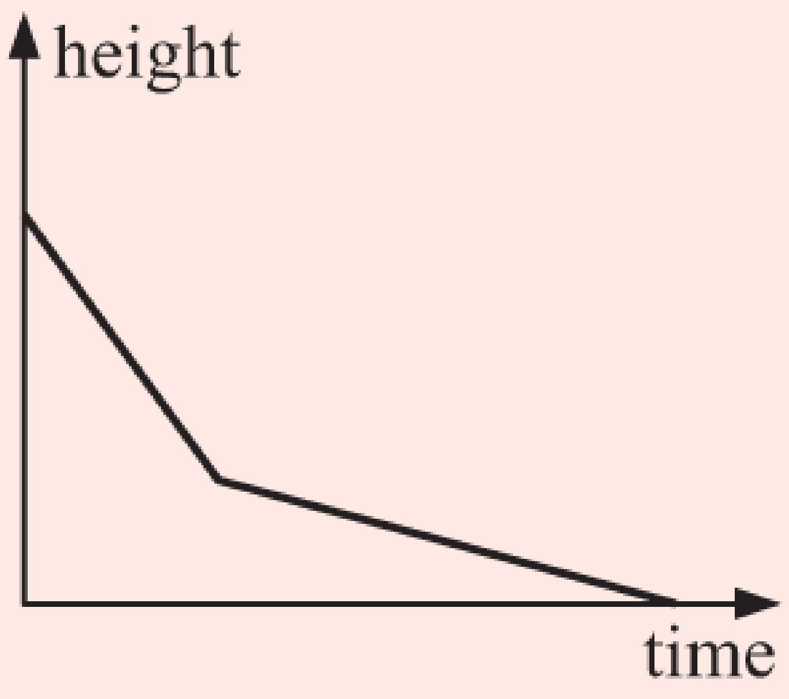
2 When a skydiver jumps out of a plane, their descent to the ground has several phases.

Initially, it is a smooth curve under constant gravity. However, as the speed of the skydiver increases, they face more and more air resistance. As a consequence, they approach terminal velocity where their speed of descent is constant.

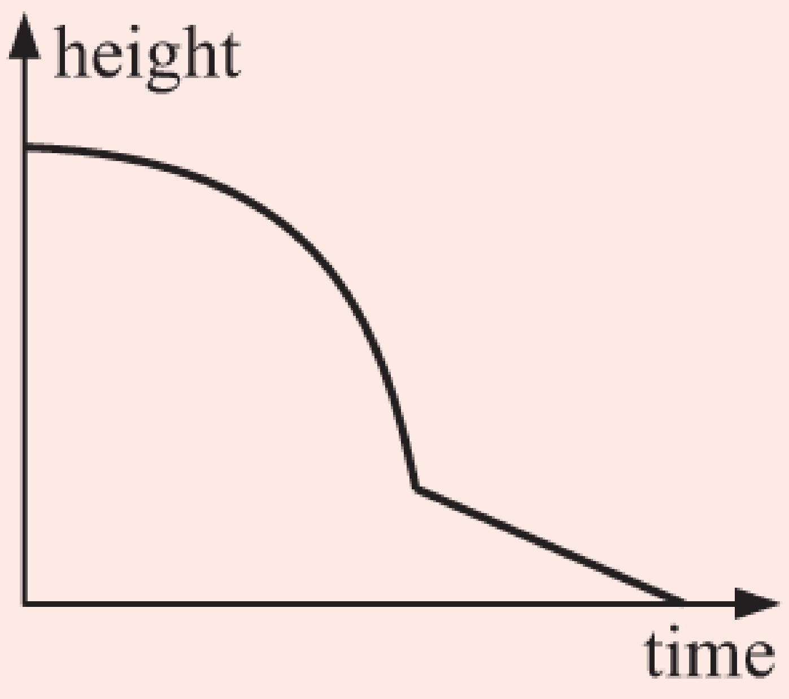
Eventually, the skydiver activates their parachute, which causes a rapid decrease in speed of descent. They will then slowly descend to the ground.

Which of the following piecewise graphs accurately describes the descent of the skydiver? Explain your answer by describing how each part of the descent corresponds to the graph.

A



B



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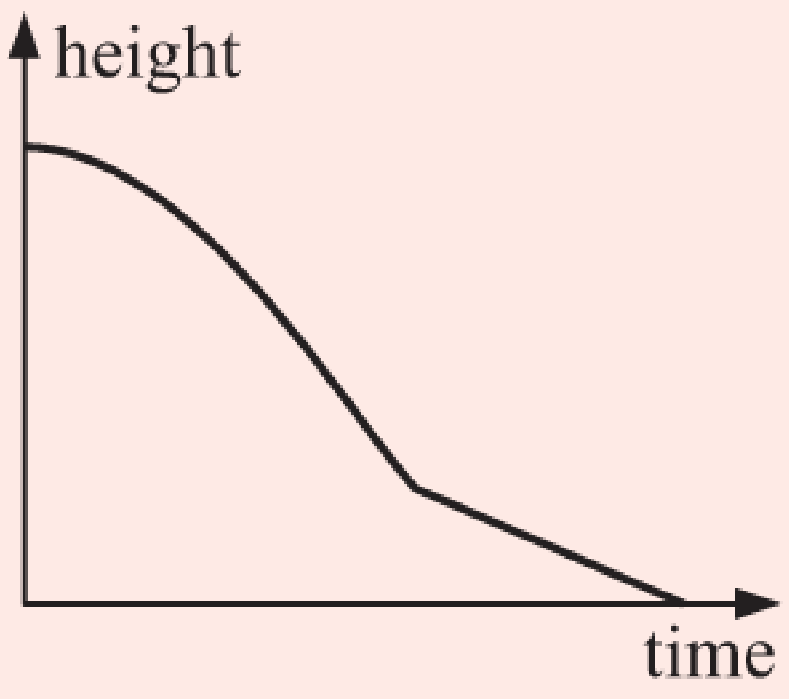


D SYSTEMS OF EQUATIONS

When we construct a mathematical model, we can often use our knowledge of mathematical functions to guess the *form* that the model should take.

For example, when a ball is thrown into the air, we can predict that its height above the ground will be modelled by a function of the form $y = at^2 + bt + c$, where t is the time in seconds, and a, b, and c are unknown **coefficients** or **parameters**.

DISCUSSION

- What do we mean by coefficients, parameters, and constants?
- Can we use these mathematical terms interchangeably without causing confusion?

If we have some data values that the model needs to fit, we can substitute the points into the function to construct a **system of equations** for the unknown parameters. By solving the system of equations simultaneously, we obtain the parameters which complete the model.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 4 **Self Tutor**

The table alongside shows the quantity V of air affected by smoke t seconds after a fire is started.

t (seconds)	1	2	3
V (L)	2	4	12

V and t are connected by a cubic model of the form $V = at^3 + bt^2 + ct$.

Find the coefficients a, b, and c.

Substituting $(1, 2)$ into the model gives $2 = a(1)^3 + b(1)^2 + c(1)$

$$\therefore \quad a + b + c = 2$$

Substituting $(2, 4)$ into the model gives $4 = a(2)^3 + b(2)^2 + c(2)$

$$\therefore \quad 8a + 4b + 2c = 4$$

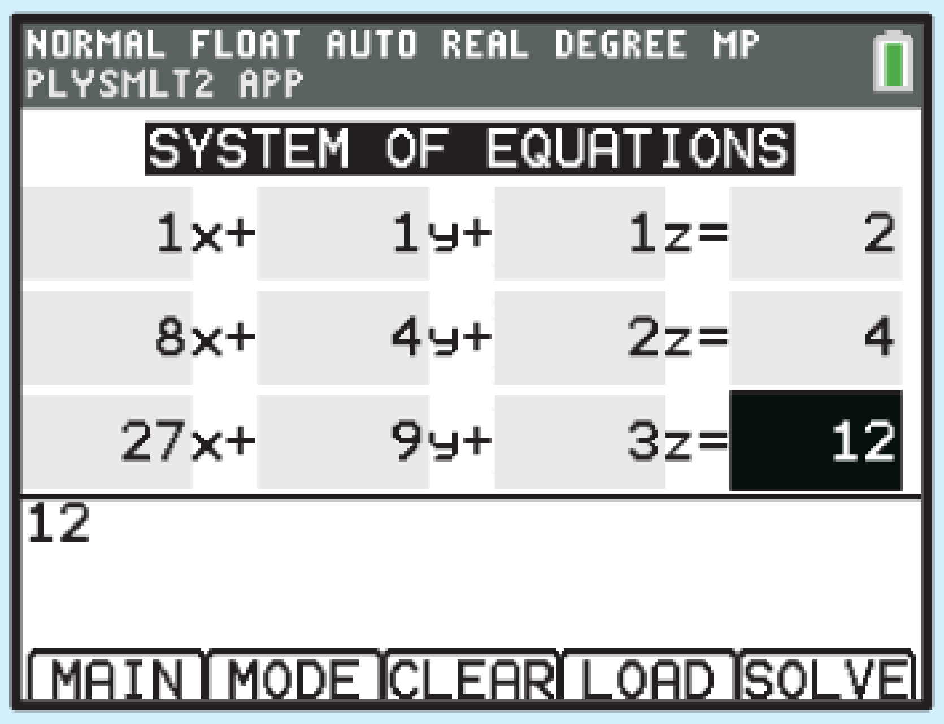
Substituting $(3, 12)$ into the model gives $12 = a(3)^3 + b(3)^2 + c(3)$

$$\therefore \quad 27a + 9b + 3c = 12$$

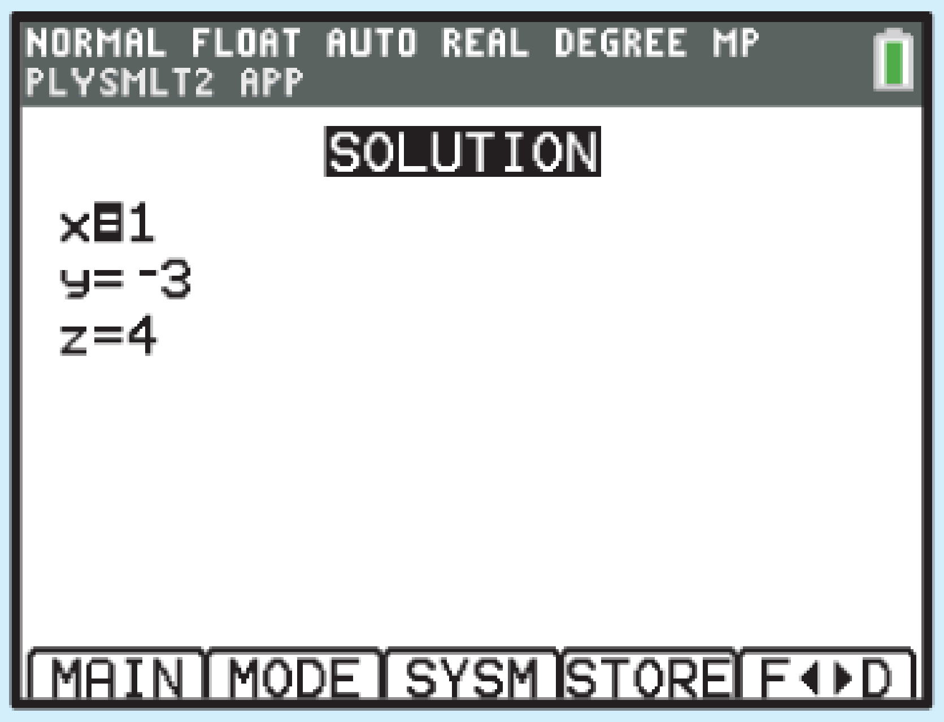
So, we have the system of equations $\begin{cases} a + b + c = 2 \\ 8a + 4b + 2c = 4 \\ 27a + 9b + 3c = 12 \end{cases}$.

Solving these equations simultaneously using technology, we find that $a = 1$, $b = -3$, and $c = 4$.

So, the model is $V = t^3 - 3t^2 + 4t$.



```
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
        SYSTEM OF EQUATIONS
   1x+      1y+      1z=      2
   8x+      4y+      2z=      4
  27x+      9y+      3z=     12
12
```

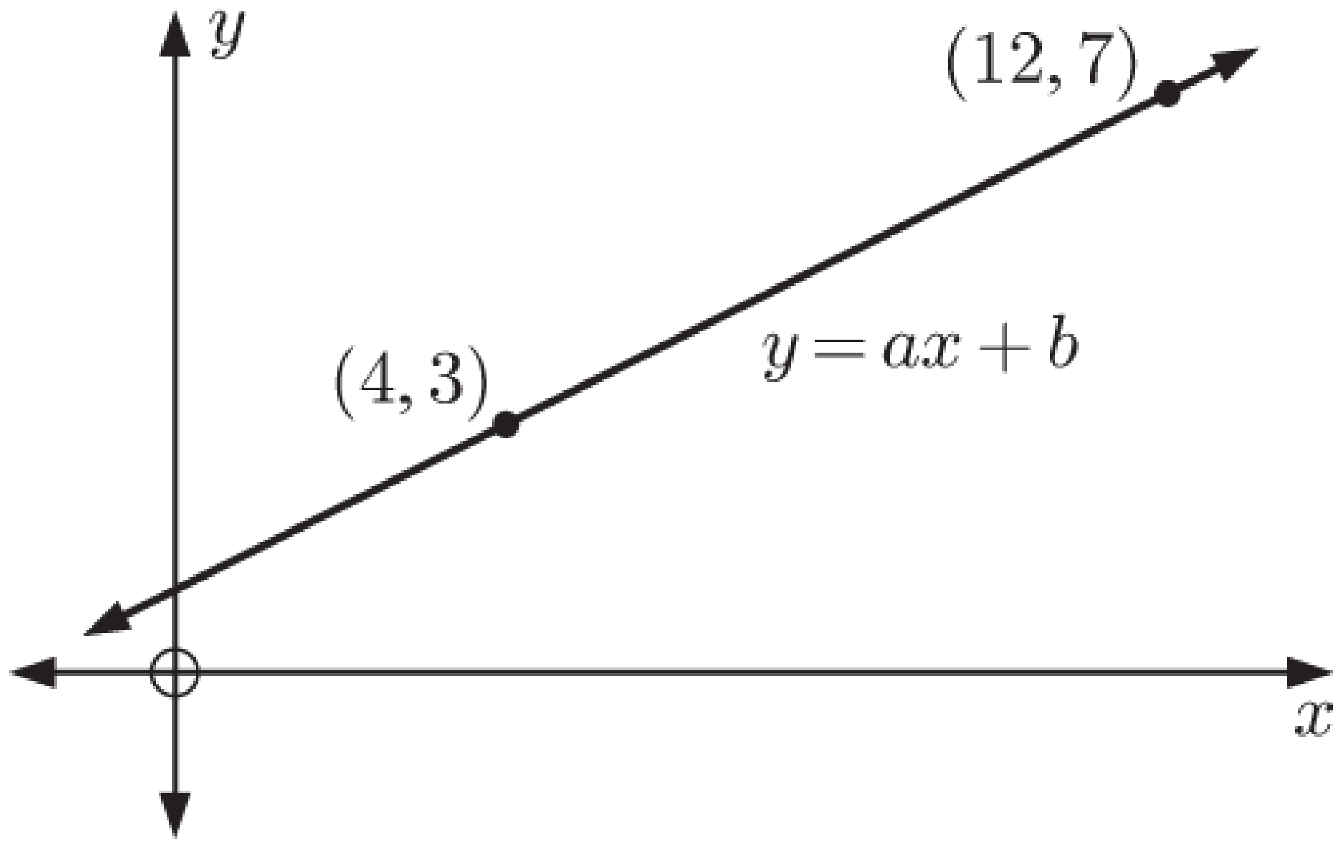


```
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
        SOLUTION
x=1
y=-3
z=4
```

EXERCISE 4D

1 **a** Use technology to find this linear model.

b Check your answer algebraically.



2 The points (1, 7), (2, 10), and (3, 11) lie on a model of the form $y = ax^2 + bx + c$.

a Construct three equations in terms of a, b, and c.

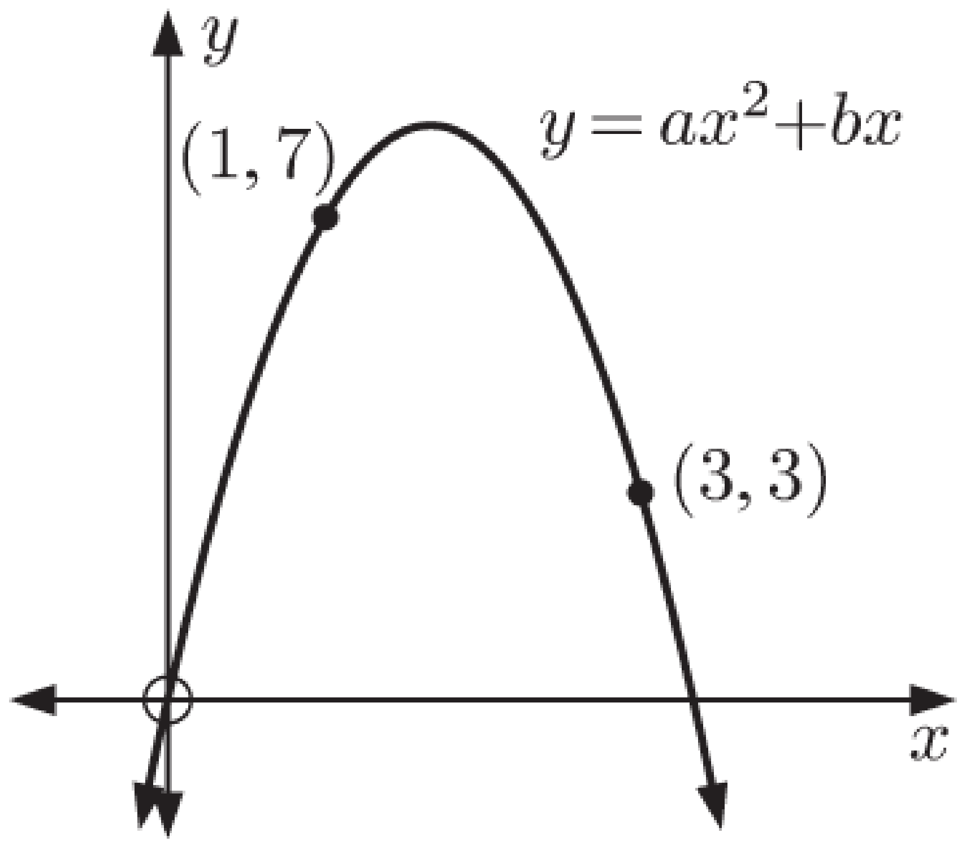
b Use technology to find a, b, and c.

c Hence determine the model.

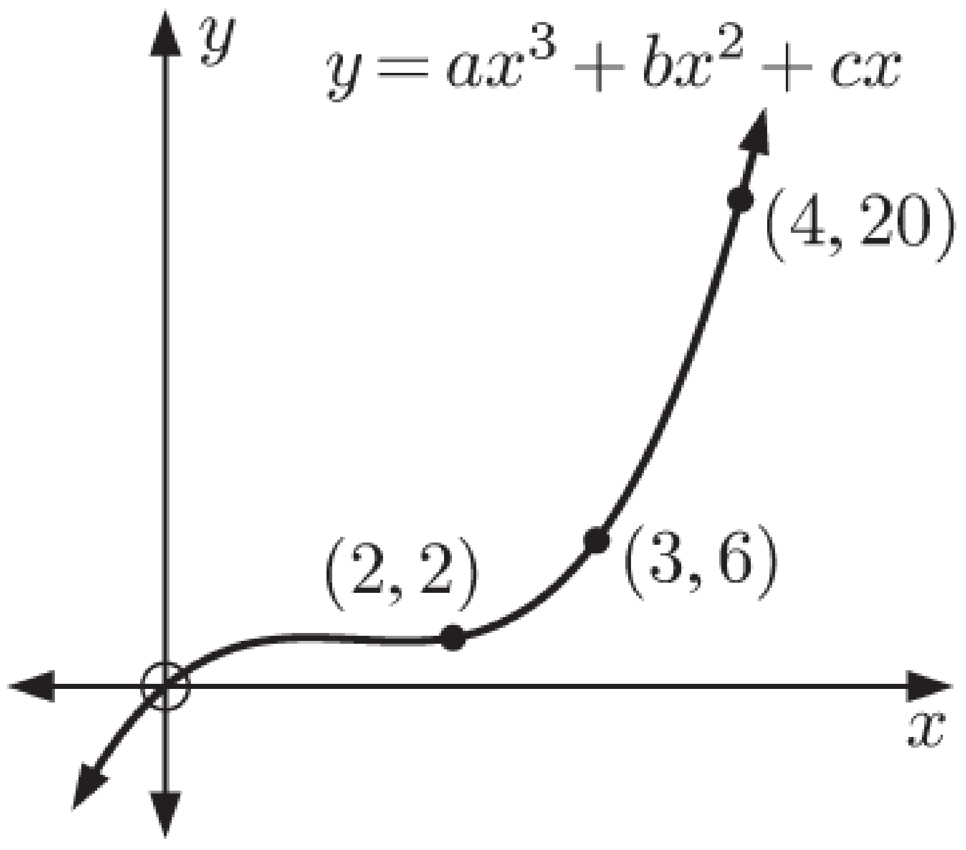
To be able to solve a system of equations, you need at least as many equations as there are unknowns.

3 Determine the model for each graph:

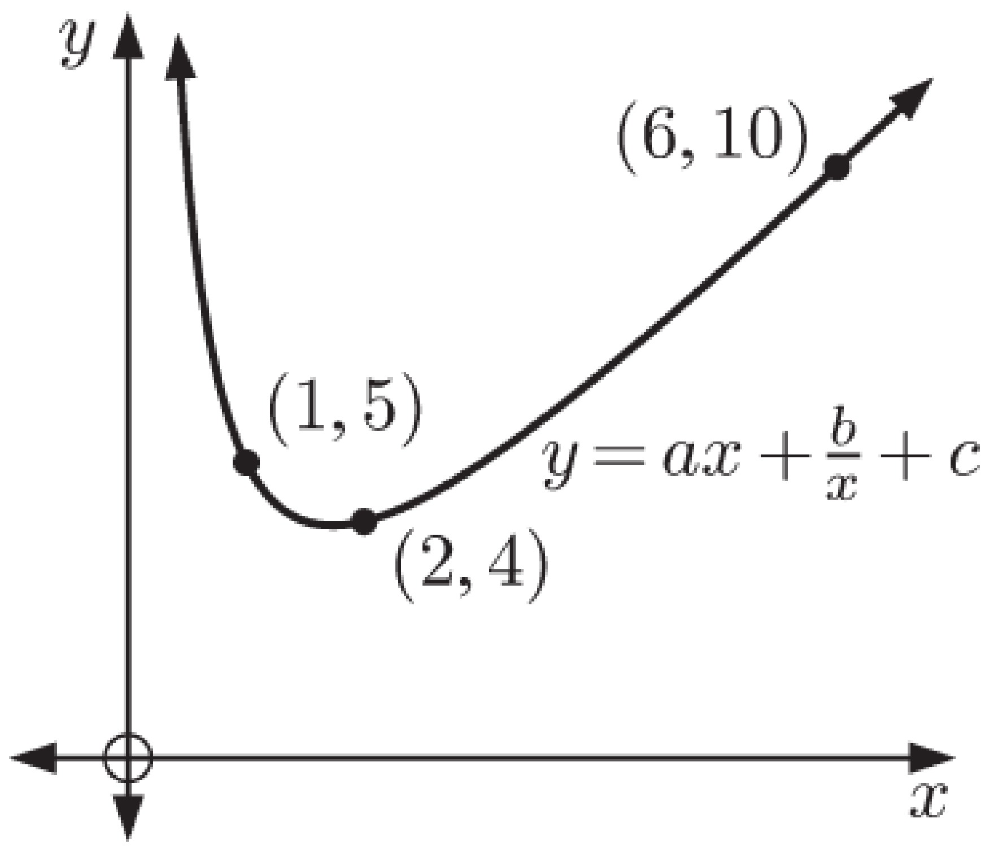
a



b



c



4 The table alongside shows the distance D m a scooter has travelled t seconds after it starts moving.

The model connecting D and t has the form $D = at^2 + bt + c\sqrt{t}$.

Find the parameters a, b, and c.

t (seconds)	1	4	9
D (metres)	1.7	7.2	23.7

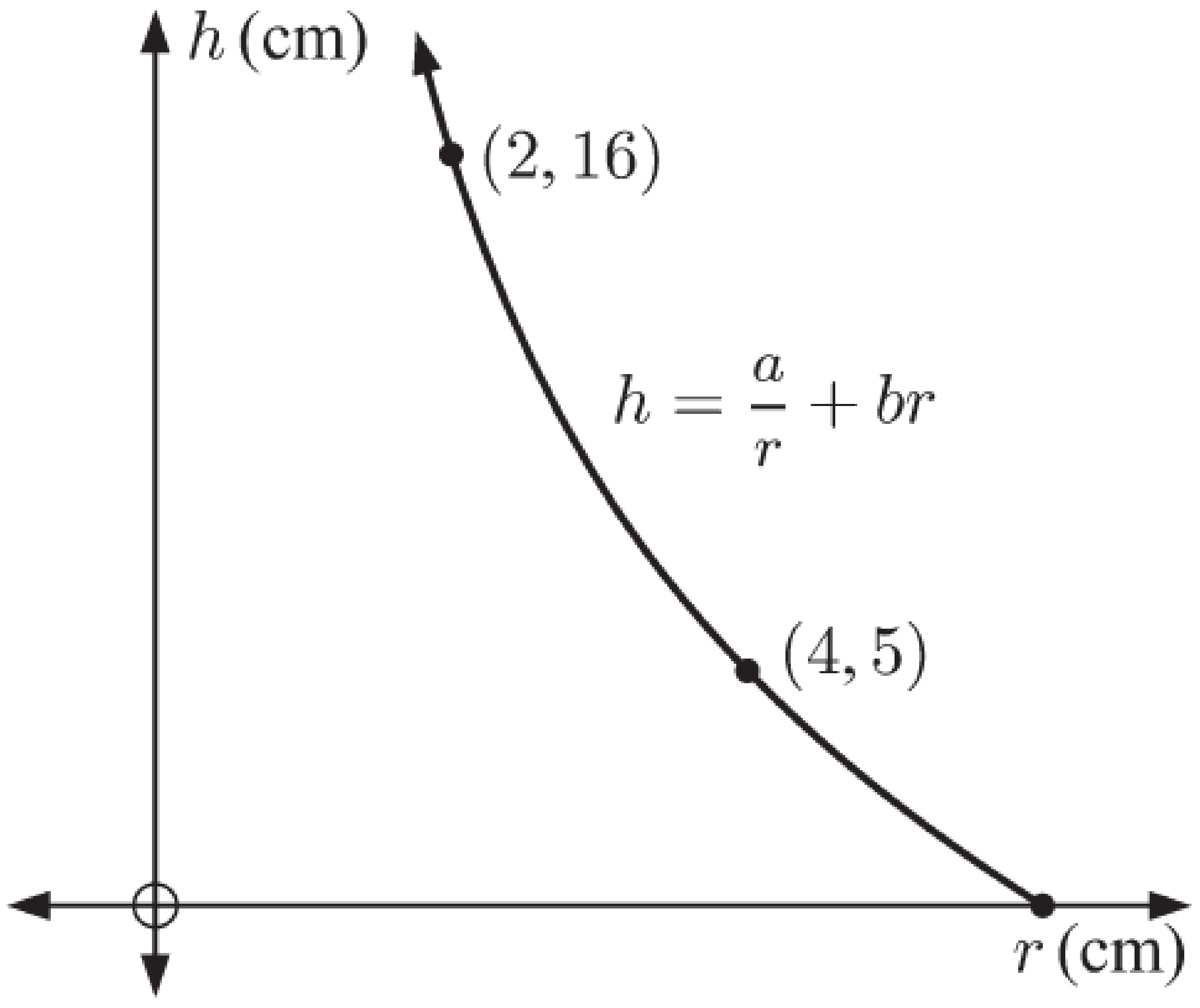
5 This graph shows the relationship between the height and radius of cylinders which have a constant surface area.

The model connecting h and r has the form $h = \frac{a}{r} + br$, where a and b are constants.

a Use the given data points to determine the model connecting h and r.

b Use the formula for the surface area of a cylinder to explain why a model of this form is reasonable.

c Find the constant surface area of these particular cylinders.

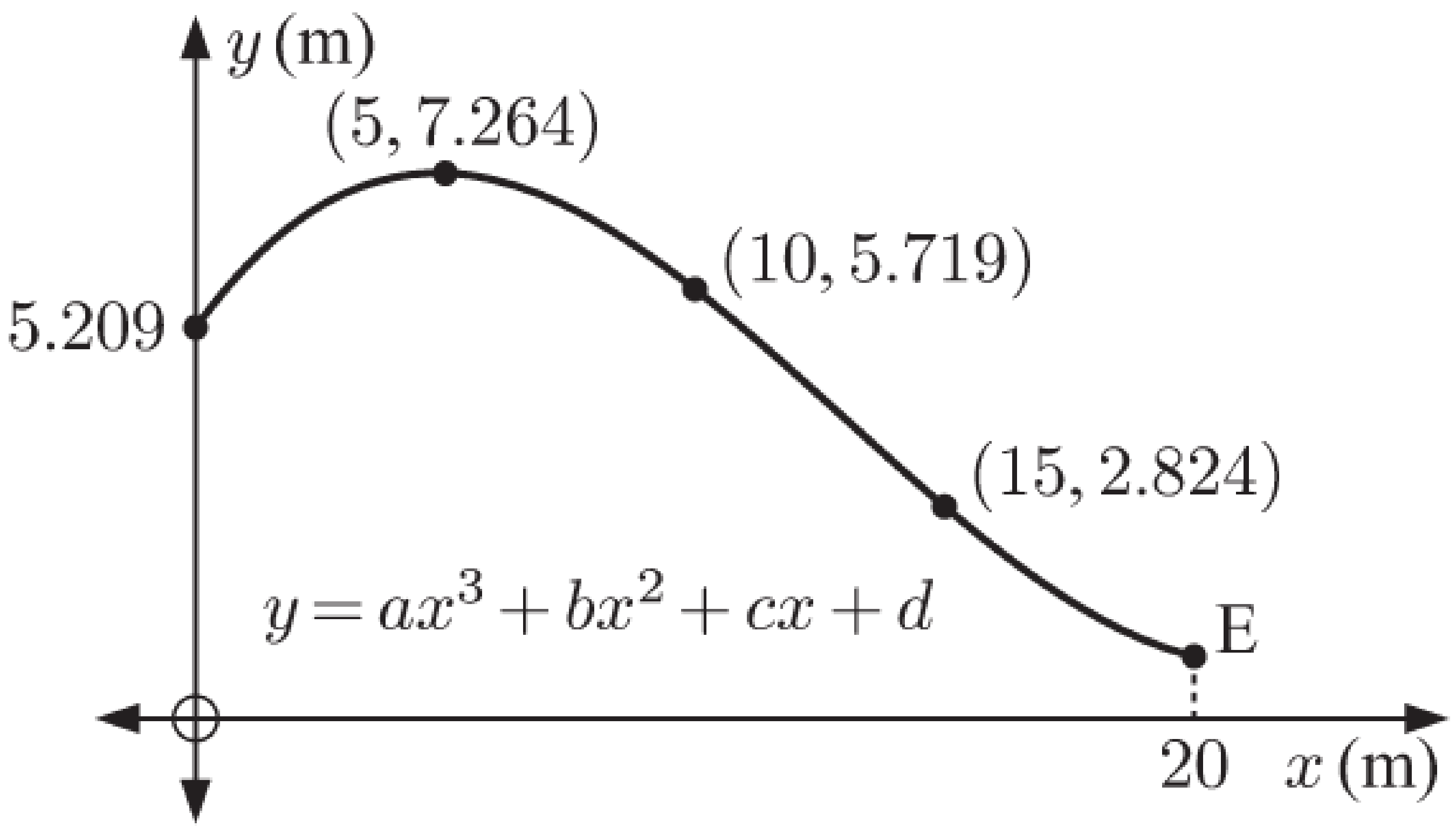


d Use your model to find the value of h when $r = 9$. Is your answer reasonable?

e For what values of r can this model be correctly applied?

6 The cross-section of a mound on a bike track is shown alongside. It can be modelled by the function
$y = ax^3 + bx^2 + cx + d$ for $0 \leqslant x \leqslant 20$.
Units are in metres.

- **a** State the value of d.
- **b** Use technology to find a, b, and c.
- **c** Find the height of the mound at point E.



REVIEW SET 4A

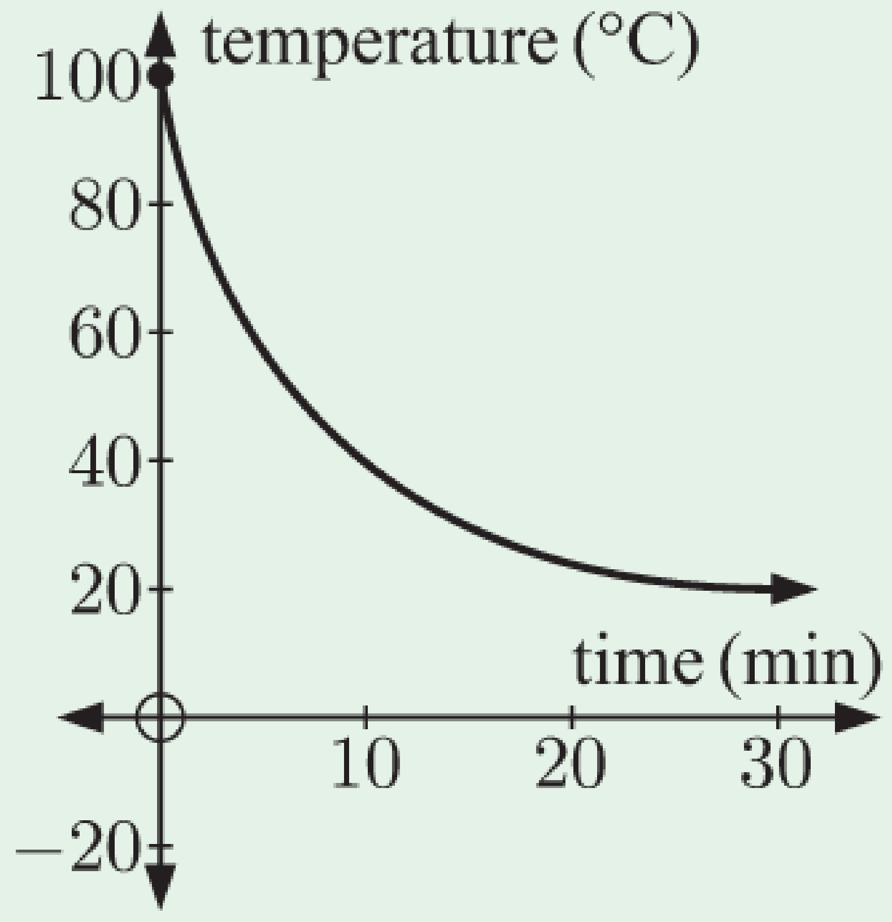
1 Ben loves to kayak. He can paddle 100 metres in 40 seconds.

- **a** Construct a model to describe how far Ben can kayak in t seconds.
- **b** Hence predict how far Ben can kayak in 10 minutes.
- **c** Do you think the actual distance Ben can kayak in 10 minutes will be more or less than your prediction? Explain your answer.

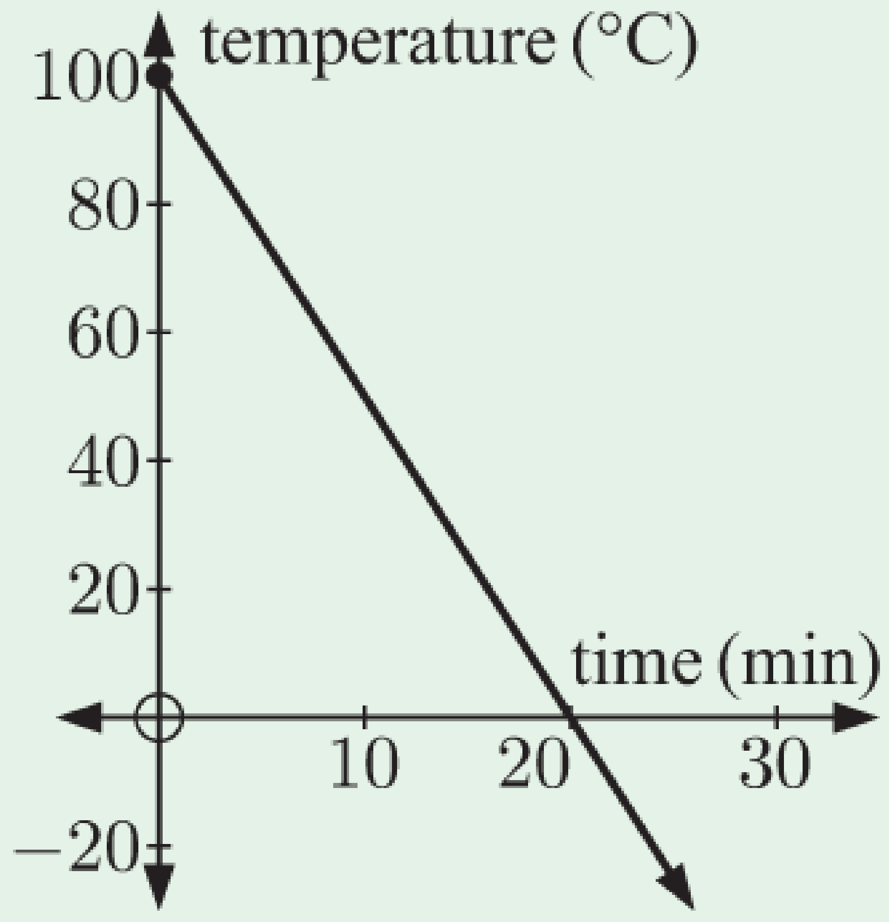
2 Some water is boiled, then poured into a cup.

- **a** Which of these graphs do you think is the most appropriate model for the temperature of the water?

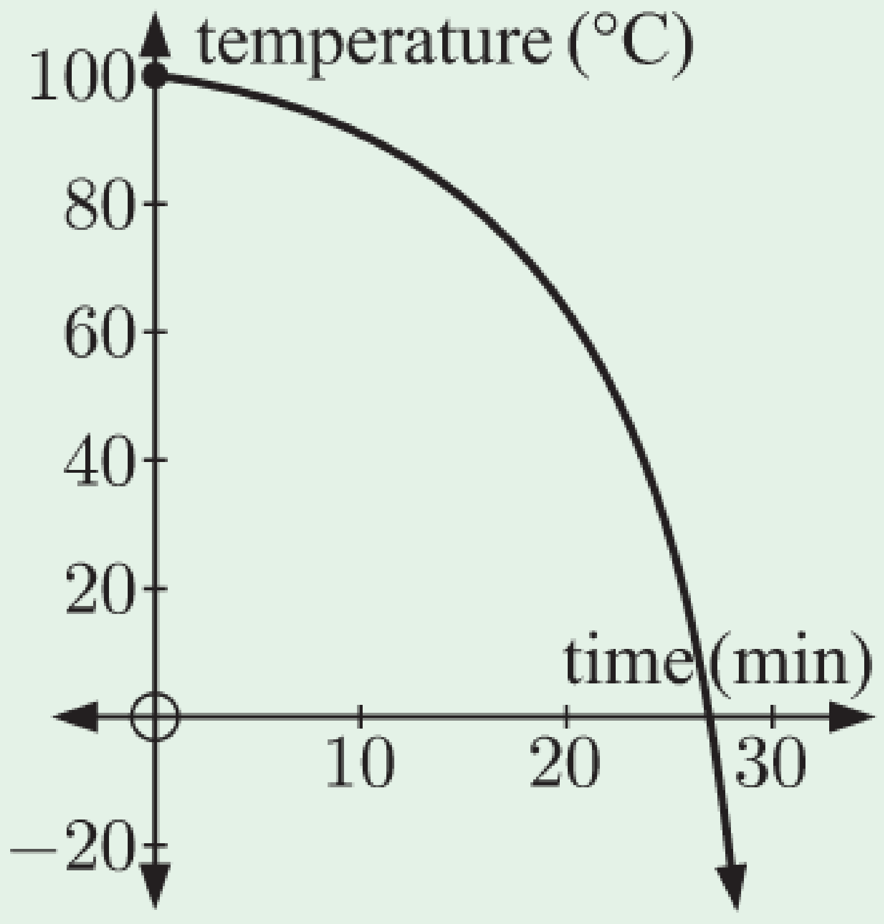
A



B

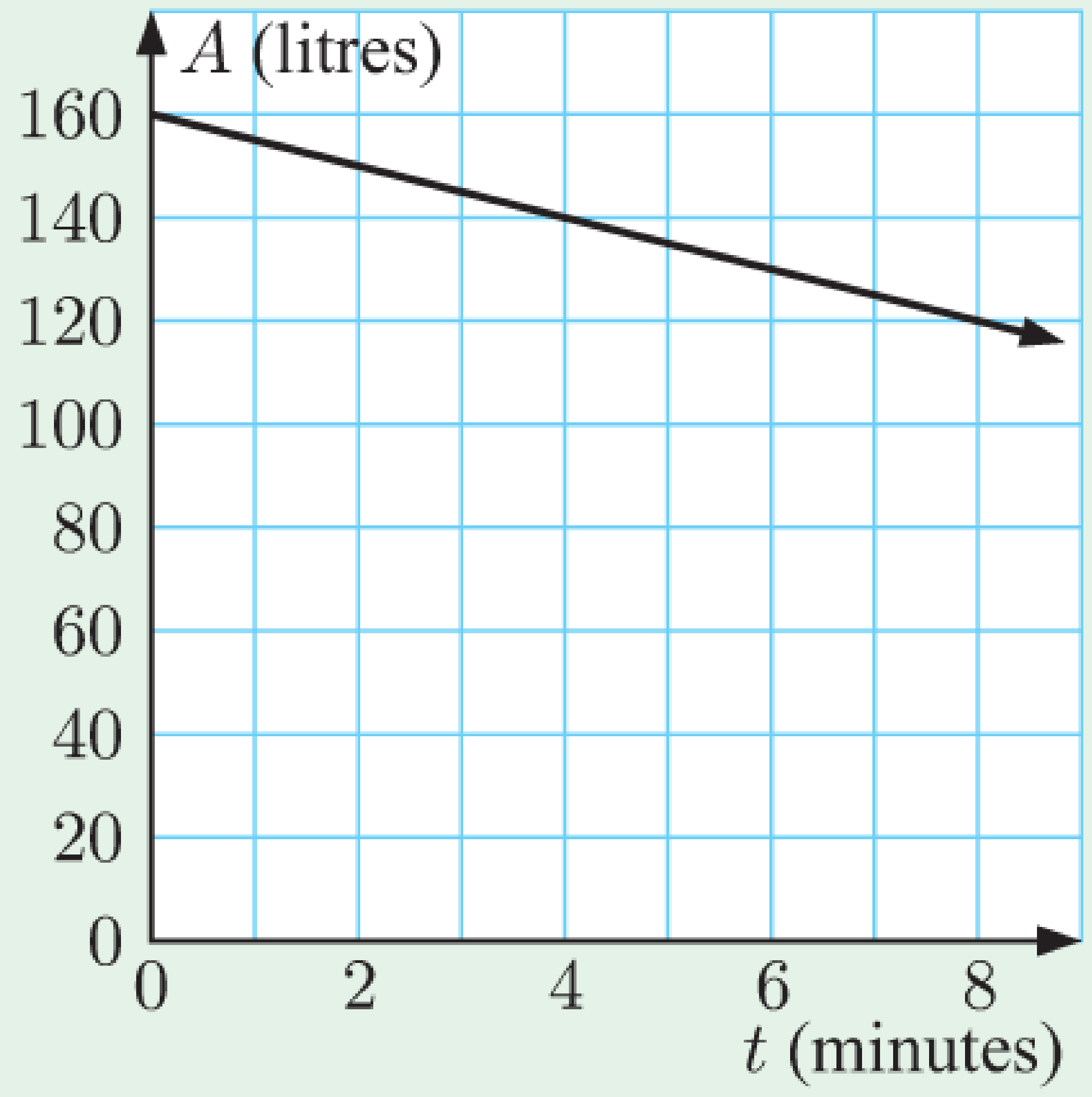


C



- **b** Use the model you selected to predict the temperature of the water after 10 minutes.

3



The amount of oil A left in a leaky barrel after t minutes is shown on the graph alongside.

- **a** Find the gradient and A-intercept of the line. Interpret your answers.
- **b** Find the model connecting A and t.
- **c** How much oil will be left after 15 minutes?
- **d** For what values of t is it reasonable to apply this model?

4 A plumber charges customers a £50 call-out fee, and then £80 for each hour he spends on the job.

a Copy and complete this table of values:

Time (t hours)	0	1	2	3	4
Cost (£C)					

b Draw the graph of C against t.

c Find the linear model connecting C and t.

d Hence find the cost of a job which takes 6 hours.

ACE PLUMBER
QUOTE
£££

5 Amy and Bernard are electricians. It would take each of them 3 days to wire a particular house.

a Assuming they were able to work without getting in each other's way, how long would it take for the two of them to do the job?

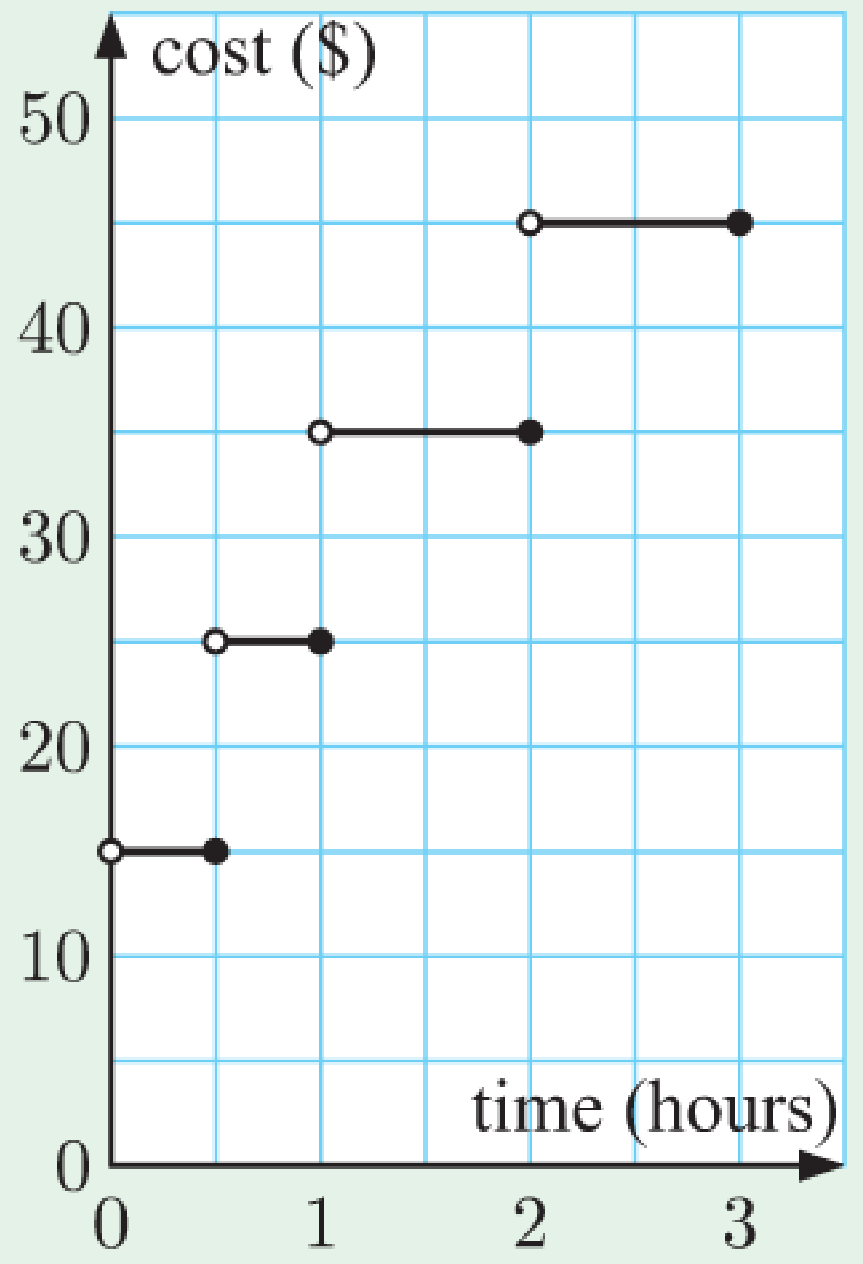
b Do you think the assumption in **a** is reasonable? If not, do you think it would take more or less time than your answer in **a**?

6 This graph shows the cost of hiring a squash court.

a Find the cost of hiring the court for:

i 45 minutes **ii** 2 hours.

b Kate and Peggy can spend no more than \$15 each to hire a court. What is the longest time they can hire the court for?



7 A triathlon consists of a 1.5 km swim, a 40 km bicycle ride, and a 10 km run. The table alongside shows the times Alana obtained for each leg of the triathlon.

Leg	*Time* (min)
Swim	36
Bicycle ride	81
Run	54

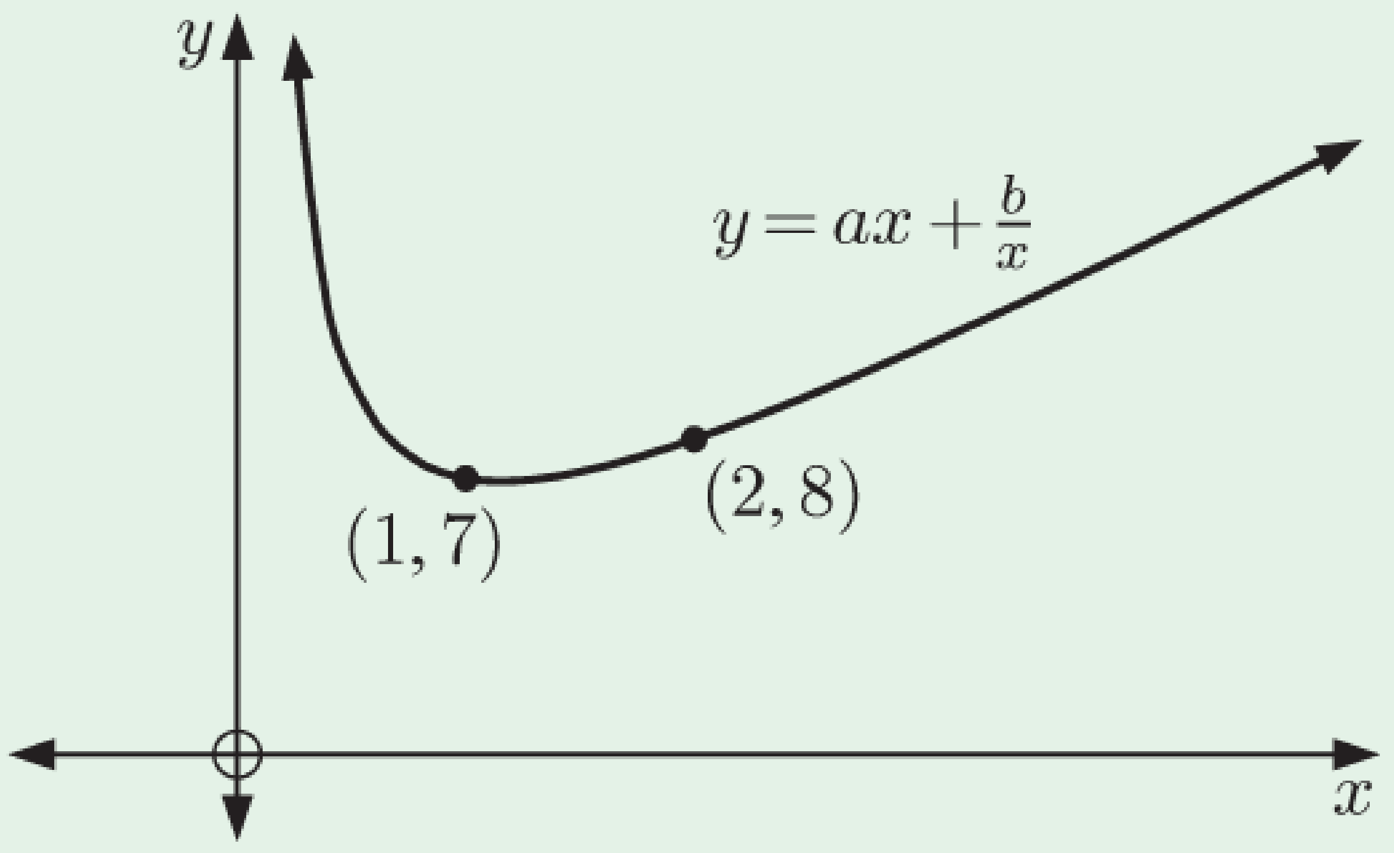
a Draw a distance-time graph to model Alana's position during the triathlon.

b Describe any assumptions you made when constructing your graph.

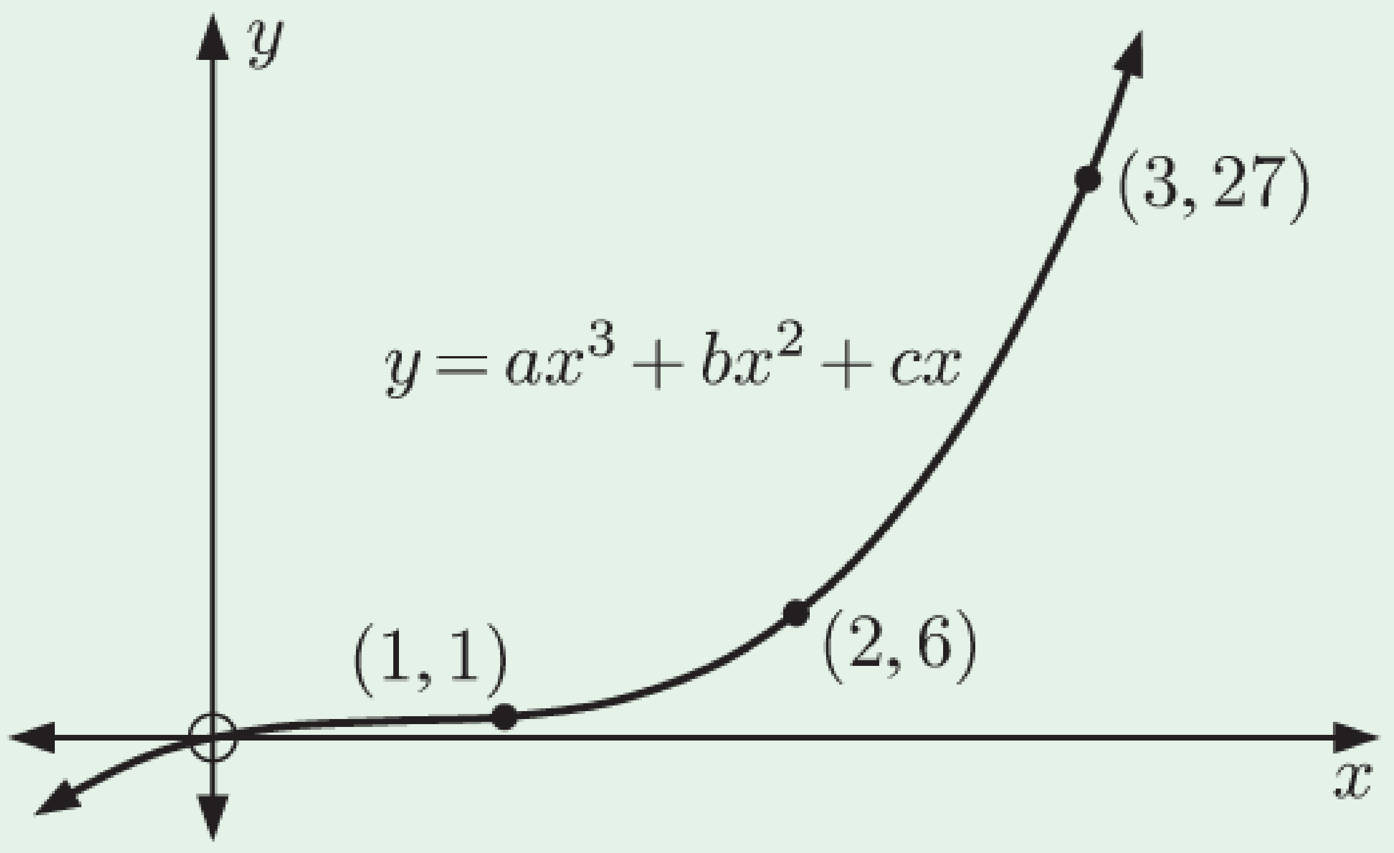
c Use your graph to estimate how far Alana had travelled after 1 hour.

8 Determine the model for each graph:

a



b



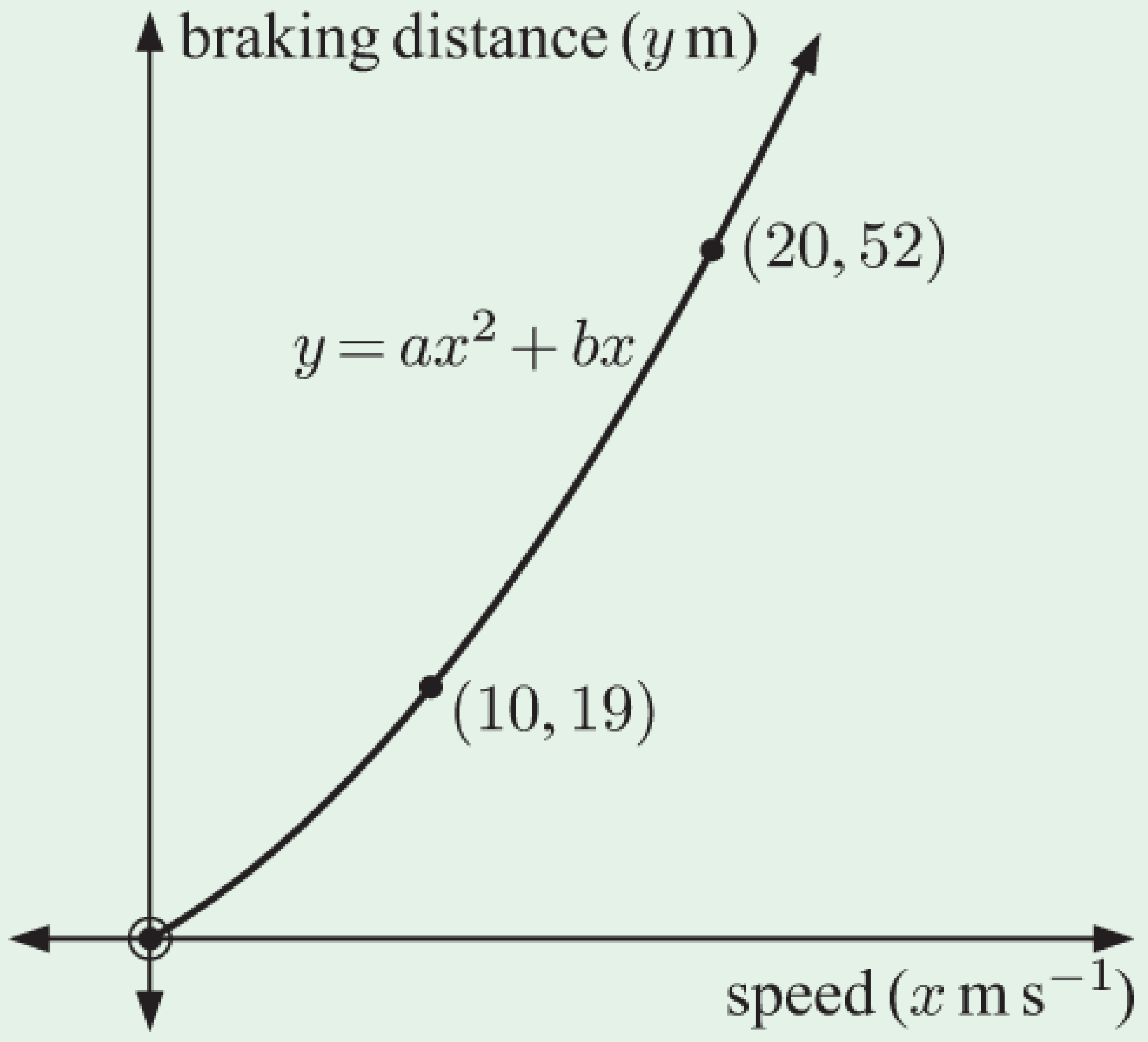
9 This graph shows the relationship between the *speed* of Daniel's car and his *braking distance*, which is the distance required to bring the car to a complete stop in an emergency.

The variables are related by the model $y = ax^2 + bx$.

a Find a and b.

b Use the model to predict Daniel's braking distance when he is travelling at 30 m s^{-1}.

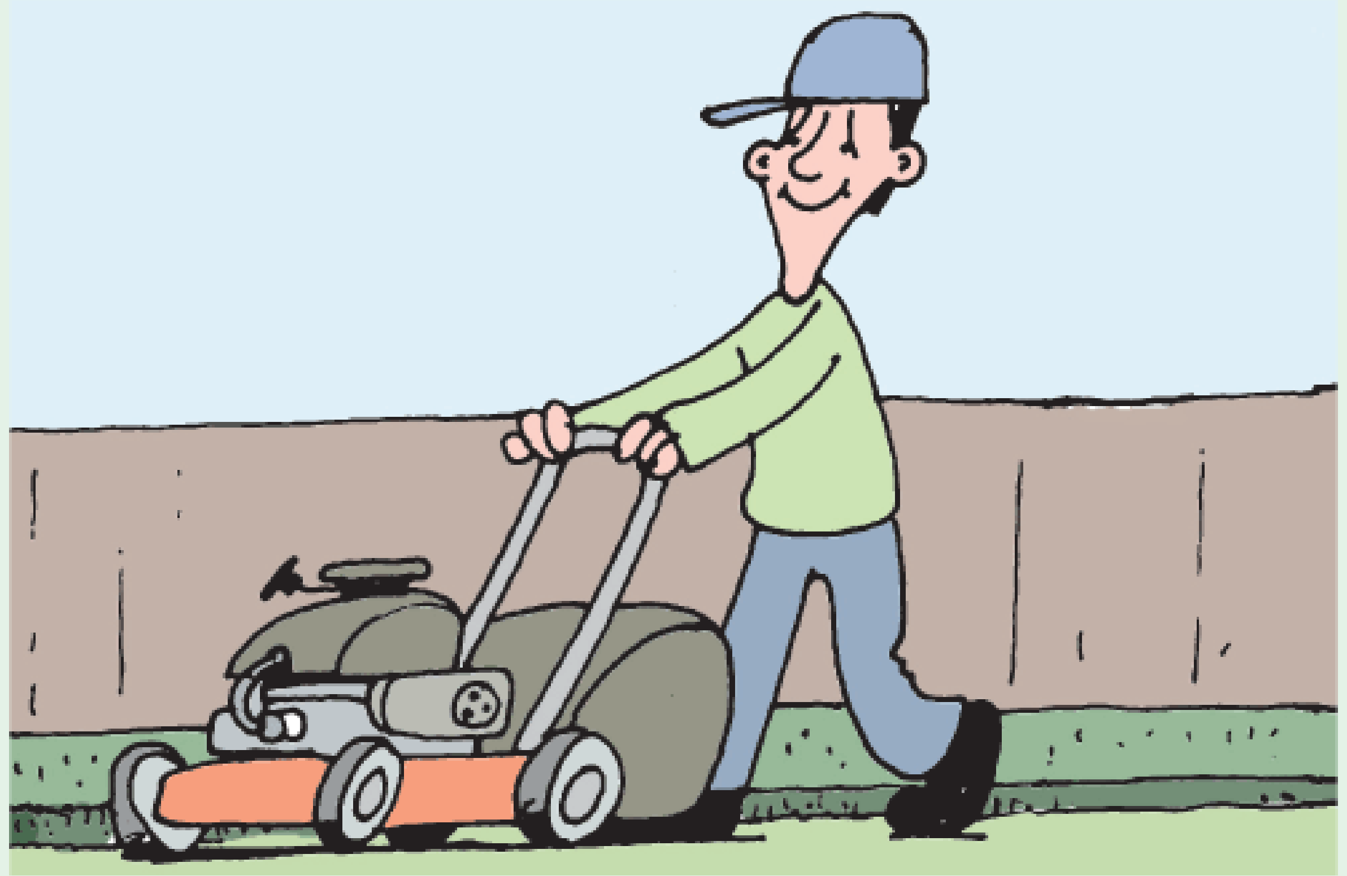
c Can we use this model to predict the braking distance for a different person? Explain your answer.



REVIEW SET 4B

1 Todd takes 3 hours to weed the garden. Sophie takes 3.5 hours to weed the garden. How long would they take to weed the garden working together? State any assumptions you make.

2 Rohan's lawn mower cuts grass to a height of 17 mm. The grass grows 3 mm each day.



a Construct a table of values for the height H of the lawn d days after it is cut, for $d = 0, 1, 2, 3, 4$.

b Draw the graph of H against d.

c Determine the linear model connecting H and d.

d Find the height of the lawn after 12 days.

e Rohan mows the lawn again when it is 8 cm high. How often does he mow the lawn?

3 Elizabeth needs to estimate the volume of usable wood in the trees growing in a plantation. She measures a sample of trees and finds the average trunk width is 45 cm. For her calculations, she assumes that the trees are perfectly cylindrical and that 80% of the wood in each tree is usable.

a Do you think these assumptions are reasonable? Explain your answer.

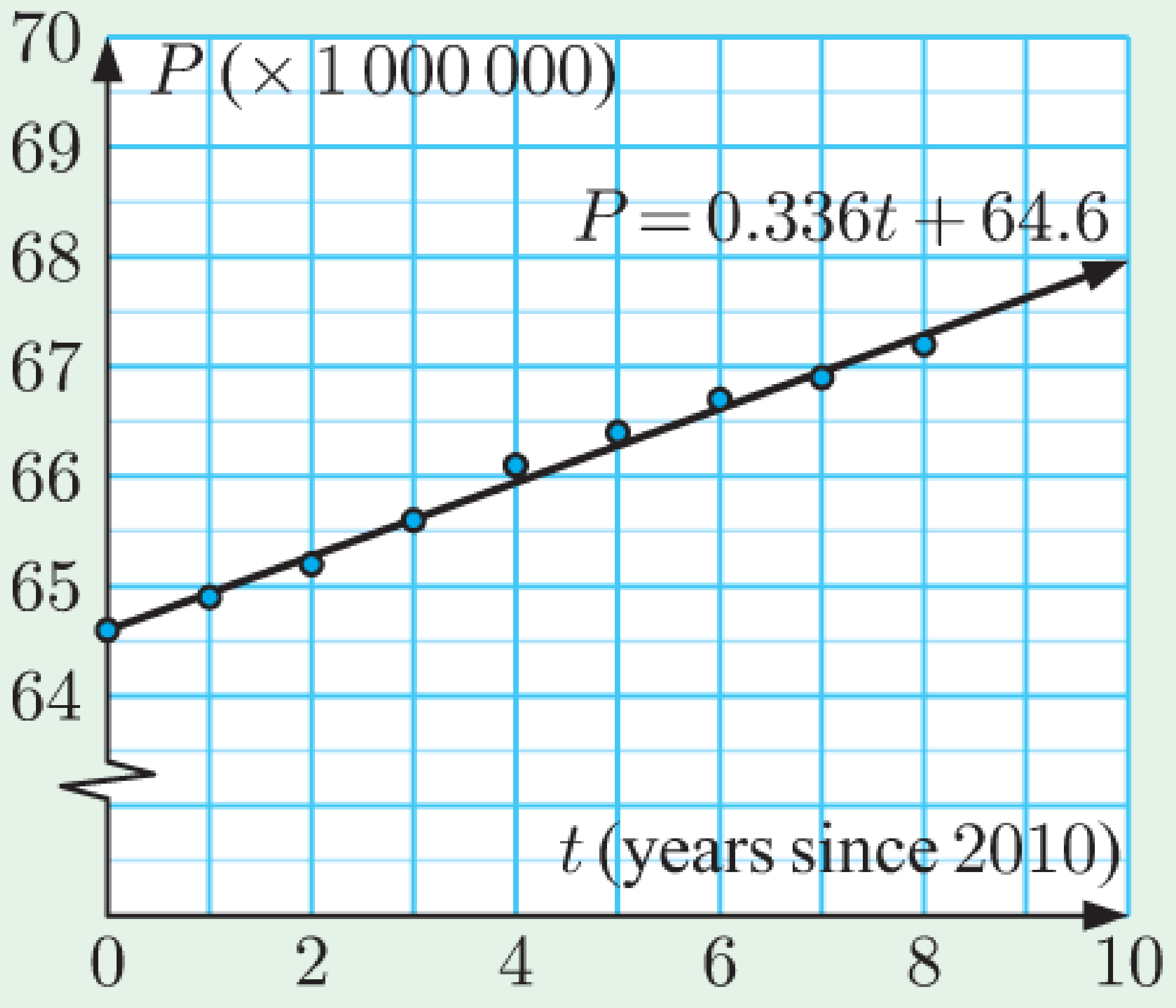
b Based on these assumptions, construct a model for the volume of usable wood in a tree that is h metres high.

c Hence predict the volume of usable wood in a 15 m high tree.

4 The graph alongside shows the population of France, in millions, since the start of 2010. The linear model $P = 0.336t + 64.6$.

a Is the linear model exact or approximate? Explain your answer.

b Use the model to predict when France's population will reach 75 million. Do you think your prediction is reasonable?



5 Shayne is looking for somewhere to stay during his vacation. He is choosing between the following places:

Bob's Beach House: $150 per night for the first 2 nights, $120 for each additional night

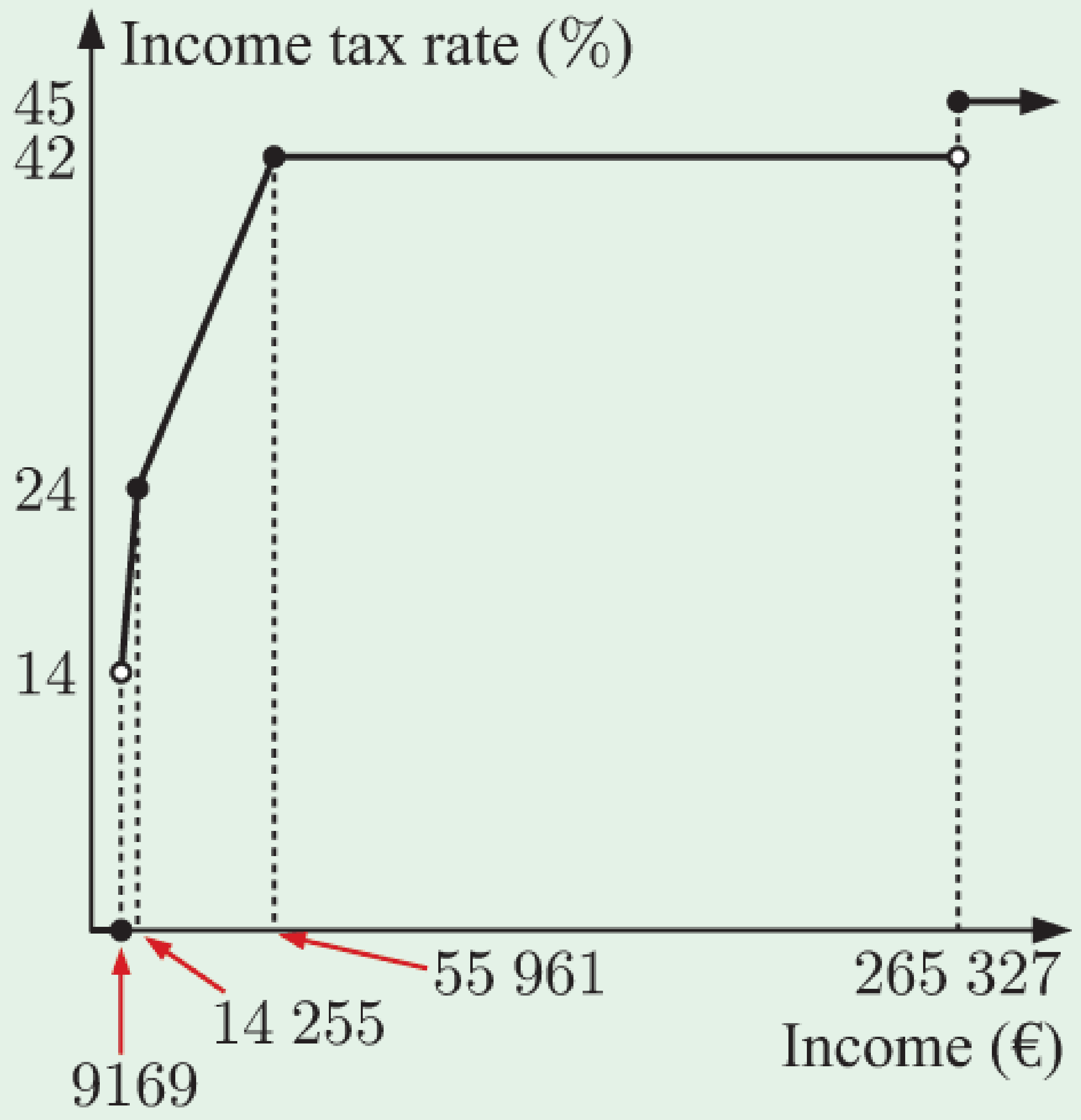
Claire's Cottage: $180 per night for the first 3 nights, $100 for each additional night

A "night" is a 24-hour period starting after 10 am.

- **a** On the same set of axes, graph *cost* against *number of nights* for each place.
- **b** Find the cost of staying for 4 nights at:
  - **i** Bob's Beach House
  - **ii** Claire's Cottage.
- **c** For what length stay is the cost the same at each place?
- **d** Shayne wants to stay for 8 nights. Which place will be cheaper, and by how much?

6 For 2019, the income tax rates for a single person resident in Germany are shown in the graph alongside.

- **a** What type of model is this?
- **b** Estimate the *rate* of income tax payable by Gert-Jan, who has an annual income of €34 500.
- **c** The tax payable on the first €55 961 is €14 729.32. Calculate the income tax payable by Nikolas, who earns €108 609 in 2019.



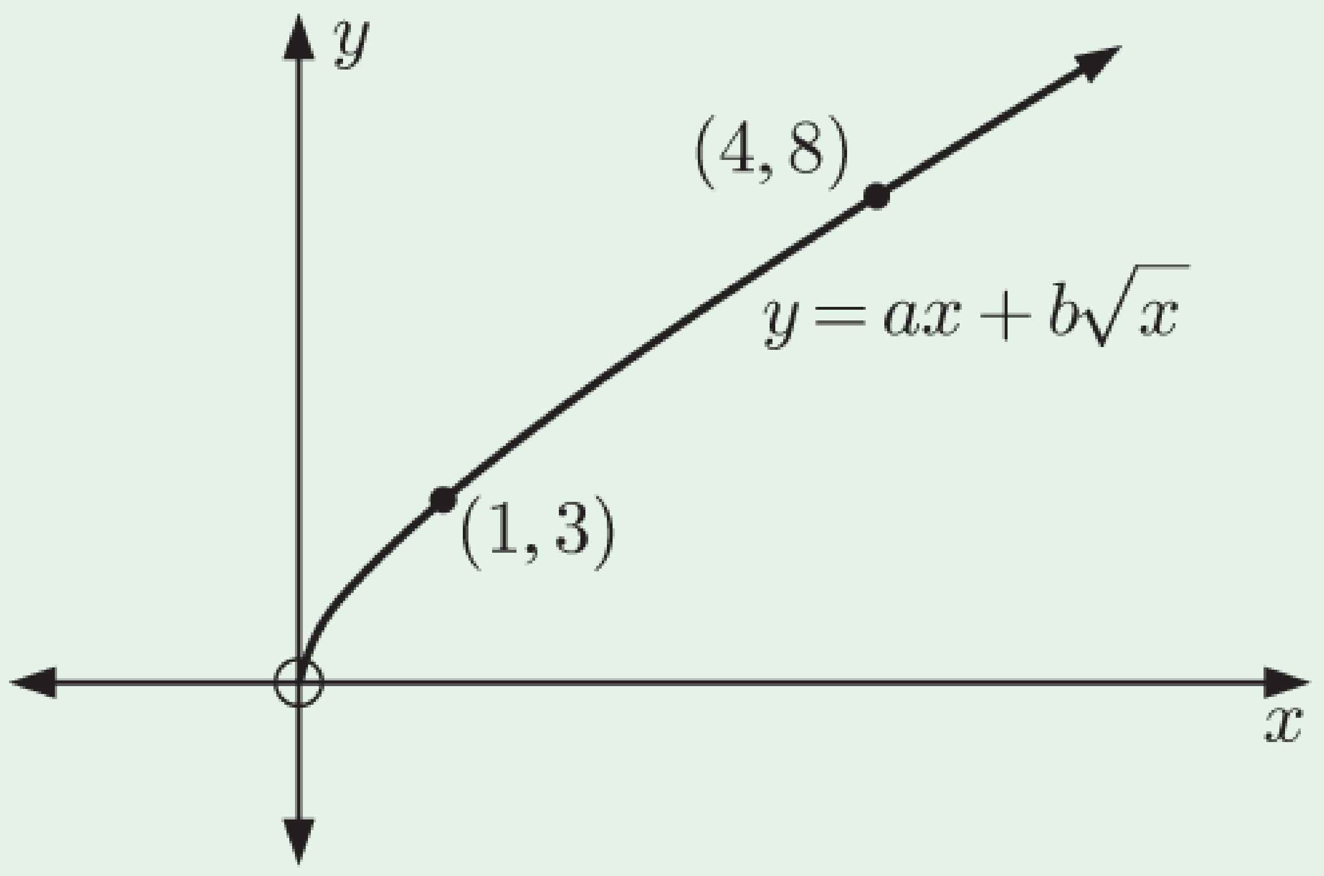
7 In a show jumping competition, a competitor is given one time penalty for every commenced four seconds by which they exceed the time allowed.

For example, if the time allowed is 75 seconds, and the competitor takes 79.4 seconds, they are 4.4 seconds over the time allowed, and so are given 2 time penalties.

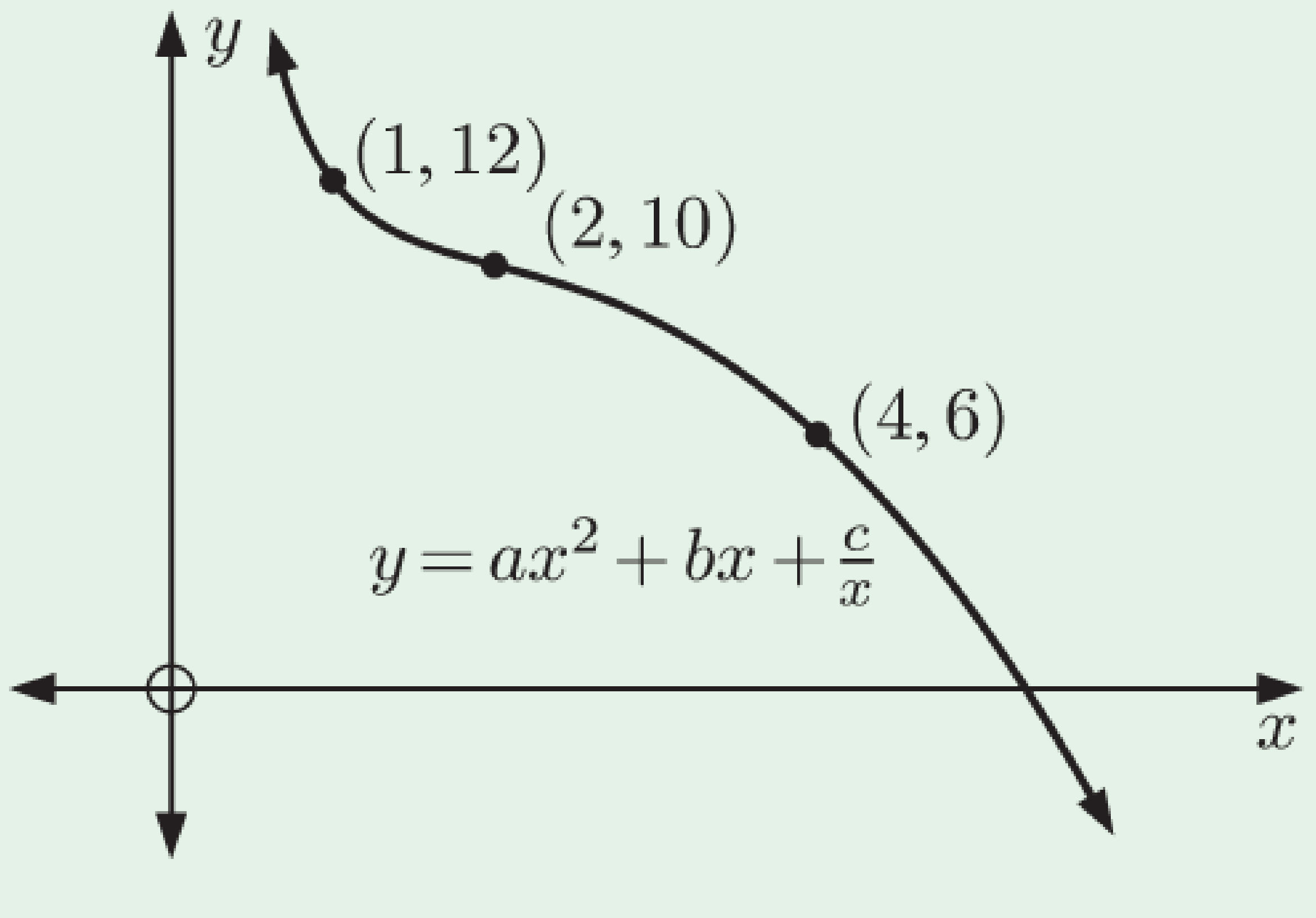
- **a** Draw a graph to display the time penalties for different times over the time allowed.
- **b** If the time allowed is 82 seconds, find the time penalties given to a rider who takes:
  - **i** 83.1 seconds
  - **ii** 87.9 seconds
  - **iii** 81.5 seconds
  - **iv** 96.3 seconds

8 Determine the model for each graph:

a

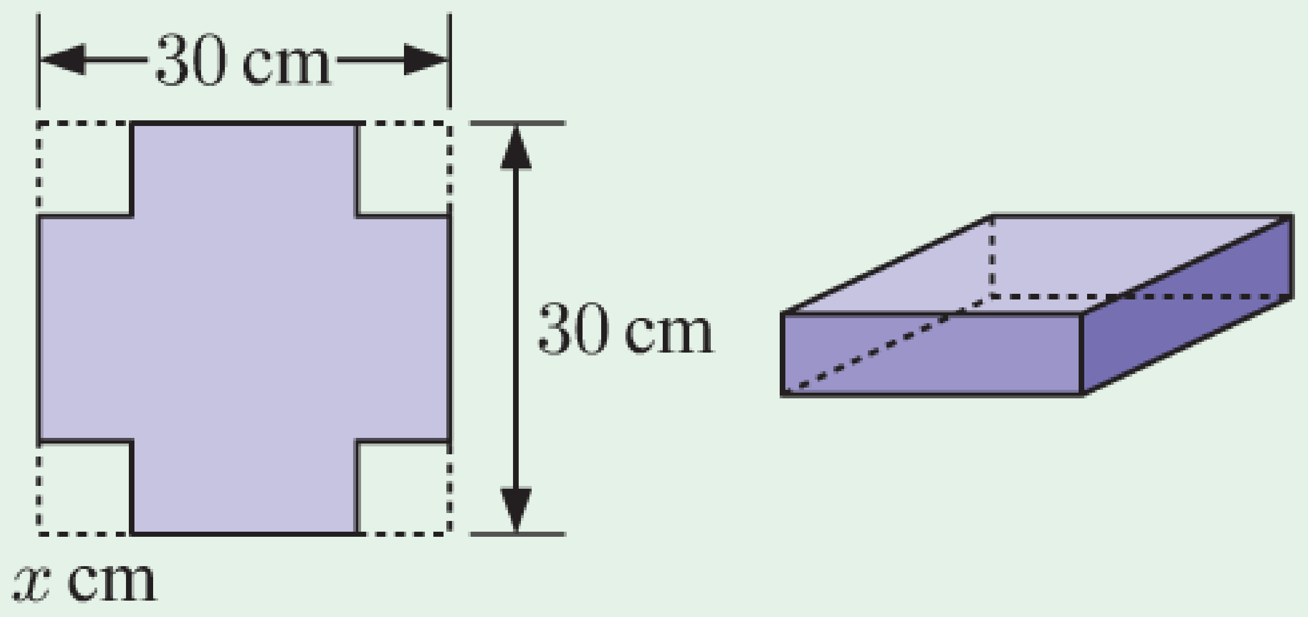


b



9 A tray is formed by cutting squares from each corner of a 30 cm $\times$ 30 cm sheet of metal, then folding the remainder.

If the squares cut out are x cm $\times$ x cm, the volume of the container can be written in the form $V = ax^3 + bx^2 + cx + d$ cm^3.



- **a** Explain why $d = 0$.
- **b** Use the data points alongside to find the values of a, b, and c.

x (cm)	2	5	10
V (cm^3)	1352	2000	1000

- **c** Use the formula for the volume of a rectangular prism to explain why a model of this form is reasonable.
- **d** For what values of x can this model be correctly applied?

Chapter 5

Bivariate statistics

Contents:

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C Line of best fit by eye
D The least squares regression line
E Spearman's rank correlation coefficient

OPENING PROBLEM

At a junior tournament, some young athletes each throw a discus. The *age* and *distance thrown* are recorded for each athlete.

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

Things to think about:

- **a** Do you think the distance an athlete can throw is related to the person's age?
- **b** What happens to the distance thrown as the age of the athlete increases?
- **c** How could you graph the data to more clearly see the relationship between the variables?
- **d** How can we *measure* the relationship between the variables?

In the previous Chapter we saw how the relationship between two variables can be described using a function. In particular, we considered linear relationships between variables, and discussed how they could be *exact* or *approximations*.

In this Chapter we consider **bivariate data**, which means data which has two variables recorded for each individual. In most real-world situations, there will not be an exact relationship between these variables. Our task is to find the model which *best fits* the data, and measure how strong the relationship between the variables is.

For example, each athlete in the **Opening Problem** has had the *two* variables *age* and *distance thrown* recorded about them. We expect the *distance thrown* will *depend* on the athlete's *age*, so *age* is the independent variable and *distance thrown* is the dependent variable.

The **independent** and **dependent** variables are sometimes called the **explanatory** and **response** variables respectively.

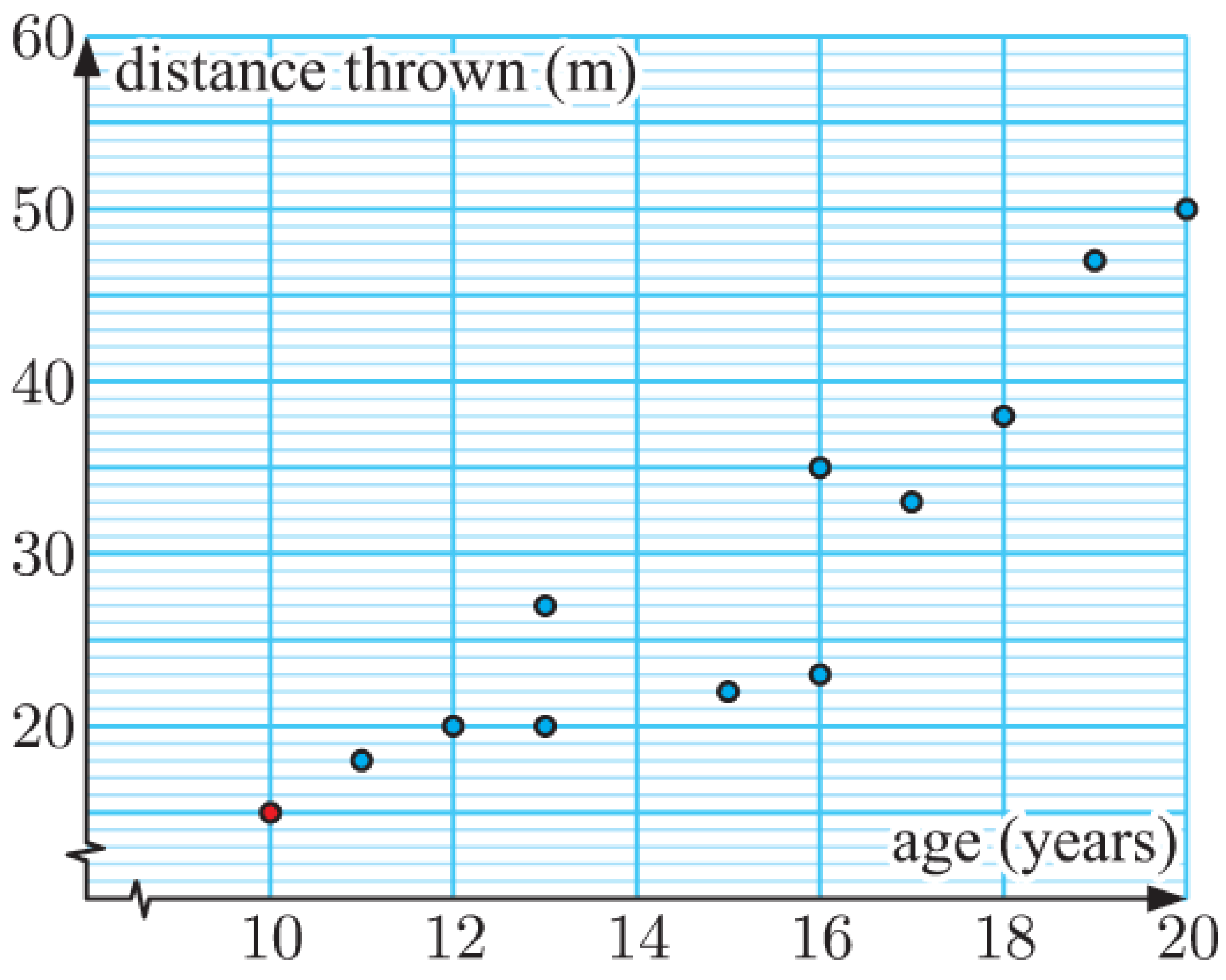
A ASSOCIATION BETWEEN NUMERICAL VARIABLES

We can observe the relationship between two numerical variables using a **scatter diagram**. We usually place the independent variable on the horizontal axis, and the dependent variable on the vertical axis.

In the **Opening Problem**, the independent variable *age* is placed on the horizontal axis, and the dependent variable *distance thrown* is placed on the vertical axis.

We then graph each data value as a point on the scatter diagram. For example, the red point represents athlete H, who is 10 years old and threw the discus 15 metres.

From the general shape formed by the dots, we can see that as the *age* increases, so does the *distance thrown*.

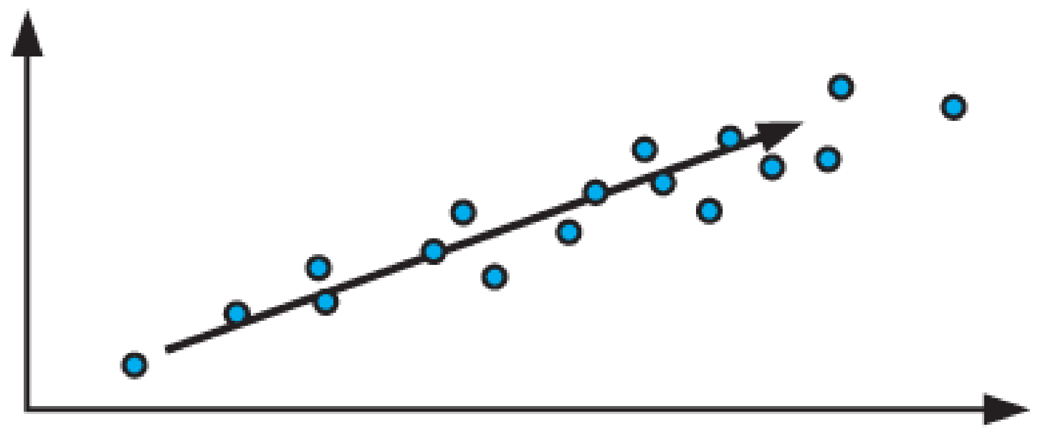


CORRELATION

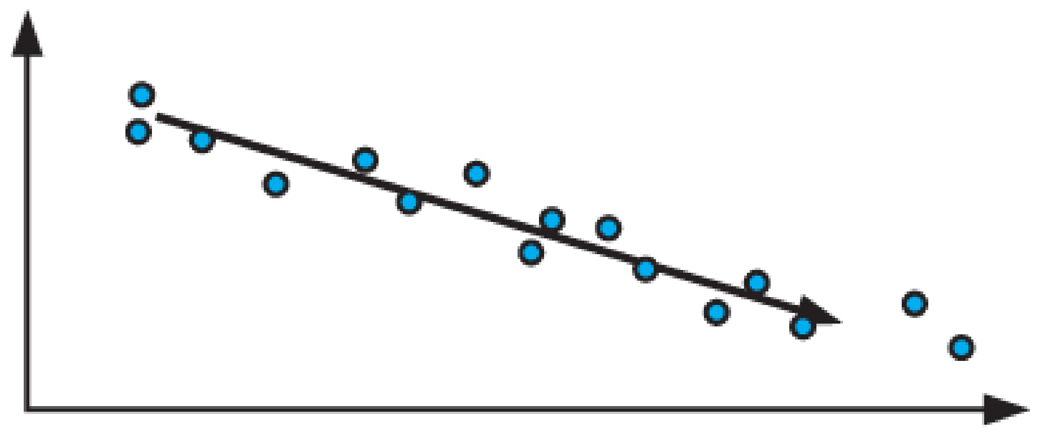
Correlation refers to the relationship or association between two numerical variables.

There are several characteristics we consider when describing the correlation between two variables: direction, linearity, strength, outliers, and causation.

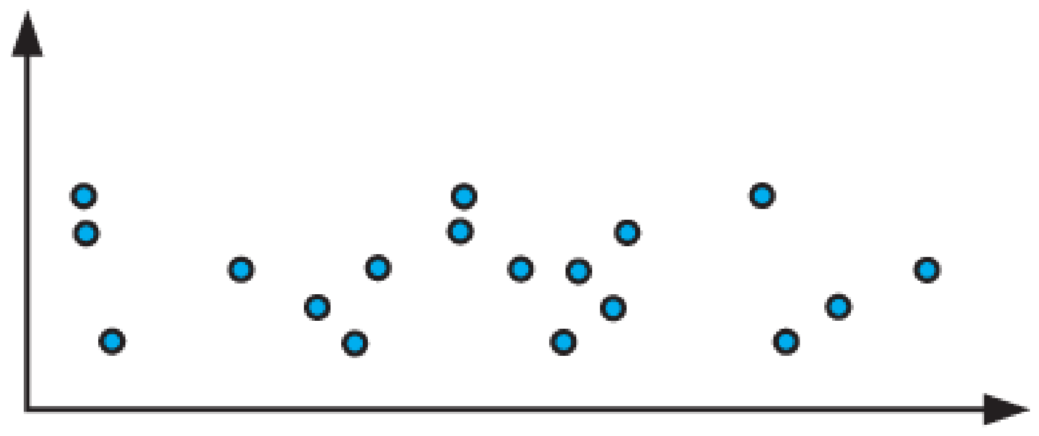
DIRECTION



For a generally *upward* trend, we say that the correlation is **positive**. An increase in the independent variable generally results in an increase in the dependent variable.



For a generally *downward* trend, we say that the correlation is **negative**. An increase in the independent variable generally results in a decrease in the dependent variable.

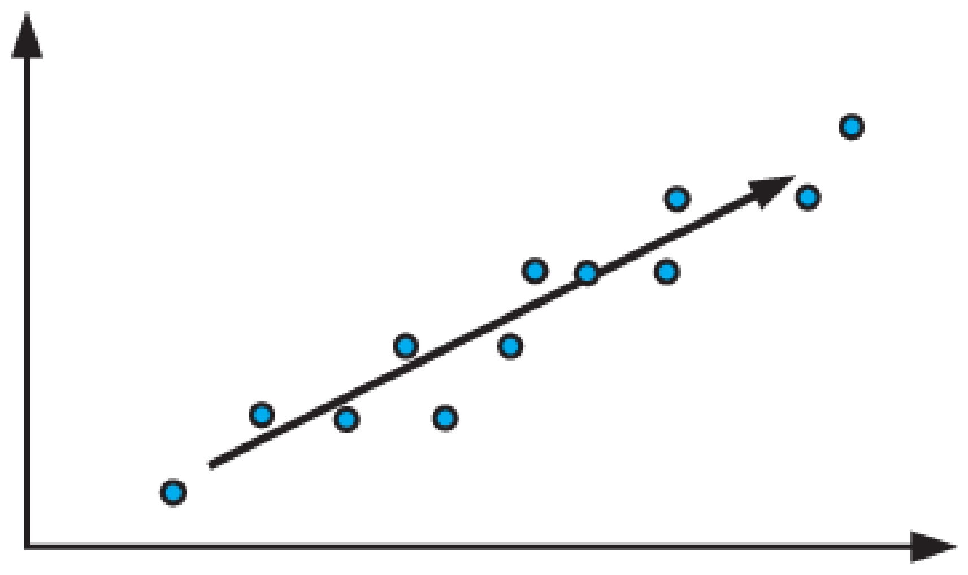


For *randomly scattered* points, with no upward or downward trend, we say there is **no correlation**.

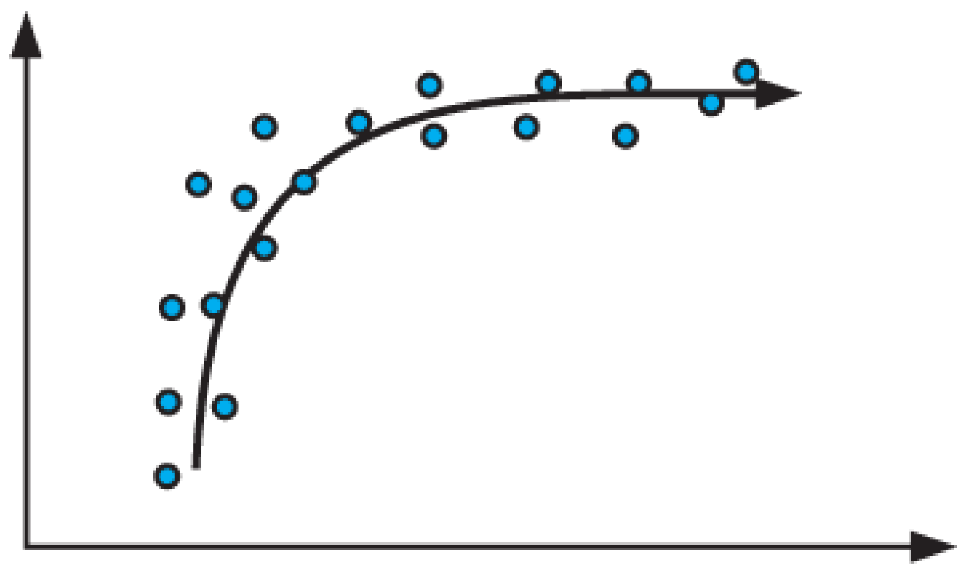
LINEARITY

When a trend exists, if the points approximately form a straight line, we say the trend is **linear**.

These points are roughly linear.



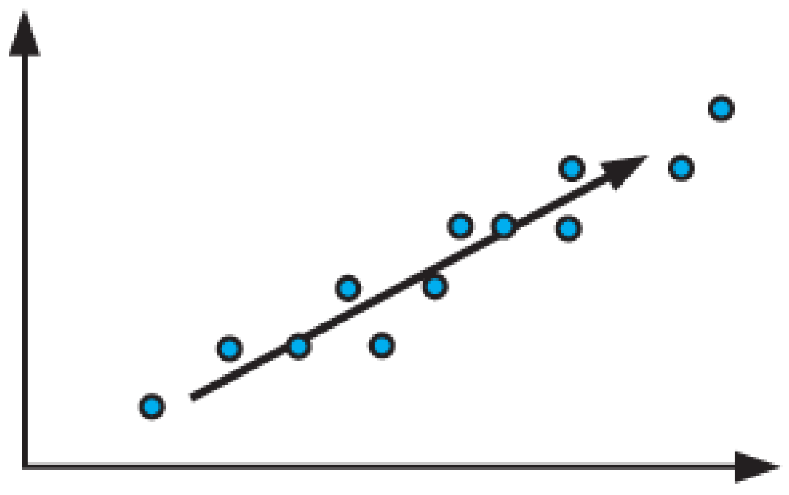
These points do not follow a linear trend.



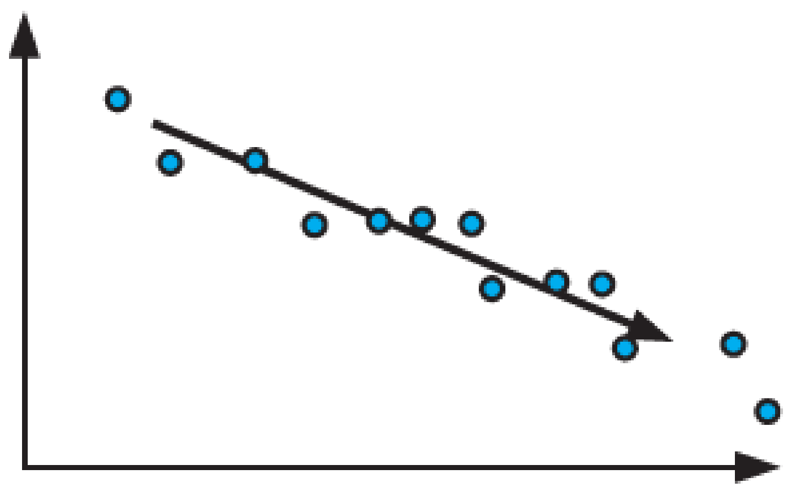
STRENGTH

To describe how closely the data follows a pattern or trend, we talk about the **strength** of correlation. It is usually described as either **strong**, **moderate**, or **weak**.

strong

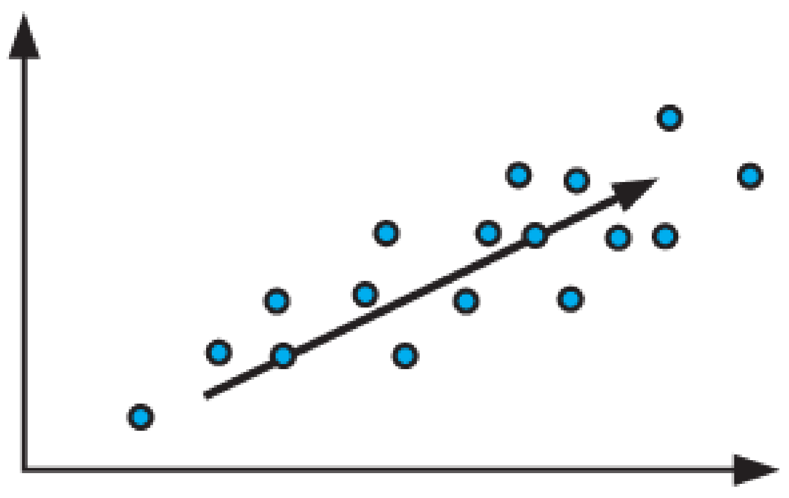


strong positive

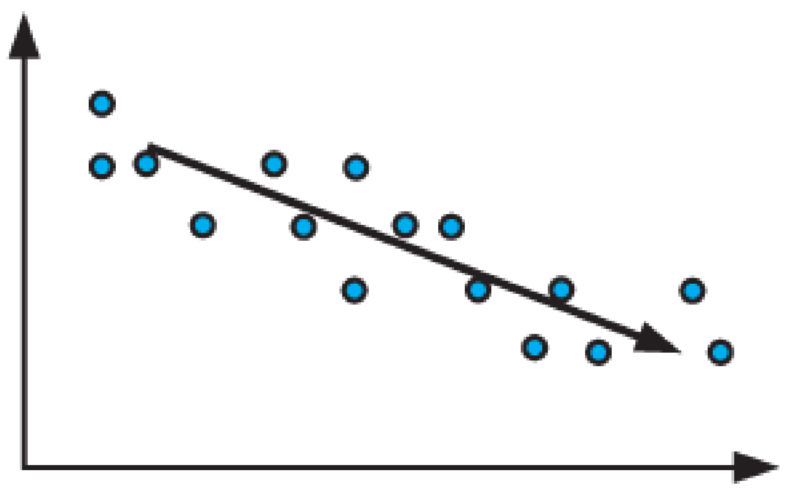


strong negative

moderate

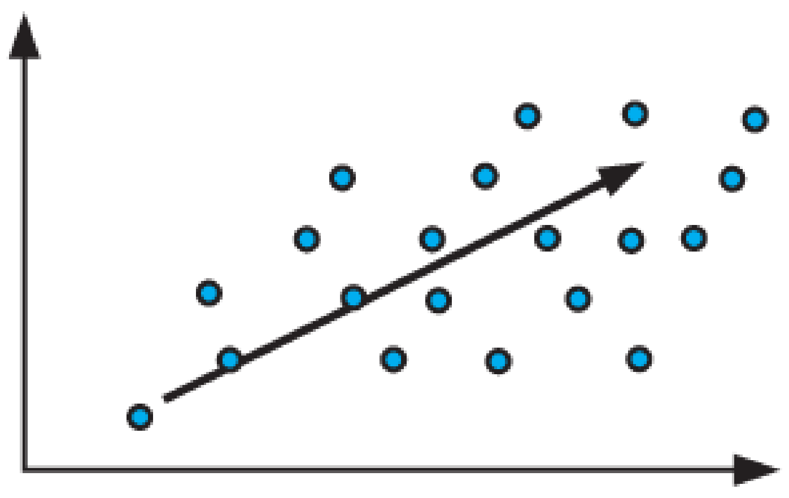


moderate positive

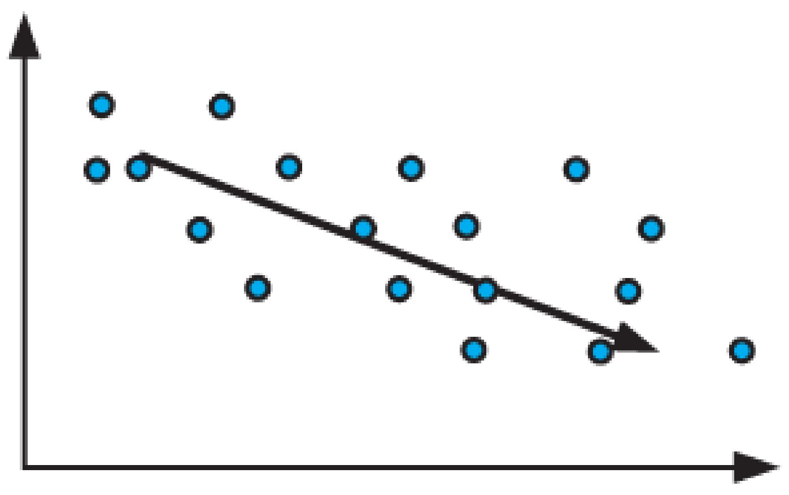


moderate negative

weak



weak positive

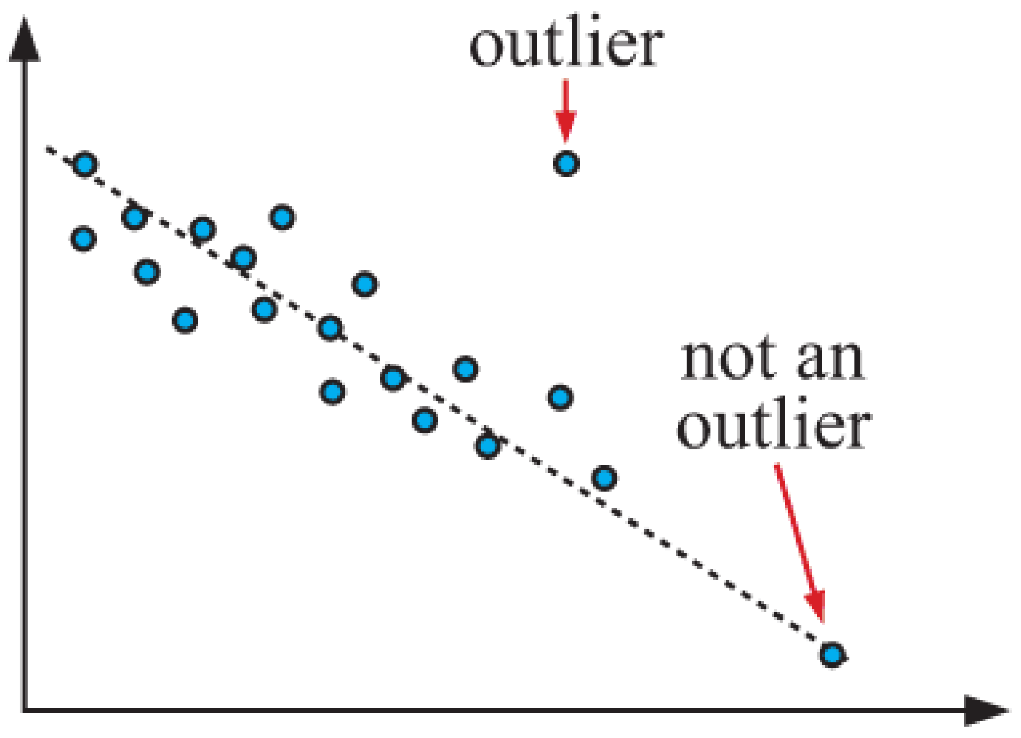


weak negative

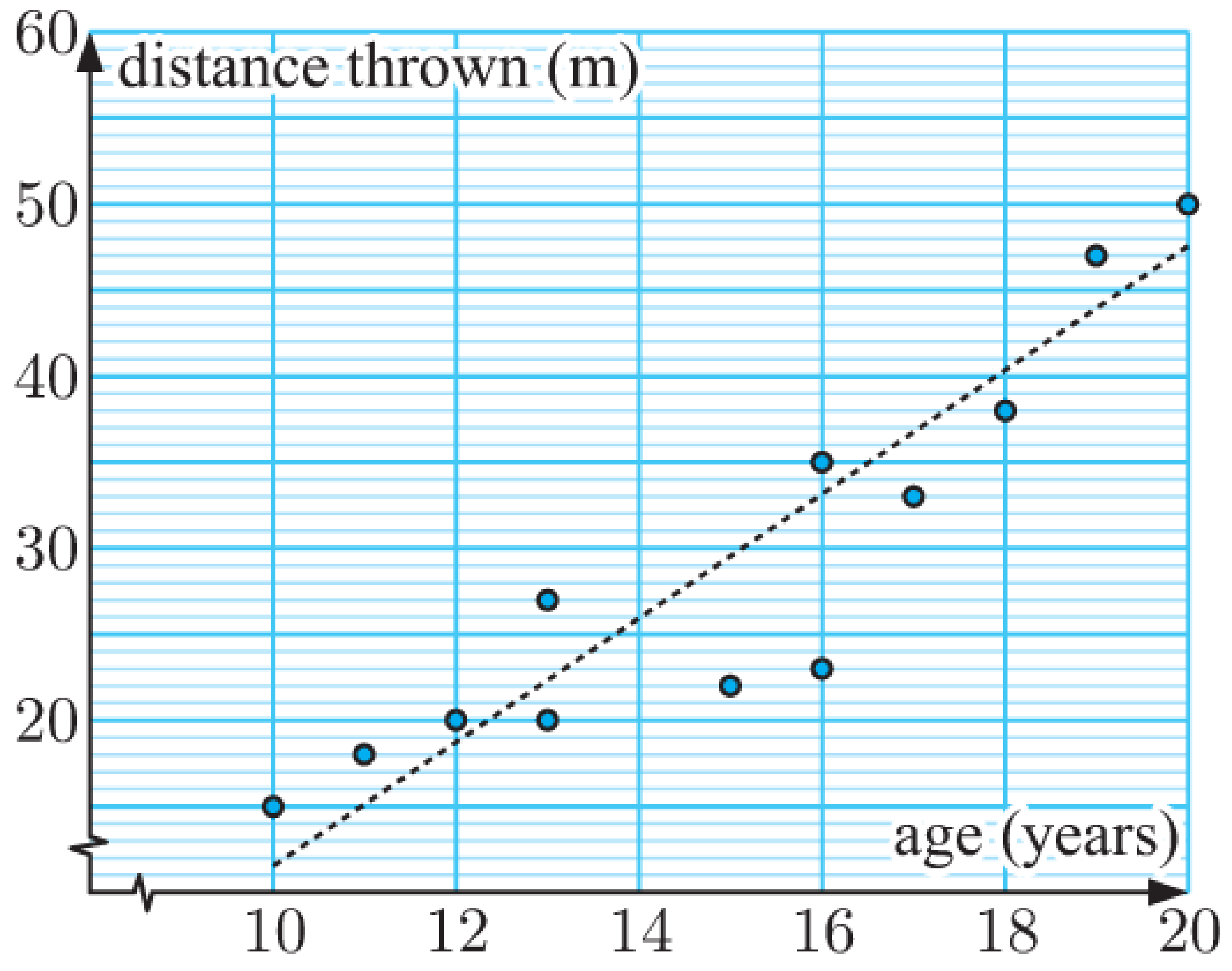
OUTLIERS

Outliers are isolated points which do not follow the trend formed by the main body of data.

If an outlier is the result of a recording or graphing error, it should be discarded. However, if the outlier is a genuine piece of data, it should be kept.



For the scatter diagram of the data in the **Opening Problem**, we can say that there is a strong positive correlation between *age* and *distance thrown*. The relationship appears to be linear, with no outliers.



CAUSALITY

Correlation between two variables does not necessarily mean that one variable *causes* the other.

For example:

- The *arm length* and *running speed* of a sample of young children were measured, and a strong, positive correlation was found between the variables.

  This does *not* mean that short arms cause a reduction in running speed, or that a high running speed causes your arms to grow long.

  Rather, there is a strong, positive correlation between the variables because both *arm length* and *running speed* are closely related to a third variable, *age*. Up to a certain age, both *arm length* and *running speed* increase with *age*.

- The number of television sets sold in London and the number of stray dogs collected in Boston were recorded over several years. A strong, positive correlation was found between the variables.

  Obviously the number of television sets sold in London was not influencing the number of stray dogs collected in Boston. It is coincidental that the variables both increased over this period of time.

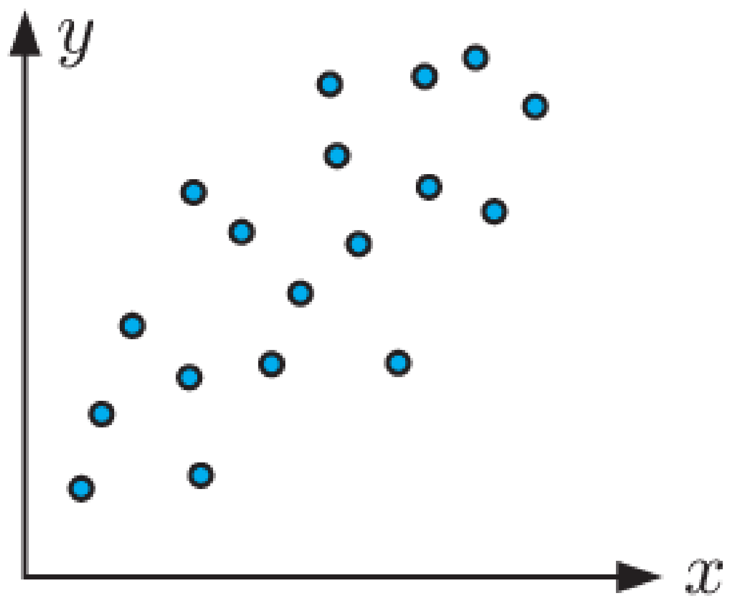
If a change in one variable *causes* a change in the other variable then we say that a **causal relationship** exists between them. In these cases, we can say that the independent variable *explains* the dependent variable. It may be more natural to use the terminology **explanatory variable** and **response variable**.

In cases where a causal relationship is not apparent, we cannot conclude that a causal relationship exists based on high correlation alone.

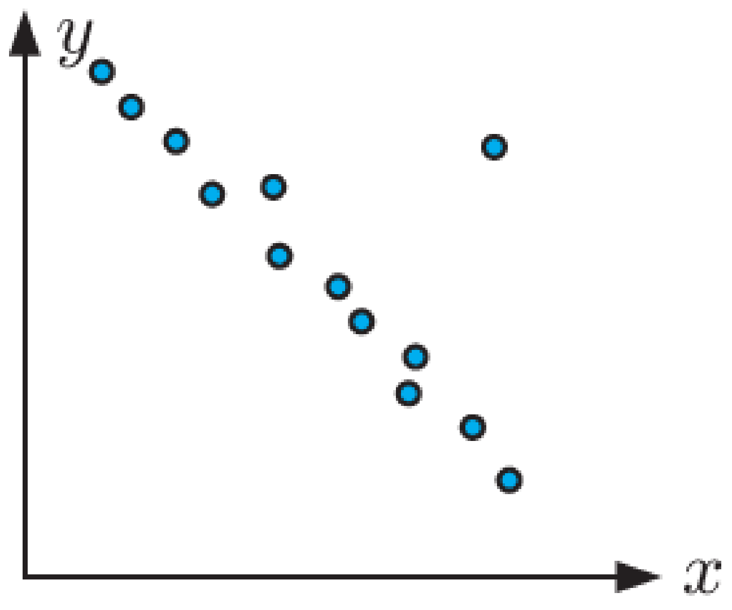
EXERCISE 5A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.

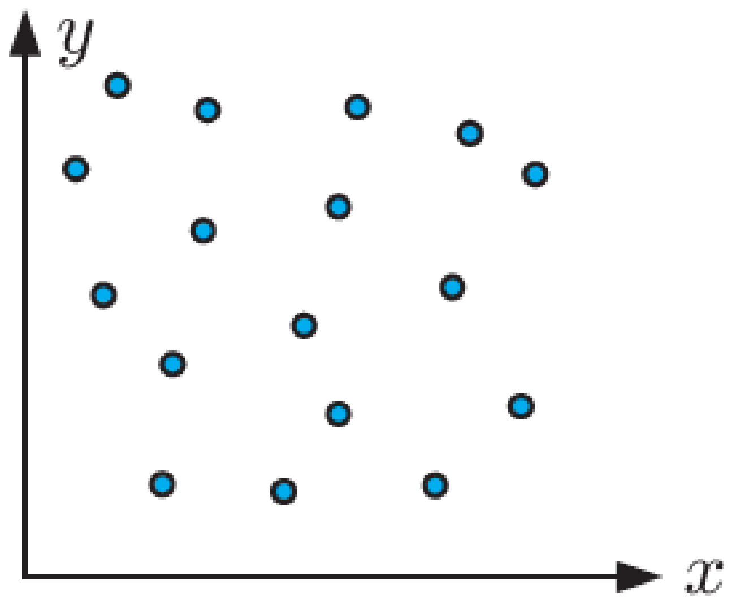
a



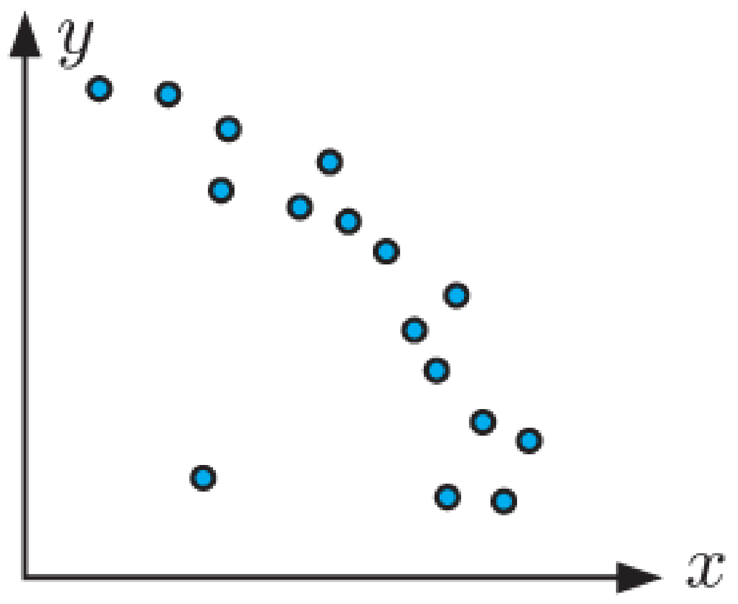
b



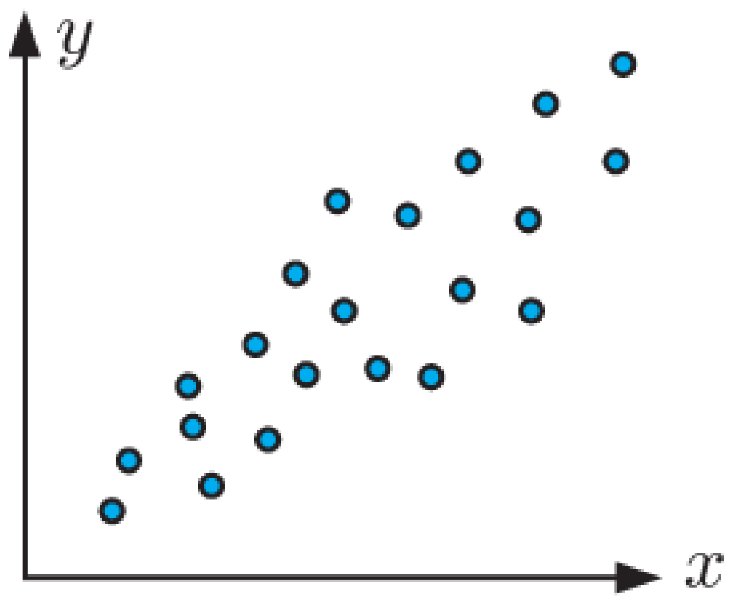
c



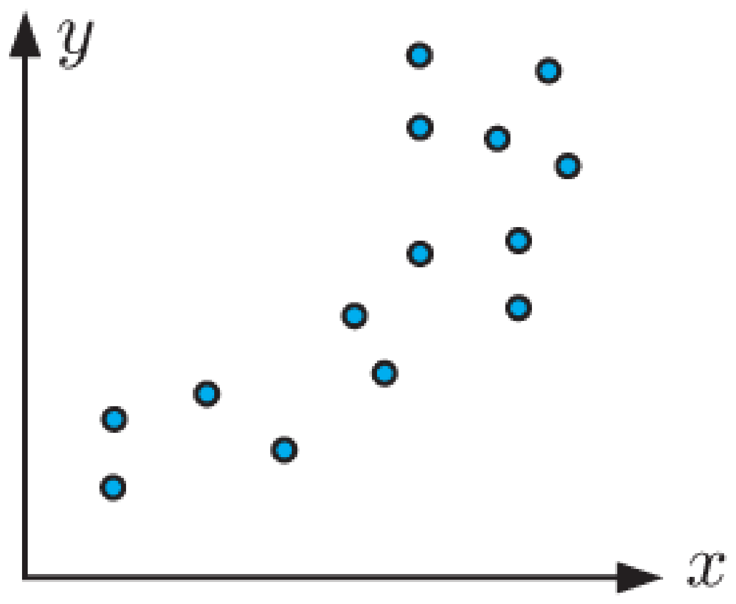
d



e



f



2 Tiffany is a hairdresser. The table below shows the number of hours she worked each day last week, and the number of customers she had.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

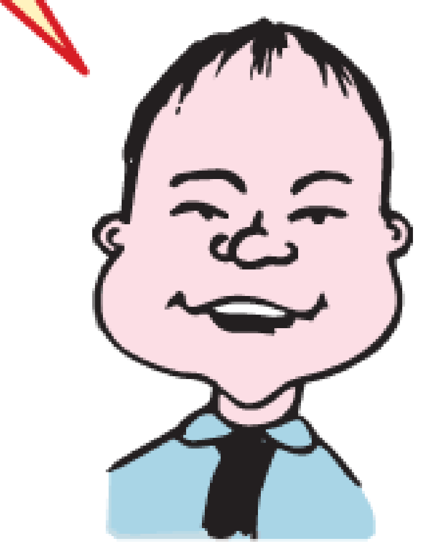
a Which is the explanatory variable, and which is the response variable?

b Draw a scatter diagram of the data.

c On which two days did Tiffany:
  i work the same number of hours
  ii have the same number of customers?

d Explain why you would expect a positive correlation between the variables.

You can use technology to help draw scatter diagrams.



GRAPHICS CALCULATOR INSTRUCTIONS

3 The scores awarded by two judges at an ice skating competition are shown in the table.

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

a Construct a scatter diagram for the data, with Judge A's scores on the horizontal axis and Judge B's scores on the vertical axis.

b Copy and complete the following comments about the scatter diagram:
There appears to be ......, ..., ... correlation between Judge A's scores and Judge B's scores.
This means that as Judge A's scores increase, Judge B's scores ......

c Would it be reasonable to conclude that an increase in Judge A's scores *causes* an increase in Judge B's scores? Explain your answer.

4 Paul owns a company which installs industrial air conditioners. The table below shows the number of workers at the company's last 10 jobs, and the time it took to complete the job.

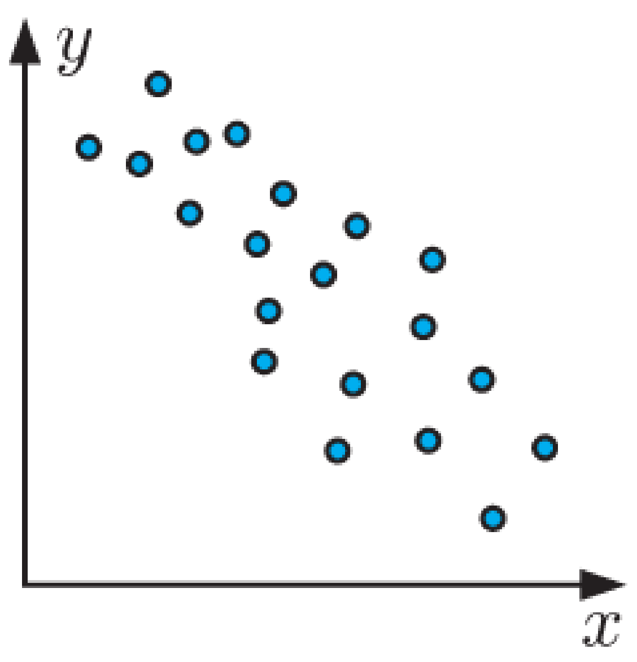
Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

- **a** Which job: **i** took the longest **ii** involved the most workers?
- **b** Draw a scatter diagram to display the data.
- **c** Describe the relationship between the variables *number of workers* and *time*.

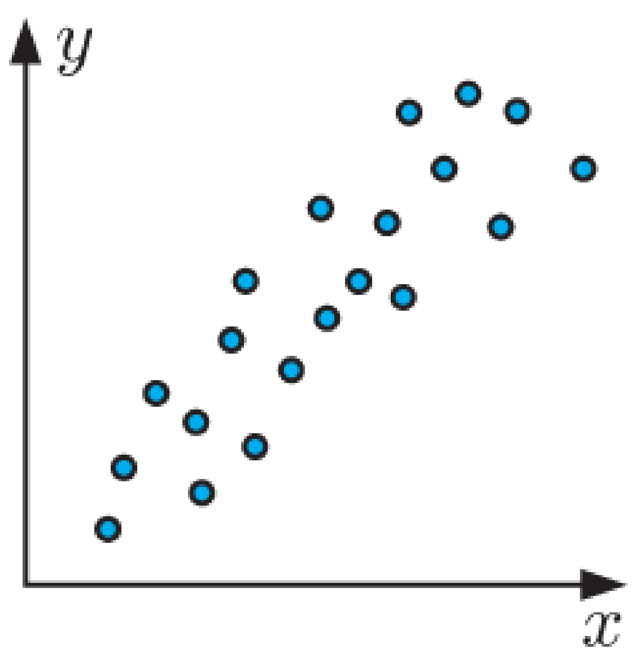
5 Choose the scatter diagram which would best illustrate the relationship between the variables x and y.

- **a** $x =$ the number of apples bought by customers, $y =$ the total cost of apples bought
- **b** $x =$ the number of pushups a student can perform in one minute,
  $y =$ the time taken for the student to run 100 metres
- **c** $x =$ the height of a person, $y =$ the weight of the person
- **d** $x =$ the distance a student travels to school, $y =$ the height of the student's uncle

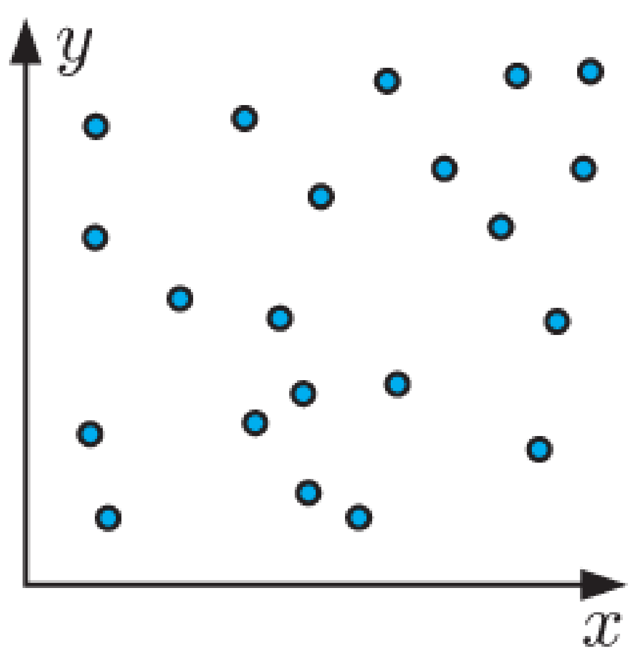
A



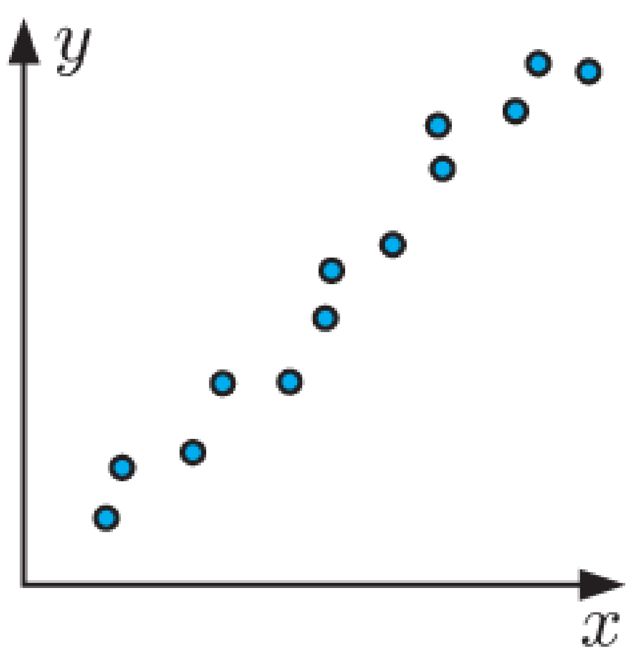
B



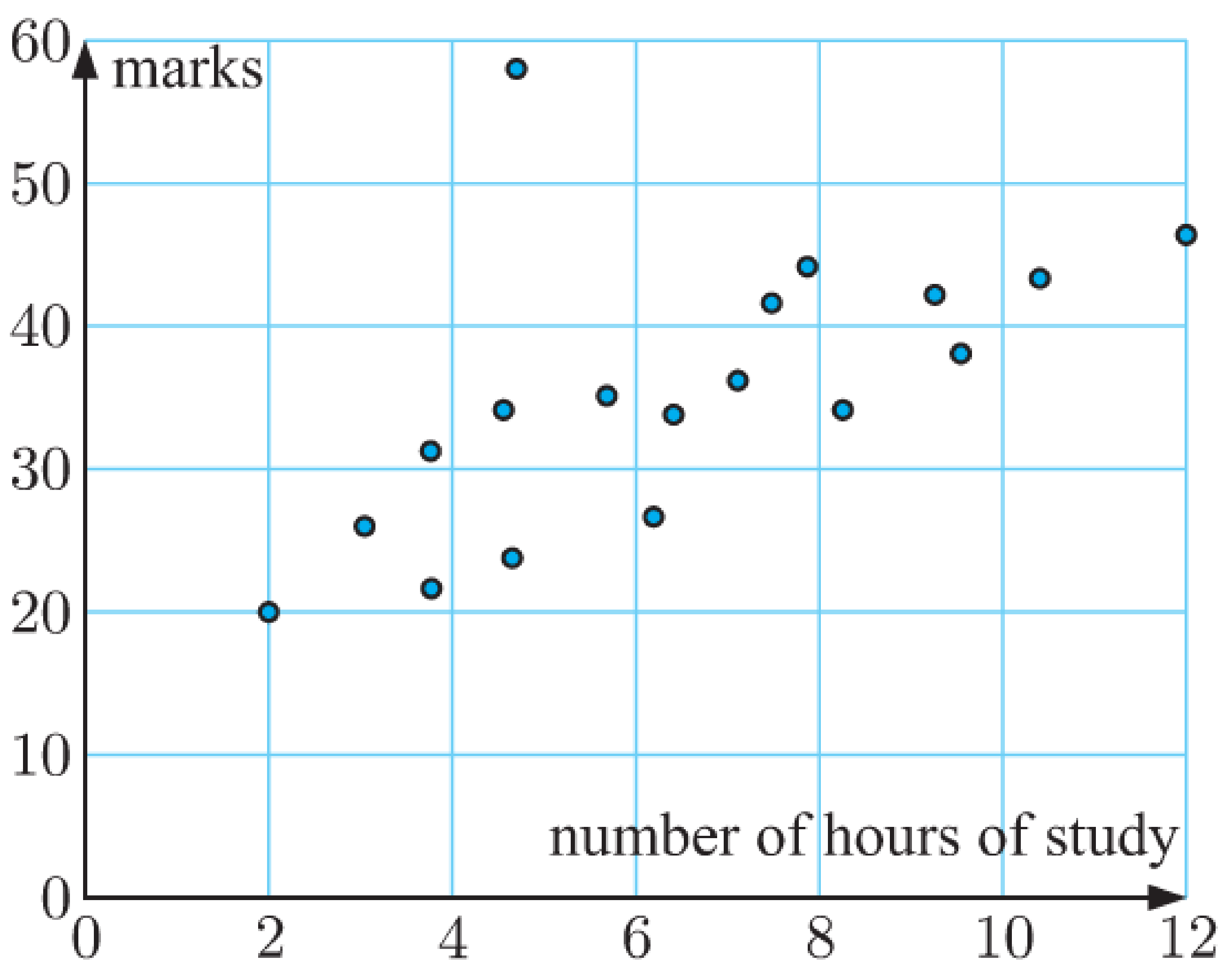
C



D



6 The scatter diagram shows the marks obtained by students in a test out of 50 marks, plotted against the number of hours each student studied for the test.



- **a** Describe the correlation between the variables.
- **b** How should the outlier be treated? Explain your answer.
- **c** Do you think there is a causal relationship between the variables? Explain your answer.

7 When the following pairs of variables were measured, a strong, positive correlation was found between each pair. Discuss whether a causal relationship exists between the variables. If not, suggest a third variable to which they may both be related.

- **a** The lengths of one's left and right feet.
- **b** The damage caused by a fire and the number of firefighters who attend it.
- **c** A company's expenditure on advertising, and the sales they make the following year.
- **d** The heights of parents and the heights of their adult children.
- **e** The numbers of hotels and numbers of service stations in rural towns.

B PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

In the previous Section, we classified the strength of the correlation between two variables as either strong, moderate, or weak. We observed the points on a scatter diagram, and judged how clearly the points formed a linear relationship.

Since this method is *subjective* and relies on the observer's opinion, it is important to get a more precise measure of the strength of linear correlation between the variables. We achieve this using **Pearson's product-moment correlation coefficient** r.

For a set of n data given as ordered pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n),

Pearson's product-moment correlation coefficient is
$$\boldsymbol{r = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2\sum(y-\overline{y})^2}}}$$

where $\overline{x}$ and $\overline{y}$ are the means of the x and y data respectively, and $\sum$ means the sum over all the data values.

You are not required to learn this formula, but you should be able to calculate the value of r using technology.

GRAPHICS CALCULATOR INSTRUCTIONS

HISTORICAL NOTE

Karl Pearson (1857 - 1936) was an English statistician who developed the product-moment correlation coefficient together with his academic advisor **Sir Francis Galton**.

Pearson made many other contributions to statistics including the use of histograms in exploratory data analysis, parameter estimation, and hypothesis testing.

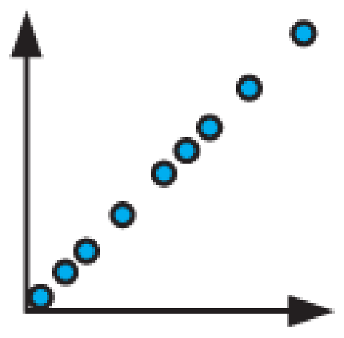
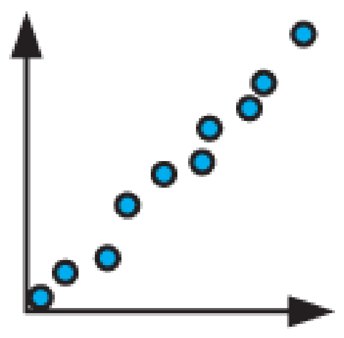
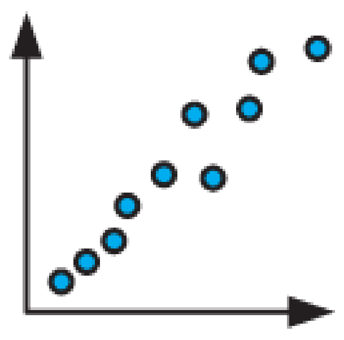
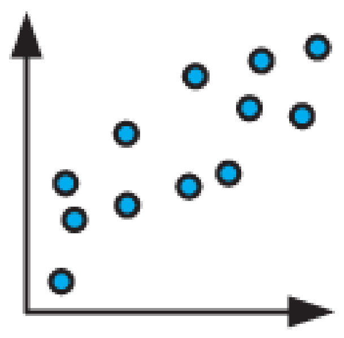
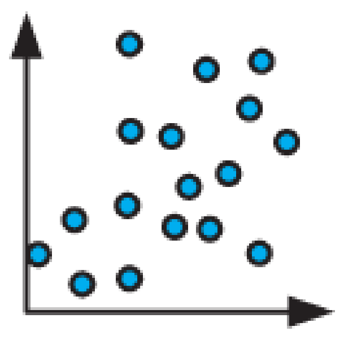
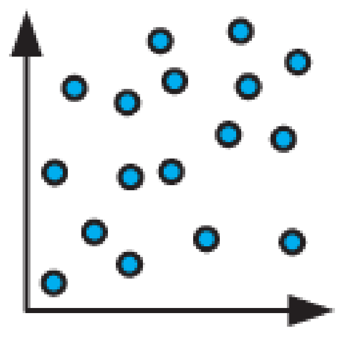
He is considered a key figure in the development of mathematical statistics.

PROPERTIES OF PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

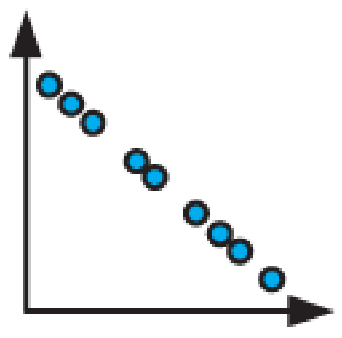
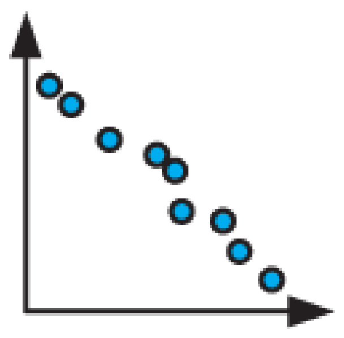
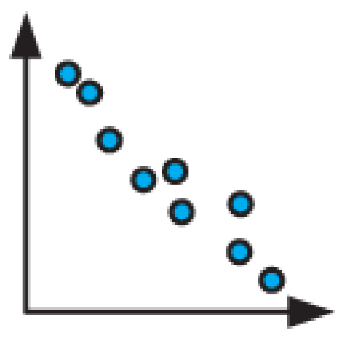
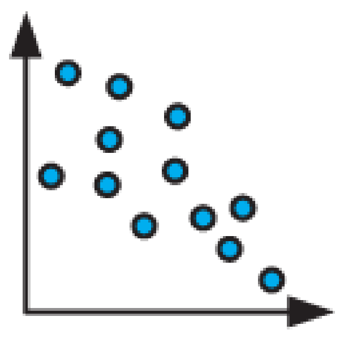
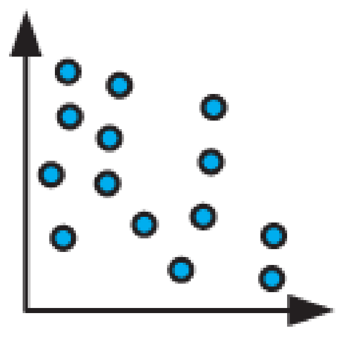
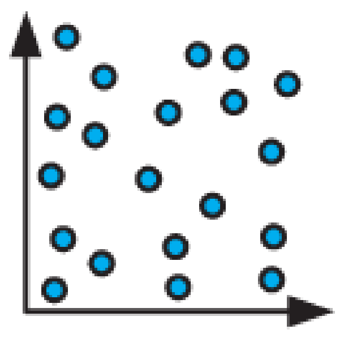
- The values of r range from -1 to $+1$.
- The **sign** of r indicates the **direction** of the correlation.
  - A positive value for r indicates the variables are **positively correlated**. An increase in one variable results in an increase in the other.
  - A negative value for r indicates the variables are **negatively correlated**. An increase in one variable results in a decrease in the other.
  - If $r = 0$ then there is **no correlation** between the variables.
- The **size** of r indicates the **strength** of the correlation.
  - A value of r close to $+1$ or -1 indicates strong correlation between the variables.
  - A value of r close to zero indicates weak correlation between the variables.

The following table is a guide for describing the strength of linear correlation using r.

Positive correlation

$r = 1$	perfect positive correlation	
$0.95 \leqslant r < 1$	very strong positive correlation	
$0.87 \leqslant r < 0.95$	strong positive correlation	
$0.7 \leqslant r < 0.87$	moderate positive correlation	
$0.5 \leqslant r < 0.7$	weak positive correlation	
$0 < r < 0.5$	very weak positive correlation	

Negative correlation

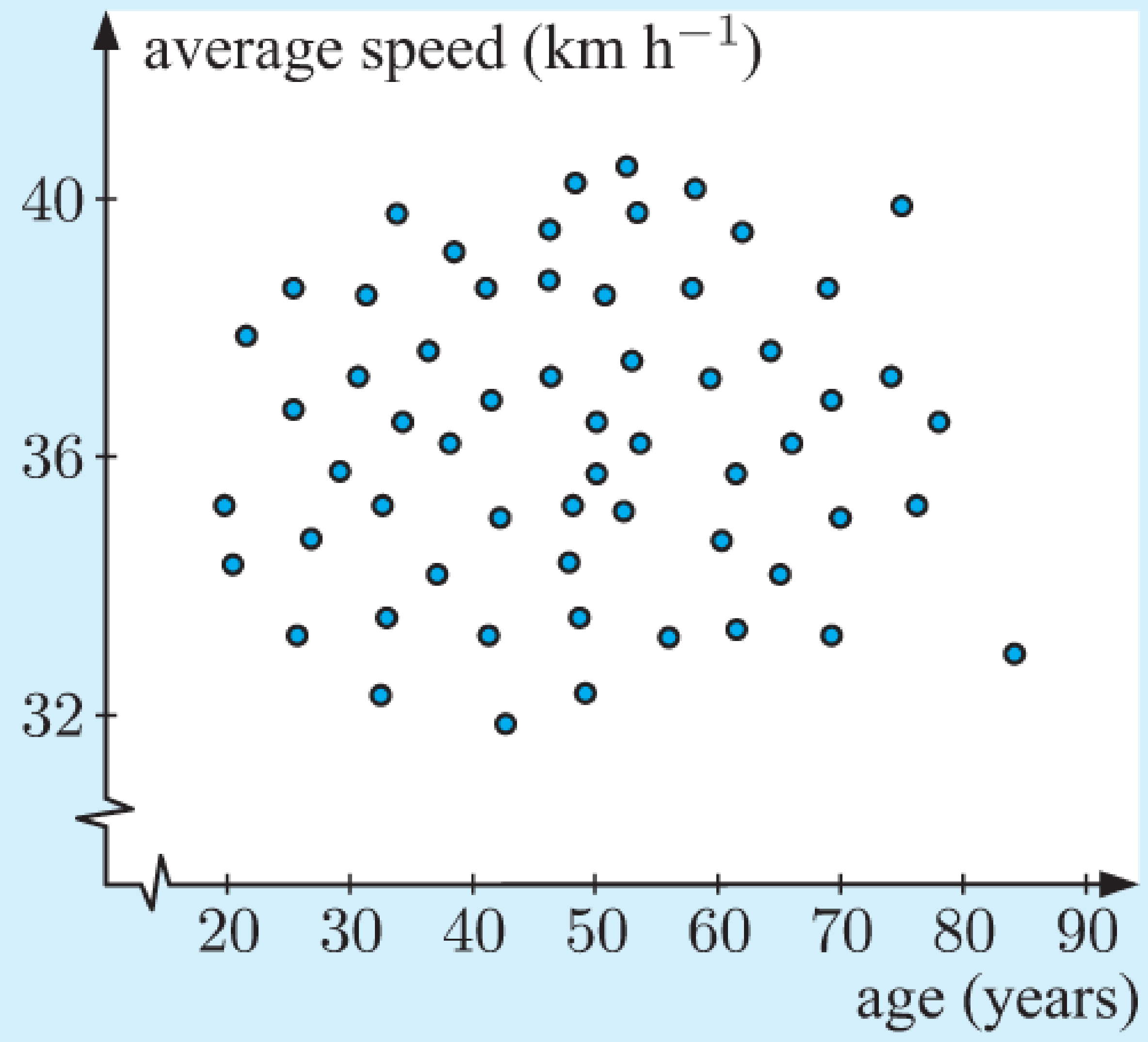
$r = -1$	perfect negative correlation	
$-1 < r \leqslant -0.95$	very strong negative correlation	
$-0.95 < r \leqslant -0.87$	strong negative correlation	
$-0.87 < r \leqslant -0.7$	moderate negative correlation	
$-0.7 < r \leqslant -0.5$	weak negative correlation	
$-0.5 < r < 0$	very weak negative correlation	

Example 1

Self Tutor

The Department of Road Safety wants to know if there is any association between *average speed* in the metropolitan area and the *age of drivers*. They commission a device to be fitted in the cars of drivers of different ages.

The results are shown in the scatter diagram. The r-value for this association is $+0.027$. Describe the association.



Since $0 < r < 0.5$, there is a very weak positive correlation between the two variables. We observe this in the graph as the points are randomly scattered.

Example 2 **Self Tutor**

The botanical gardens have been trying a new chemical to control the number of beetles infesting their plants. The results of one of their tests are shown in the table.

Sample	*Quantity of chemical* (g)	*Number of surviving beetles*
A	2	11
B	5	6
C	6	4
D	3	6
E	9	3

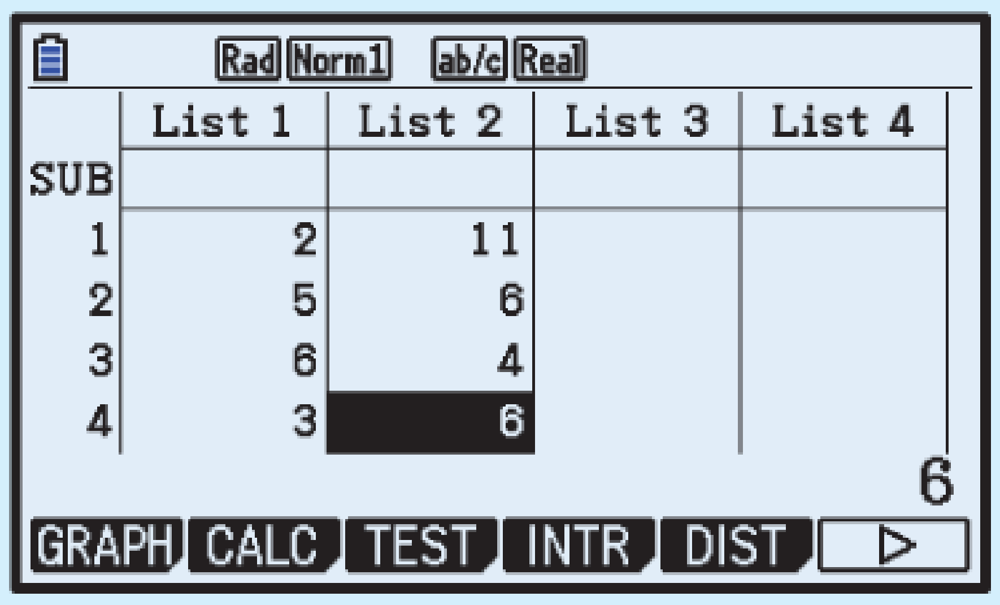
a Draw a scatter diagram for the data.

b Determine the correlation coefficient r.

c Describe the correlation between the *quantity of chemical* and the *number of surviving beetles*.

We first enter the data into separate lists:

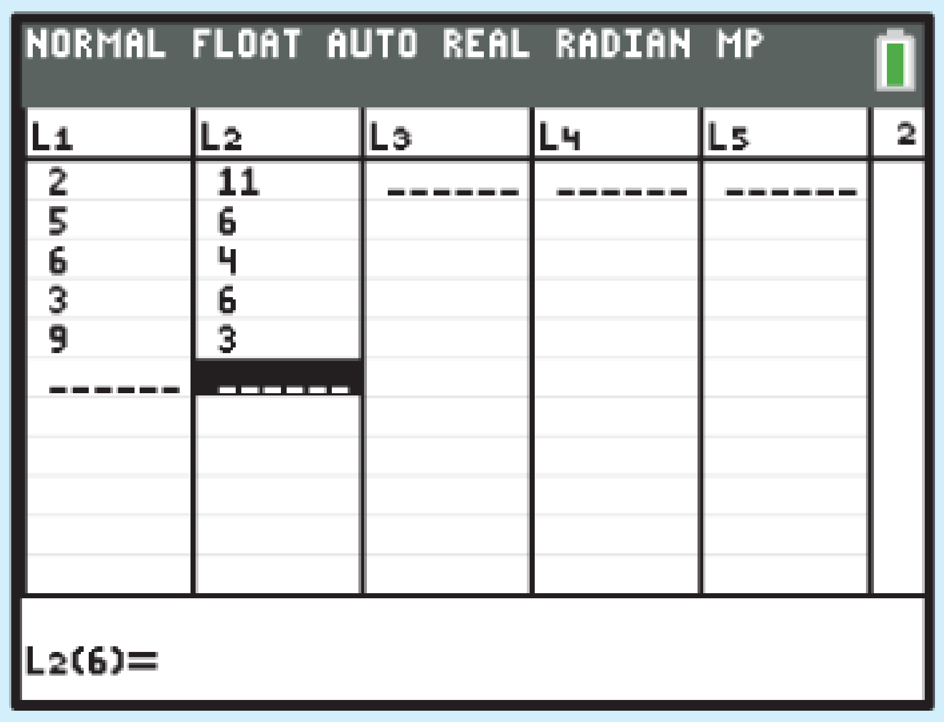
Casio fx-CG50



```
     List 1  List 2  List 3  List 4
SUB
  1       2      11
  2       5       6
  3       6       4
  4       3       6
                                  6
```

Rad Norm1 ab/c Real
GRAPH CALC TEST INTR DIST ▷

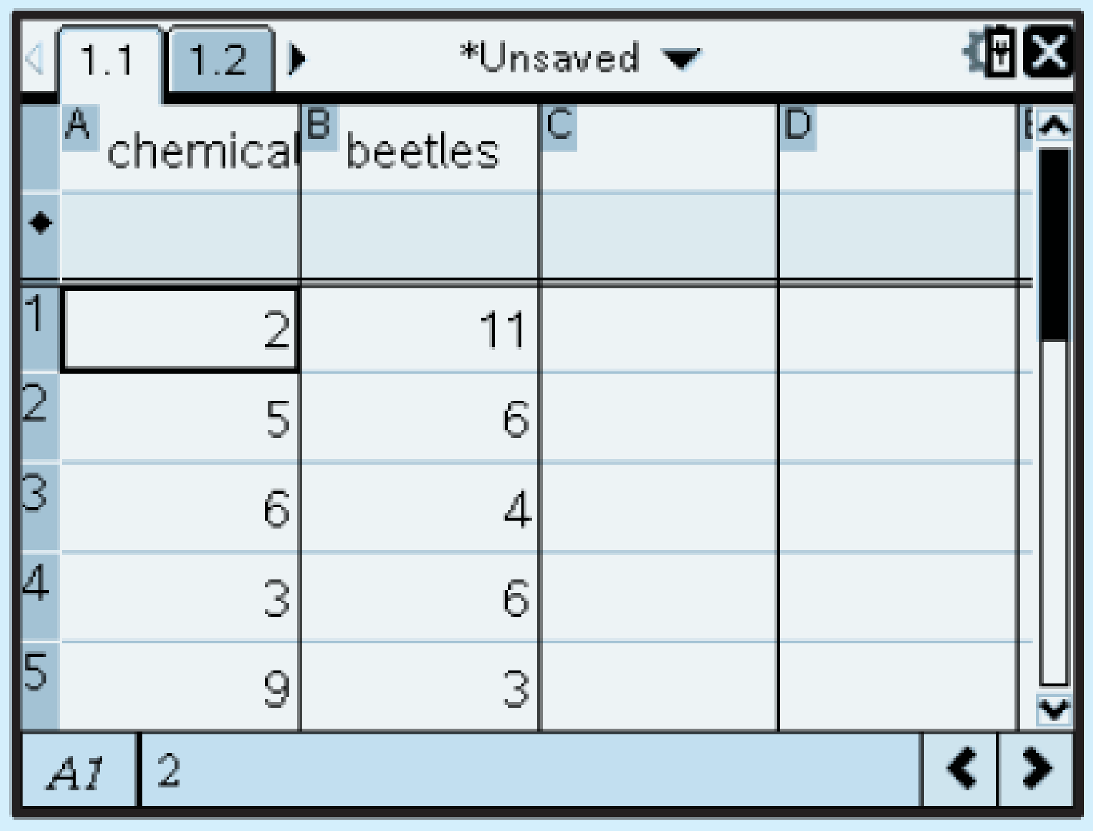
TI-84 Plus CE



```
L1   L2   L3   L4   L5
2    11   ---- ---- ----
5    6
6    4
3    6
9    3
---- ----
L2(6)=
```

NORMAL FLOAT AUTO REAL RADIAN MP

TI-*n*spire

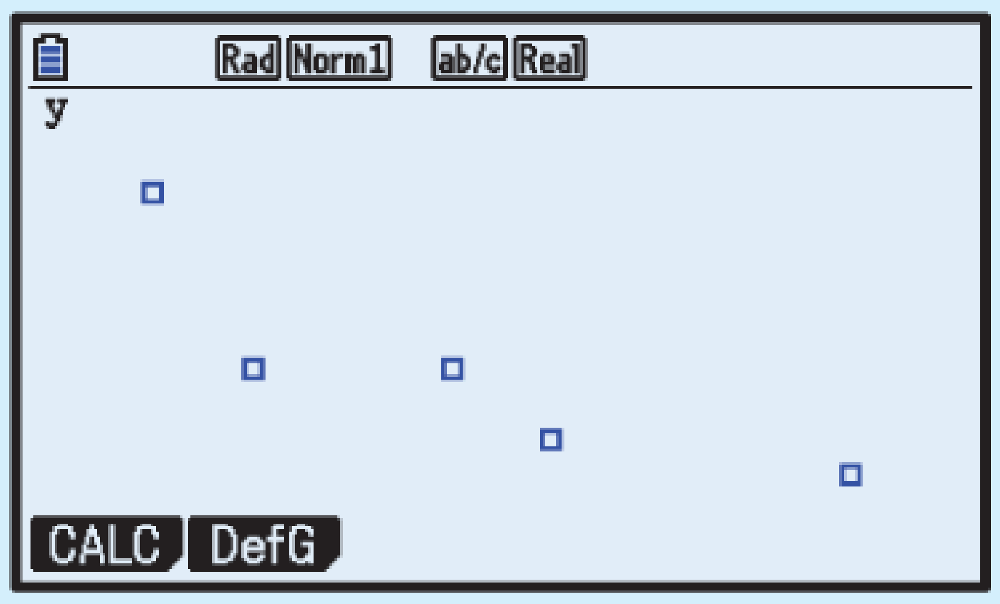


```
A chemical  B beetles  C    D
1        2        11
2        5         6
3        6         4
4        3         6
5        9         3
A1 2
```

1.1 1.2 *Unsaved

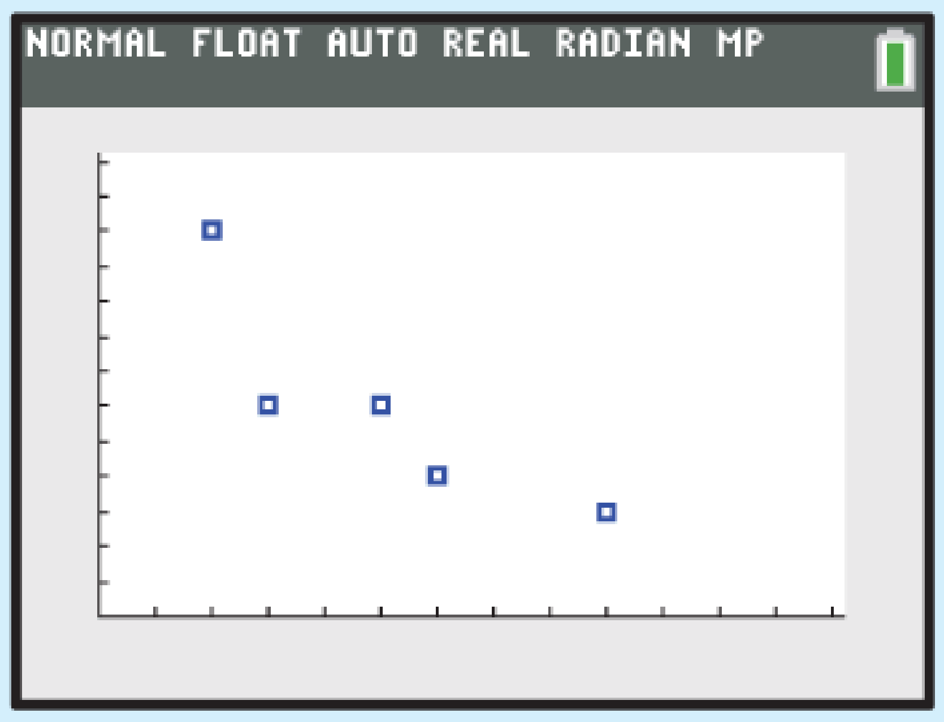
a

Casio fx-CG50



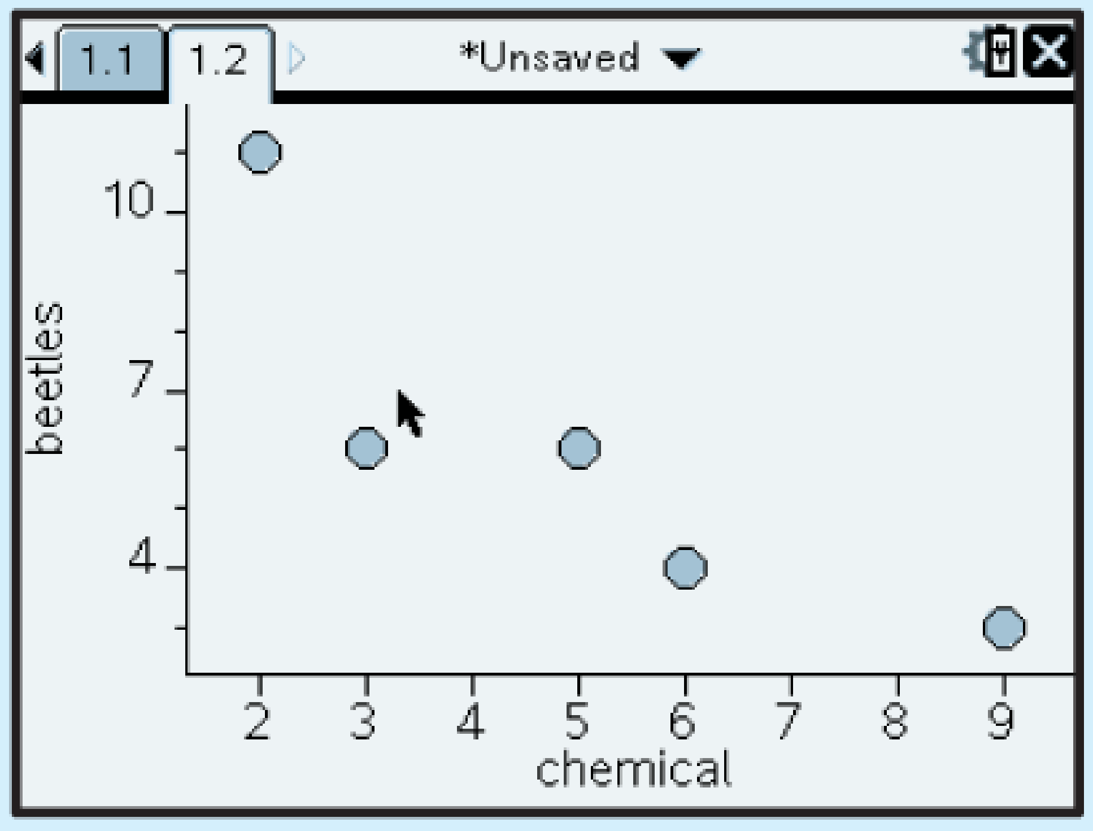
Rad Norm1 ab/c Real
y
CALC DefG

TI-84 Plus CE



NORMAL FLOAT AUTO REAL RADIAN MP

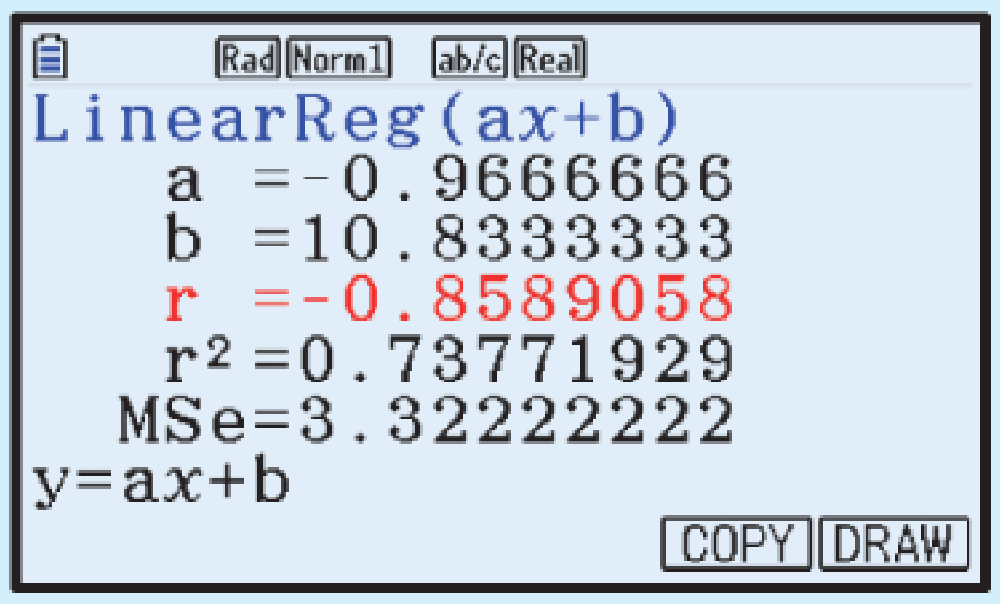
TI-*n*spire



1.1 1.2 *Unsaved
10
7
4
beetles
2 3 4 5 6 7 8 9
chemical

b

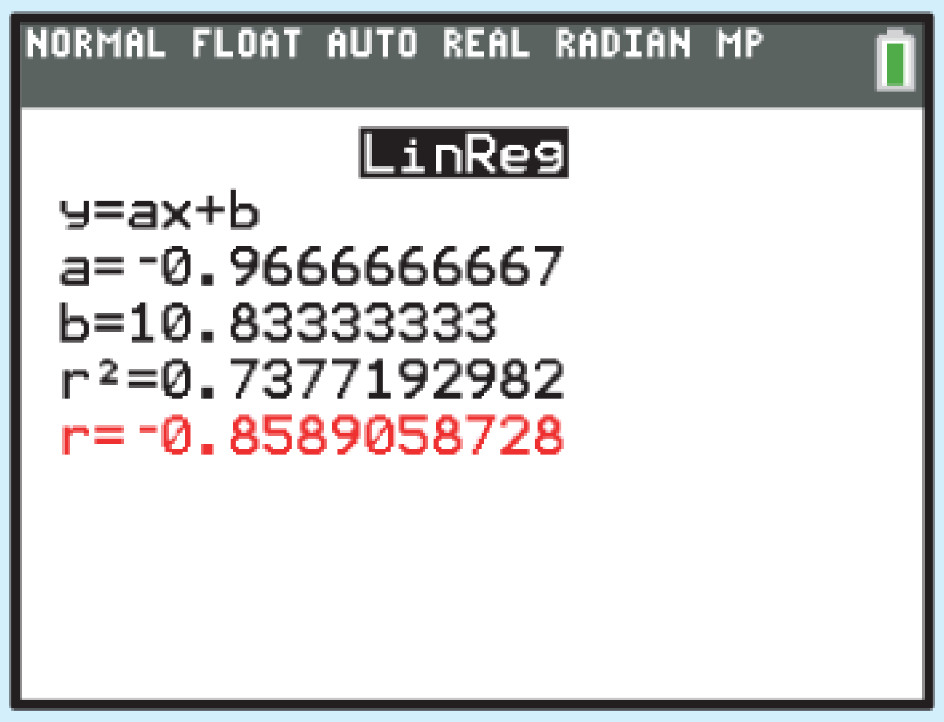
Casio fx-CG50



```
LinearReg(ax+b)
    a =-0.9666666
    b =10.8333333
    r =-0.8589058
   r² =0.73771929
  MSe=3.32222222
y=ax+b
```

Rad Norm1 ab/c Real
COPY DRAW

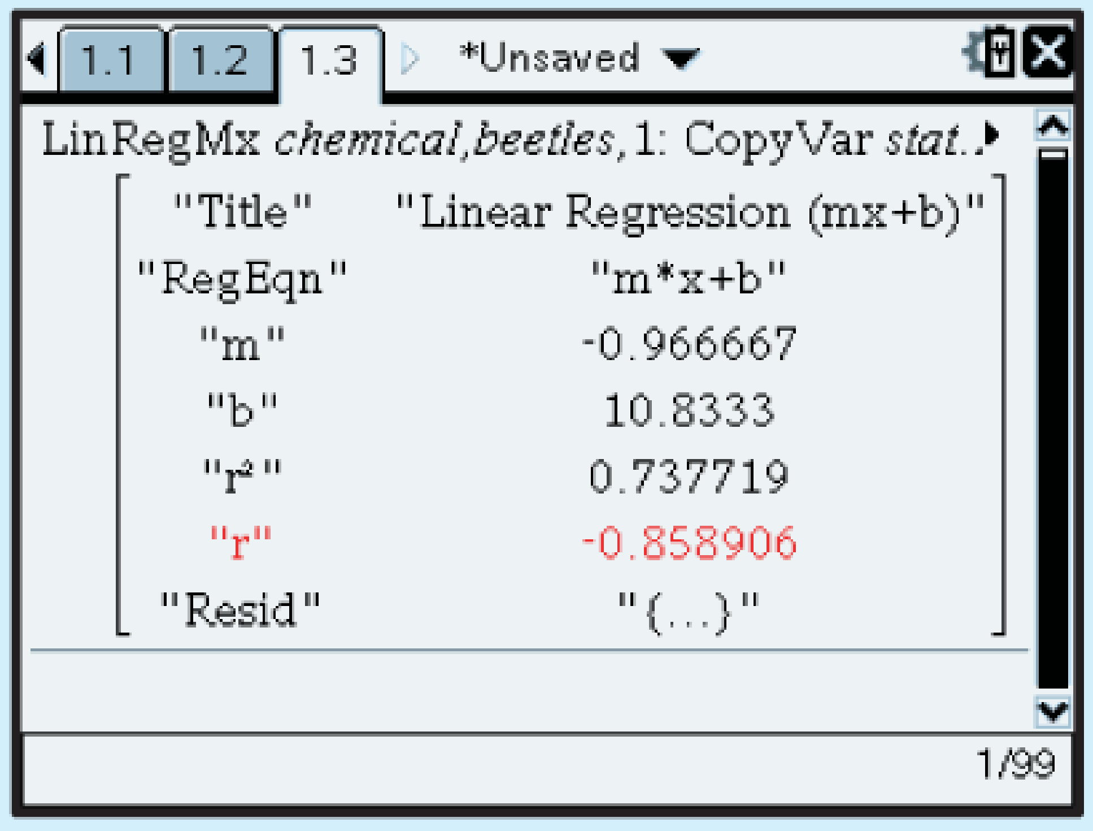
TI-84 Plus CE



```
LinReg
y=ax+b
a=-0.9666666667
b=10.83333333
r²=0.7377192982
r=-0.8589058728
```

NORMAL FLOAT AUTO REAL RADIAN MP

TI-*n*spire



```
LinRegMx chemical,beetles,1: CopyVar stat.
   "Title"    "Linear Regression (mx+b)"
  "RegEqn"          "m*x+b"
     "m"           -0.966667
     "b"            10.8333
    "r²"            0.737719
     "r"           -0.858906
   "Resid"          "{...}"
                                    1/99
```

1.1 1.2 1.3 *Unsaved

So, $r \approx -0.859$.

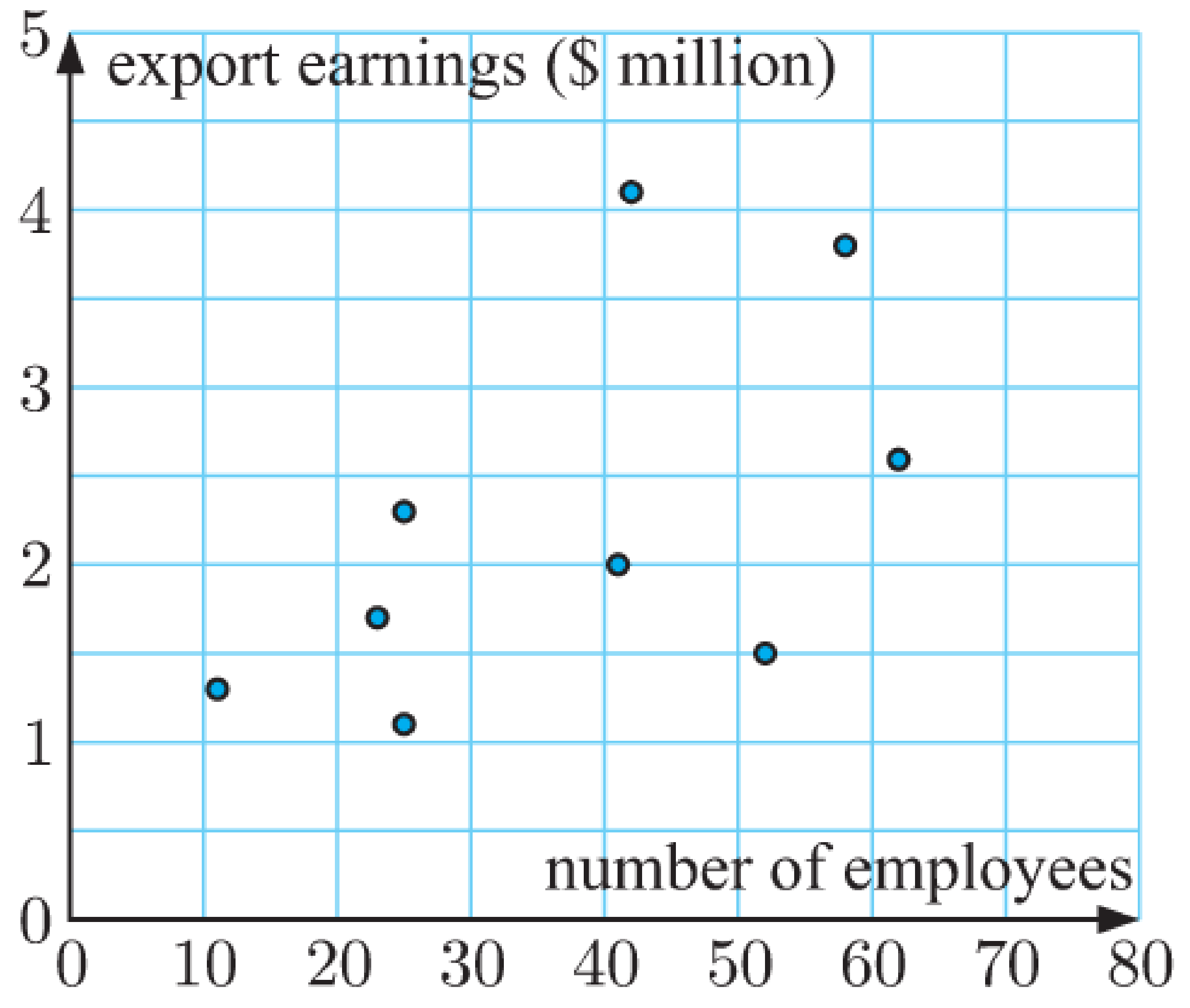
c There is a moderate negative correlation between the *quantity of chemical used* and the *number of surviving beetles*.

In general, the more chemical that is used, the fewer beetles that survive.

EXERCISE 5B

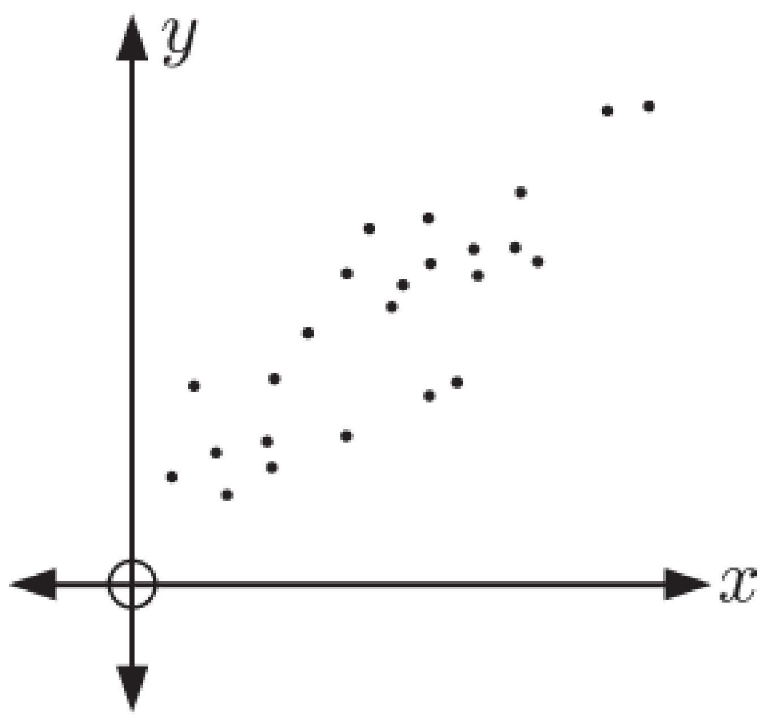
1 In a recent survey, the Department of International Commerce compared the *number of employees of a company* with its *export earnings*. A scatter diagram of their data is shown alongside. The corresponding value of r is 0.556.

Describe the association between the variables.

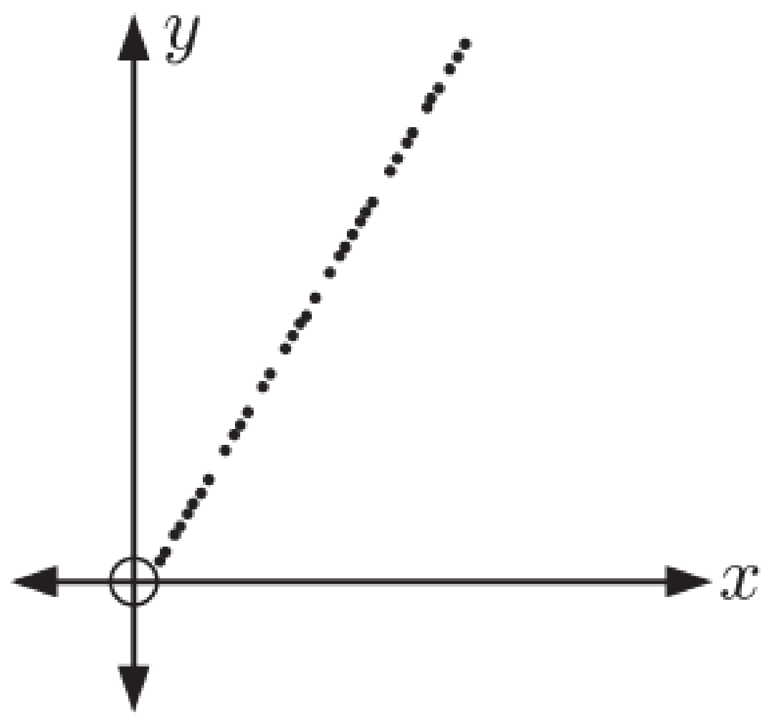


2 Match each scatter diagram with the correct value of r.

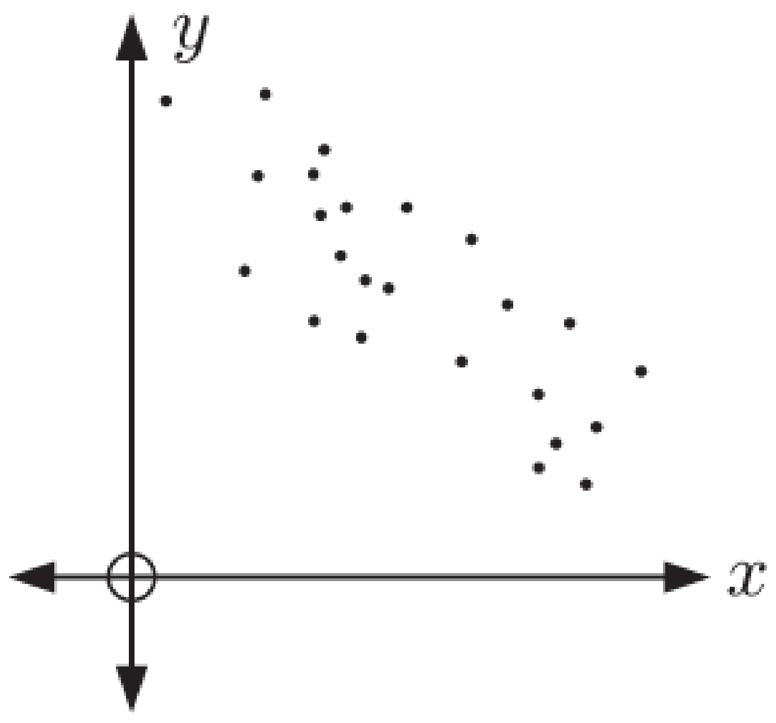
a



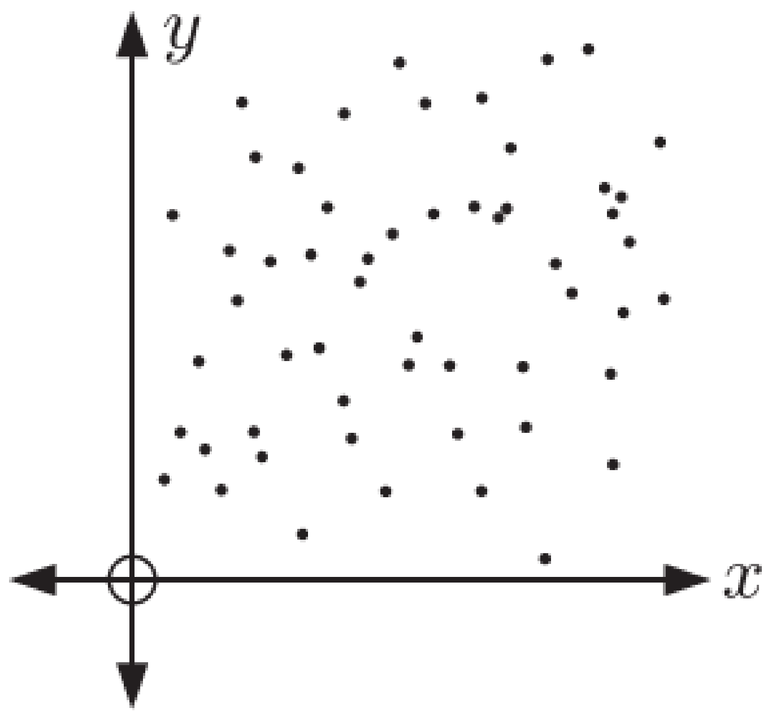
b



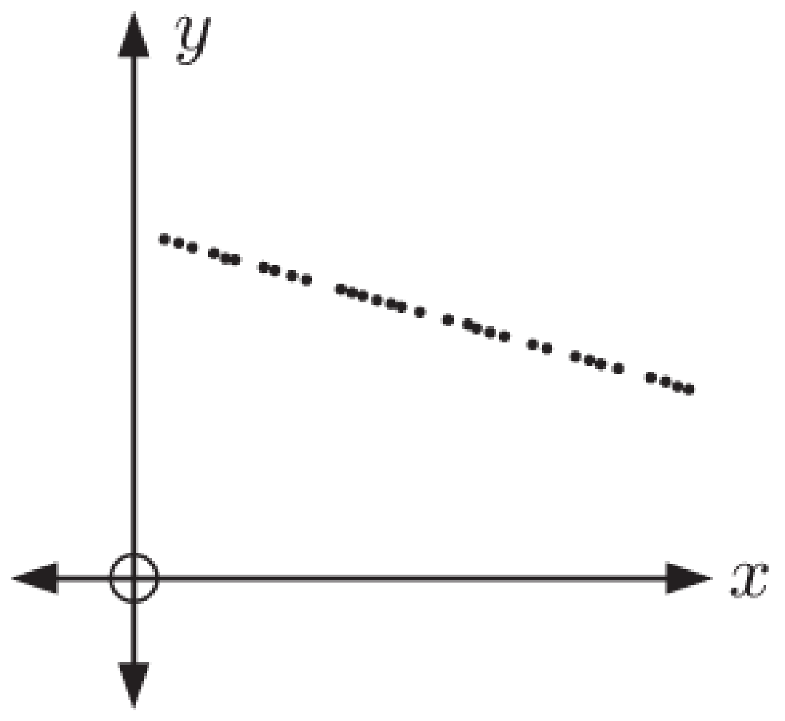
c



d



e



A $r = 1$ **B** $r = 0.6$ **C** $r = 0$ **D** $r = -0.7$ **E** $r = -1$

3 For each of the following data sets:

- **i** Draw a scatter diagram for the data.
- **ii** Calculate Pearson's product-moment correlation coefficient r.
- **iii** Describe the linear correlation between x and y.

a

x	1	2	3	4	5	6
y	3	2	5	5	9	6

b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

4 A selection of students was asked how many phone calls and text messages they received the previous day. The results are shown alongside.

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the linear correlation between *phone calls received* and *text messages received*.

d Give a reason why this correlation may occur.

5 Consider the **Opening Problem** on page **102**.

- **a** Calculate r for the data.
- **b** Hence describe the association between the variables.

6 Jill does her washing every Saturday and hangs her clothes out to dry. She notices that the clothes dry faster some days than others. She investigates the relationship between the temperature and the time her clothes take to dry:

Temperature (x °C)	25	32	27	39	35	24	30	36	29	35
Drying time (y minutes)	100	70	95	25	38	105	70	35	75	40

- **a** Draw a scatter diagram for the data.
- **b** Calculate r.
- **c** Describe the correlation between *temperature* and *drying time*.

7 This table shows the number of supermarkets in 10 towns, and the number of car accidents that have occurred in these towns in the last month.

Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37

- **a** Draw a scatter diagram for the data.
- **b** Calculate r.
- **c** Identify the outlier in the data.
- **d** It was found that the outlier was due to an error in the data collection process.
  - **i** Recalculate r with the outlier removed.
  - **ii** Describe the relationship between the variables.
  - **iii** Discuss the effect of removing the outlier on the value of r.
- **e** Do you think there is a causal relationship between the variables? Explain your answer.

8 A health researcher notices that the incidence of Multiple Sclerosis (MS) is higher in some parts of the world than in others.
To investigate further, she records the *latitude* and *incidence of MS per 100 000 people* of 20 countries.

Latitude (degrees)	55	25	41	22	47	37	56	14	34	25
MS incidence per 100 000	165	95	75	20	180	140	230	15	45	65

Latitude (degrees)	27	65	10	24	4	56	46	8	50	40
MS incidence per 100 000	30	140	5	15	2	290	95	8	160	105

- **a** Draw a scatter diagram for the data.
- **b** Calculate the value of r.
- **c** Describe the relationship between the variables.
- **d** Is the incidence of MS higher near the equator, or near the poles?

Higher latitudes occur near the poles. Lower latitudes occur near the equator.

ACTIVITY 1 COMPARING HEIGHT AND FOOT LENGTH

In this Activity, you will explore the relationship between the *height* and *foot length* of the students in your class.

You will need: ruler, tape measure

What to do:

1. Predict whether there will be positive correlation, no correlation, or negative correlation between the *height* and *foot length* of the students in your class.
2. Measure the height and foot length of each student in your class. Record your measurements in a table like the one below:

Student	*Height* (cm)	*Foot length* (cm)

3. Use technology to draw a scatter diagram for the data.
4. Calculate Pearson's product-moment correlation coefficient r for the data.
5. Describe the relationship between *height* and *foot length*. Was your prediction correct?
6. Do you think that a high value of r indicates a causal relationship in this case?

C LINE OF BEST FIT BY EYE

If there is a sufficiently strong linear correlation between two variables, we can draw a line of best fit to illustrate their relationship. In general, it is only worth drawing a line of best fit if the correlation between the variables is strong. There is no fixed rule, but we suggest that a line of best fit is not appropriate if $|r| < 0.85$.

If we draw the line just by observing the points, we call it a **line of best fit by eye**. This line will vary from person to person.

We draw a line of best fit connecting variables x and y as follows:

Step 1: Calculate the mean of the x values $\overline{x}$, and the mean of the y values $\overline{y}$.

Step 2: Mark the **mean point** $(\overline{x}, \overline{y})$ on the scatter diagram.

Step 3: Draw a line through the mean point which fits the trend of the data, and so that about the same number of data points are above the line as below it.

Consider again the data from the **Opening Problem**:

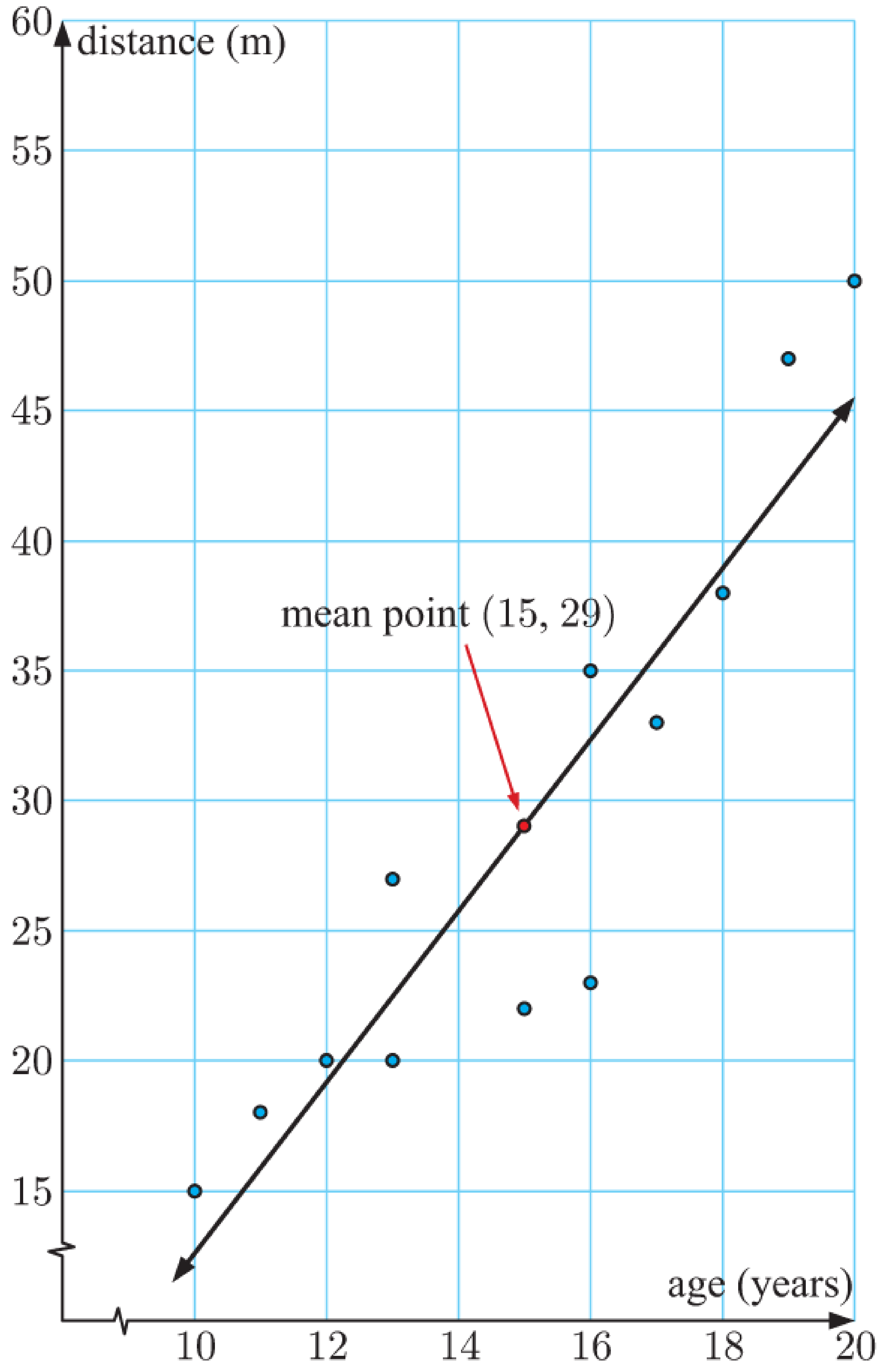
Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

We have seen that there is a strong positive linear correlation between *age* and *distance thrown*.

We can therefore model the data using a line of best fit.

The mean age is 15 years and the mean distance thrown is 29 m. We therefore draw our line of best fit through the mean point (15, 29).

We can use the line of best fit to estimate the value of y for any given value of x, and vice versa.



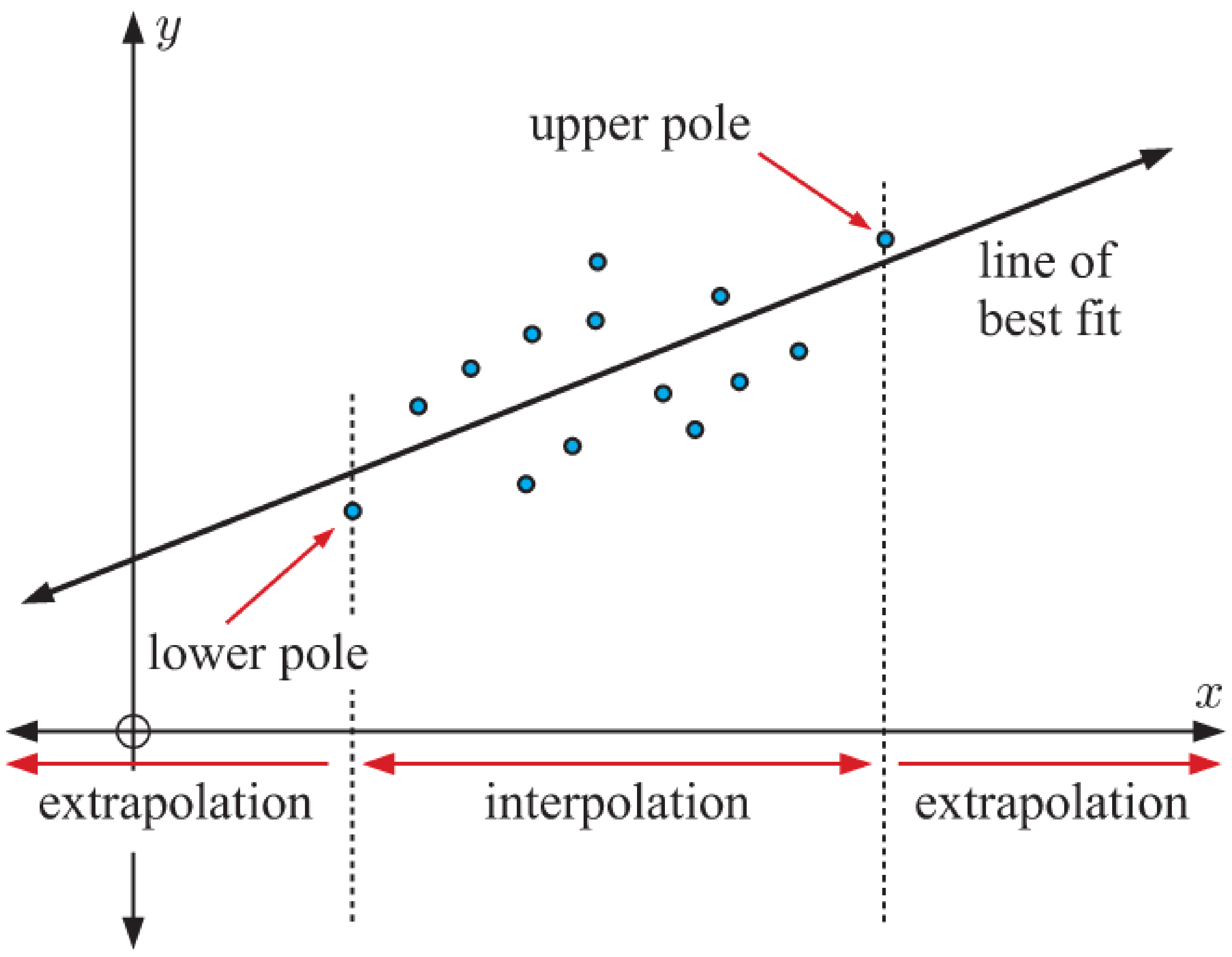
We draw the line through the mean point so it follows the trend of the data and there are about the same number of points above the line as below the line.

INTERPOLATION AND EXTRAPOLATION

Consider the data in the scatter diagram alongside. The data with the highest and lowest values are called the **poles**.

The line of best fit for the data is also drawn on the scatter diagram. We can use this line to predict the value of one variable for a given value of the other.

- If we predict a y value for an x value **in between** the poles, we say we are **interpolating** in between the poles.
- If we predict a y value for an x value **outside** the poles, we say we are **extrapolating** outside the poles.



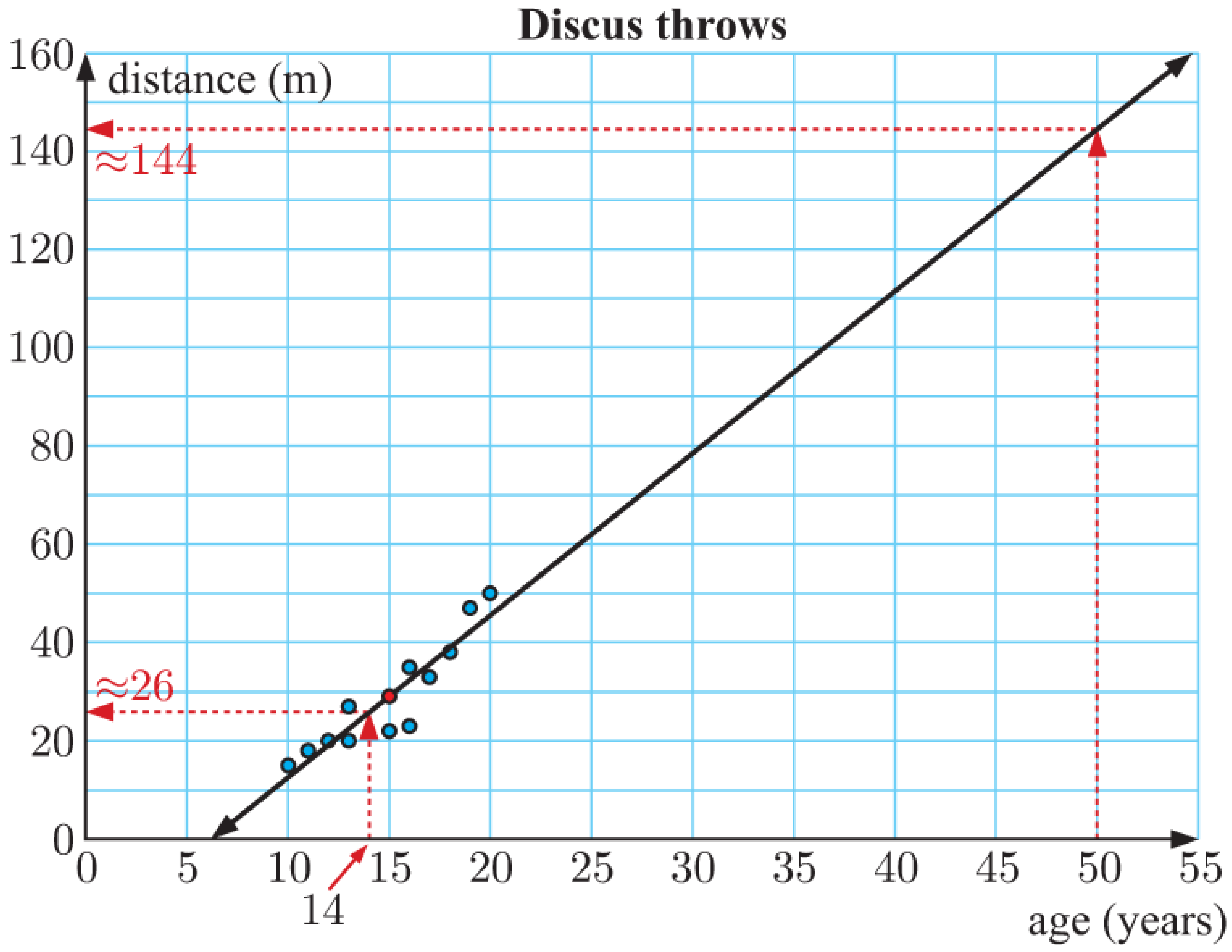
The accuracy of an interpolation depends on how well the linear model fits the data. This can be gauged by the correlation coefficient and by ensuring that the data is randomly scattered around the line of best fit.

The accuracy of an extrapolation depends not only on how well the model fits, but also on the assumption that the linear trend will continue past the poles. The validity of this assumption depends greatly on the situation we are looking at.

For example, consider the line of best fit for the data in the **Opening Problem**. It can be used to predict the distance a discus will be thrown by an athlete of a particular age.

The age 14 is within the range of ages in the original data, so it is reasonable to predict that a 14 year old will be able to throw the discus 26 m.

However, it is unlikely that the linear trend shown in the data will continue far beyond the poles. For example, according to the model, a 50 year old might throw the discus 144 m. This is almost twice the current world record of 76.8 m, so it would clearly be an unreasonable prediction.



Example 3

Self Tutor

On a hot day, six cars were left in the sun in a car park. The length of time each car was left in the sun was recorded, as well as the temperature inside the car at the end of the period.

Car	A	B	C	D	E	F
Time (x minutes)	50	5	25	40	15	45
Temperature (y °C)	47	28	36	42	34	41

a Calculate $\overline{x}$ and $\overline{y}$.

b Draw a scatter diagram for the data.

c Locate the mean point $(\overline{x}, \overline{y})$ on the scatter diagram, then draw a line of best fit through this point.

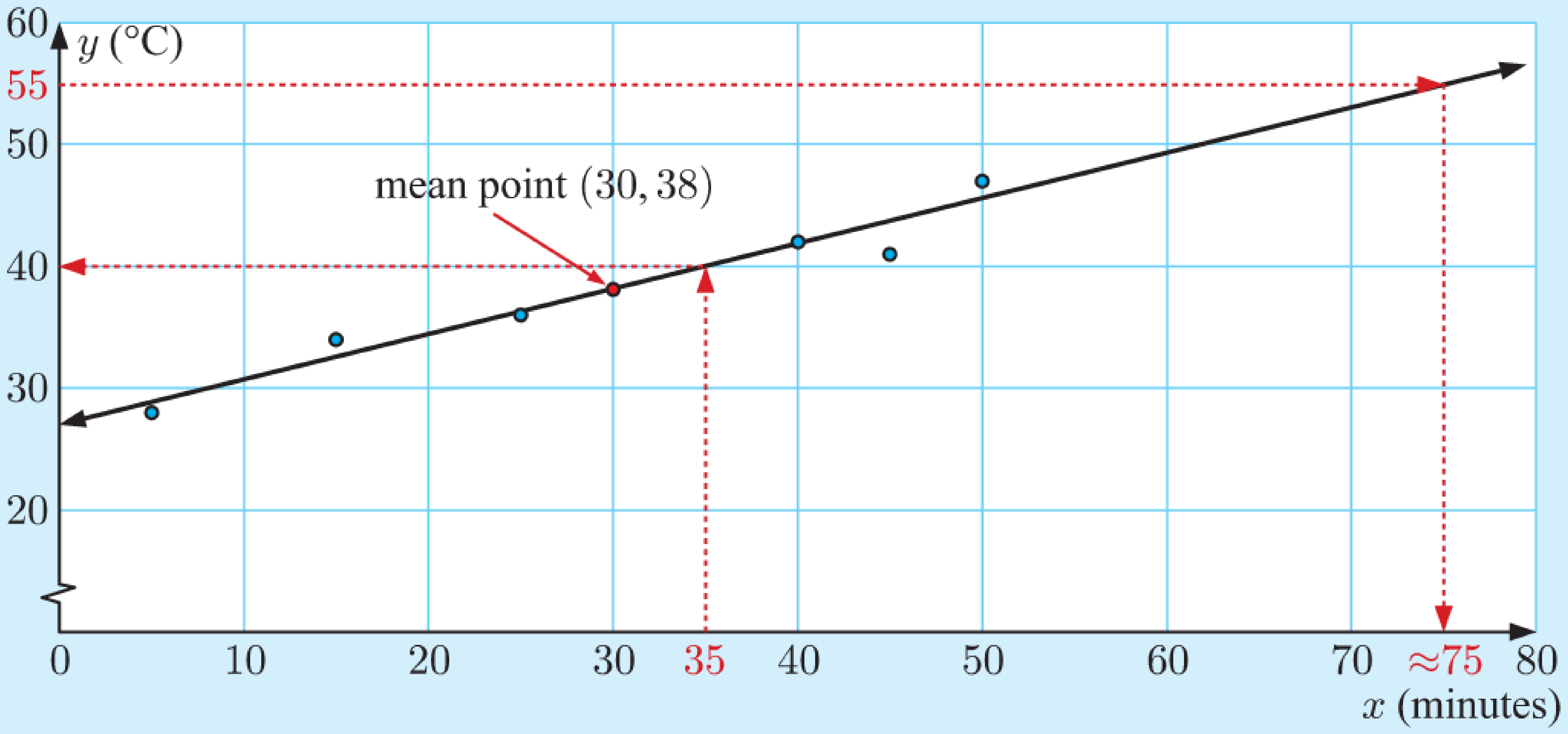
d Predict the temperature of a car which has been left in the sun for 35 minutes.

e Predict how long it would take for a car's temperature to reach 55°C.

f Comment on the reliability of your predictions in **d** and **e**.

a $\overline{x} = \dfrac{50+5+25+40+15+45}{6} = 30, \qquad \overline{y} = \dfrac{47+28+36+42+34+41}{6} = 38$

b, c



d When $x = 35$, $y \approx 40$.
The temperature of a car left in the sun for 35 minutes will be approximately 40°C.

e When $y = 55$, $x \approx 75$.
It would take approximately 75 minutes for a car's temperature to reach 55°C.

f The prediction in **d** is reliable, as the data appears linear, and this is an interpolation.
The prediction in **e** may be unreliable, as it is an extrapolation and the linear trend displayed by the data may not continue beyond the 50 minute mark.

EXERCISE 5C

1 Consider the data set alongside.

x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10

- **a** Draw a scatter diagram for the data.
- **b** Does the data appear to be positively or negatively correlated?
- **c** Calculate r for the data.
- **d** Describe the strength of the relationship between x and y.
- **e** Calculate the mean point $(\overline{x}, \overline{y})$.
- **f** Locate the mean point, then use it in drawing a line of best fit.
- **g** Estimate the value of y when $x = 22$.

2 Fifteen students were weighed and their pulse rates were measured:

Weight (x kg)	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
Pulse rate (y beats per min)	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58

- **a** Draw a scatter diagram for the data.
- **b** Calculate r.
- **c** Describe the relationship between *weight* and *pulse rate*.
- **d** Calculate the mean point $(\overline{x}, \overline{y})$.
- **e** Locate the mean point on the scatter diagram, then use it in drawing a line of best fit.
- **f** Estimate the pulse rate of a 50 kg student. Comment on the reliability of your estimate.

3 The trunk widths and heights of the trees in a garden are given below:

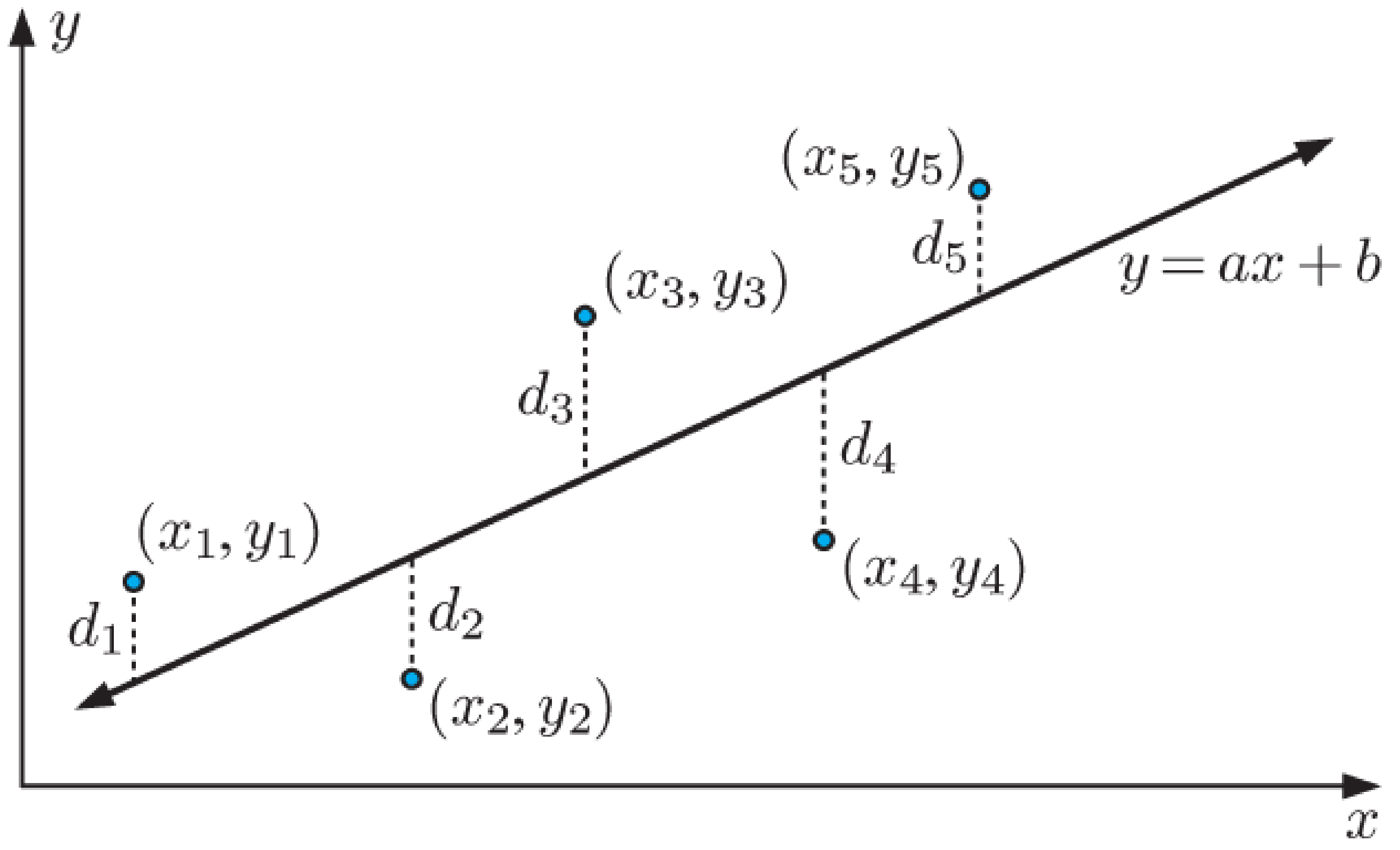
Trunk width (x cm)	35	47	72	40	15	87	20	66	57	24	32
Height (y m)	11	18	24	12	3	30	22	21	17	5	10

- **a** Draw a scatter diagram for the data.
- **b** Which of the points is an outlier?
- **c** How would you describe the tree represented by the outlier?
- **d** Calculate the mean point $(\overline{x}, \overline{y})$.
- **e** Locate the mean point on the scatter diagram, then draw a line of best fit through the mean point.
- **f** Predict the height of a tree with trunk width 120 cm. Comment on the reliability of your prediction.
- **g** Predict the trunk width of a tree with height 10 m. Comment on the reliability of your prediction.

D THE LEAST SQUARES REGRESSION LINE

The problem with drawing a line of best fit by eye is that the line drawn will vary from one person to another. For consistency, we use a method known as **linear regression** to find the equation of the line which best fits the data. The most common method is the method of "**least squares**".

In least squares linear regression, we minimise the sum of the squares of the vertical distances between each data point and the **regression line**.



In this course you will not be required to find the equation of the least squares regression line by hand.

Instead, you can use your **graphics calculator** or the **statistics package**.

STATISTICS PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

Example 4 **Self Tutor**

The annual income and average weekly grocery bill for a selection of families is shown below:

Income (x thousand pounds)	55	36	25	47	60	64	42	50
Grocery bill (y pounds)	120	90	60	160	190	250	110	150

a Construct a scatter diagram to illustrate the data.

b Use technology to find the equation of the regression line.

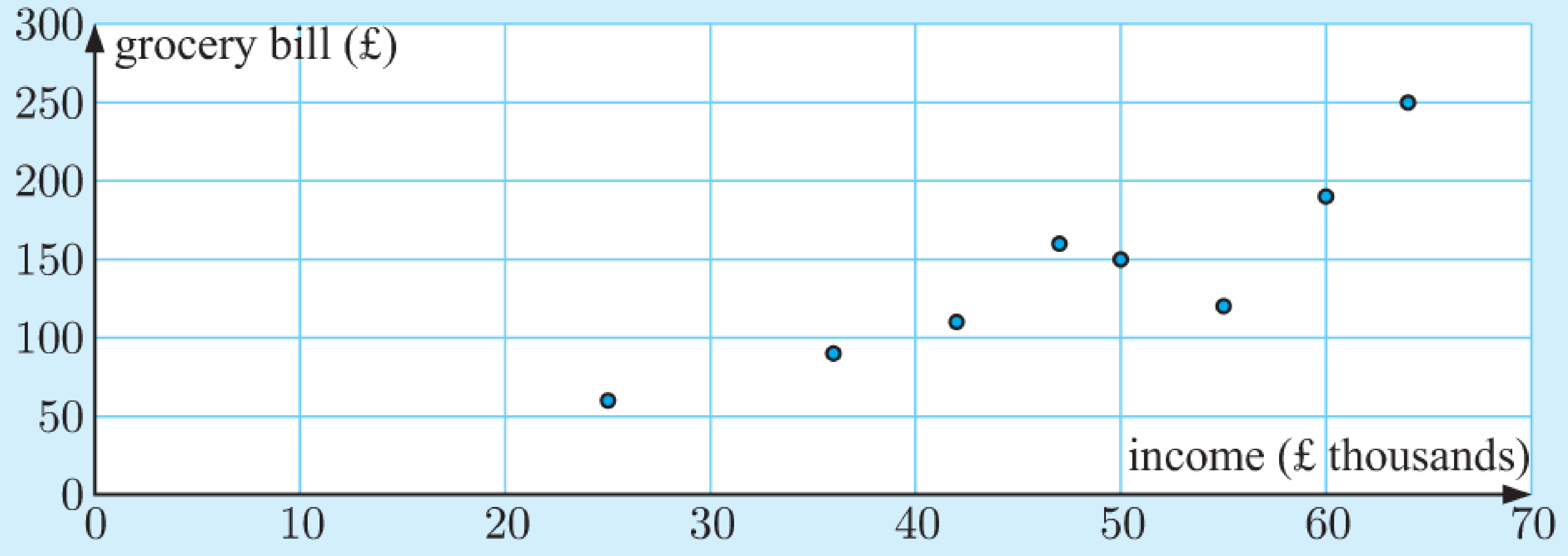
c State and interpret the gradient of the regression line.

d Estimate the weekly grocery bill for a family with an annual income of £95 000.

e Estimate the annual income of a family whose weekly grocery bill is £100.

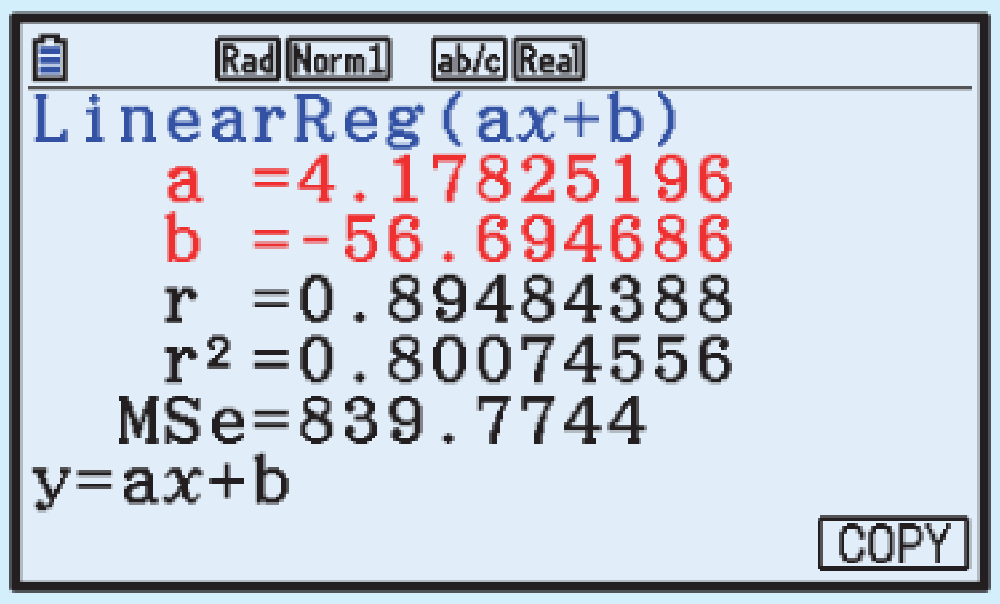
f Comment on whether the estimates in **d** and **e** are likely to be reliable.

a



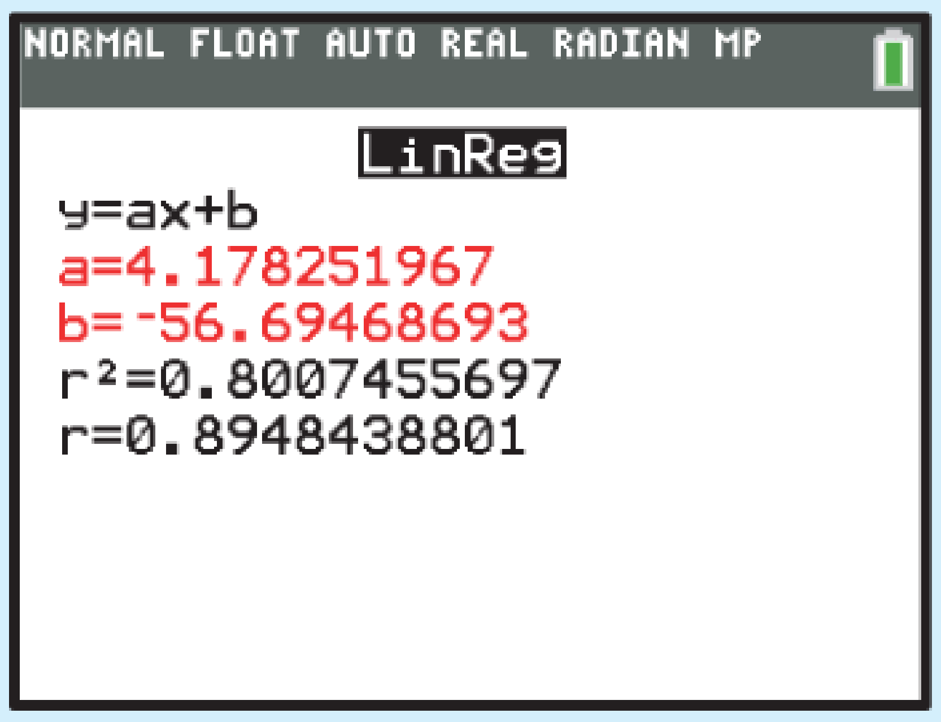
b

Casio fx-CG50



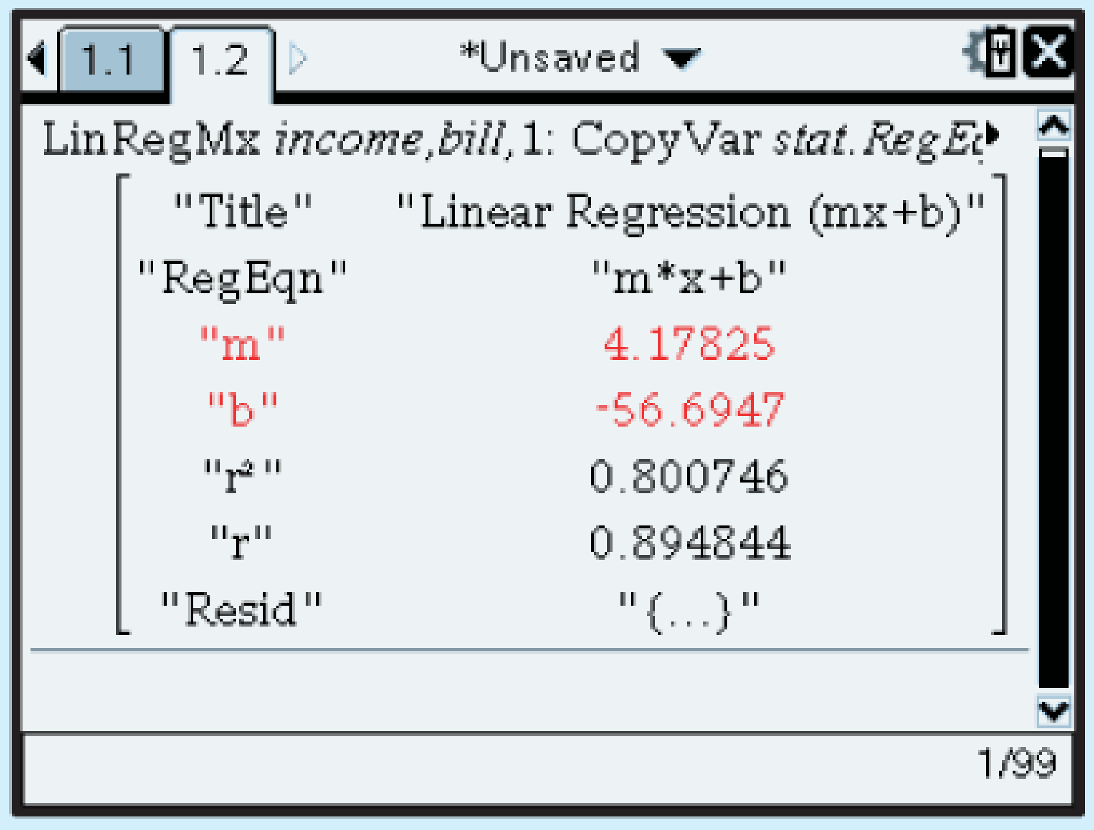
```
LinearReg(ax+b)
    a =4.17825196
    b =-56.694686
    r =0.89484388
    r² =0.80074556
   MSe=839.7744
y=ax+b
```

TI-84 Plus CE



```
LinReg
y=ax+b
a=4.178251967
b=-56.69468693
r²=0.8007455697
r=0.8948438801
```

TI-*n*spire



```
LinRegMx income,bill,1: CopyVar stat.RegEq
  "Title"     "Linear Regression (mx+b)"
  "RegEqn"    "m*x+b"
  "m"         4.17825
  "b"         -56.6947
  "r²"        0.800746
  "r"         0.894844
  "Resid"     "{...}"
```

Using technology, the regression line is $y \approx 4.18x - 56.7$

c The gradient of the regression line ≈ 4.18. This means that for every additional £1000 of income, a family's weekly grocery bill will increase by an average of £4.18.

d When $x = 95$, $y \approx 4.18(95) - 56.7 \approx 340$

So, we expect a family with an income of £95 000 to have a weekly grocery bill of approximately £340.

e When $y = 100$, $100 \approx 4.18x - 56.7$

$$\therefore\ 156.7 \approx 4.18x \qquad \{\text{adding } 56.7 \text{ to both sides}\}$$

$$\therefore\ x \approx 37.5 \qquad \{\text{dividing both sides by } 4.18\}$$

So, we expect a family with a weekly grocery bill of £100 to have an annual income of approximately £37 500.

f The estimate in **d** is an extrapolation, so the estimate may not be reliable.

The estimate in **e** is an interpolation and there is strong linear correlation between the variables. We therefore expect this estimate to be reliable.

EXERCISE 5D

1 Consider the data set below.

x	10	4	6	8	9	5	7	1	2	3
y	20	6	8	13	20	12	13	4	2	7

GRAPHICS CALCULATOR INSTRUCTIONS

- **a** Draw a scatter diagram for the data.
- **b** Use technology to find the equation of the regression line, and plot the line on your calculator.
- **c** Use **b** to draw the regression line on your scatter diagram.

2 Steve wanted to see whether there was any relationship between the temperature when he leaves for work in the morning, and the time it takes for him to get to work.

He collected data over a 14 day period:

Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39

- **a** Draw a scatter diagram for the data.
- **b** Calculate r.
- **c** Describe the relationship between the variables.
- **d** Is it reasonable to fit a linear model to this data? Explain your answer.

3 The table below shows the price of petrol and the number of customers per hour for sixteen petrol stations.

Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
Number of customers (y)	45	42	25	48	43	15	19	10

Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
Number of customers (y)	30	23	42	50	12	24	32	17

a Calculate Pearson's product-moment correlation coefficient for the data.

b Describe the relationship between the *petrol price* and the *number of customers*.

c Use technology to find the equation of the regression line.

d State and interpret the gradient of the regression line.

e Estimate the number of customers per hour for a petrol station which sells petrol at 115.9 cents per litre.

f Estimate the petrol price at a petrol station which has 40 customers per hour.

g Comment on the reliability of your estimates in **e** and **f**.

4 To investigate whether speed cameras have an impact on road safety, data was collected from several cities. The number of speed cameras in operation was recorded for each city, as well as the number of accidents over a 7 day period.

Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32

a Construct a scatter diagram to display the data.

b Calculate r for the data.

c Describe the relationship between the *number of speed cameras* and the *number of car accidents*.

d Find the equation of the regression line.

e State and interpret the gradient and y-intercept of the regression line.

f Estimate the number of car accidents in a city with 10 speed cameras.

5 The table below contains information about the *maximum speed* and *ceiling* (maximum altitude obtainable) for nineteen World War II fighter planes. The maximum speed is given in km h^{-1}, and the ceiling is given in km.

Maximum speed	*Ceiling*
460	8.84
420	10.06
530	10.97
530	9.906
490	9.448
530	10.36
680	11.73

Maximum speed	*Ceiling*
680	10.66
720	11.27
710	12.64
660	11.12
780	12.80
730	11.88

Maximum speed	*Ceiling*
670	12.49
570	10.66
440	10.51
670	11.58
700	11.73
520	10.36

a Draw a scatter diagram for the data.

b Calculate r.

c Describe the association between *maximum speed* (x) and *ceiling* (y).

d Use technology to find the regression line, and draw the line on your scatter diagram.

e State and interpret the gradient of the regression line.

f Estimate the ceiling for a fighter plane with a maximum speed of 600 $\text{km}\,\text{h}^{-1}$.

g Estimate the maximum speed for a fighter plane with a ceiling of 11 km.

6 A group of children was asked the numbers of hours they spent exercising and watching television each week.

Exercise (x hours per week)	4	1	8	7	10	3	3	2
Television (y hours per week)	12	24	5	9	1	18	11	16

a Draw a scatter diagram for the data.

b Calculate r.

c Describe the correlation between *time exercising* and *time watching television*.

d Find the equation of the regression line, and draw the line on your scatter diagram.

e State and interpret the gradient and y-intercept of the regression line.

f **i** One of the children in the group exercised for 7 hours each week. How much television does this child watch weekly?

  ii Use the regression line to predict the amount of television watched each week by a child who exercises for 7 hours each week.

  iii Compare your answers to **i** and **ii**.

7 The yield of pumpkins on a farm depends on the quantity of fertiliser used.

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4

a Draw a scatter diagram for the data, and identify the outlier.

b What effect do you think the outlier has on:

  i the strength of correlation of the data

  ii the gradient of the regression line?

c Calculate the correlation coefficient:

  i with the outlier included

  ii without the outlier.

d Calculate the equation of the regression line:

  i with the outlier included

  ii without the outlier.

e If you wish to estimate the yield when 15 g per m^2 of fertiliser is used, which regression line from **d** should be used? Explain your answer.

f Can you explain what may have caused the outlier? Do you think the outlier should be kept when analysing the data?

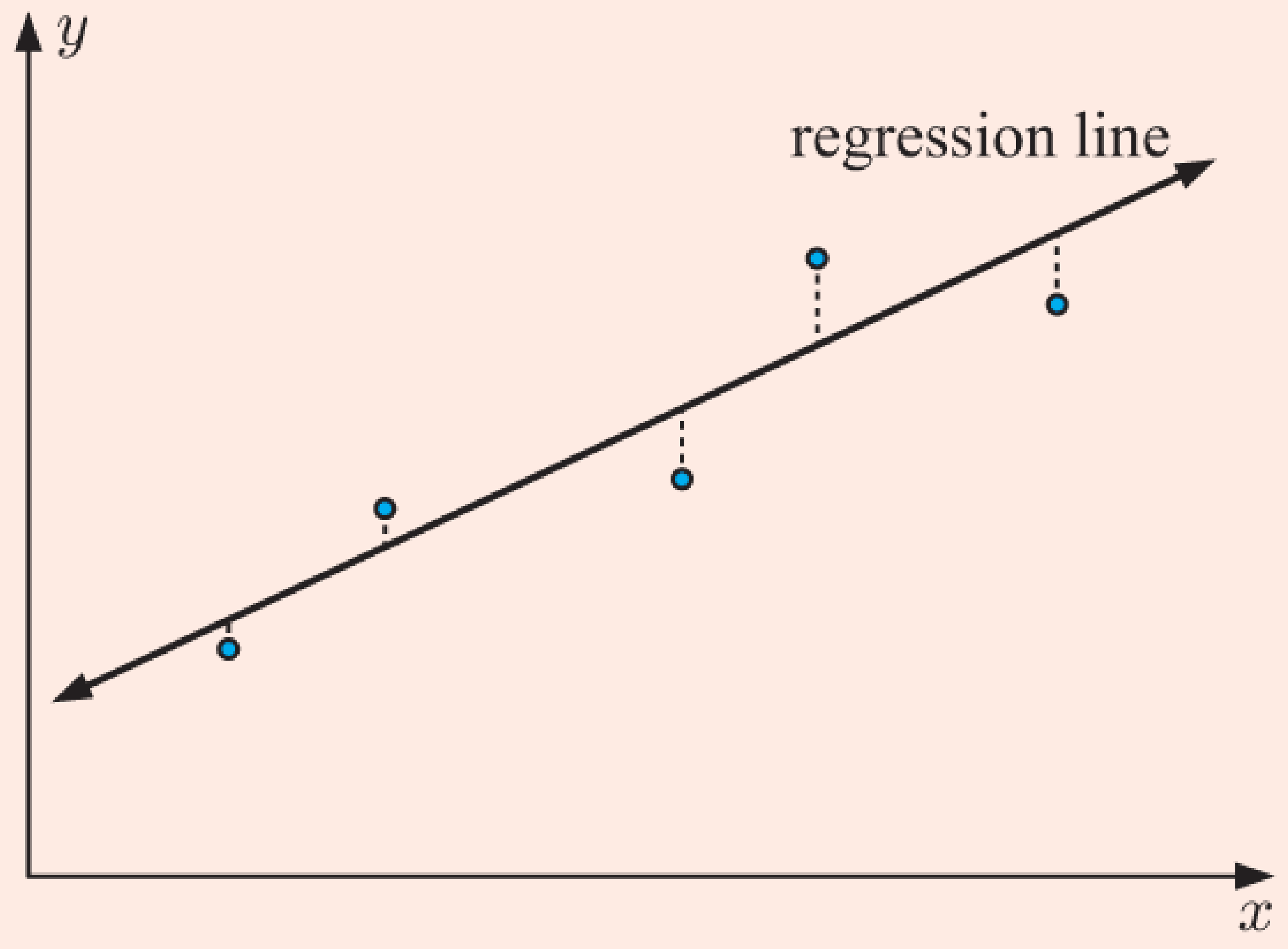
ACTIVITY 2 RESIDUAL PLOTS

In addition to the *correlation coefficient* and the *linearity* of a scatter diagram, we can use a **residual plot** to decide whether a linear model is appropriate. Click on the icon to explore these graphs.

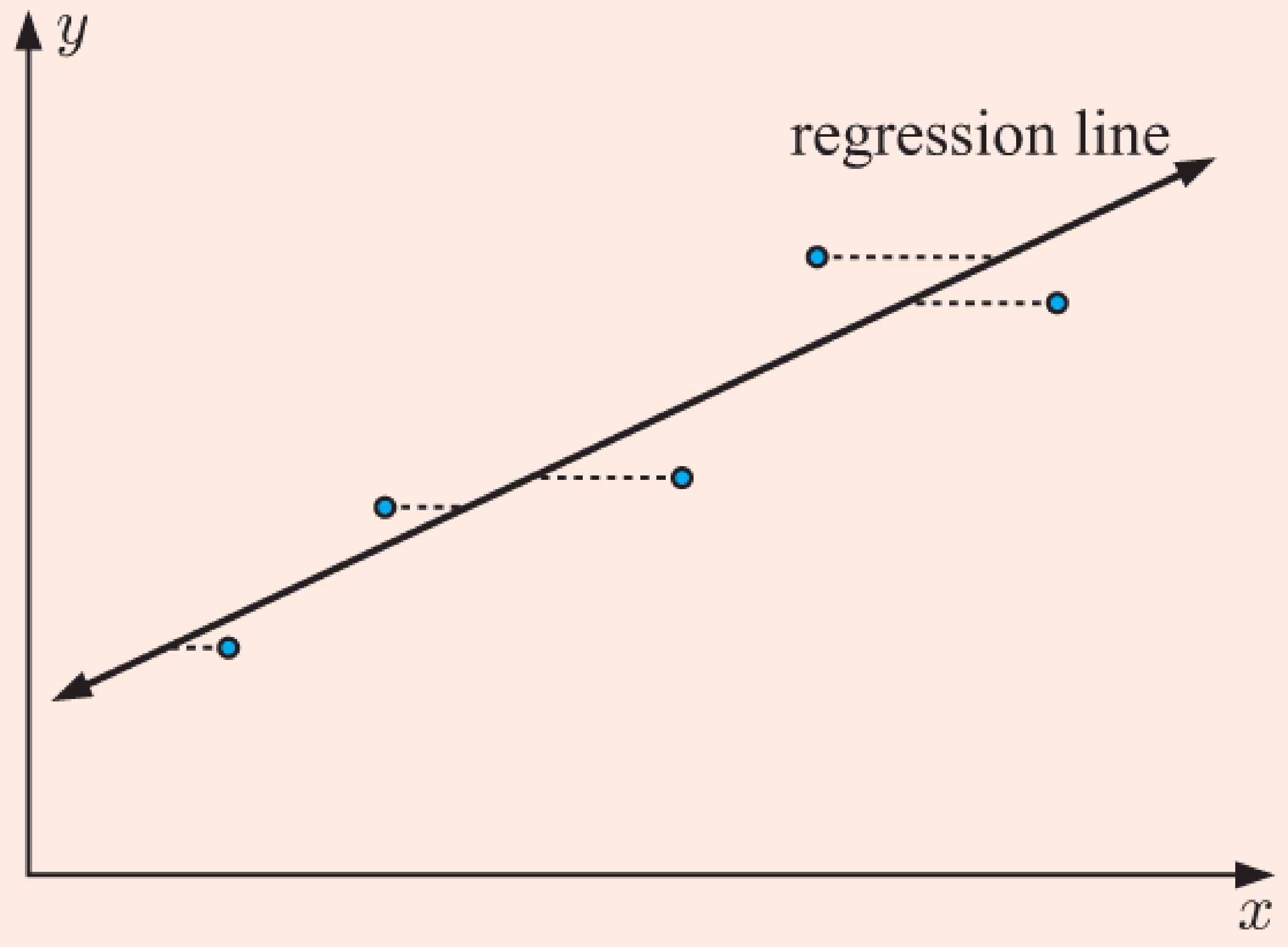
RESIDUAL PLOTS

DISCUSSION

When we generated our least squares regression line, our aim was to minimise the *vertical* distances between the data points and the line. This means that when we use the regression line to estimate a value of y given a value of x, we can be sure that the error in our estimate is minimised.



The graph alongside shows the same data points and regression line. It also shows the *horizontal* distance each point is from the line.



1 Do you think that this regression line also minimises the horizontal distances between the data points and the line?
How would the regression line be different if we tried to minimise the horizontal distances instead?

2 Suppose we use a regression line to estimate x given a value of y.
- **a** Can we be sure that the error in this estimate has been minimised?
- **b** Will this estimate be as reliable as an estimate of y given a value of x?

THEORY OF KNOWLEDGE

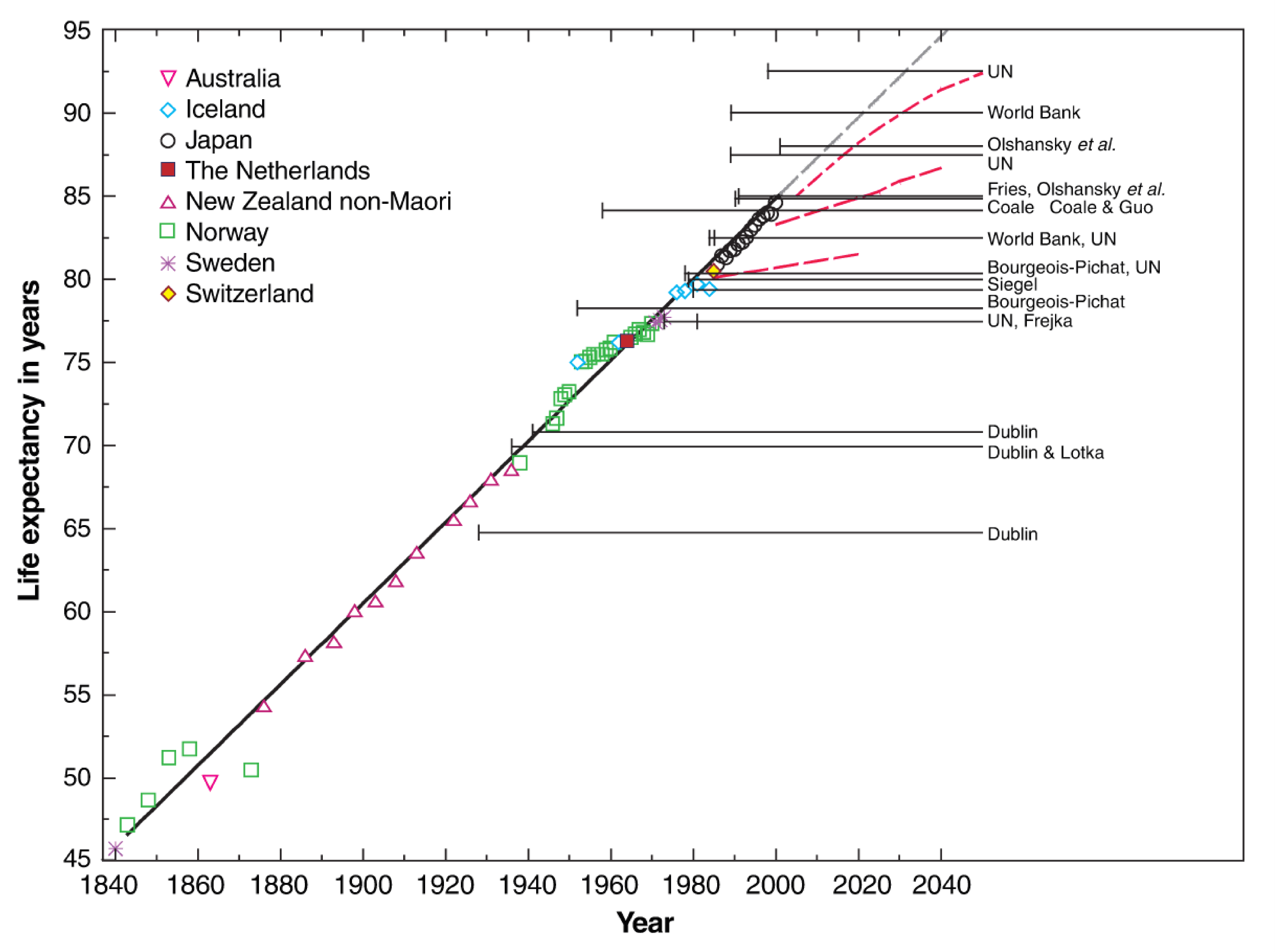
The use of extrapolation for predicting the future leads to debate on many global issues. Even when data shows a strong linear correlation, we need to consider whether it is reasonable for the trend to continue in the long term.

For example, the graph below is based on the article by Oeppen and Vaupel (2002)[1]. It shows female life expectancy from 1840 to the early 2000s, and the country with the highest female life expectancy at each point in time.

Notice that:

- The linear regression trend line is drawn in black, and extrapolated in grey.
- The horizontal black lines show asserted "ceilings" on life expectancy. The vertical line at the left end shows the year of publication.
- The dashed red lines denote projections of female life expectancy in Japan published by the United Nations (UN) in 1986, 1999, and 2001.

Record female life expectancy from 1840 to the present - Oeppen and Vaupel (2002)



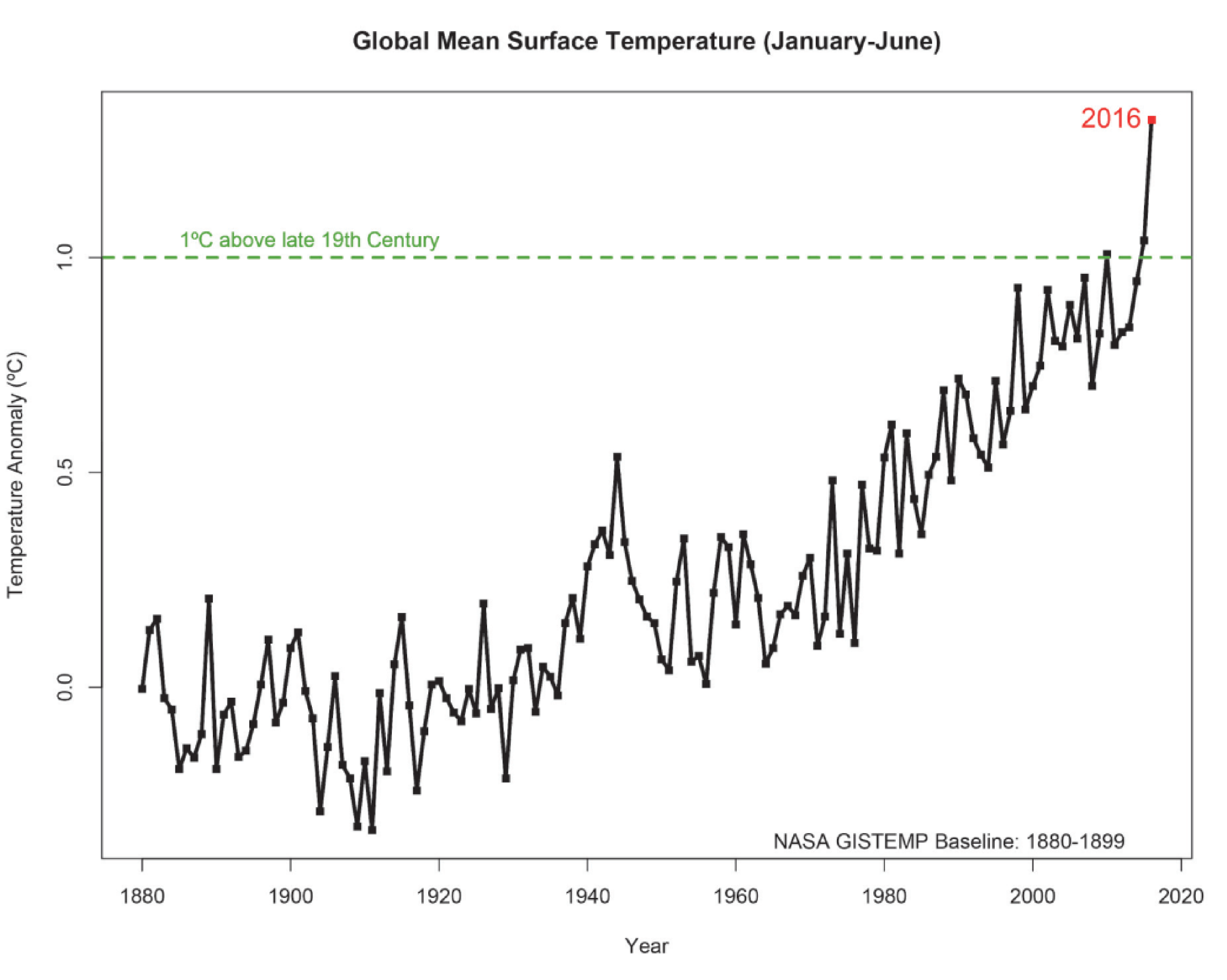
1 Discuss the relationship between the variables.

2 Use the regression line to predict female life expectancy in the year 2100. Do you think this is realistic?

3 Discuss the "ceilings" suggested by publishers over time. Is there evidence to suggest that human life expectancy will approach a limiting "ceiling"?

4 Discuss the accuracy of the UN projections for females in Japan from 1986 to 1999. Is there reason to expect the latest projection will be more reliable?

The graph below shows data from the NASA Goddard Institute for Space Studies[2]. The data for each point is for the first six months of the corresponding year.



5 Discuss the relationship between the variables. Is it reasonable to use a linear model to describe the mean surface temperature of the Earth over time? Is it reasonable to even conclude that the mean surface temperature of the Earth is increasing?

6 How can we predict the mean surface temperature of the Earth in the future?

7 Is mathematical extrapolation valid evidence for dictating environmental policy?

References:

[1] Oeppen and Vaupel, *Broken limits to life expectancy*, Science, **296**, 5570, 1029-1031, 2002.

[2] www.nasa.gov/feature/goddard/2016/climate-trends-continue-to-break-records

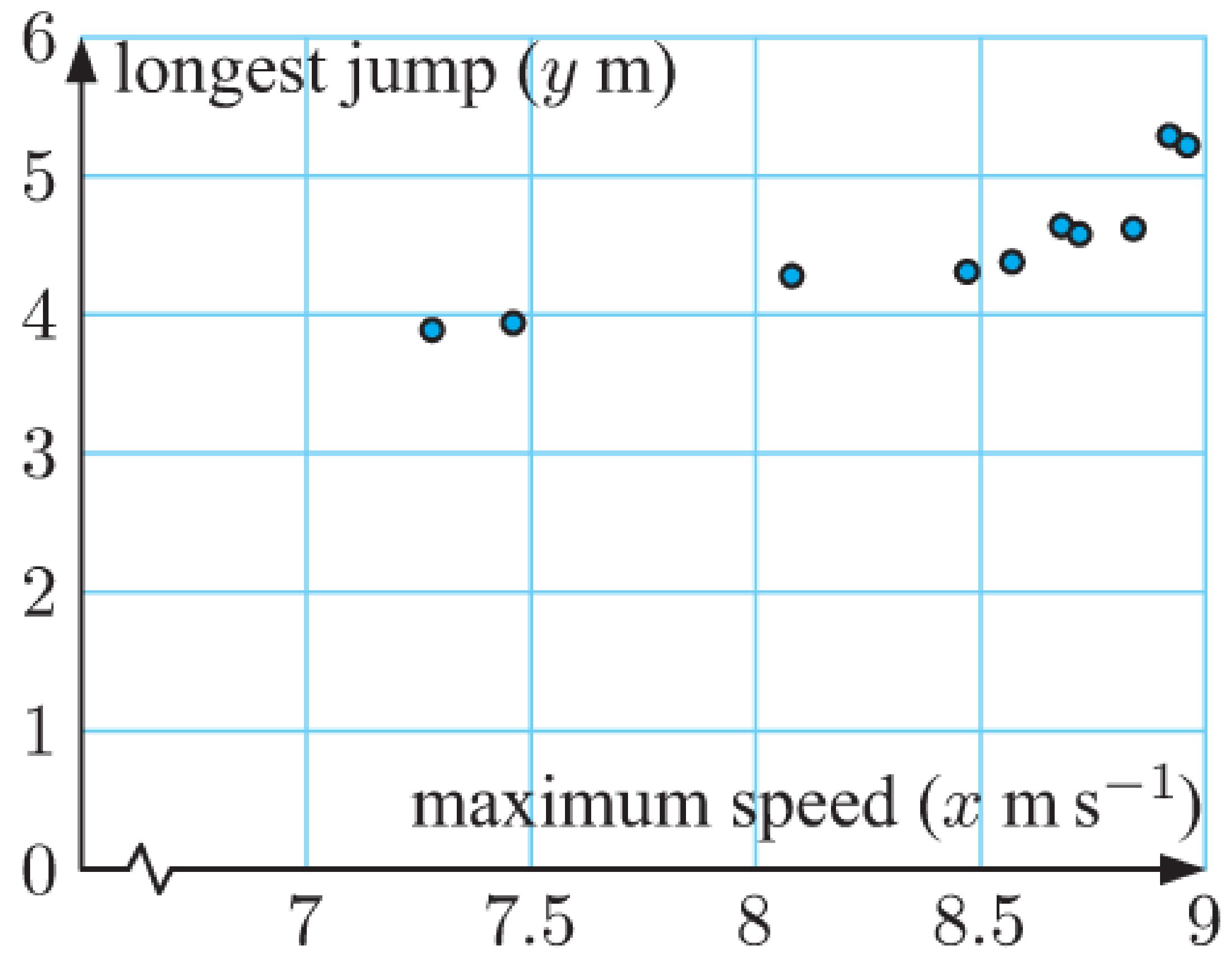
E SPEARMAN'S RANK CORRELATION COEFFICIENT

The table below shows data from 10 young athletes competing in a track and field competition. It includes the length of their longest jump, and the maximum speed they reached on their approach.

Athlete	A	B	C	D	E	F	G	H	I	J
Maximum speed (x m s^{-1})	8.68	8.92	8.57	8.47	8.96	8.84	8.72	7.46	7.28	8.08
Longest jump (y m)	4.64	5.29	4.38	4.31	5.22	4.62	4.58	3.94	3.89	4.28

The scatter diagram of the data is shown alongside.

Pearson's correlation coefficient for the data ≈ 0.864, which suggests a strong (but not perfect) positive correlation between the variables.

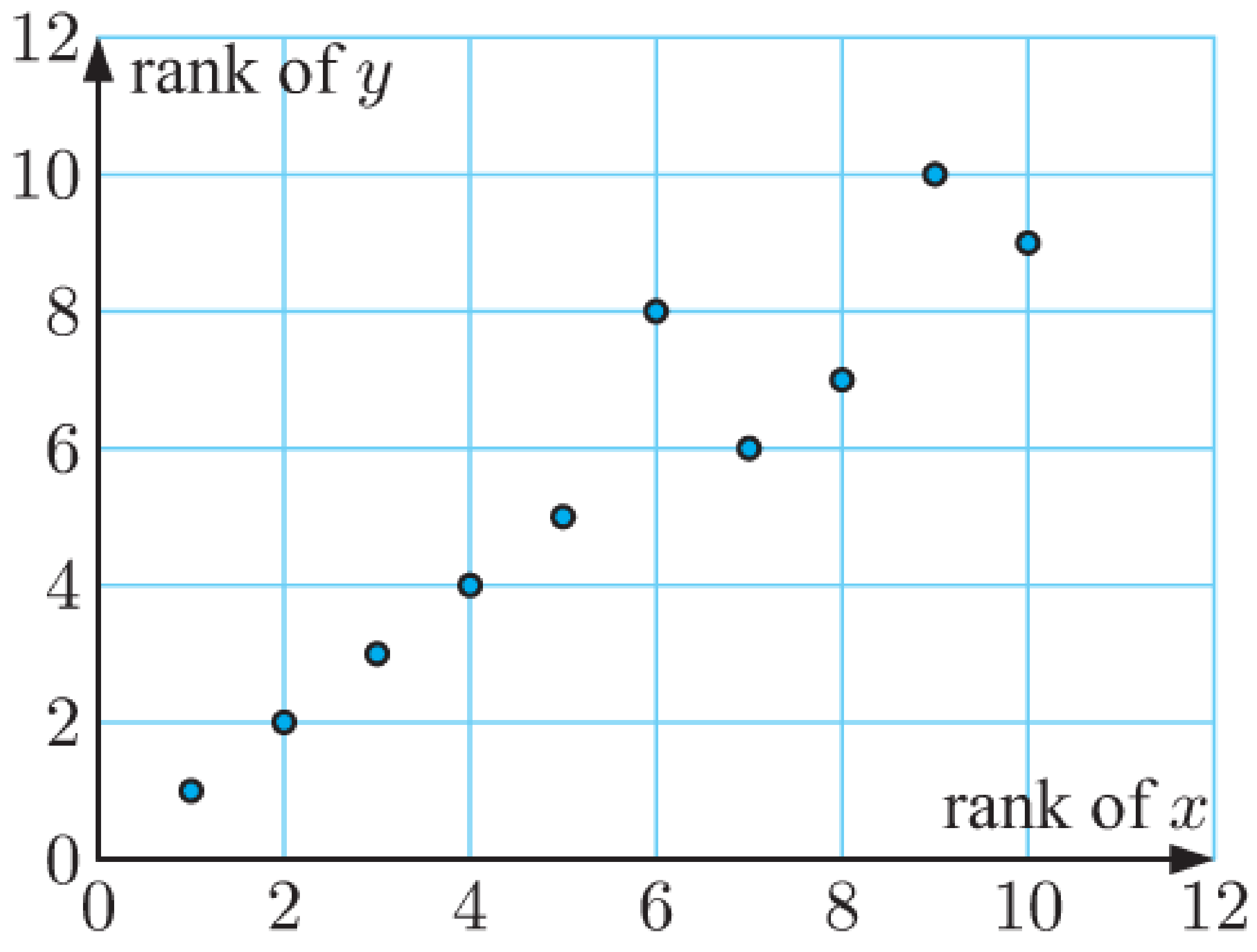


In a case like this, we may not be convinced that a linear model is appropriate for the data. We may therefore wish to focus just on the upward or downward *trend* of the data. One way to do this is to consider the relationship between each variable's **ranks** instead of the raw values.

For the above example, we can assign each x-value and each y-value a rank. A rank of 1 corresponds to the smallest value for the variable, rank 2 corresponds to the second smallest, and so on.

Athlete	A	B	C	D	E	F	G	H	I	J
Maximum speed (x m s^{-1})	8.68	8.92	8.57	8.47	8.96	8.84	8.72	7.46	7.28	8.08
rank of x	6	9	5	4	10	8	7	2	1	3
Longest jump (y m)	4.64	5.29	4.38	4.31	5.22	4.62	4.58	3.94	3.89	4.28
rank of y	8	10	5	4	9	7	6	2	1	3

The scatter diagram for the ranked data is shown alongside. For this data we observe a more clear linear trend with Pearson's correlation coefficient ≈ 0.952. This indicates a strong upward *but not necessarily linear* trend in the original data.



Spearman's rank correlation coefficient of a bivariate data set is defined as the Pearson product-moment correlation coefficient of the variables' **ranks**.

To distinguish between the two correlation coefficients, it is common to use r_p and r_s for Pearson's product-moment correlation and Spearman's rank correlation respectively.

Since Spearman's correlation only considers the relationship between the ranks and not the data itself, r_s is often used instead of r_p when the data is clearly non-linear, but has an upward or downward trend.

Example 5 **Self Tutor**

The population of black bears on a particular island has been recorded every 5 years since 1978:

Years since 1978 (t)	0	5	10	15	20	25	30	35	40
Population (P)	1030	1836	2678	3077	3394	3501	3479	3418	3511

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient r_p for this data.

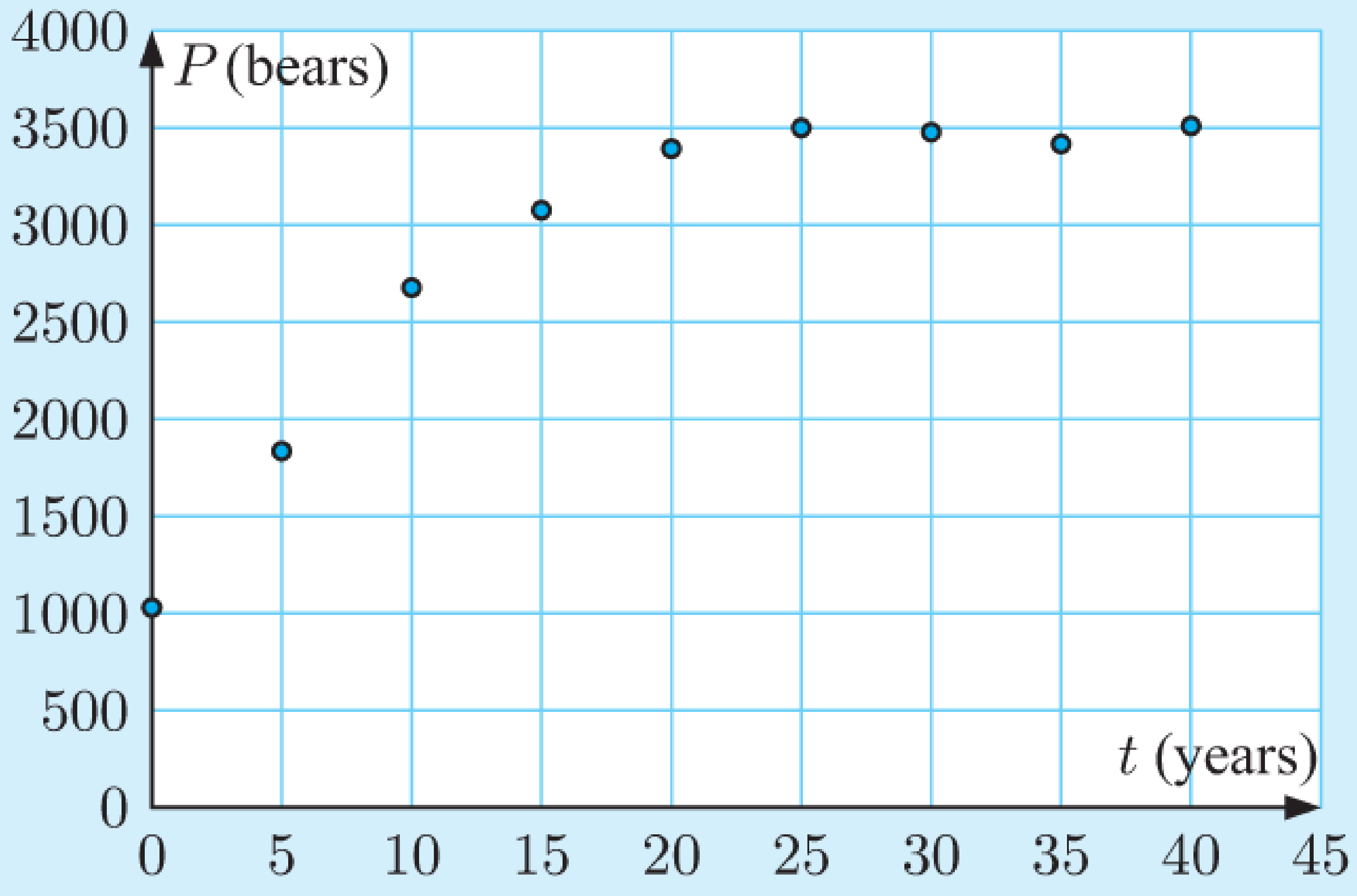
c Find the ranks for each variable and draw a scatter diagram of the ranks.

d Calculate Spearman's rank correlation coefficient, r_s.

e Which correlation coefficient is more appropriate to use for this data? Explain your answer.

f Describe the correlation between the variables.

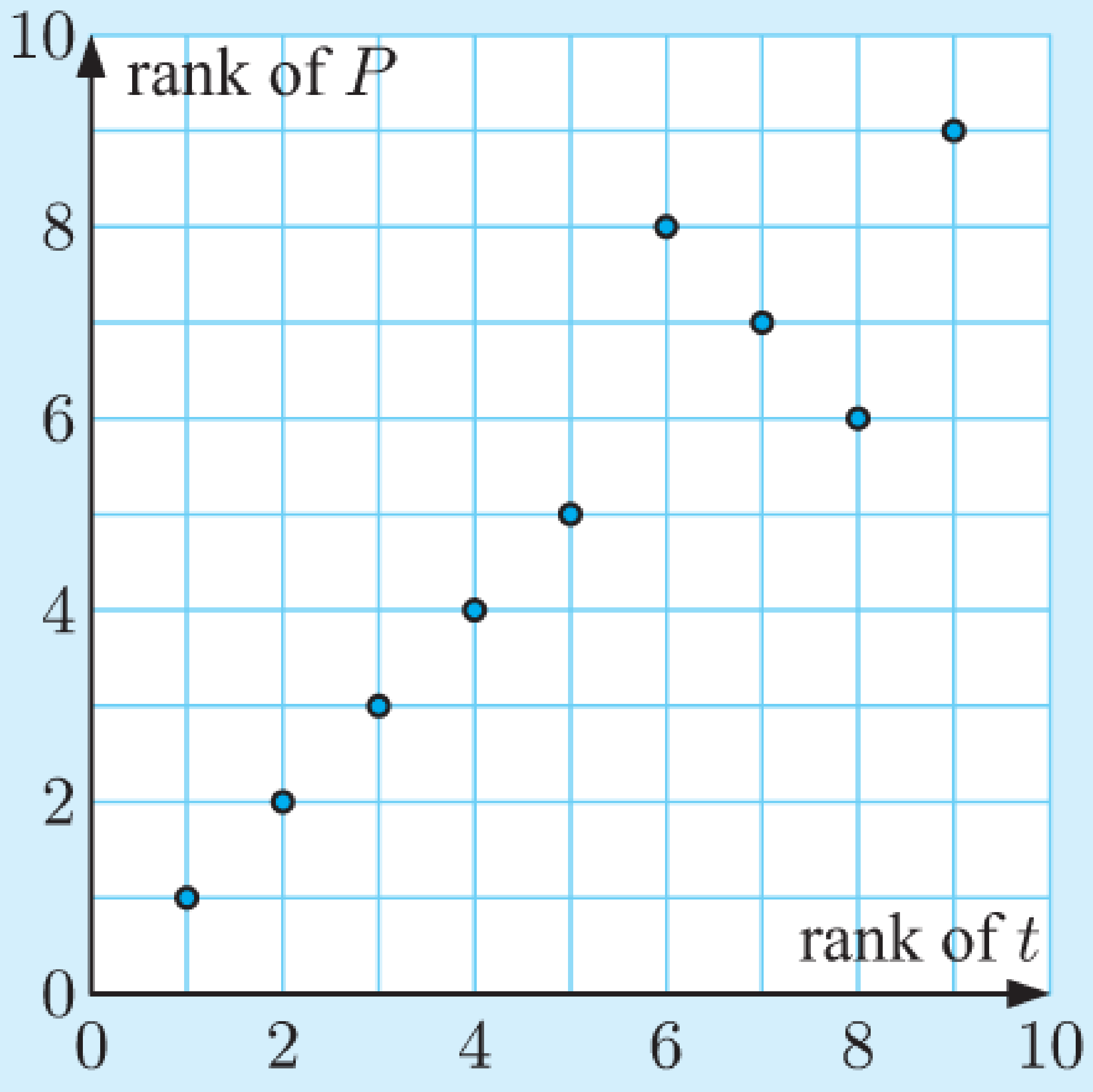
a



b Using technology, $r_p \approx 0.860$.

c

t	0	5	10	15	20	25	30	35	40
rank of t	1	2	3	4	5	6	7	8	9
P	1030	1836	2678	3077	3394	3501	3479	3418	3511
rank of P	1	2	3	4	5	8	7	6	9



d Using technology, $r_s \approx 0.933$.

e Looking at the scatter diagram of the raw data, the relationship between the variables is clearly *non-linear*. Spearman's rank correlation coefficient is therefore more appropriate.

f Based on the value of r_s, there is a strong, positive non-linear correlation between the variables.

RANKS OF TIED VALUES

In some cases we have x or y-values that are equal. These are called **ties**. For example, in the data alongside we have one tie in the y-values.

x	3	2	5	4	8	6	7
y	3	4	6	8	7	7	1

The values in a tie must be given the same rank. To do this, the unique values are assigned ranks which leave out values for the ties.

x	3	2	5	4	8	6	7
rank of x	2	1	4	3	7	5	6
y	3	4	6	8	7	7	1
rank of y	2	3	4	7			1

ranks 5 and 6

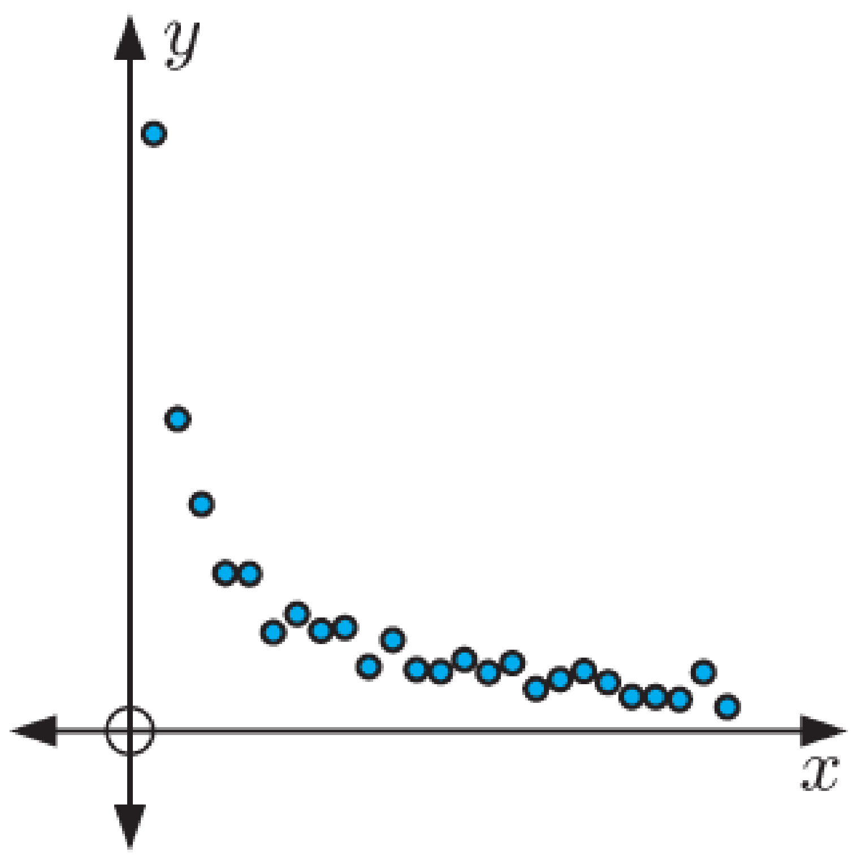
We then assign the ties the **average** of the ranks that were left out for them. In this case the ranks left out were 5 and 6, so we give the two shaded 7s the rank $= \frac{5+6}{2} = 5.5$.

x	3	2	5	4	8	6	7
rank of x	2	1	4	3	7	5	6
y	3	4	6	8	7	7	1
rank of y	2	3	4	7	5.5	5.5	1

EXERCISE 5E

1 Match each scatter diagram with the correct rank scatter diagram:

a



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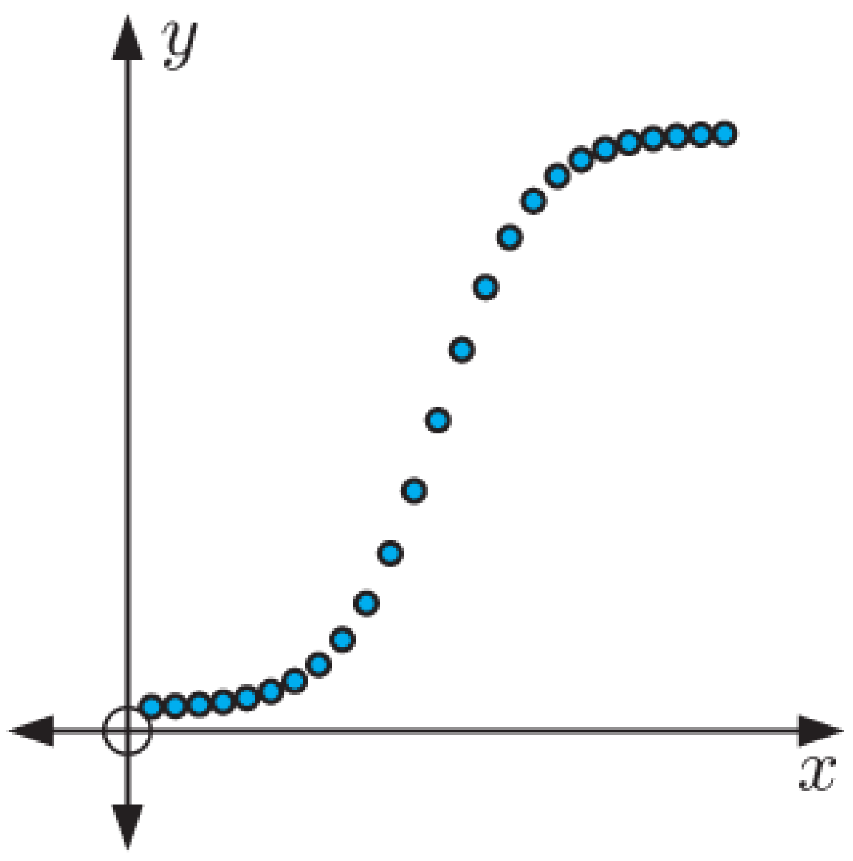

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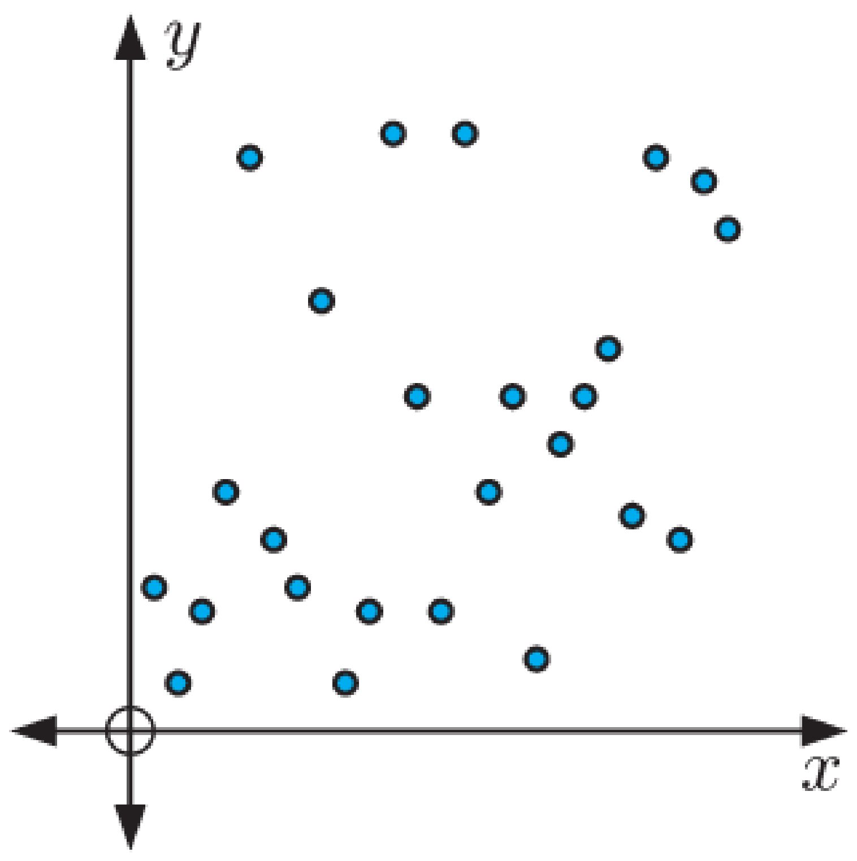
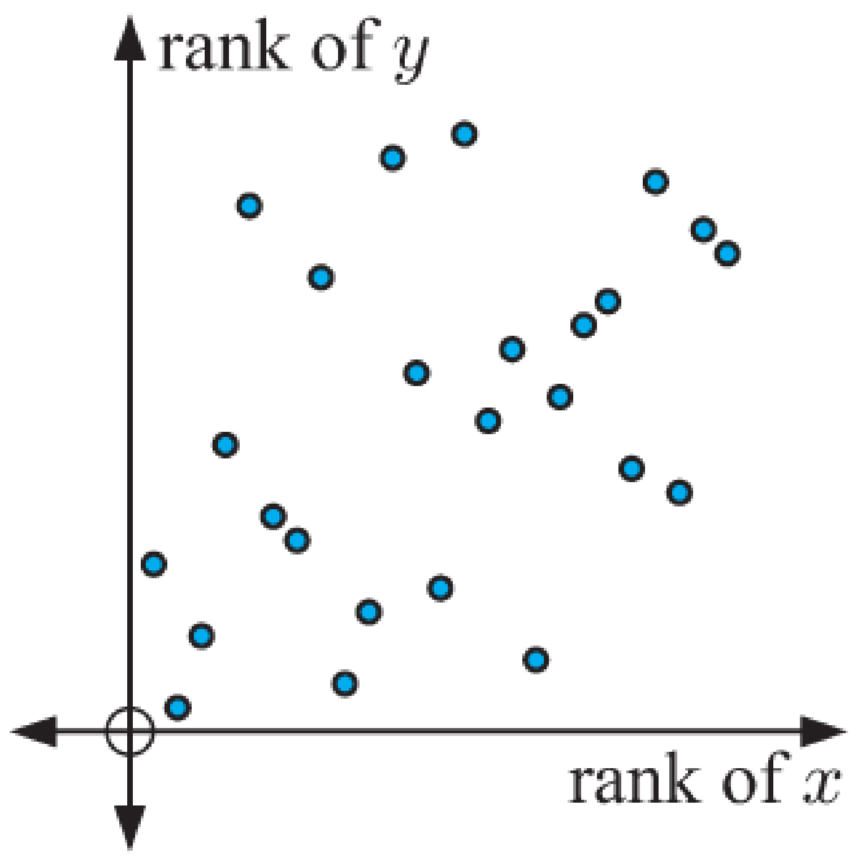
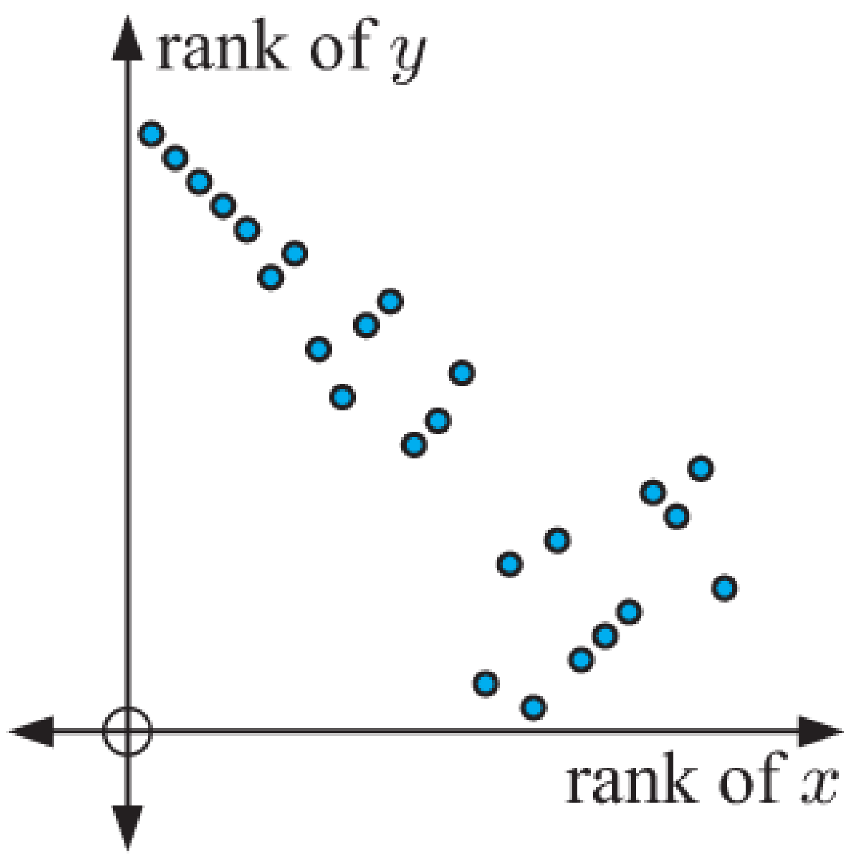


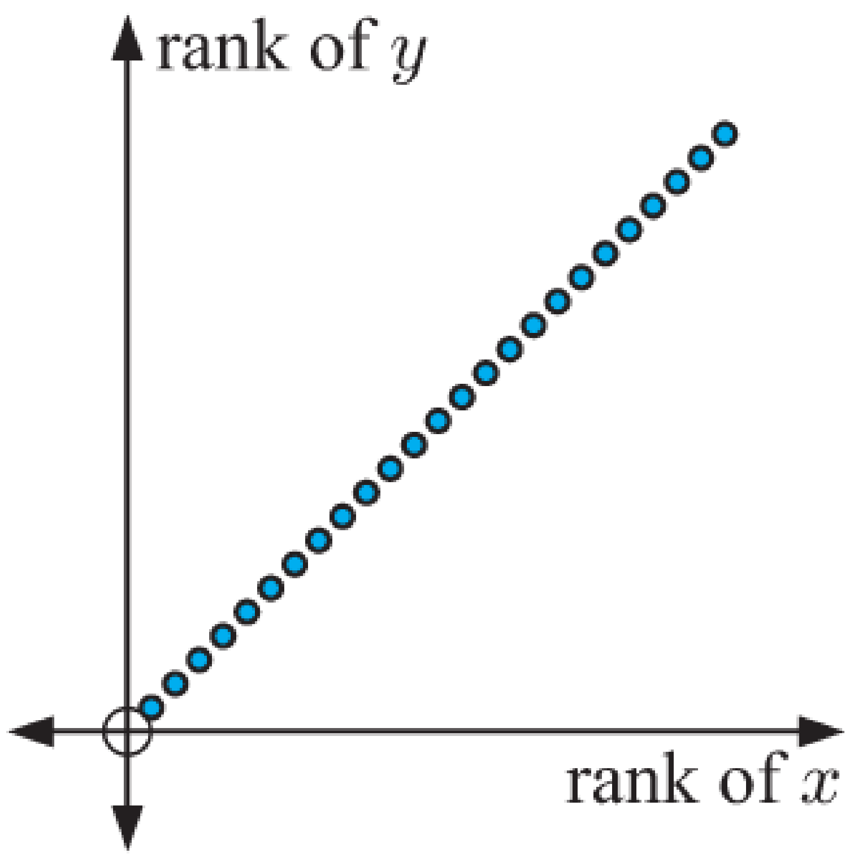
A



B



C

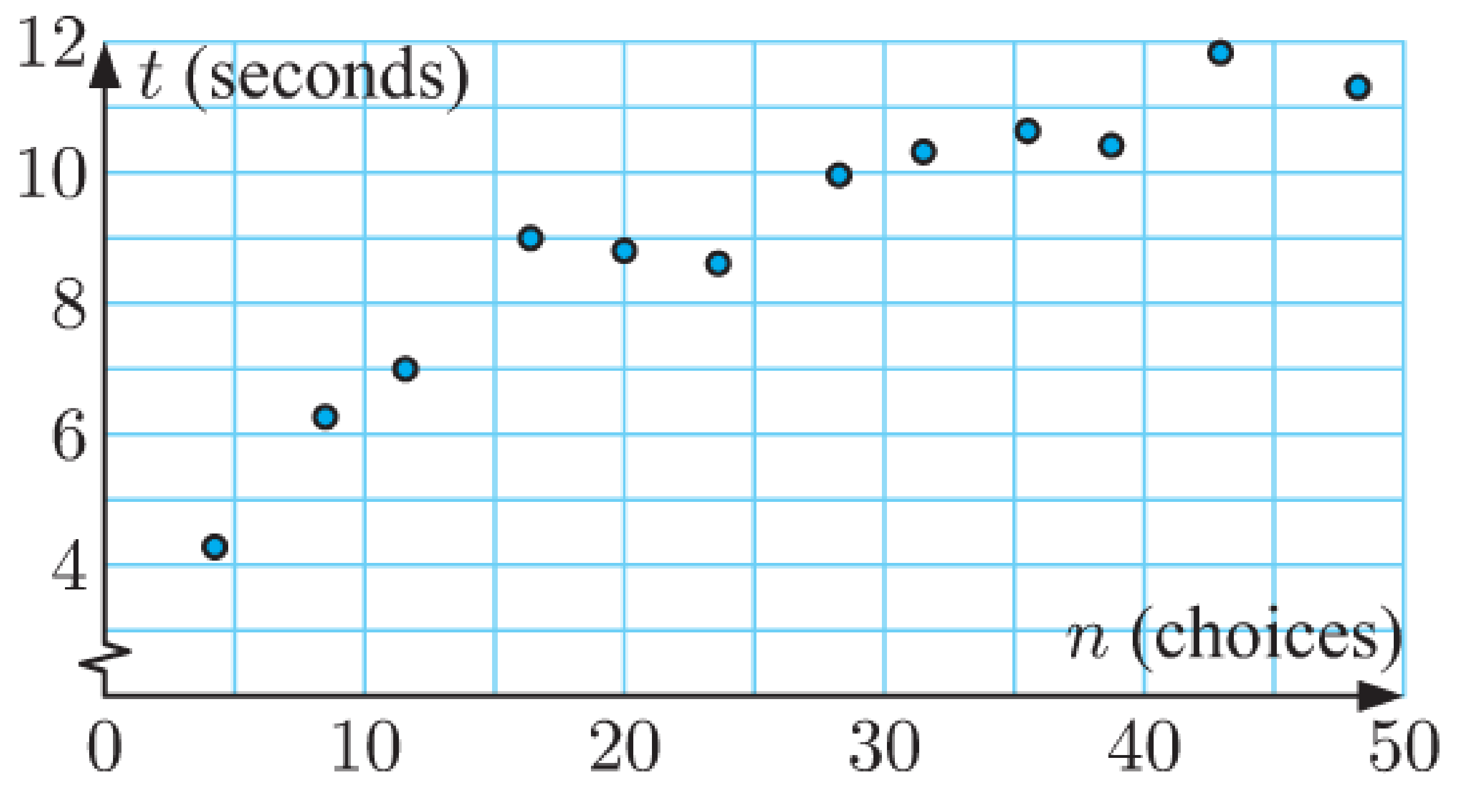


2 For a particular data set, Spearman's rank correlation $r_s \approx -0.7$.

a Is the trend in the data positive or negative?

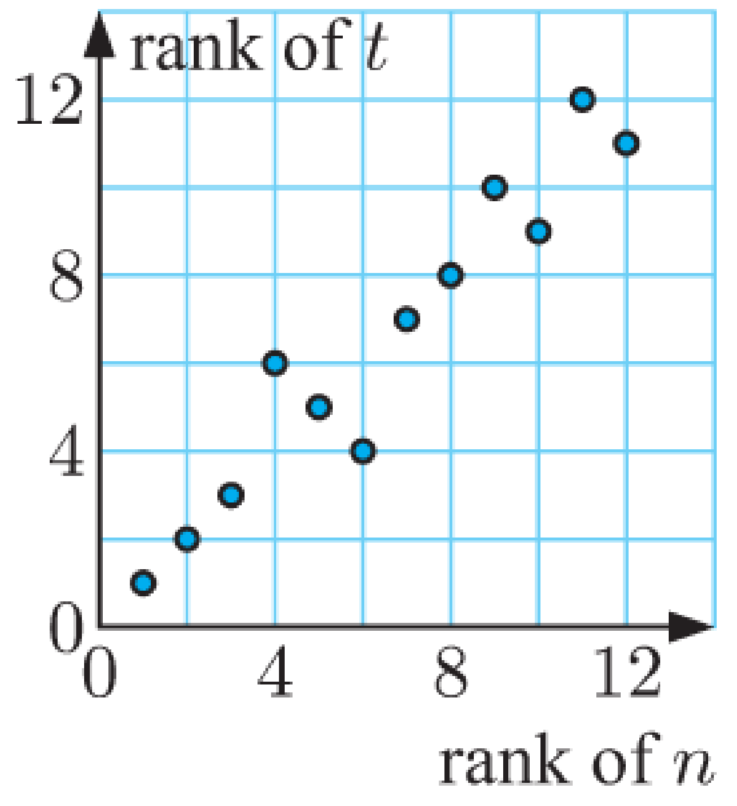
b Can we determine the *linearity* of the data from the value of r_s alone?

3 The scatter diagram alongside shows the time t that a person takes to make a selection given a set of n choices.

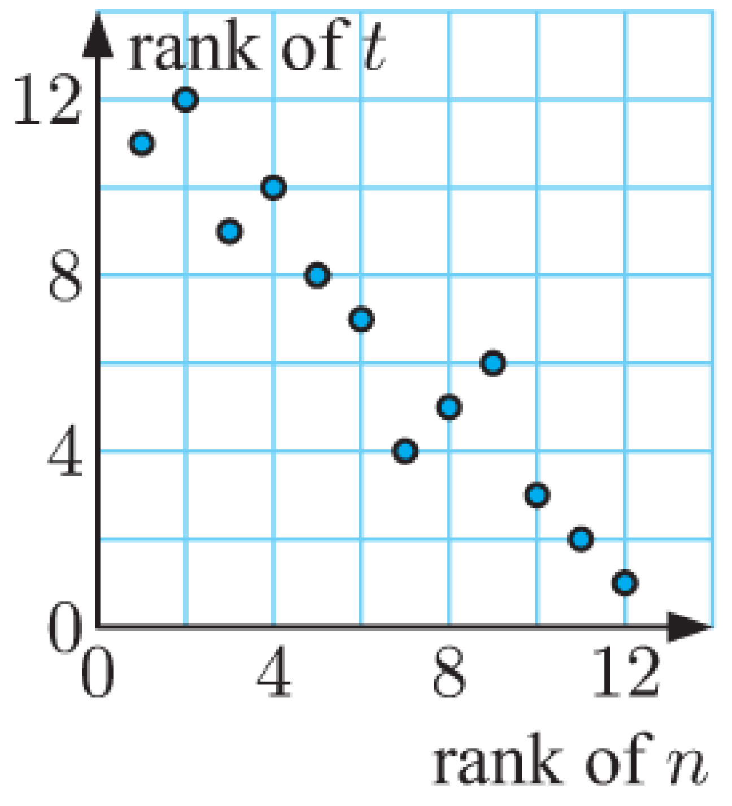


a Which of the following scatter diagrams is the scatter diagram of the ranks?

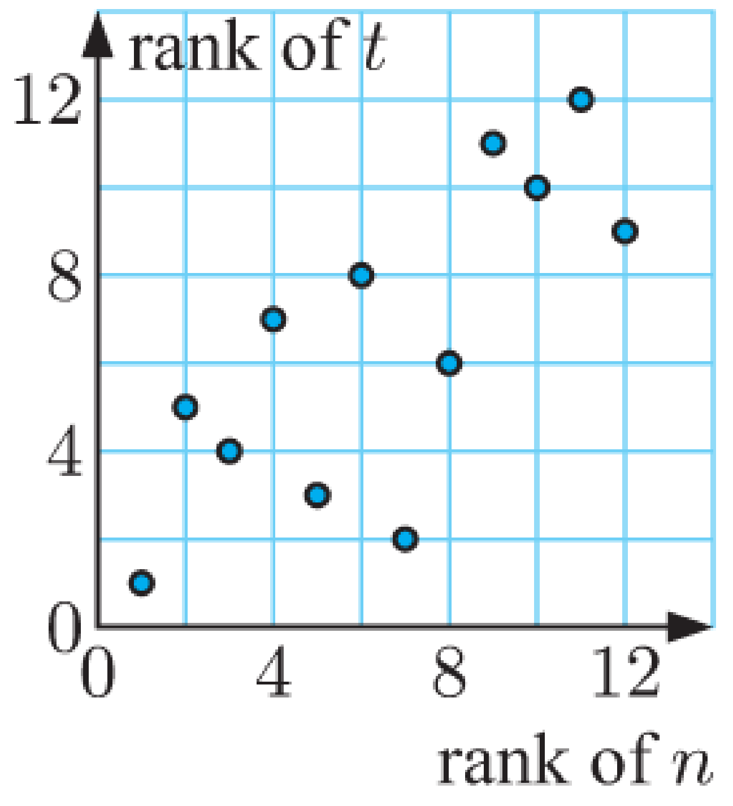
A



B



C



b *Hence* identify the correct value of Spearman's rank correlation coefficient for this data:

A $r_s \approx 0.958$ **B** $r_s \approx -0.958$ **C** $r_s \approx 0.755$

4 In a 60 minute Art lesson, students were asked to make as many paper cranes as possible. The table shows how long it took each student to make a paper crane, and how many cranes they made during the lesson:

Time taken (t min)	6	9.5	4	5	8	7.5	11	6.5
Cranes made (C)	9	5	16	11	6	7	4	8

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient.

c Find the ranks for each of the variables and draw a scatter diagram for the ranks.

d Calculate Spearman's rank correlation coefficient.

e Which correlation coefficient is more appropriate to use for this data? Explain your answer.

f Describe the correlation between the variables.

5 A local council collected data for the *area* of its parks and the number of maple trees each contains:

Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28

a Draw a scatter diagram for this data.

b Use the scatter diagram to explain why you would expect Pearson's correlation and Spearman's correlation coefficients to have similar values.

c Calculate r_p and r_s to verify your answer to **b**.

6 Ten students in a typing contest were given one minute to type as many words as possible. The table below shows how many words each student typed, and how many errors they made:

Number of words (x)	40	53	20	65	35	60	85	49	35	76
Number of errors (y)	11	15	2	20	4	22	30	16	27	25

- **a** Calculate r_p and r_s.
- **b** Draw a scatter diagram for this data and hence identify the outlier.
- **c** Recalculate r_p and r_s without the outlier.
- **d** Which correlation coefficient is more affected by the presence of the outlier?

7 Consider the long jump data from the start of this Section.

- **a** Use the longest jumps to determine the *placing* of each athlete in the competition.
- **b** Out of *longest jump* or *placing*, which is the independent variable?
- **c** Explain why:
  - **i** the variable *placing* is already ranked
  - **ii** Spearman's correlation coefficient for the variables *longest jump* and *placing* must be exactly -1.

ACTIVITY 3 ANSCOMBE'S QUARTET

Anscombe's quartet is a collection of four bivariate data sets which have interesting statistical properties.

It was first described in 1973 by the English statistician **Francis Anscombe** (1918 - 2001). At the time, computers were becoming increasingly popular in statistics, as they allowed for more large scale and complex computations to be done within a reasonable amount of time. However, many common statistical packages primarily performed numerical calculations rather than produce graphs. Such output was often limited to those with advanced programming skills.

In his 1973 article, Anscombe stressed that:

> "*A computer should make* both *calculations* and *graphs. Both sorts of output should be studied; each will contribute to understanding.*"

Francis Anscombe
Photo courtesy of
Yale University.

The data values for Anscombe's quartet are given in the tables below:

Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

Enter the data into your **graphics calculator** or click on the icon to access the data in the **statistics package**.

STATISTICS PACKAGE

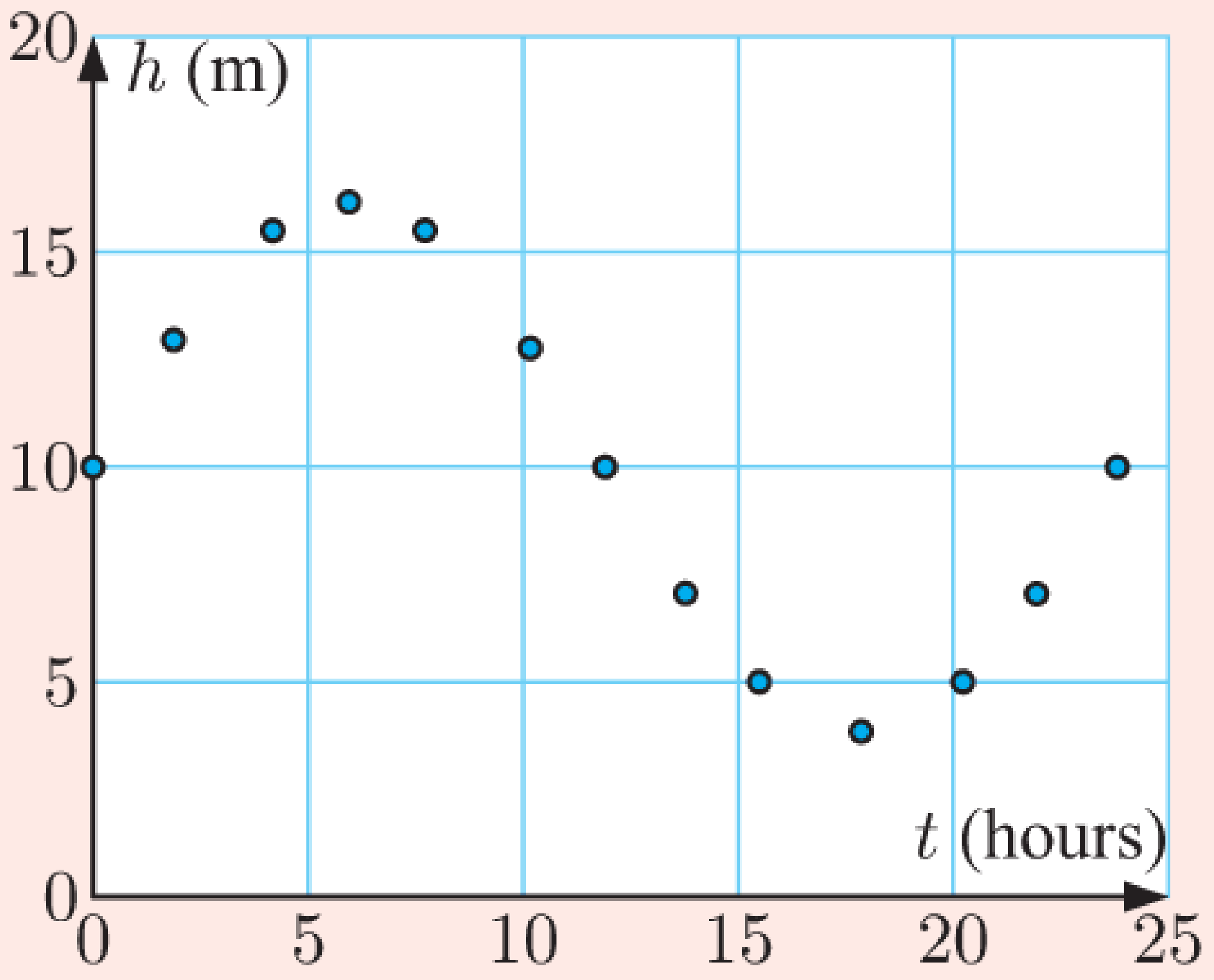
What to do:

1. For each data set, use technology to calculate:
   - **a** the mean of each variable
   - **b** the population variance of each variable.

   Comment on your answers.
2. Find the regression line and Pearson's product-moment correlation for each data set. What do you notice?
3. Construct a scatter diagram for each data set, and plot the corresponding regression line on the same set of axes.
4. How do your calculations in **1** and **2** compare to your graphs in **3**? Is a linear model necessarily appropriate for each data set?
5. Calculate Spearman's rank correlation coefficient for each data set and comment on your results.
6. Discuss why it is important to consider both graphs *and* descriptive statistics when analysing data.

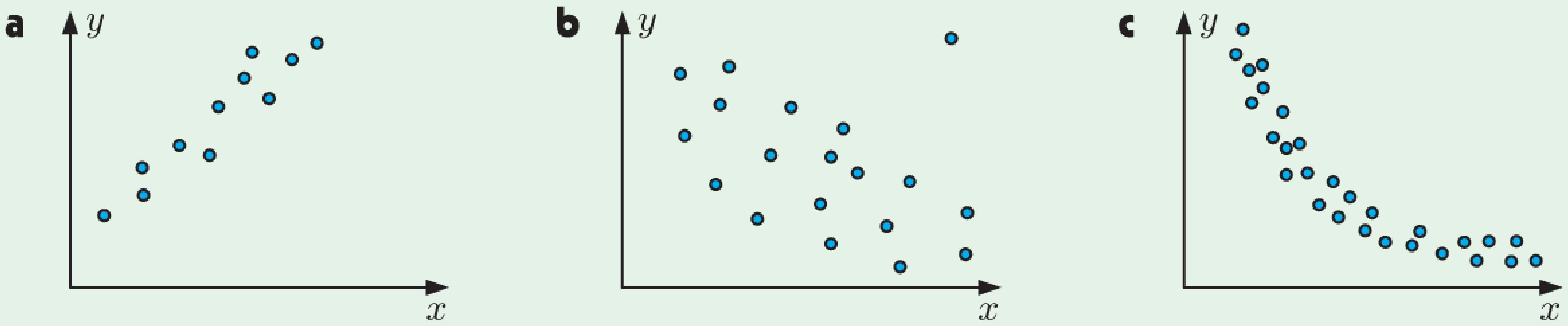
   For example, consider the data in the scatter diagram alongside.

   Is either Pearson's product-moment correlation or Spearman's rank correlation appropriate for measuring the strength of the relationship between these variables?



REVIEW SET 5A

1. For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.



2. Kerry wants to investigate the relationship between the *water bill* and the *electricity bill* for the houses in her neighbourhood.
   - **a** Do you think the correlation between the variables is likely to be positive or negative? Explain your answer.
   - **b** Is there a causal relationship between the variables? Justify your answer.

3 The table below shows the ticket and beverage sales for each day of a 12 day music festival:

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient r_{p}.

c Find the ranks for each variable and draw a scatter diagram for the ranks.

d Calculate Spearman's rank correlation coefficient r_{s} for these variables.

e Describe the correlation between *ticket sales* and *beverage sales*.

4 A garden centre manager believes that during March, the number of customers is related to the temperature at noon. Over a fortnight, the number of customers and the noon temperature were recorded.

Temperature (x°C)	23	25	28	30	30	27	25	28	32	31	33	29	27
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76

a Draw a scatter diagram of the data.

b Calculate Pearson's product-moment correlation coefficient r_{p}.

c Calculate Spearman's rank correlation coefficient r_{s}.

d Identify the outlier.

e Remove the outlier, and re-calculate r_{p} and r_{s}.

f Which correlation coefficient is most affected by the presence of the outlier?

g Describe the association between the *number of customers* and the *noon temperature* at the garden centre.

5 A clothing store recorded the length of time customers were in the store and the amount they spent.

Time (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
Money (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a Find the mean for each variable.

b Draw a scatter diagram for the data. Plot the mean point, and draw a line of best fit by eye.

c Describe the relationship between *time in the store* and the *money spent*.

6 Tomatoes are sprayed with a pesticide-fertiliser mix. The table below shows the *yield of tomatoes* per bush for various *spray concentrations*.

Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes per bush (y)	67	90	103	120	124	150	82

a Draw a scatter diagram to display the data.

b Determine the value of r and interpret your answer.

c Is there an outlier present that is affecting the correlation?

d The outlier was found to be a recording error. Remove the outlier from the data set, and recalculate r. Is it reasonable to now fit a linear model?

e Determine the equation of the regression line.

f State and interpret the gradient and y-intercept of the regression line.

g Use your line to estimate:

  i the yield if the spray concentration is 7 mL per L

  ii the spray concentration if the yield is 200 tomatoes per bush.

h Comment on the reliability of your estimates in **g**.

7 The ages and heights of children at a playground are given below:

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157

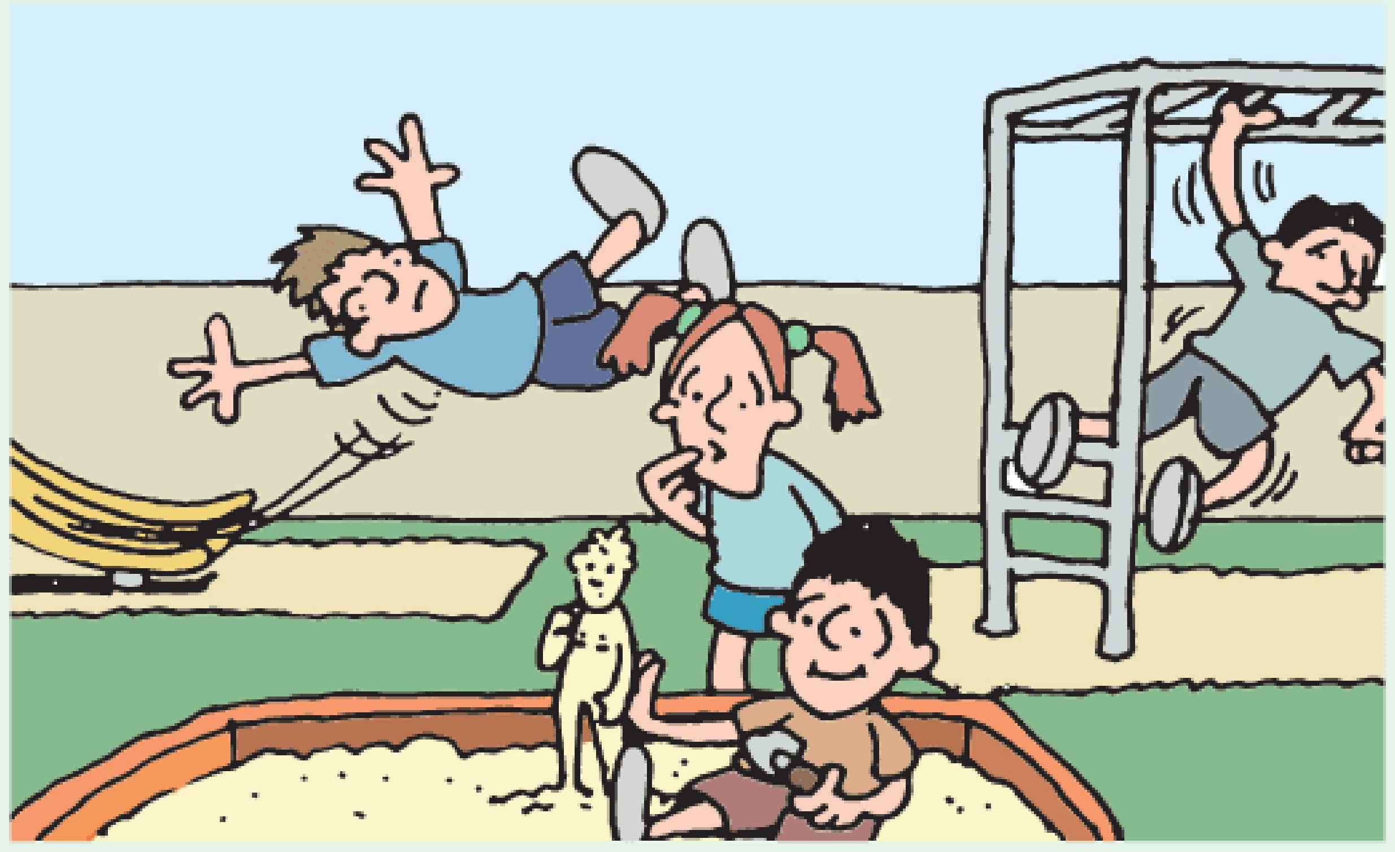
a Draw a scatter diagram for the data.

b Use technology to find the regression line.

c State and interpret the gradient of the regression line.

d Use the regression line to predict the height of a 5 year old child.

e Based on the given data, at what age would you expect a child to reach 140 cm in height?



REVIEW SET 5B

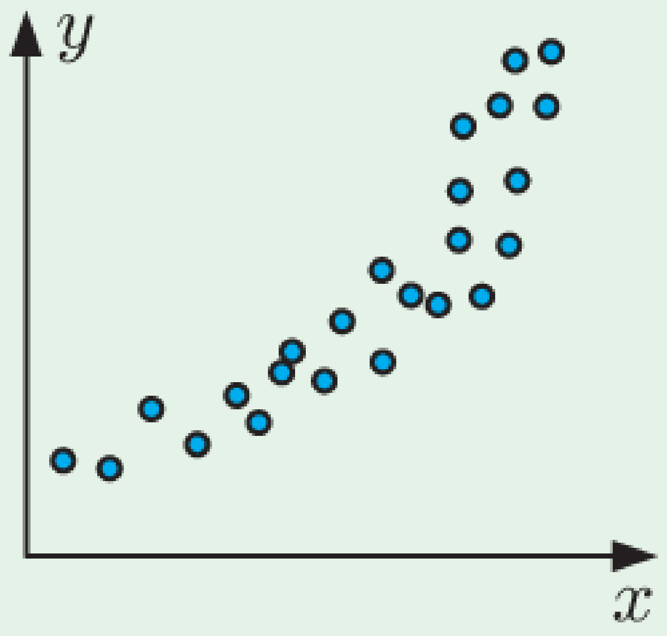
1 For each pair of variables, discuss whether the correlation between the variables is likely to be positive or negative, and whether a causal relationship exists between the variables:

a *price of tickets* and *number of tickets sold*

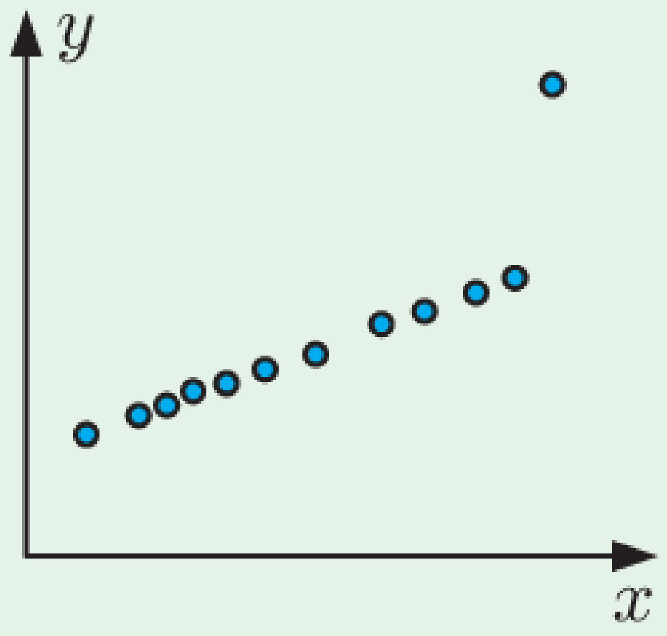
b *ice cream sales* and *number of shark attacks*.

2 Match each scatter diagram to the correct rank correlation coefficient:

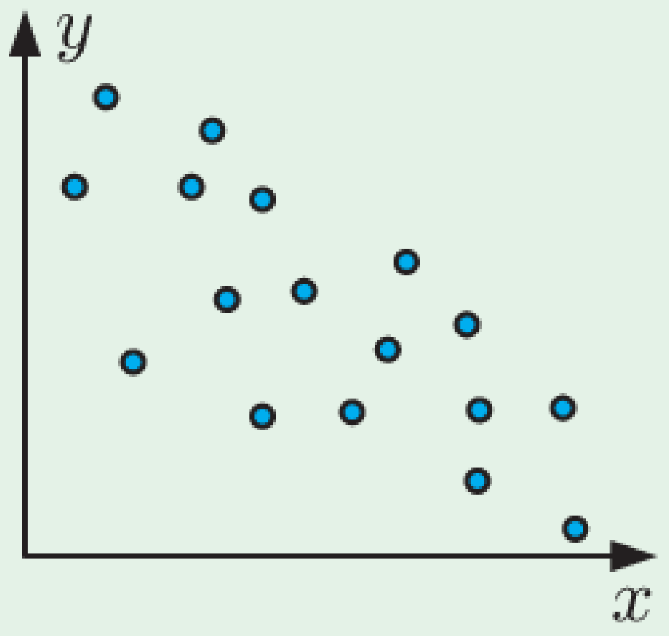
a



b



c



A $r_s = 1$ **B** $r_s = -0.4$ **C** $r_s = 0.7$

3 A group of students is comparing their results for a Mathematics test and an Art project:

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66

a Construct a scatter diagram for the data.

b Describe the relationship between the Mathematics and Art marks.

c Calculate the correlation coefficient r between the variables.

4 Safety authorities advise drivers to travel three seconds behind the car in front of them. This gives the driver a greater chance of avoiding a collision if the car in front has to brake quickly or is itself involved in an accident.

A test was carried out to find out how long it would take a driver to bring a car to rest from the time a red light was flashed. The following results were recorded for a particular driver in the same car under the same test conditions.

Speed (v km h^{-1})	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a Find the mean point $(\overline{v}, \overline{t})$.

b Draw a scatter diagram of the data. Add the mean point and draw a line of best fit by eye.

c Hence estimate the stopping time for a speed of:

i 55 km h^{-1} **ii** 110 km h^{-1}

d Which of your estimates in **c** is more likely to be reliable?

5 A craft shop sells canvasses in a variety of sizes. The table below shows the area and price of each canvas type.

Area (x cm^2)	100	225	300	625	850	900
Price (£y)	6	12	13	24	30	35

a Construct a scatter diagram for the data.

b Calculate the product-moment correlation coefficient r.

c Describe the correlation between *area* and *price*.

d Find the regression line and draw it on your scatter diagram.

e Estimate the price of a canvas with area 1200 cm^2. Discuss whether your estimate is likely to be reliable.

6 Eight identical flower beds contain petunias. The different beds were watered different numbers of times each week, and the number of flowers each bed produced was recorded in the table below:

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250

a Draw a scatter diagram for the data, and describe the correlation between the variables.

b Find the equation of the regression line.

c Is it likely that a causal relationship exists between these two variables? Explain your answer.

d Plot the regression line on the scatter diagram.

e Violet has two beds of petunias. She waters one of the beds 5 times a fortnight and the other 10 times a week.

i How many flowers can she expect from each bed?

ii Discuss which of your estimates is likely to be more reliable.

7 A drinks vendor varies the price of Supa-fizz on a daily basis. He records the number of sales of the drink as shown:

Price (\$$p$)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431

- **a** Produce a scatter diagram for the data.
- **b** Are there any outliers? If so, should they be included in the analysis?
- **c** Calculate the equation of the regression line.
- **d** State and interpret the gradient of the regression line.
- **e** Do you think the regression line would give a reliable prediction of sales if Supa-fizz was priced at 50 cents? Explain your answer.

8 A bird bath is filled with water. Over time, the water evaporates as shown in the table below:

Time (t hours)	3	6	9	12	15	18	21	24
Water remaining (V litres)	6.7	3.6	2	1.1	0.6	0.32	0.18	0.10

- **a** Draw a scatter diagram of V against t.
- **b** Calculate Pearson's product-moment correlation r_p for the data.
- **c** By inspecting the data and your scatter diagram, state the value of Spearman's rank correlation r_s.
- **d** Which correlation coefficient is more appropriate for this data? Explain your answer.

Chapter 6

Quadratic functions

Contents:

A Quadratic functions
B Graphs from tables of values
C Axes intercepts
D Graphs of the form $y = ax^2$
E Graphs of quadratic functions
F Axis of symmetry
G Vertex
H Finding a quadratic from its graph
I Intersection of graphs
J Quadratic models

OPENING PROBLEM

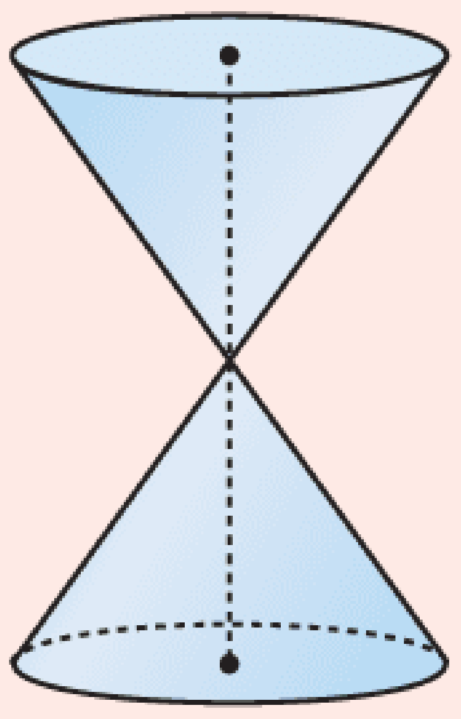
When an athlete throws a javelin, its height above the ground when it has travelled a horizontal distance of x m, is given by $H(x) = -0.015x^2 + x + 1.7$ metres.

Things to think about:

- **a** What does the graph of $H(x)$ look like?
- **b** From what height was the javelin released?
- **c** How high is the javelin after it has travelled 20 metres horizontally?
- **d** What is the maximum height reached by the javelin?
- **e** For what values of x is it reasonable to use this model?
- **f** The javelin was released when the athlete was 3.4 m behind the line. What distance was recorded for the throw?

In this Chapter we will study **quadratic functions** and investigate their graphs which are called **parabolas**. There are many examples of parabolas in everyday life, including water fountains, bridges, and radio telescopes.

ACTIVITY 1 CONIC SECTIONS



A cone is *right-circular* if its apex is directly above the centre of the base.

Suppose we have two right-circular cones, and we place one upside-down on the first. Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result, called **conic sections**:

- a point
- a line
- a line-pair
- a circle
- an ellipse
- a parabola
- a hyperbola

Click on the icon to explore the conic sections.

DEMO

You should observe how the parabola results when cutting the cone parallel to its slant edge.

A QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

FINDING y GIVEN x

For any value of x, the corresponding value of y can be found by substitution.

Example 1 **Self Tutor**

If $y = -2x^2 + 3x + 1$ find the value of y when:

a $x = 0$ **b** $x = 2$ **c** $x = -3$.

a When $x = 0$,

$$\begin{aligned} y &= -2(0)^2 + 3(0) + 1 \\ &= 0 + 0 + 1 \\ &= 1 \end{aligned}$$

b When $x = 2$,

$$\begin{aligned} y &= -2(2)^2 + 3(2) + 1 \\ &= -8 + 6 + 1 \\ &= -1 \end{aligned}$$

c When $x = -3$,

$$\begin{aligned} y &= -2(-3)^2 + 3(-3) + 1 \\ &= -18 - 9 + 1 \\ &= -26 \end{aligned}$$

SUBSTITUTING POINTS

We can test whether an ordered pair (x, y) satisfies a quadratic function by substituting the x-coordinate into the function, and seeing whether the result matches the y-coordinate.

Example 2 **Self Tutor**

Determine whether the given point satisfies the quadratic function:

a $y = 3x^2 + 2x$ $\quad (2, 16)$ **b** $y = -x^2 - 2x + 1$ $\quad (-3, 1)$

a When $x = 2$,

$$\begin{aligned} y &= 3(2)^2 + 2(2) \\ &= 12 + 4 \\ &= 16 \end{aligned}$$

$\therefore$ $(2, 16)$ satisfies the function $y = 3x^2 + 2x$.

b When $x = -3$,

$$\begin{aligned} y &= -(-3)^2 - 2(-3) + 1 \\ &= -9 + 6 + 1 \\ &= -2 \end{aligned}$$

$\therefore$ $(-3, 1)$ does not satisfy the function $y = -x^2 - 2x + 1$.

FINDING x GIVEN y

When we substitute a value for y into a quadratic function, we are left with a quadratic equation. Solving the quadratic equation gives us the values of x corresponding to that y-value. There may be 0, 1, or 2 solutions.

Example 3 **Self Tutor**

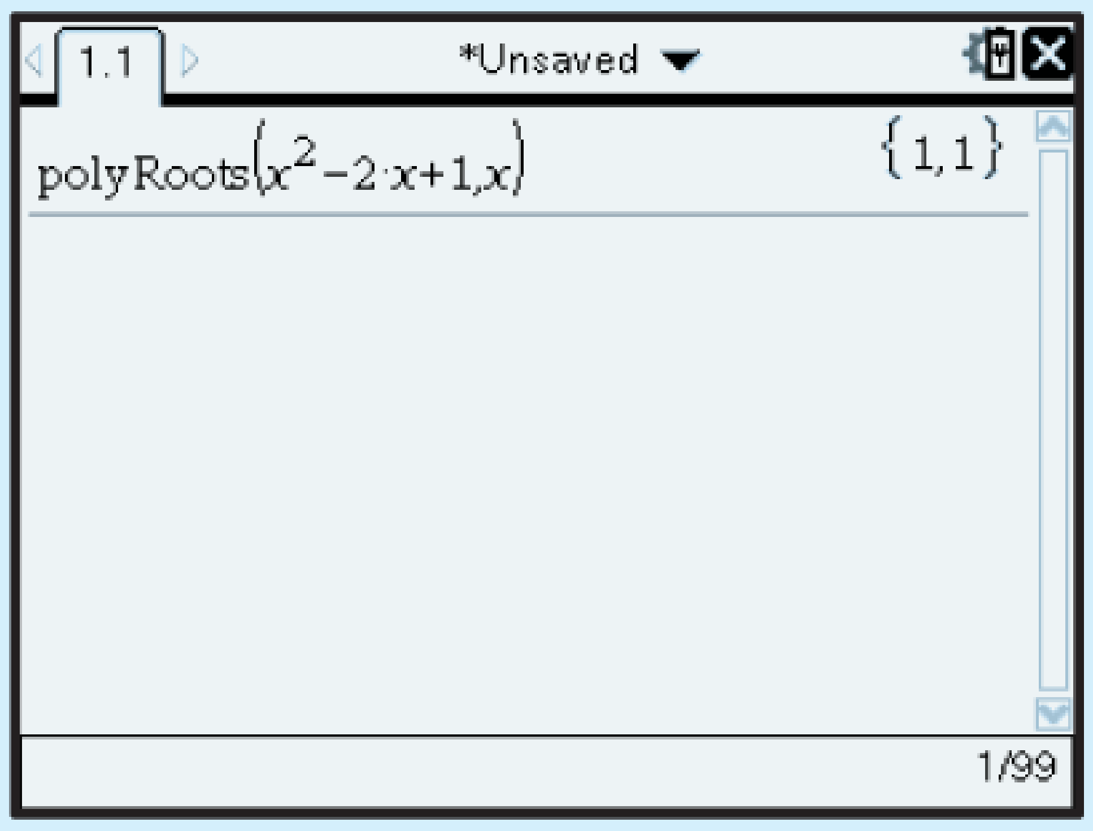
If $y = x^2 - 2x + 3$, find the value(s) of x when:

a $y = 2$ **b** $y = 18$.

a If $y = 2$ then

$x^2 - 2x + 3 = 2$

$\therefore\ x^2 - 2x + 1 = 0$



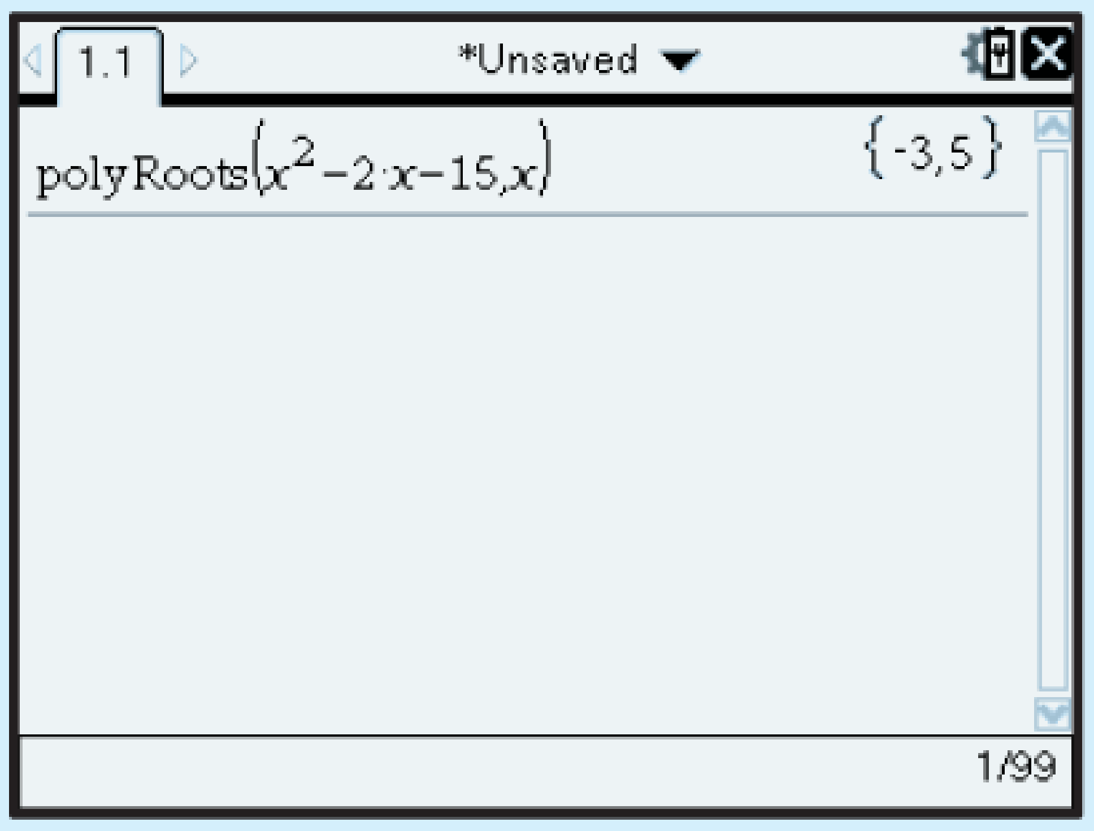
polyRoots$(x^2-2\cdot x+1,x)$ $\{1,1\}$

$\therefore\ x = 1$

b If $y = 18$ then

$x^2 - 2x + 3 = 18$

$\therefore\ x^2 - 2x - 15 = 0$



polyRoots$(x^2-2\cdot x-15,x)$ $\{-3,5\}$

$\therefore\ x = -3$ or 5

EXERCISE 6A

1 Which of the following are quadratic functions?

a $y = 2x^2 - 4x + 10$

b $y = 8x + 3$

c $y = -2x^2$

d $y = \frac{1}{3}x + 6 - x^2$

e $2y + x - 3 = 0$

f $y - 2x^2 = 3x - 1$

2 For each of the following functions, find the value of y for the given value of x:

a $y = x^2 + 3x - 7$ when $x = 1$

b $y = -2x^2 + 5x + 2$ when $x = -2$

c $y = 3x^2 - 2x - 5$ when $x = 3$

d $y = -3x^2 + 7x - 2$ when $x = -1$.

3 Copy and complete each table of values:

a $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y					

b $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y					

4 **a** If $f(x) = x^2 + 3x - 7$, find $f(2)$ and $f(-1)$.

b If $f(x) = 2x^2 - x + 1$, find $f(0)$ and $f(-3)$.

c If $g(x) = -3x^2 - 2x + 4$, find $g(3)$ and $g(-2)$.

5 Determine whether the given point satisfies the quadratic function:

a $y = 2x^2 + 5$ $(0,\ 4)$

b $y = x^2 - 3x + 2$ $(2,\ 0)$

c $f(x) = -x^2 + 2x - 5$ $(-1,\ -8)$

d $y = -2x^2 - x + 6$ $(3,\ -15)$

e $y = 3x^2 - 4x + 10$ $(2,\ 10)$

f $f(x) = -\frac{1}{2}x^2 + 4x - 1$ $(2,\ 5)$

6 For each quadratic function, find the value(s) of x for the given value of y:

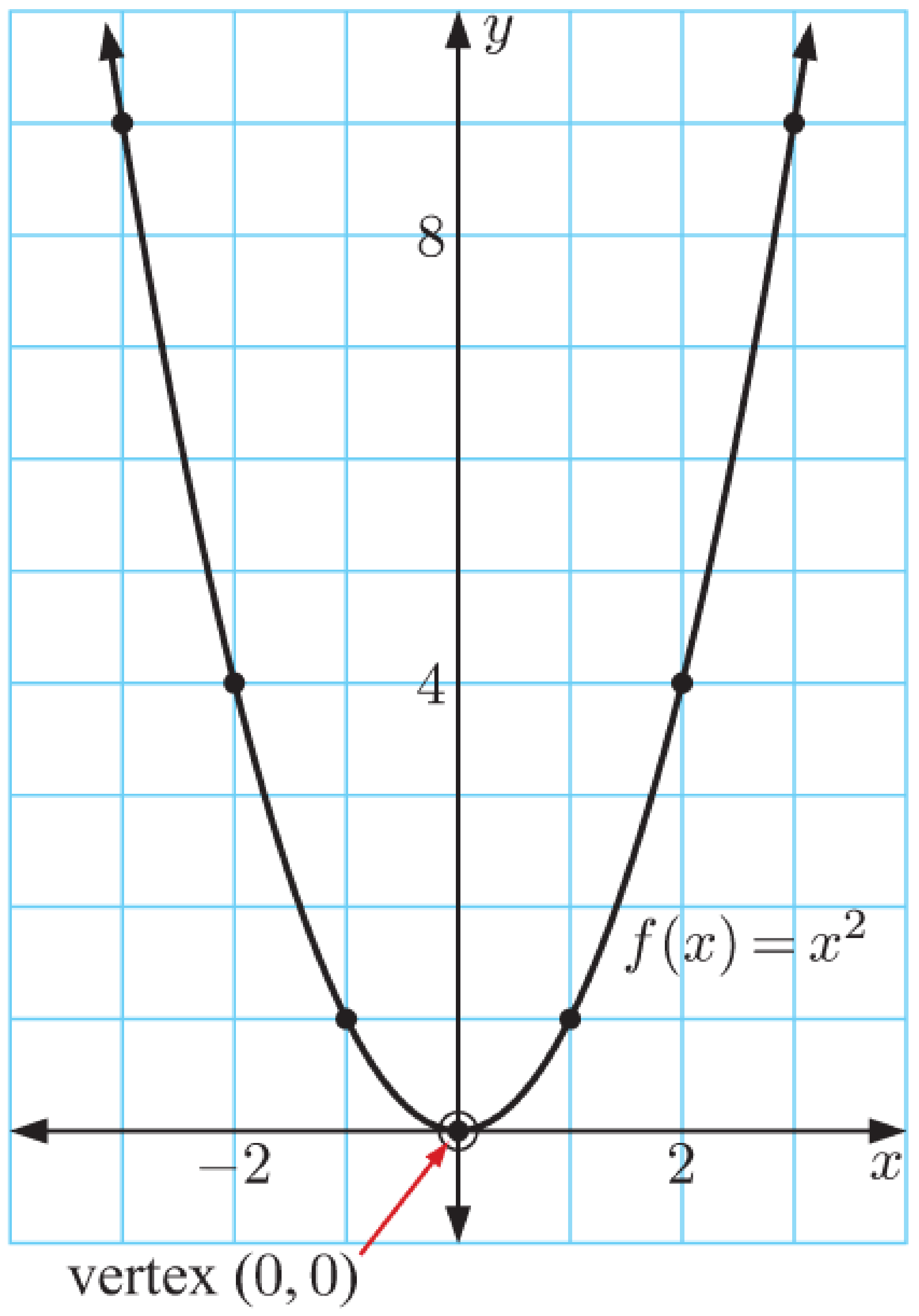
- **a** $y = x^2 + 3x + 6$ when $y = 4$
- **b** $y = x^2 - 4x + 7$ when $y = 3$
- **c** $y = x^2 - 6x + 1$ when $y = -4$
- **d** $y = 2x^2 + 5x + 1$ when $y = 4$
- **e** $y = \frac{1}{2}x^2 + \frac{5}{2}x - 2$ when $y = 1$
- **f** $y = -\frac{1}{2}x^2 + 2x - 1$ when $y = 2$.

B GRAPHS FROM TABLES OF VALUES

The simplest quadratic function is $f(x) = x^2$.

Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9



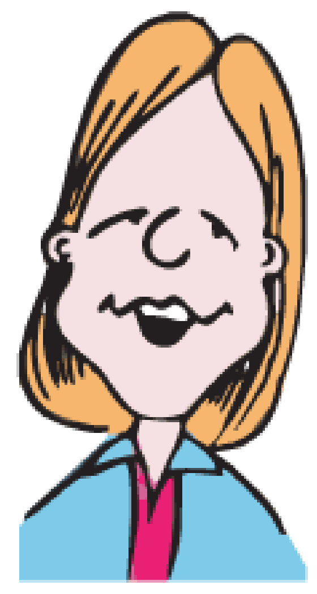
The curve formed is called a **parabola**. The graphs of all quadratic functions have this same basic shape.

Notice that the curve $f(x) = x^2$:

- opens upwards
- has a **vertex** or **turning point** at (0, 0)
- is **symmetric** about the y-axis.

  For example, we can see from the table of values that: $f(-3) = f(3)$
  $f(-2) = f(2)$
  $f(-1) = f(1)$

The **vertex** of a parabola is where the graph is at its maximum or minimum.



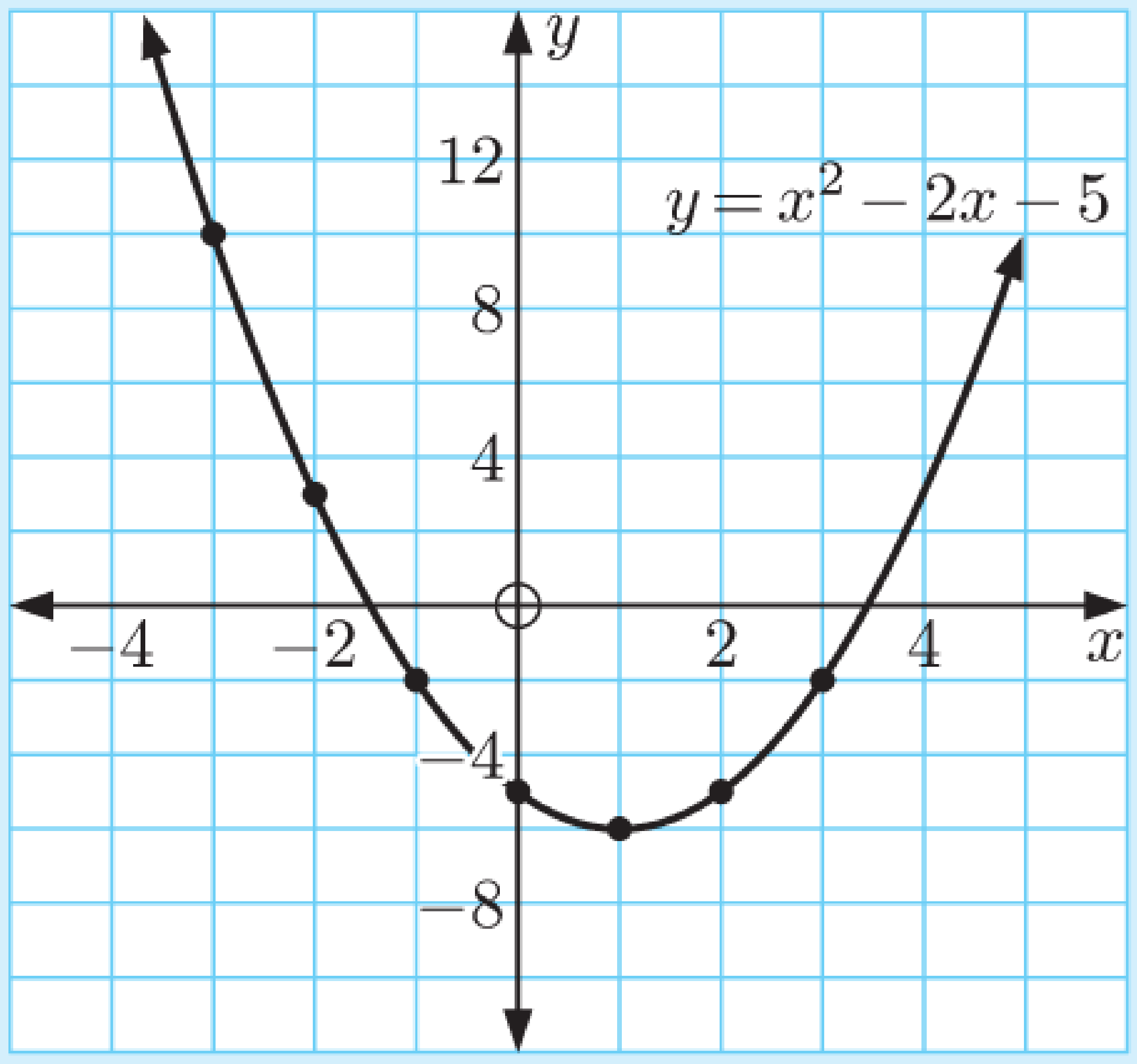
Example 4 **Self Tutor**

Draw the graph of $y = x^2 - 2x - 5$ from a table of values.

When $x = -3$, $y = (-3)^2 - 2(-3) - 5$
$= 9 + 6 - 5$
$= 10$

We can perform similar calculations for other values of x, to produce a table of values:

x	-3	-2	-1	0	1	2	3
y	10	3	-2	-5	-6	-5	-2



EXERCISE 6B

1 Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ for each of the following functions, and hence graph the function.

- **a** $y = x^2 + 2x - 2$
- **b** $f(x) = x^2 - 3$
- **c** $y = x^2 - 2x$
- **d** $f(x) = -x^2 + x + 2$
- **e** $y = x^2 - 4x + 4$
- **f** $f(x) = -2x^2 + 3x + 10$

Use the **graphing package** or your **graphics calculator** to check your answers.

GRAPHING PACKAGE

When drawing a graph from a table of values, plot the points then join them with a smooth curve.

2 **a** Copy and complete this table of values:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 2$							
$x^2 - 2$							

b Hence sketch $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 2$ on the same set of axes.

c Comment on your results.

3 **a** Copy and complete this table of values:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$(x+2)^2$							
$(x-2)^2$							

b Hence sketch $y = x^2$, $y = (x+2)^2$, and $y = (x-2)^2$ on the same set of axes.

c Comment on your results.

4 **a** Copy and complete this table of values:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$							
$3x^2$							

b Hence sketch $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on the same set of axes.

c Comment on your results.

5 **a** Copy and complete this table of values:

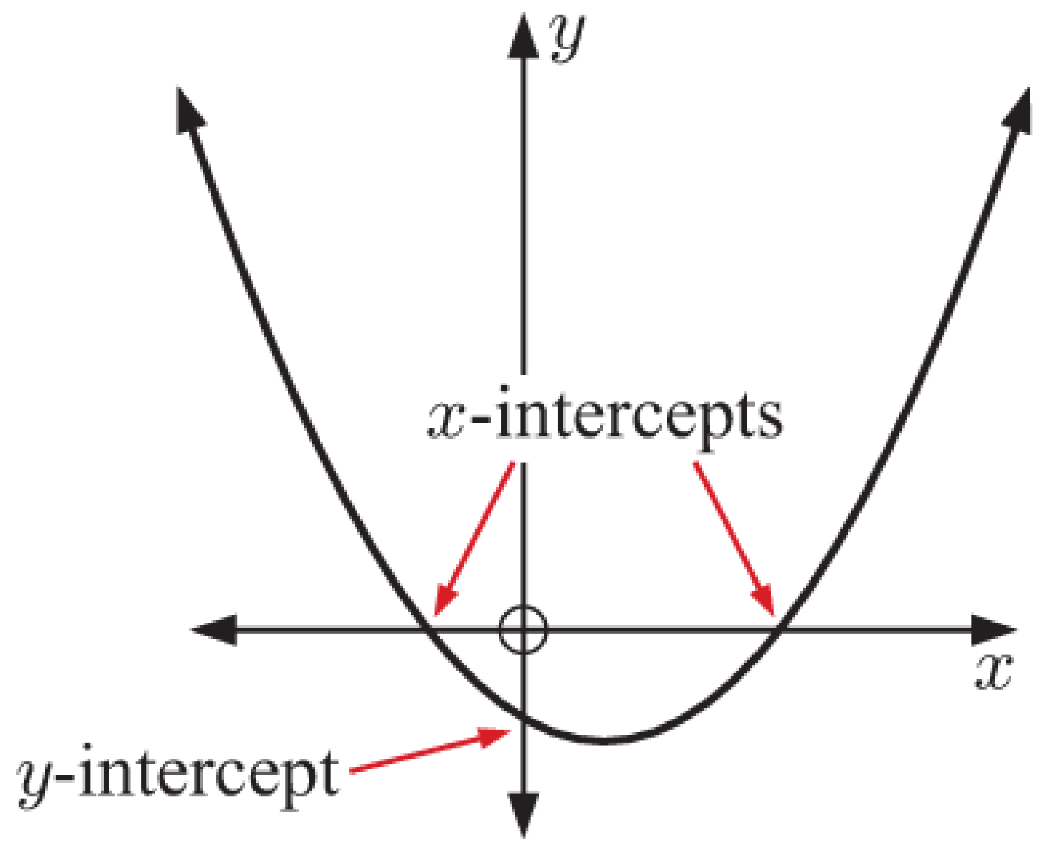
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-x^2$							
$(-x)^2$							

b Hence sketch $y = x^2$, $y = -x^2$, and $y = (-x)^2$ on the same set of axes.

c Comment on your results.

C AXES INTERCEPTS

The axes intercepts are an important property of quadratic functions.



The **x-intercepts** are values of x where the graph meets the x-axis.

The **y-intercept** is the value of y where the graph meets the y-axis.

INVESTIGATION 1 AXES INTERCEPTS

What to do:

1 For each quadratic function, use the **graphing package** or your **graphics calculator** to:

i draw the graph **ii** find the y-intercept **iii** find any x-intercepts.

GRAPHING PACKAGE

a $y = x^2 - 3x - 4$ **b** $y = -x^2 + 2x + 8$ **c** $y = 2x^2 - 3x$

d $y = -2x^2 + 2x - 3$ **e** $y = (x-1)(x-3)$ **f** $y = -(x+2)(x-3)$

g $y = 3(x+1)(x+4)$ **h** $y = 2(x-2)^2$ **i** $y = -3(x+1)^2$

2 From your observations in question **1**:

a State the y-intercept of a quadratic function in the form $y = ax^2 + bx + c$.

b State the x-intercepts of a quadratic function in the form $y = a(x-\alpha)(x-\beta)$.

c Comment on the x-intercepts and graph of quadratic functions in the form $y = a(x-\alpha)^2$.

THE y-INTERCEPT

The y-intercept is found by letting $x = 0$. For a quadratic function of the form $y = ax^2 + bx + c$, the y-intercept is the constant term c.

We can see this since, letting $x = 0$,

$$\begin{aligned} y &= a(0)^2 + b(0) + c \\ &= 0 + 0 + c \\ &= c \end{aligned}$$

Example 5 **Self Tutor**

Find the y-intercept of:

a $y = 2x^2 - 6x + 5$ **b** $y = (x+3)(x-6)$

a When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5.

b When $x = 0$, $y = (3)(-6)$

$= -18$

$\therefore$ the y-intercept is -18.

THE x-INTERCEPTS

The x-intercepts are found by letting $y = 0$. For this reason they are called the **zeros** of the function.

Consider a quadratic function in the factored form $y = a(x-\alpha)(x-\beta)$, $a \neq 0$.

Letting $y = 0$, $a(x-\alpha)(x-\beta) = 0$

$\therefore$ $x - \alpha = 0$ or $x - \beta = 0$ {null factor law}

$\therefore$ $x = \alpha$ or $x = \beta$

The x-intercepts of $y = a(x-\alpha)(x-\beta)$ are α and β.

For quadratic functions given in other forms, we find the x-intercepts using technology.

Example 6 **Self Tutor**

Find the x-intercepts of:

a $y = 3(x-4)(x+2)$

b $y = x^2 - 2x - 35$

a When $y = 0$, $3(x-4)(x+2) = 0$

$\therefore$ $x - 4 = 0$ or $x + 2 = 0$ {null factor law}

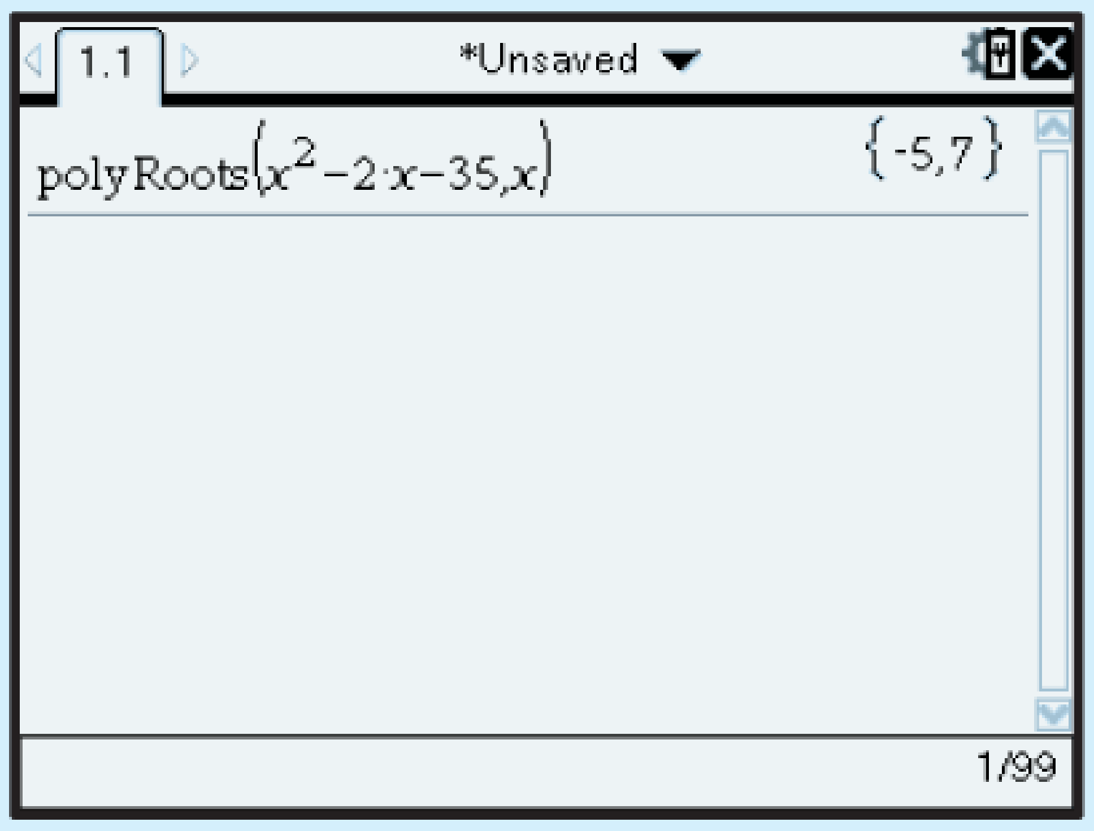
$\therefore$ $x = 4$ or -2

$\therefore$ the x-intercepts are 4 and -2.

b When $y = 0$, $x^2 - 2x - 35 = 0$

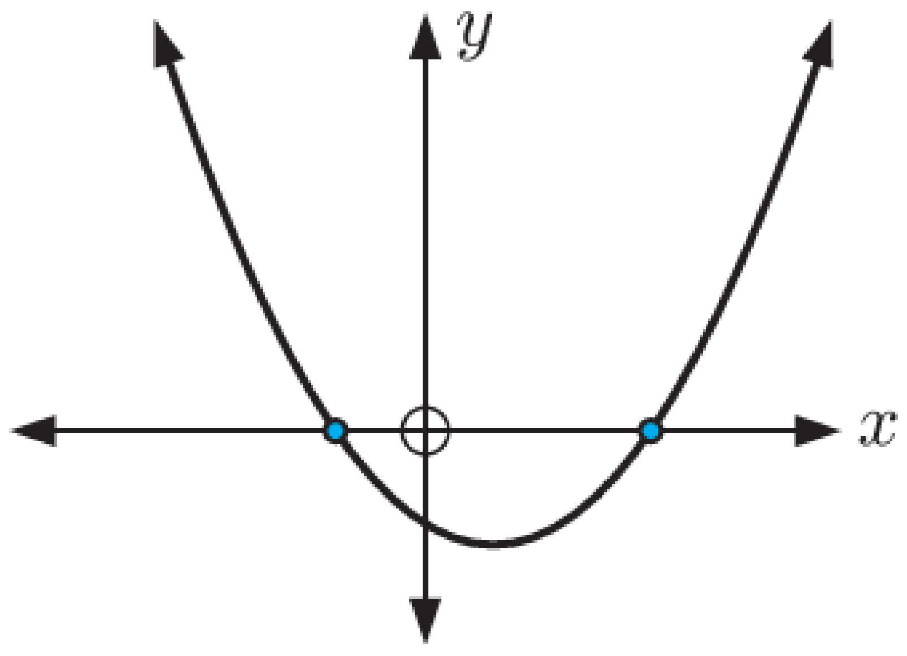
$\therefore$ $x = -5$ or 7 {technology}

$\therefore$ the x-intercepts are -5 and 7.

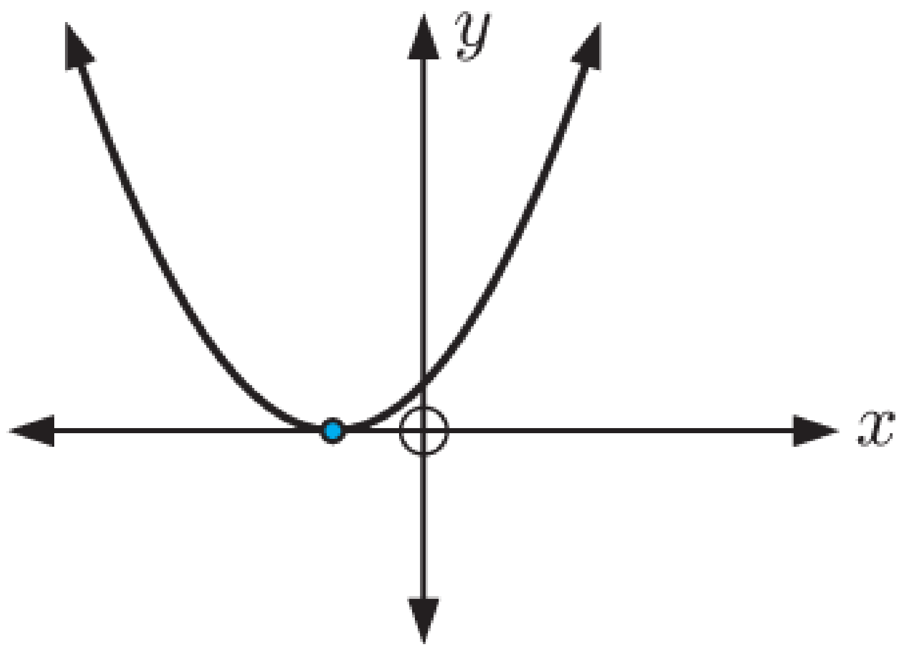


polyRoots$(x^2-2\cdot x-35,x)$ $\{-5,7\}$

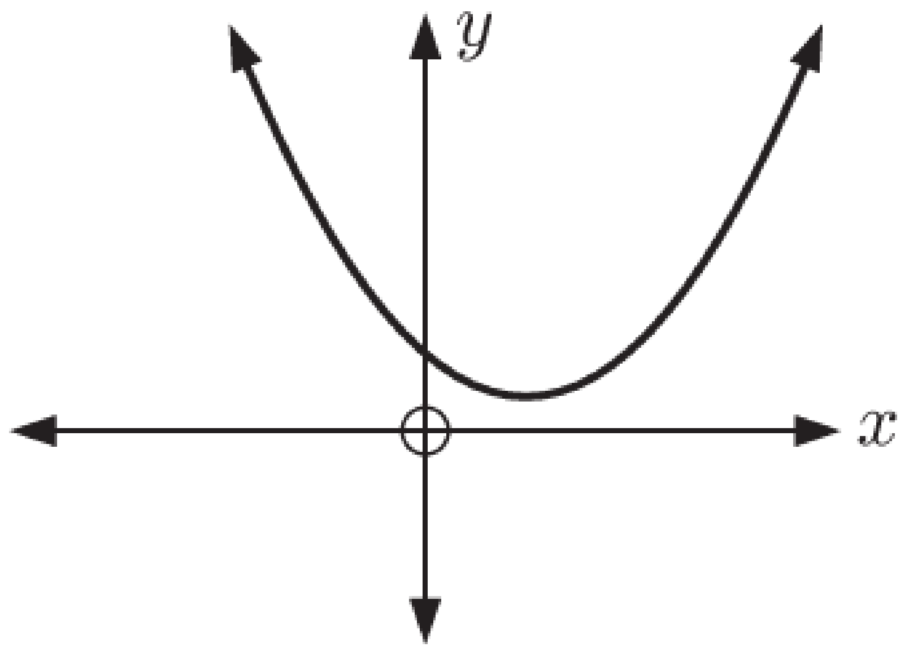
Quadratic equations can have two, one, or zero solutions. This means that quadratic functions can have two, one, or zero x-intercepts.



two x-intercepts



one x-intercept



no x-intercepts

EXERCISE 6C

1 State the y-intercept of the function:

a $y = x^2 + 2x + 3$

b $y = 2x^2 + 5x - 1$

c $y = -x^2 - 3x - 4$

d $f(x) = 3x^2 - 10x + 1$

e $y = 3x^2 + 5$

f $y = 4x^2 - x$

g $y = 8 - x - 2x^2$

h $f(x) = 2x - x^2 - 5$

i $y = 6x^2 + 2 - 5x$

2 Find the y-intercept of the function:

a $y=(x+1)(x+3)$ **b** $f(x)=(x-2)(x+3)$ **c** $y=(x-7)^2$

d $y=(2x+5)(3-x)$ **e** $y=x(x-4)$ **f** $f(x)=-(x+4)(x-5)$

3 Find the x-intercept(s) of the function:

a $y=(x-2)(x-5)$ **b** $y=(x-3)(x+4)$ **c** $y=2(x+6)(x+3)$

d $f(x)=-(x-7)(x+1)$ **e** $y=x(x-8)$ **f** $y=-3(x+5)(x-5)$

g $y=(2x-3)(x+1)$ **h** $y=(3x+1)(2x-5)$ **i** $y=(x+4)^2$

j $f(x)=7(x-2)^2$ **k** $y=-4(x+1)^2$ **l** $y=(4x+3)^2$

4 How many zeros does a quadratic function have if it:

a cuts the x-axis twice **b** *touches* the x-axis

c lies entirely below the x-axis?

5 Find the zeros of the following functions, if they exist:

a $y=x^2-x-6$ **b** $y=x^2-16$ **c** $y=x^2+5$

d $f(x)=3x-x^2$ **e** $y=x^2-12x+36$ **f** $y=x^2+x-7$

g $y=-2x^2+x-5$ **h** $y=-6x^2+x+5$ **i** $f(x)=3x^2+x-1$

6 Find the axes intercepts of the function:

a $y=(x+2)(x-1)$ **b** $f(x)=(x+3)^2$ **c** $y=(x+5)(x-2)$

d $y=(3x-2)(x-5)$ **e** $y=-x^2+7x-8$ **f** $y=-x^2-8x-16$

g $y=x^2-7x$ **h** $f(x)=-2x^2+3x+7$ **i** $y=2x^2-18$

j $y=-x^2+2x-9$ **k** $y=4x^2-4x-3$ **l** $y=-5x^2+2x+11$

D GRAPHS OF THE FORM $y=ax^2$

INVESTIGATION 2 GRAPHS OF THE FORM $y=ax^2$

In this Investigation we consider quadratic functions of the form $y=ax^2$. We explore how the *sign* and *size* of a affect the graph of the function.

GRAPHING PACKAGE

What to do:

1 Use the **graphing package** or your **graphics calculator** to graph the following functions on the same set of axes.

a $y=x^2$, $y=\frac{1}{2}x^2$, $y=2x^2$, and $y=3x^2$

b $y=-x^2$, $y=-\frac{1}{2}x^2$, $y=-2x^2$, and $y=-3x^2$.

DEMO

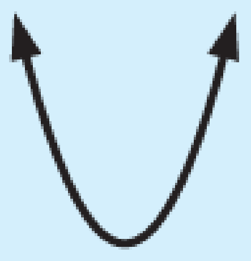
2 Use the interactive demonstration to explore other functions of the form $y=ax^2$.

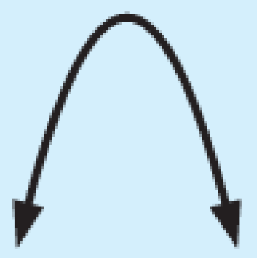
3 What effect does the *sign* of a have on the graph?

4 What effect does the *size* of a have on the graph?

From the **Investigation** we make the following observations:

$y = ax^2$ has vertex $(0,\ 0)$ for all $a \neq 0$.

If $a > 0$, the graph opens upwards. It has the shape .

If $a < 0$, the graph opens downwards. It has the shape .

If $a < -1$ or $a > 1$, the graph is "thinner" than $y = x^2$.

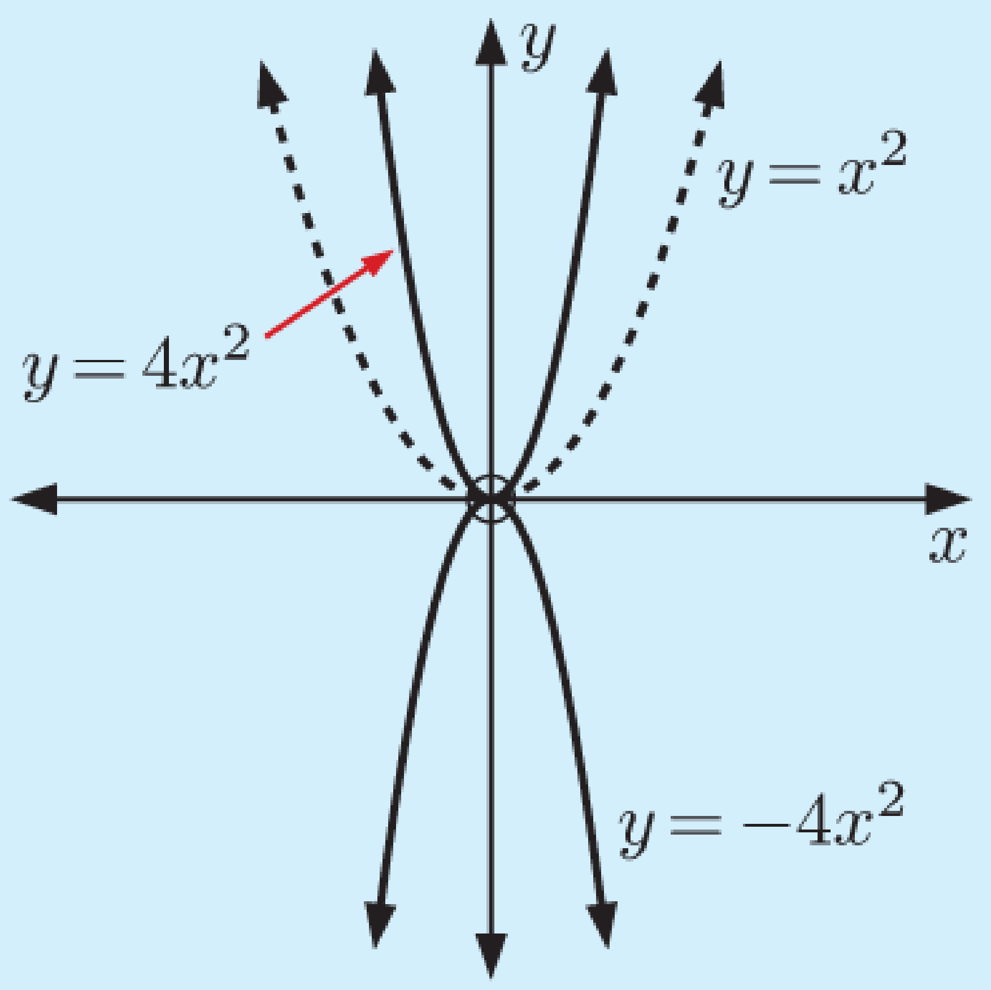
If $-1 < a < 1$, $a \neq 0$, the graph is "wider" than $y = x^2$.

Example 7 **Self Tutor**

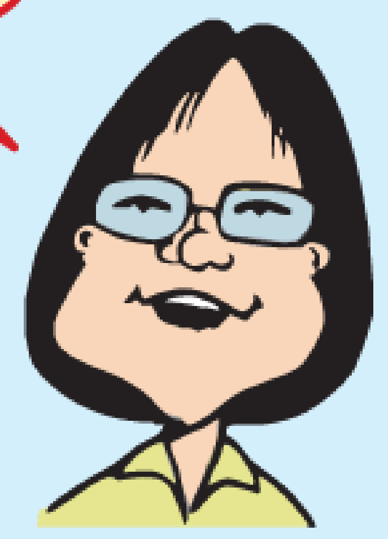
Sketch $y = x^2$, $y = 4x^2$, and $y = -4x^2$ on the same set of axes.

$y = 4x^2$ is "thinner" than $y = x^2$.

$y = -4x^2$ has the same shape as $y = 4x^2$ but opens downwards.



If $a > 0$, the vertex is the **minimum**.
If $a < 0$, the vertex is the **maximum**.



EXERCISE 6D

GRAPHING PACKAGE

1 For each of the following, sketch $y = x^2$ and the function on the same set of axes. In each case comment on the direction in which the graph opens, and the shape of the graph.

a $y = 5x^2$ **b** $y = -5x^2$ **c** $y = \frac{2}{3}x^2$

d $y = -\frac{2}{3}x^2$ **e** $y = -3x^2$ **f** $y = \frac{1}{4}x^2$

Use the **graphing package** or your **graphics calculator** to check your answers.

2 State the coordinates of the vertex of each function, and explain whether the vertex is a maximum or a minimum turning point.

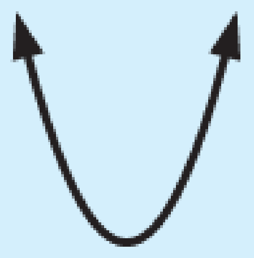
a $y = 3x^2$ **b** $y = -6x^2$ **c** $y = -\frac{1}{3}x^2$

E GRAPHS OF QUADRATIC FUNCTIONS

We can sketch the graph of a quadratic function by considering the value of a and the axes intercepts.

Example 8 **Self Tutor**

Sketch the graph of $y = x^2 - 3x - 10$ by considering the value of a and the axes intercepts.

Since $a = 1$ which is > 0, the parabola has shape  .

When $x = 0$, $y = -10$

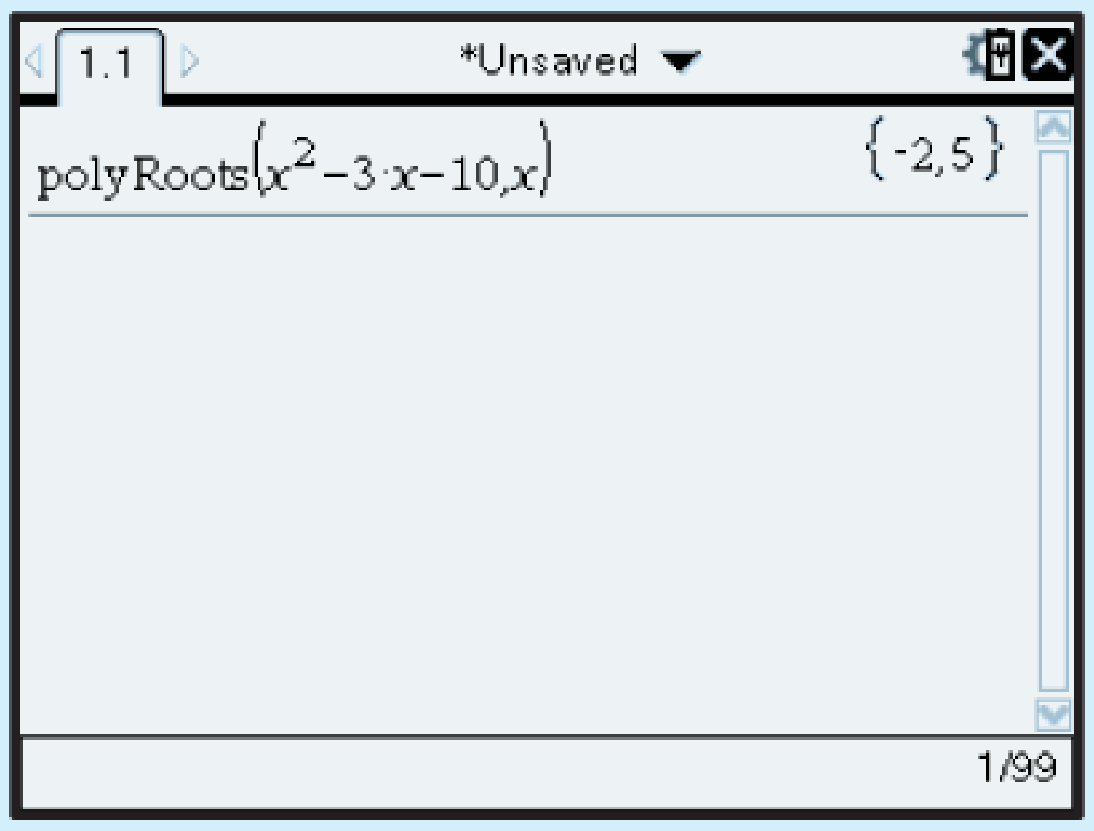
$\therefore$ the y-intercept is -10.

When $y = 0$,

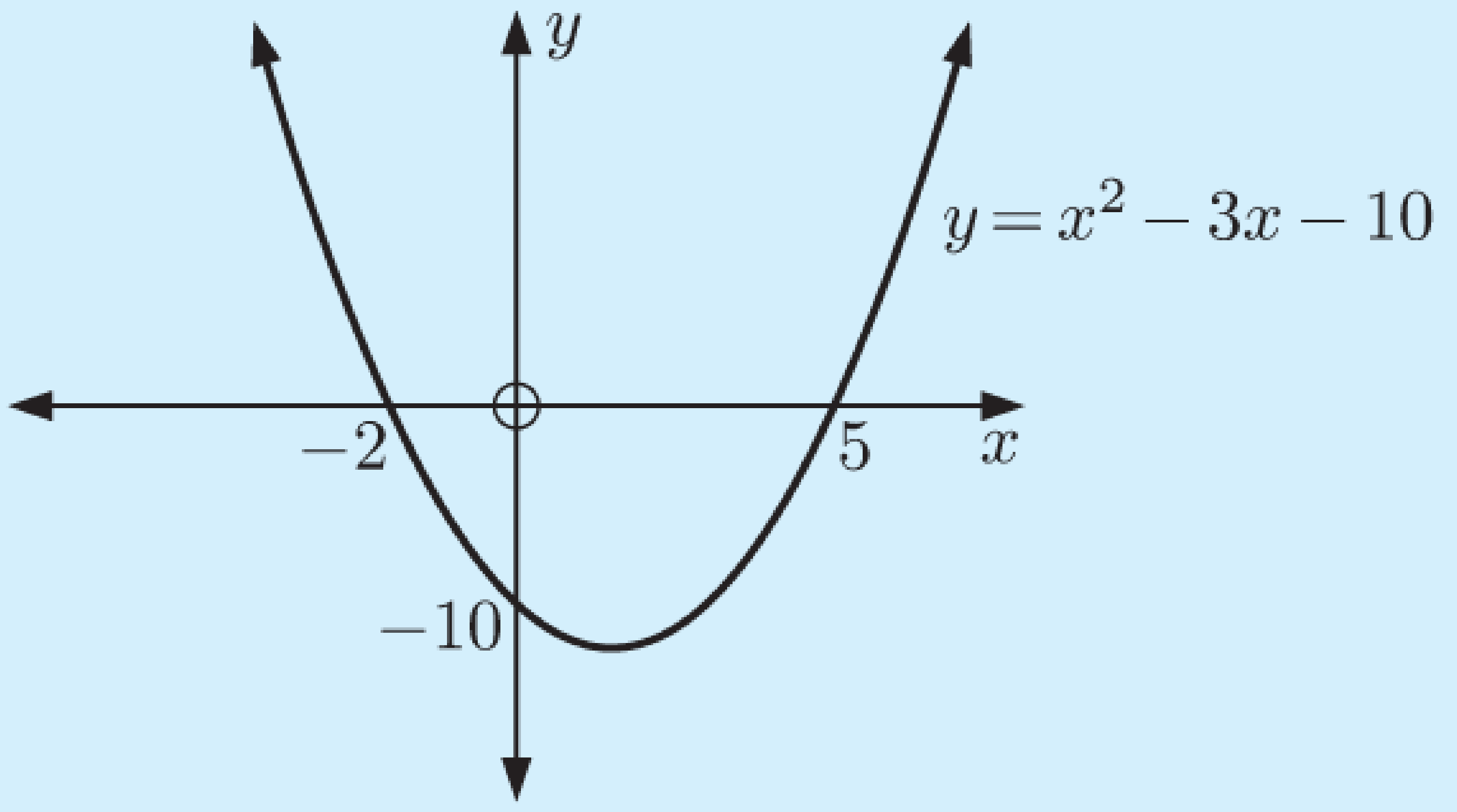
$x^2 - 3x - 10 = 0$

$\therefore$ $x = -2$ or 5 {using technology}

$\therefore$ the x-intercepts are -2 and 5.



polyRoots$(x^2-3\cdot x-10,x)$ $\{-2,5\}$



A sketch must show the general shape and key features of the graph.

EXERCISE 6E

1 Sketch the graph of the quadratic function with:

- **a** x-intercepts -1 and 3, and y-intercept -2
- **b** x-intercepts -5 and 2, and y-intercept 4
- **c** x-intercepts 1 and 6, and y-intercept -7
- **d** x-intercept 3 and y-intercept 5
- **e** x-intercept -1 and y-intercept -4.

If a quadratic function has only one x-intercept then its graph must *touch* the x-axis.

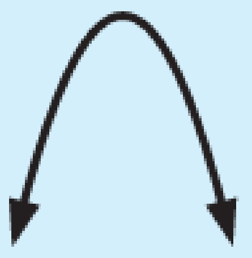
2 Sketch the graph of each function by considering the value of a and the axes intercepts:

- **a** $y = x^2 + x - 12$
- **b** $y = x^2 + 4x - 5$
- **c** $y = -x^2 + 6x - 9$
- **d** $y = x^2 + 8x + 16$
- **e** $y = -x^2 + 4x + 12$
- **f** $y = -x^2 + 6x + 4$
- **g** $y = 2x^2 + 2x - 24$
- **h** $y = -2x^2 - 3x + 9$
- **i** $y = -4x^2 + 20x - 25$

Example 9

Self Tutor

Sketch the graph of $y = -3(x+3)(x-4)$ by considering the value of a and the axes intercepts.

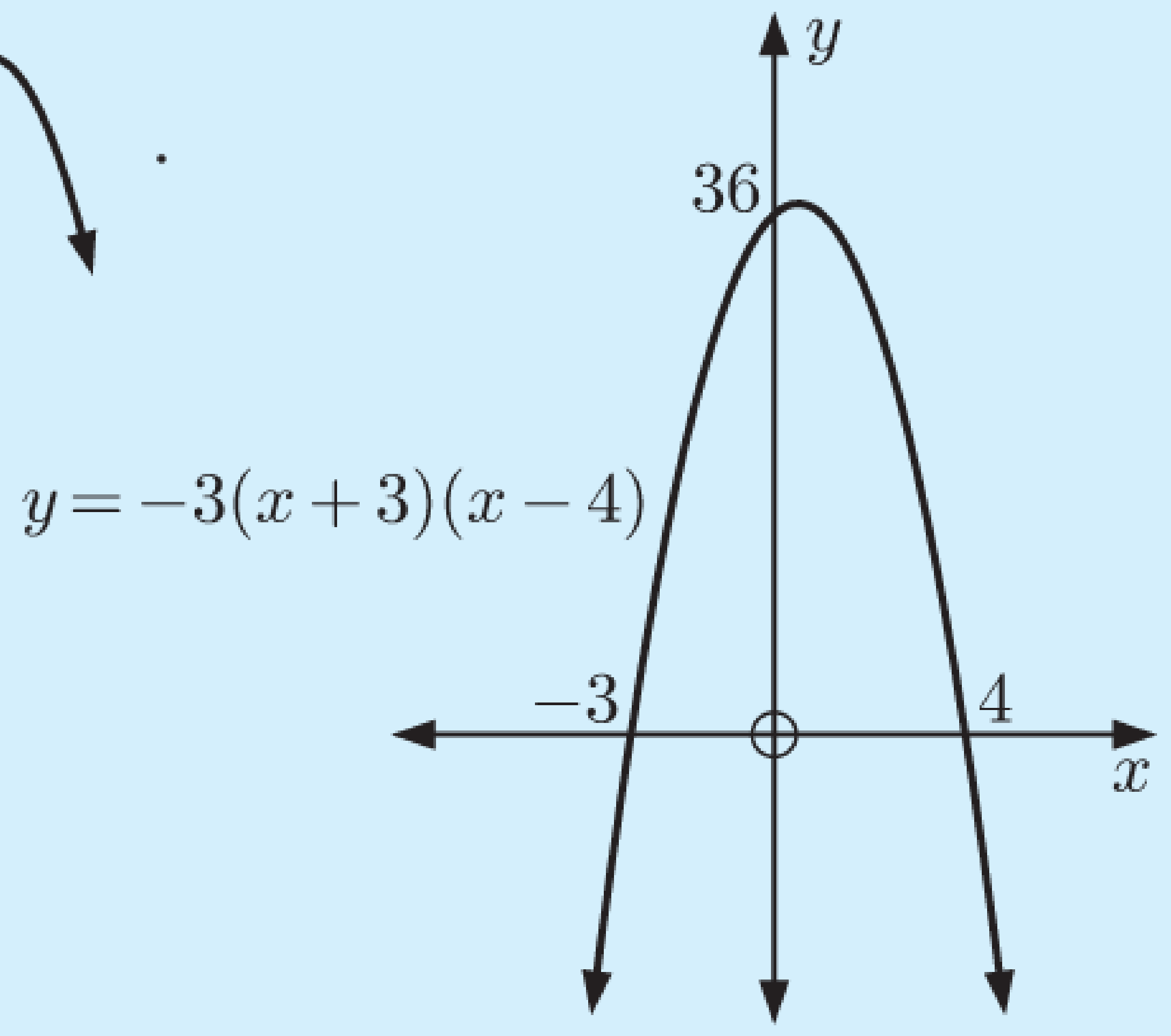
Since $a = -3$ which is < 0, the parabola has shape  .

When $x = 0$, $y = -3(3)(-4)$
$= 36$

$\therefore$ the y-intercept is 36

When $y = 0$, $-3(x+3)(x-4) = 0$
$\therefore$ $x = -3$ or $x = 4$

$\therefore$ the x-intercepts are -3 and 4.



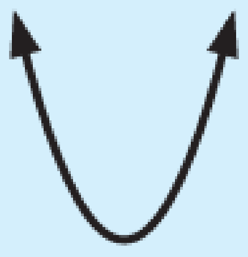
3 Sketch the graph of each function by considering the value of a and the axes intercepts:

a $y = (x-4)(x+2)$
b $y = -(x-4)(x+2)$
c $f(x) = 2(x+3)(x+5)$
d $y = -3(x+1)(x+5)$
e $f(x) = (3x-2)(x+4)$
f $f(x) = -(2x-1)(x+2)$

Example 10

Self Tutor

Sketch the graph of $f(x) = 4(x-2)^2$ by considering the value of a and the axes intercepts.

Since $a = 4$ which is > 0, the parabola has shape  .

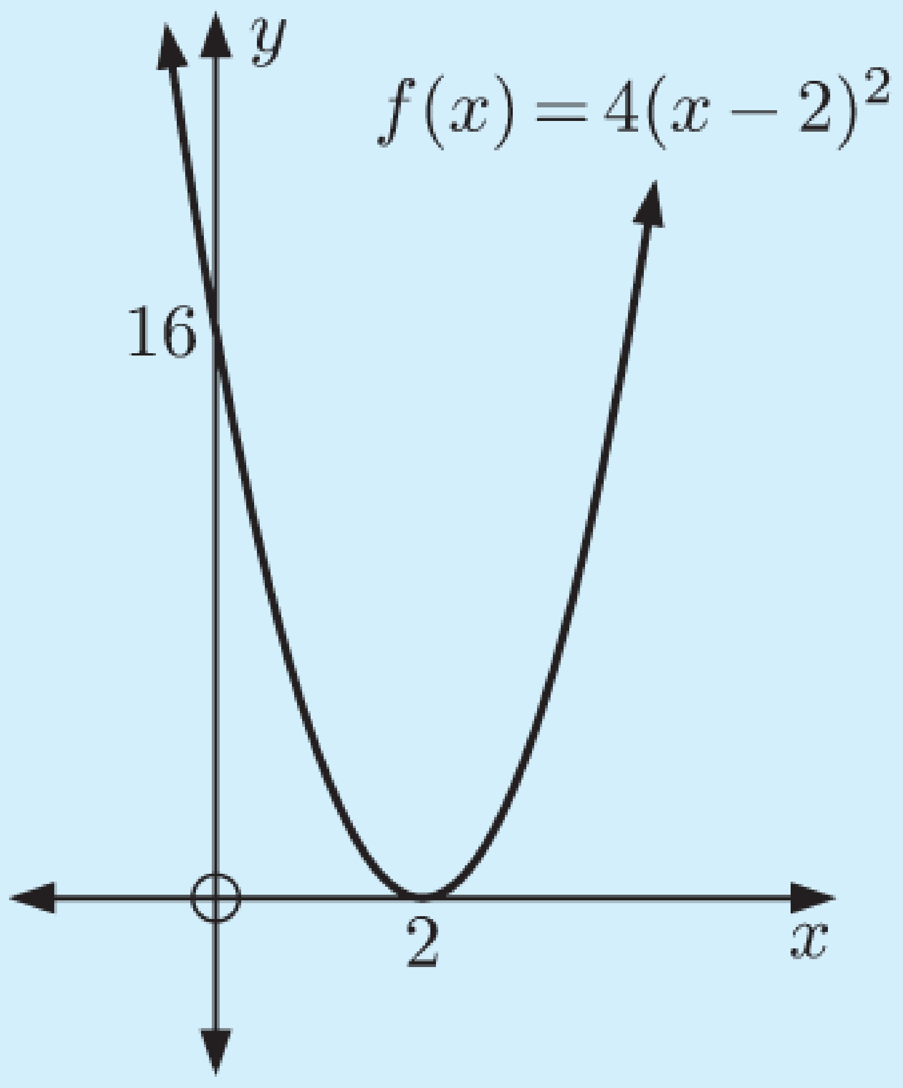
$f(0) = 4(0-2)^2 = 16$

$\therefore$ the y-intercept is 16.

When $f(x) = 0$, $4(x-2)^2 = 0$
$\therefore$ $x = 2$

$\therefore$ the x-intercept is 2.

There is only one x-intercept, so the graph *touches* the x-axis.



4 Sketch the graph of each function by considering the value of a and the axes intercepts:

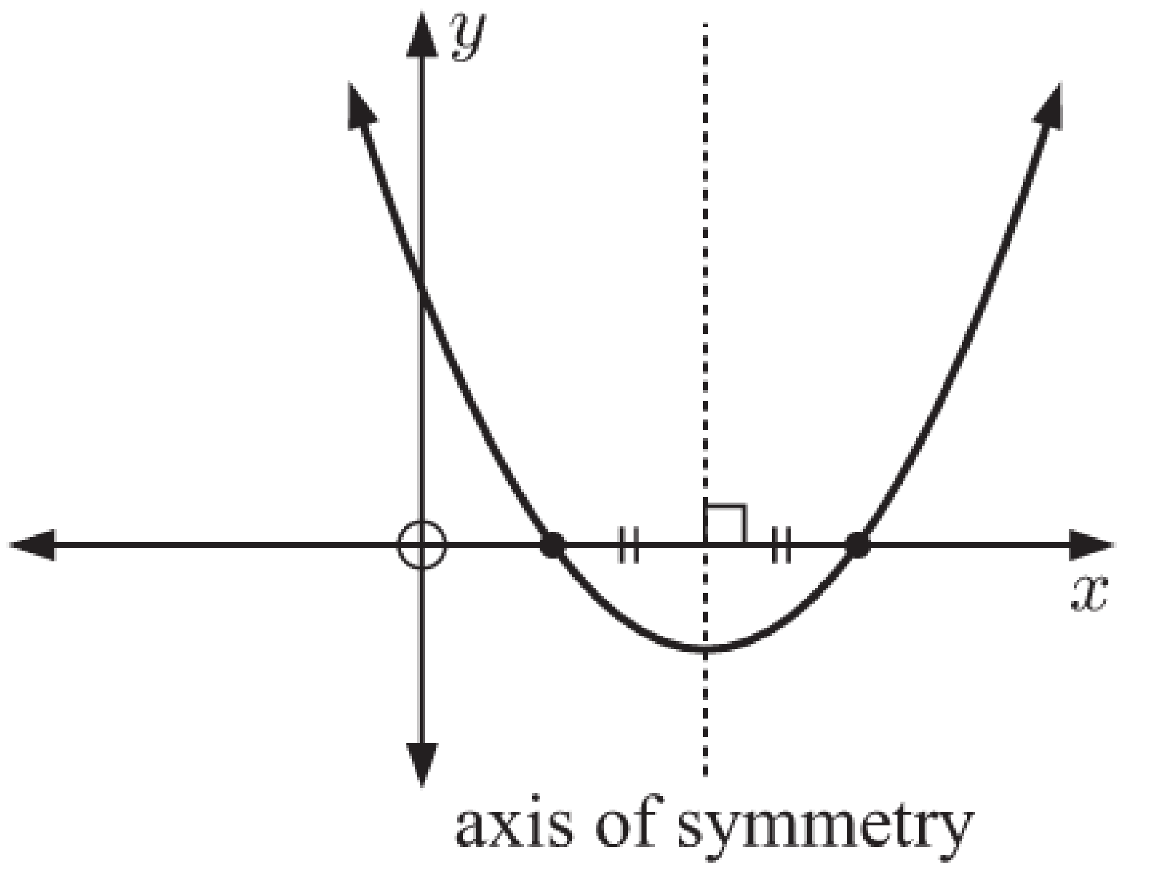
a $y = 3(x-1)^2$
b $y = 2(x+3)^2$
c $y = -\frac{1}{4}(x+2)^2$

F AXIS OF SYMMETRY

The graph of any quadratic function is symmetric about a vertical line called the **axis of symmetry**.

Since the axis of symmetry is vertical, its equation will have the form $x = k$.

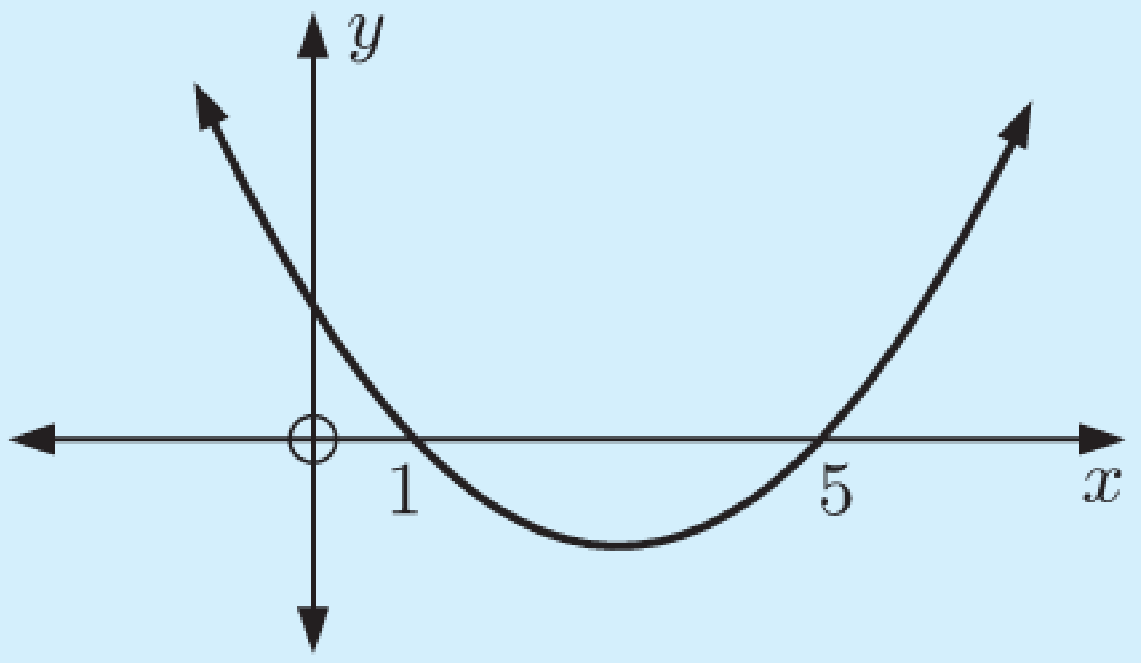
If a quadratic function has two x-intercepts, then the axis of symmetry lies halfway between them.



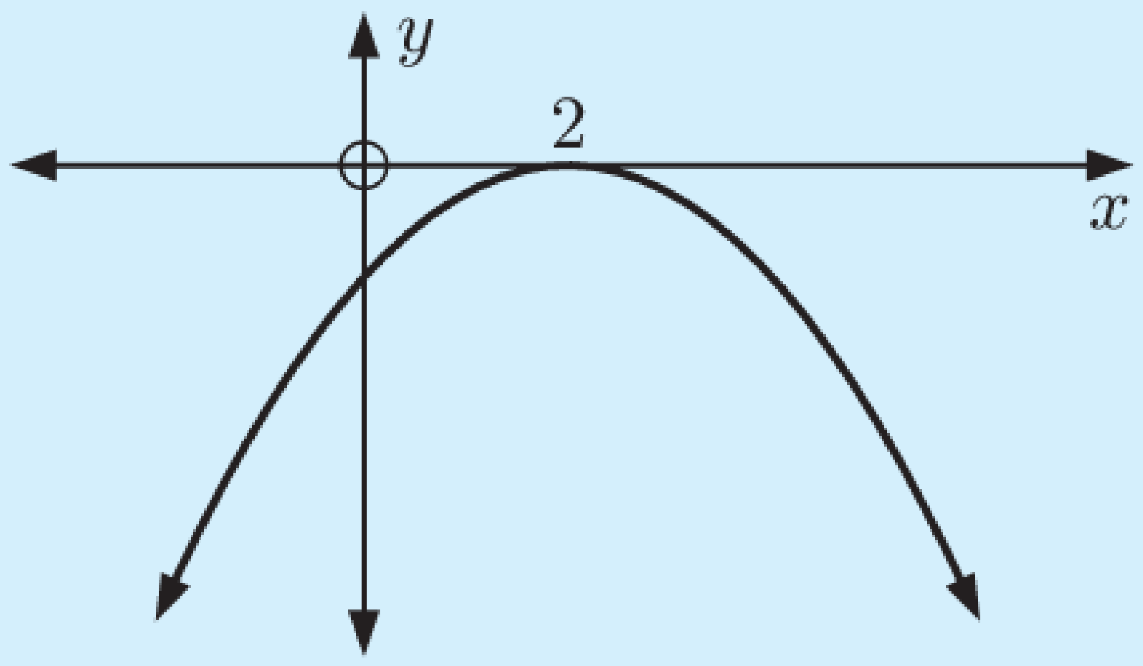
Example 11 **Self Tutor**

Find the equation of the axis of symmetry for the following quadratic functions:

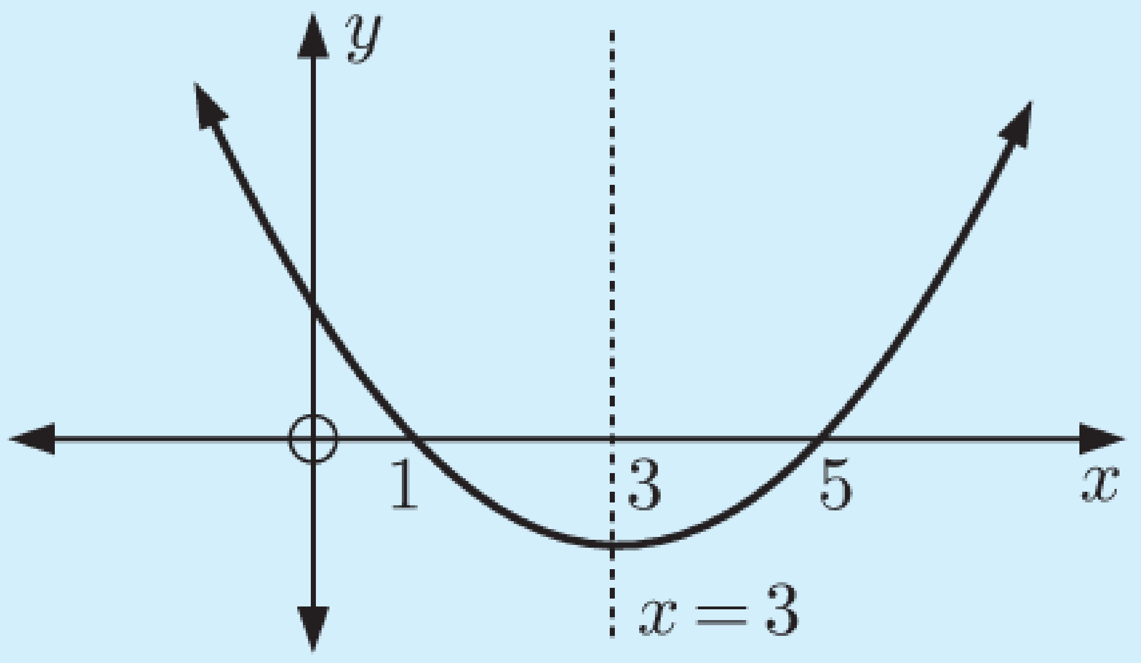
a



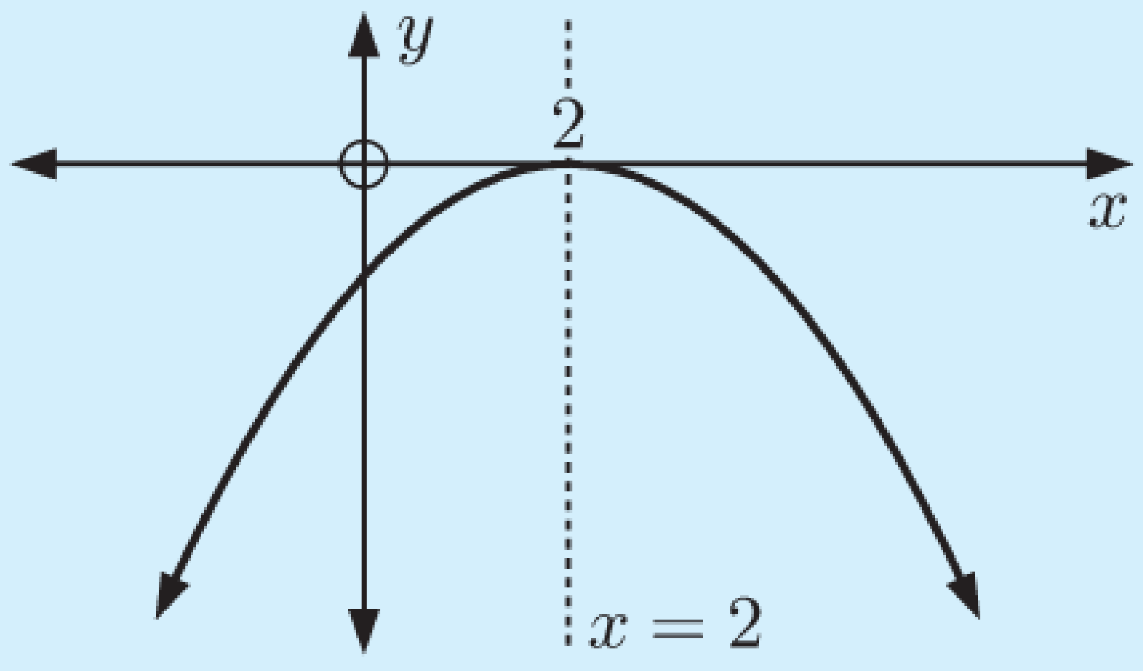
b



a The x-intercepts are 1 and 5.
3 is halfway between 1 and 5, so the axis of symmetry is $x = 3$.



b The only x-intercept is 2, so the axis of symmetry is $x = 2$.

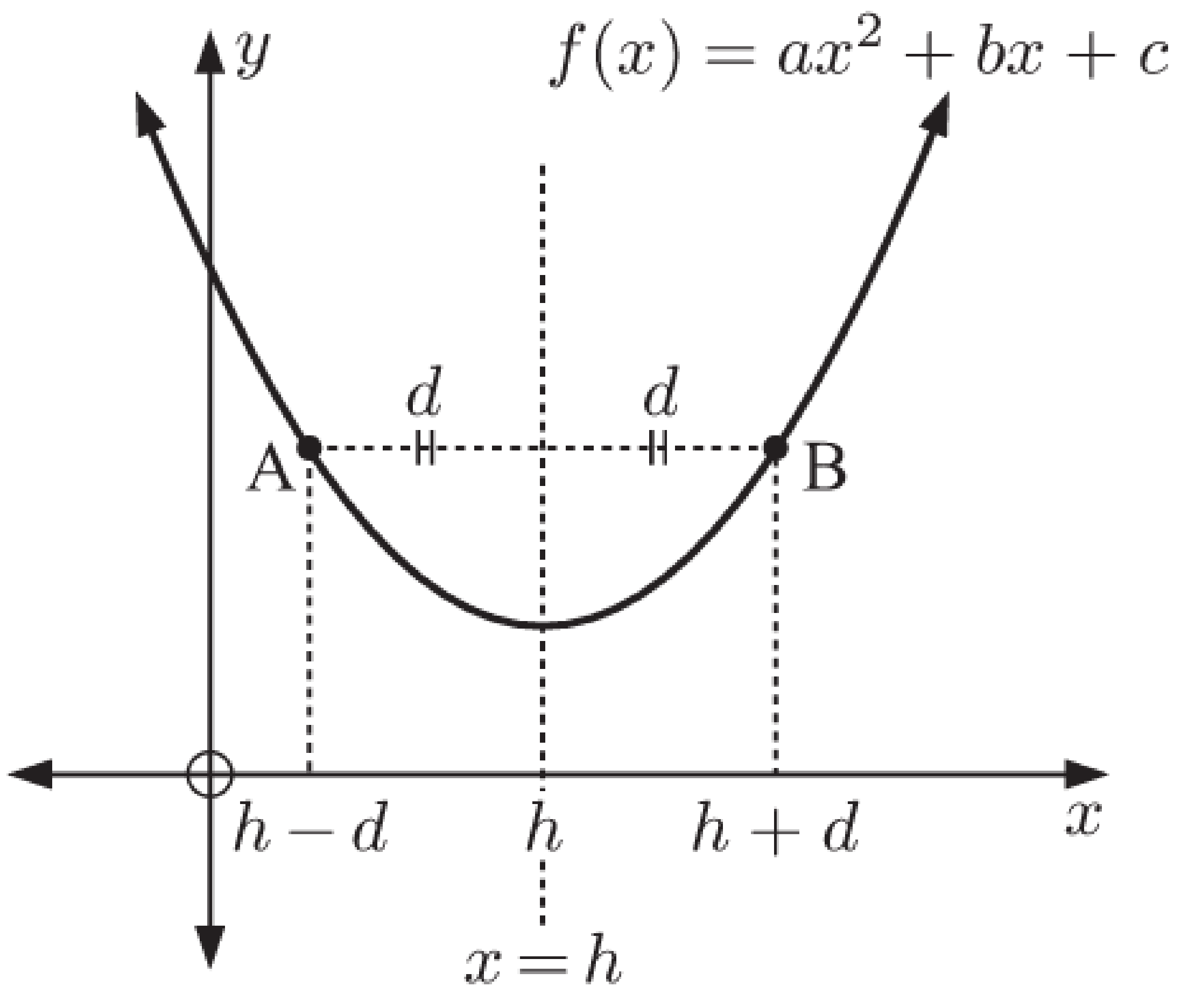


AXIS OF SYMMETRY OF $y = ax^2 + bx + c$

When quadratic functions are given in expanded form, for example $y = 2x^2 + 9x + 4$, we cannot easily identify the x-intercepts. We have also seen that some quadratic functions do not have any x-intercepts. We therefore need a method for finding the axis of symmetry of a function without using x-intercepts.

Suppose the quadratic function $f(x) = ax^2 + bx + c$ has axis of symmetry $x = h$.

Let A and B be two points on $f(x)$ which are d units either side of the axis of symmetry. By the symmetry of the function, they must have the same y-coordinate.



$$\therefore \quad f(h-d) = f(h+d)$$

$$\therefore \quad a(h-d)^2 + b(h-d) + \cancel{c} = a(h+d)^2 + b(h+d) + \cancel{c}$$

$$\therefore \quad a(\cancel{h^2} - 2hd + \cancel{d^2}) + \cancel{bh} - bd = a(\cancel{h^2} + 2hd + \cancel{d^2}) + \cancel{bh} + bd$$

$$\therefore \quad \overset{-2}{\cancel{-4}}ah\cancel{d} = \cancel{2}b\cancel{d}$$

$$\therefore \quad h = -\frac{b}{2a}$$

The equation of the axis of symmetry of $y = ax^2 + bx + c$ is $\boldsymbol{x = -\frac{b}{2a}}$.

Example 12 **Self Tutor**

Find the equation of the axis of symmetry of $y = 3x^2 + 4x - 5$.

$y = 3x^2 + 4x - 5$ has $a = 3$, $b = 4$, $c = -5$.

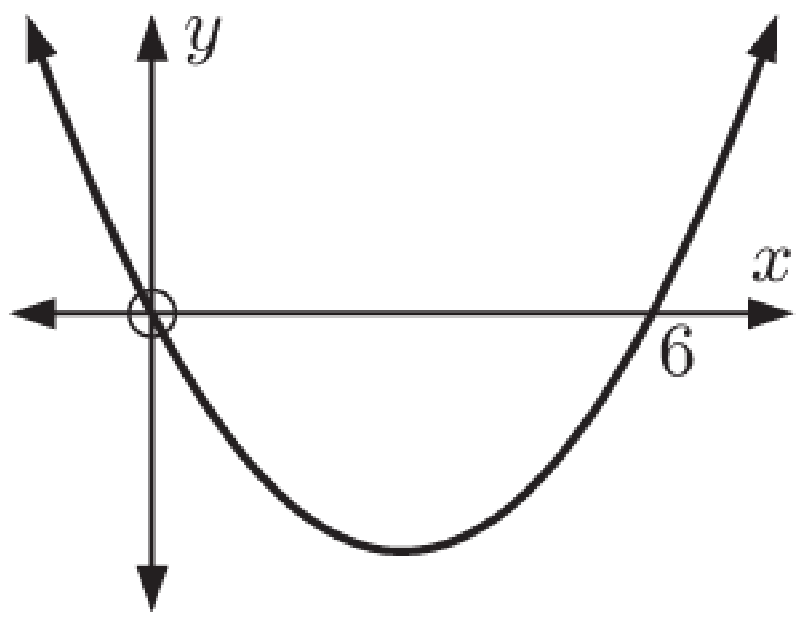
Now $-\frac{b}{2a} = -\frac{4}{2 \times 3} = -\frac{2}{3}$

$\therefore$ the axis of symmetry has equation $x = -\frac{2}{3}$.

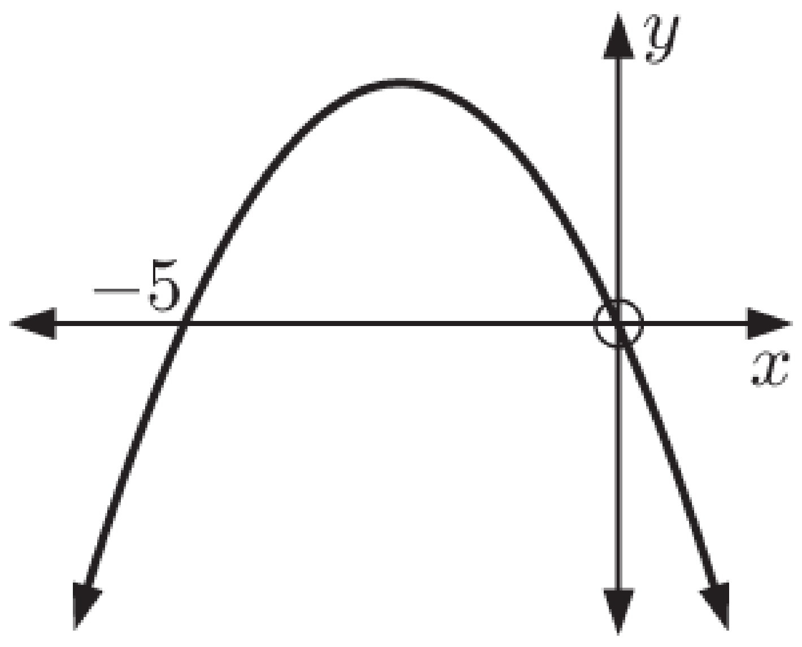
EXERCISE 6F

1 Find the axis of symmetry for each function:

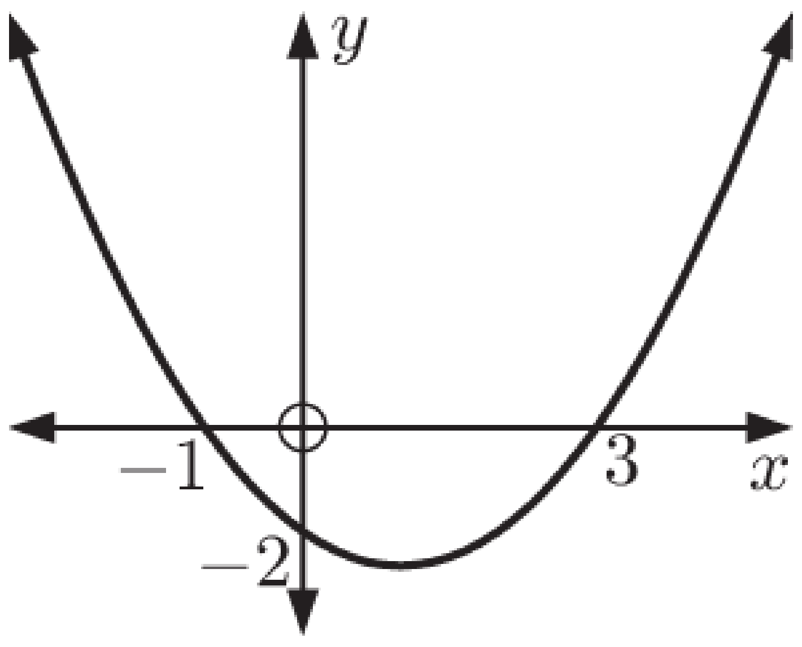
a



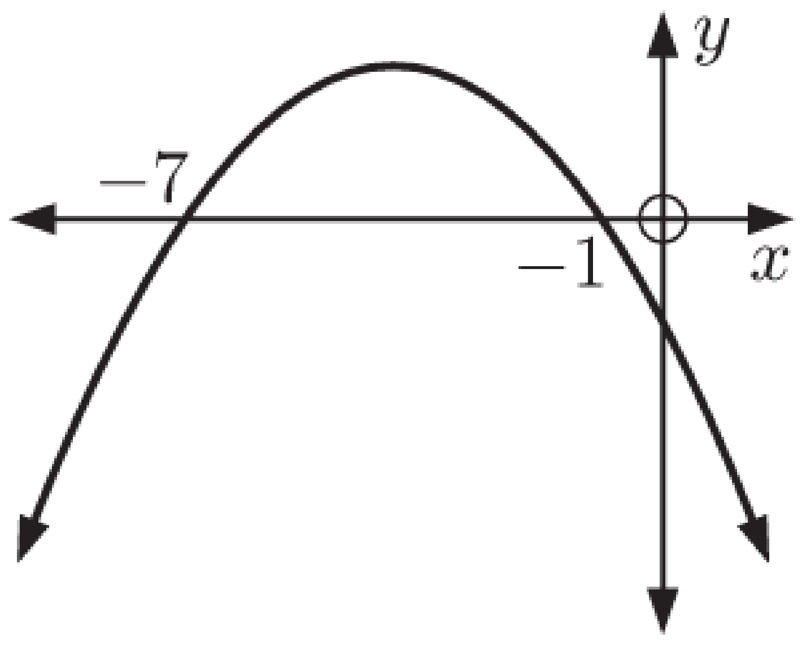
b



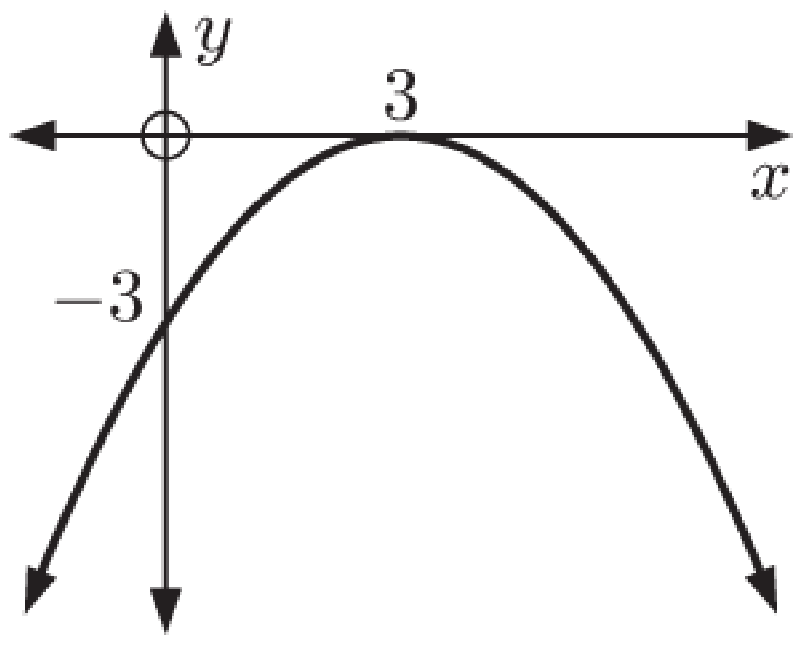
c



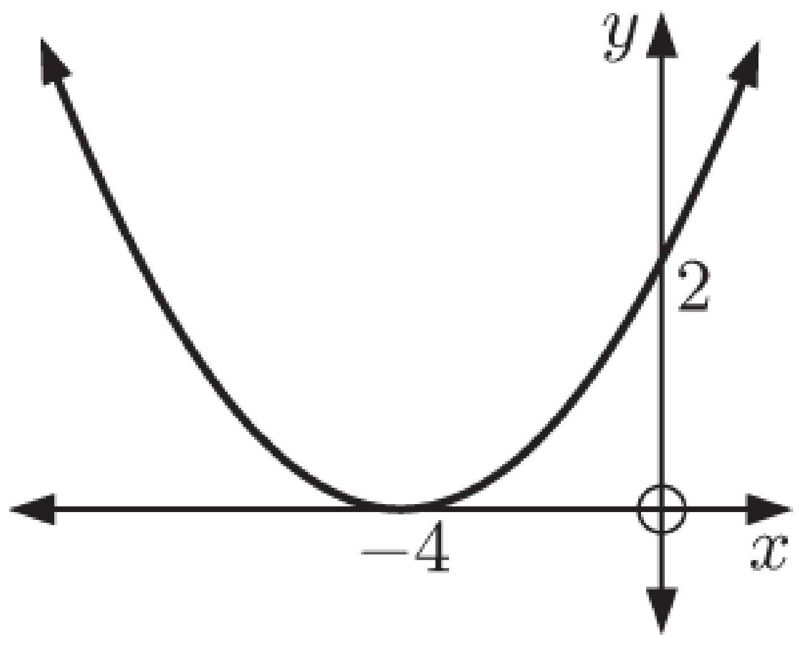
d



e



f



2 Find the axis of symmetry for each function:

a $y = (x-2)(x-6)$
b $y = x(x+4)$
c $y = -(x+3)(x-5)$
d $y = (x-3)(x-8)$
e $y = 2(x-5)^2$
f $y = -3(x+2)^2$

3 A quadratic function has axis of symmetry $x = -3$, and one of its x-intercepts is 4. Find the other x-intercept.

4 Find the axis of symmetry of each function:

a $y = x^2 + 6x + 2$
b $y = x^2 - 8x - 1$
c $f(x) = 2x^2 + 5x - 3$
d $y = -x^2 + 3x - 7$
e $y = 2x^2 - 5$
f $y = -5x^2 + 7x$
g $y = -3x^2 - x + 4$
h $y = 10x - 3x^2$
i $f(x) = \frac{1}{8}x^2 + x - 1$

5 **a** Use x-intercepts to find the axis of symmetry of $y = (x+2)(x-5)$.

b Check your answer to **a** by writing the function in the form $y = ax^2 + bx + c$, and then evaluating $-\frac{b}{2a}$.

6 The quadratic function $f(x) = ax^2 + 6x - 4$ has axis of symmetry $x = -2$. Find the value of a.

7 **a** Find the axis of symmetry of:

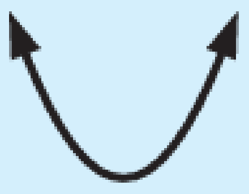
i $y = 2x^2 + 5x + 1$ **ii** $y = 2x^2 + 5x + 7$ **iii** $y = 2x^2 + 5x - 4$

Comment on your answers.

b For the quadratic function $y = ax^2 + bx + c$, the axis of symmetry $x = -\frac{b}{2a}$ depends on a and b, but not on c. Explain this result using transformations.

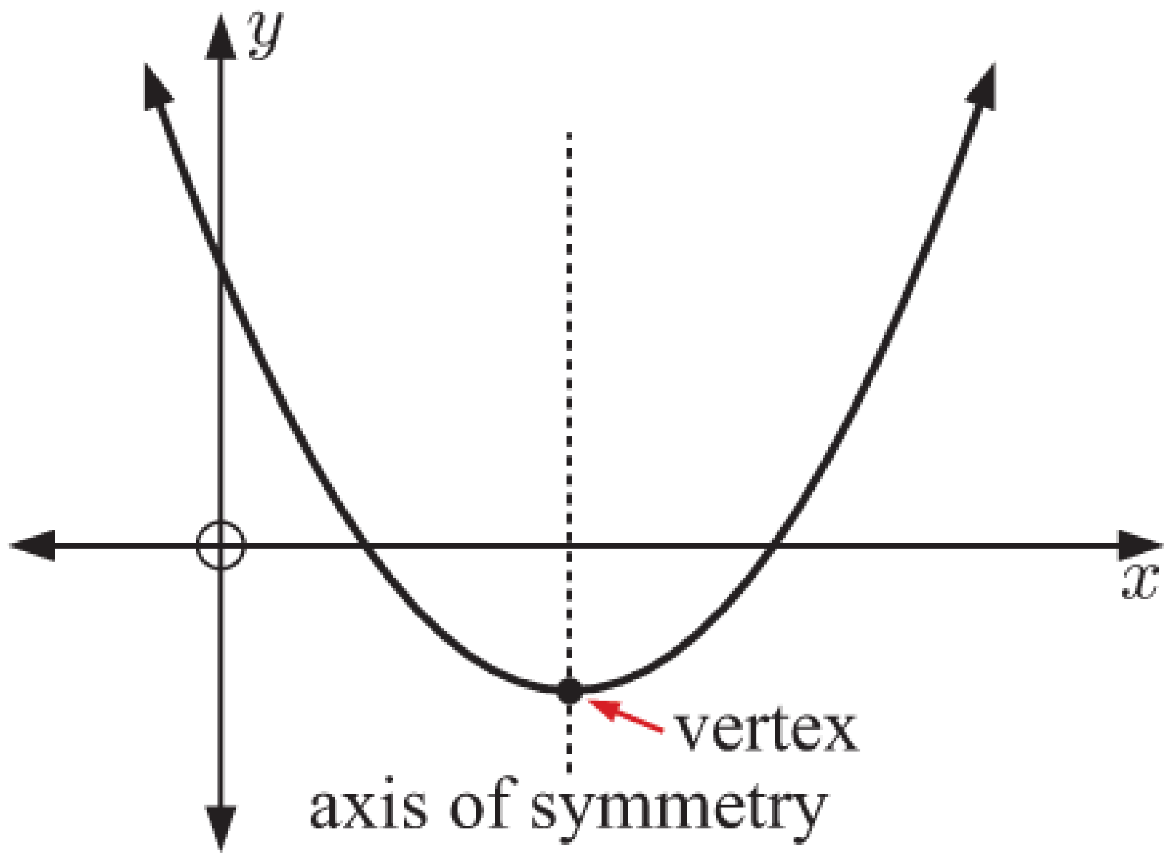
G VERTEX

The **vertex** or **turning point** of the quadratic function $y = ax^2 + bx + c$ is the point at which the function has:

- a **maximum value** for $a < 0$, *or*
- a **minimum value** for $a > 0$  .

The vertex of a quadratic function always lies on the **axis of symmetry**, so the axis of symmetry gives us the x-coordinate of the vertex.

The y-coordinate is found by substituting this value of x into the function.



Example 13 **Self Tutor**

Find the coordinates of the vertex of $f(x) = x^2 + 6x + 4$.

$a = 1, \quad b = 6, \quad c = 4$

Now $-\frac{b}{2a} = -\frac{6}{2 \times 1} = -3$

$\therefore$ the axis of symmetry is $x = -3$.

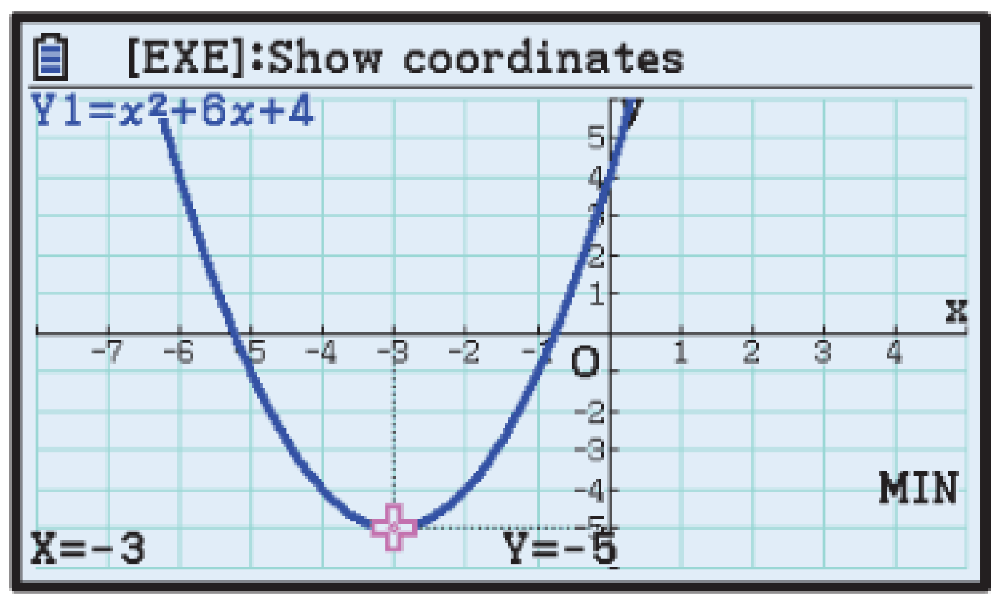
$$\begin{aligned} f(-3) &= (-3)^2 + 6(-3) + 4 \\ &= 9 - 18 + 4 \\ &= -5 \end{aligned}$$

So, the vertex is $(-3, -5)$.

The vertex is a minimum turning point since $a > 0$.

We can also use technology to find the vertex of a quadratic function.

GRAPHICS CALCULATOR INSTRUCTIONS



[EXE]:Show coordinates

Y1=x²+6x+4

X=-3 Y=-5

MIN

Example 14 **Self Tutor**

Consider the quadratic function $y = -(x+5)(x-1)$.

a Find the axes intercepts.

b Find the equation of the axis of symmetry.

c Find the coordinates of the vertex.

d Sketch the function, showing all important features.

e State the domain and range of the function.

a When $x = 0$, $y = -(5)(-1) = 5$

$\therefore$ the y-intercept is 5.

When $y = 0$,

$-(x+5)(x-1) = 0$

$\therefore x = -5$ or 1

$\therefore$ the x-intercepts are -5 and 1.

b The axis of symmetry is halfway between the x-intercepts -5 and 1.

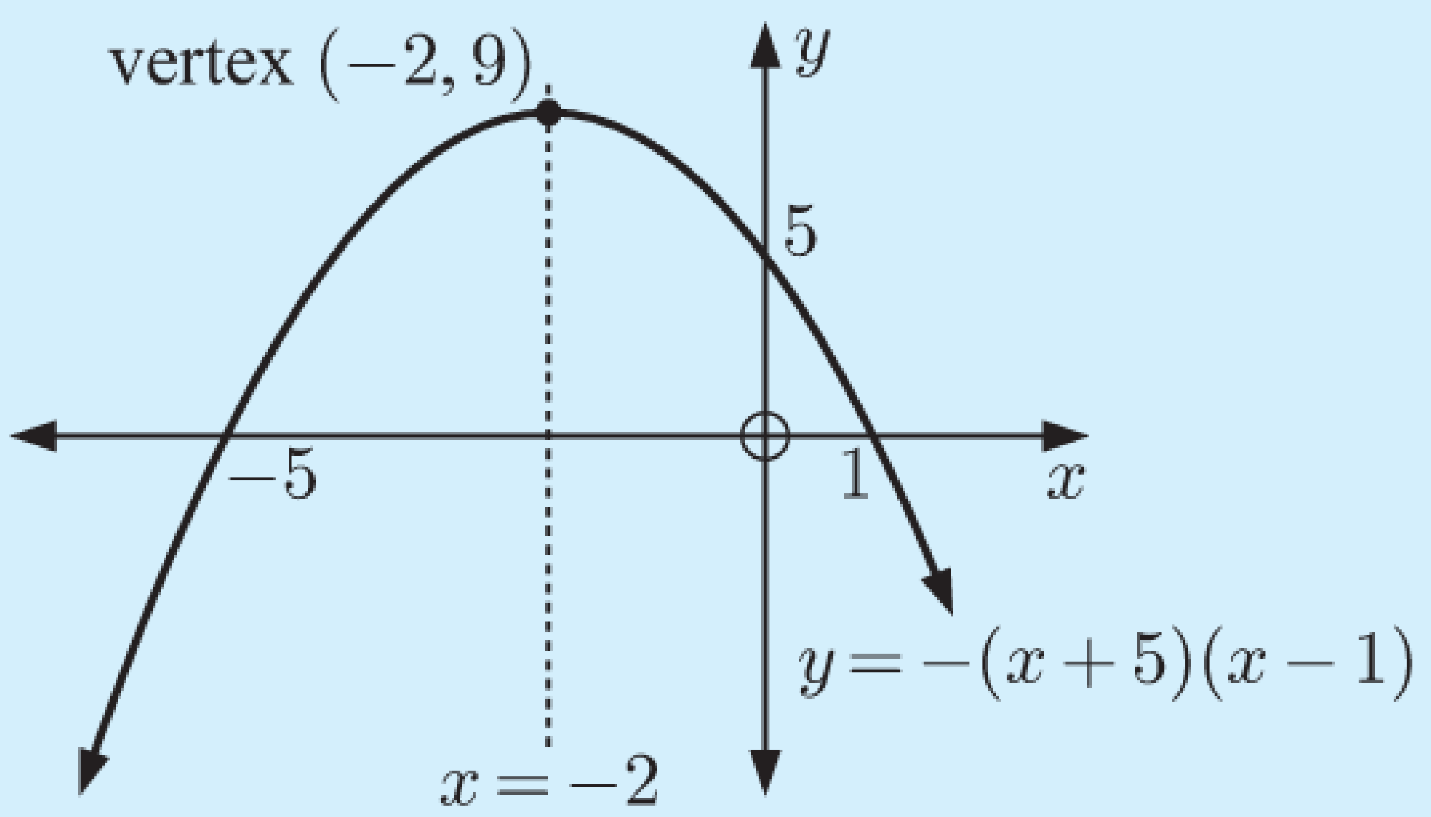
So, the axis of symmetry is $x = -2$.

c When $x = -2$,

$$\begin{aligned} y &= -(-2+5)(-2-1) \\ &= -(3)(-3) \\ &= 9 \end{aligned}$$

So, the vertex is $(-2, 9)$.

d

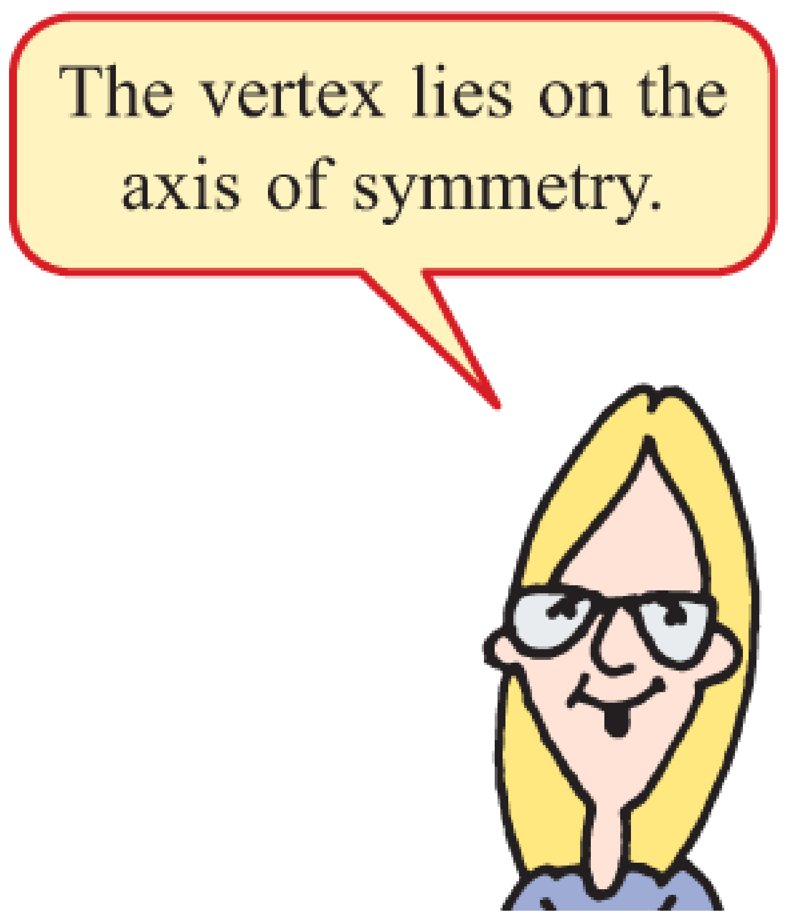


e The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \leqslant 9\}$.

EXERCISE 6G

1 Locate the turning point or vertex for each of the following quadratics. In each case, state whether the vertex is a maximum turning point or a minimum turning point.


The vertex lies on the axis of symmetry.

a $y = x^2 - 4x + 2$

b $y = (x+3)(x-1)$

c $y = 2x^2 + 4$

d $y = -3x^2 + 1$

e $y = (x-4)(x+2)$

f $y = -x^2 - 4x - 9$

g $y = 2x^2 + 6x - 1$

h $y = -2(x+3)(x-4)$

i $y = -\frac{1}{2}x^2 + x - 5$

j $y = \frac{1}{4}x^2 - 7x + 6$

2 For each of the following quadratics:

i Find the axes intercepts.

ii Find the axis of symmetry.

iii Find the coordinates of the vertex.

iv Sketch the quadratic.

v State the domain and range.

a $y = (x-1)(x-7)$

b $y = -x^2 - 6x - 8$

c $y = 6x - x^2$

d $y = -(x-1)(x-2)$

e $y = 2x^2 + 4x - 24$

f $y = -3x^2 + 4x - 1$

g $y = 2x^2 - 5x + 2$

h $y = (2x-5)(2x+1)$

i $y = -\frac{1}{4}x^2 + 2x - 3$

3 Find:

- **a** the minimum value of $y = x^2 - 8x + 5$
- **b** the maximum value of $y = -x^2 - 10x + 4$
- **c** the minimum value of $y = 3x^2 + 3x - 2$
- **d** the maximum value of $y = -\frac{1}{2}x^2 + 7x - 4$.

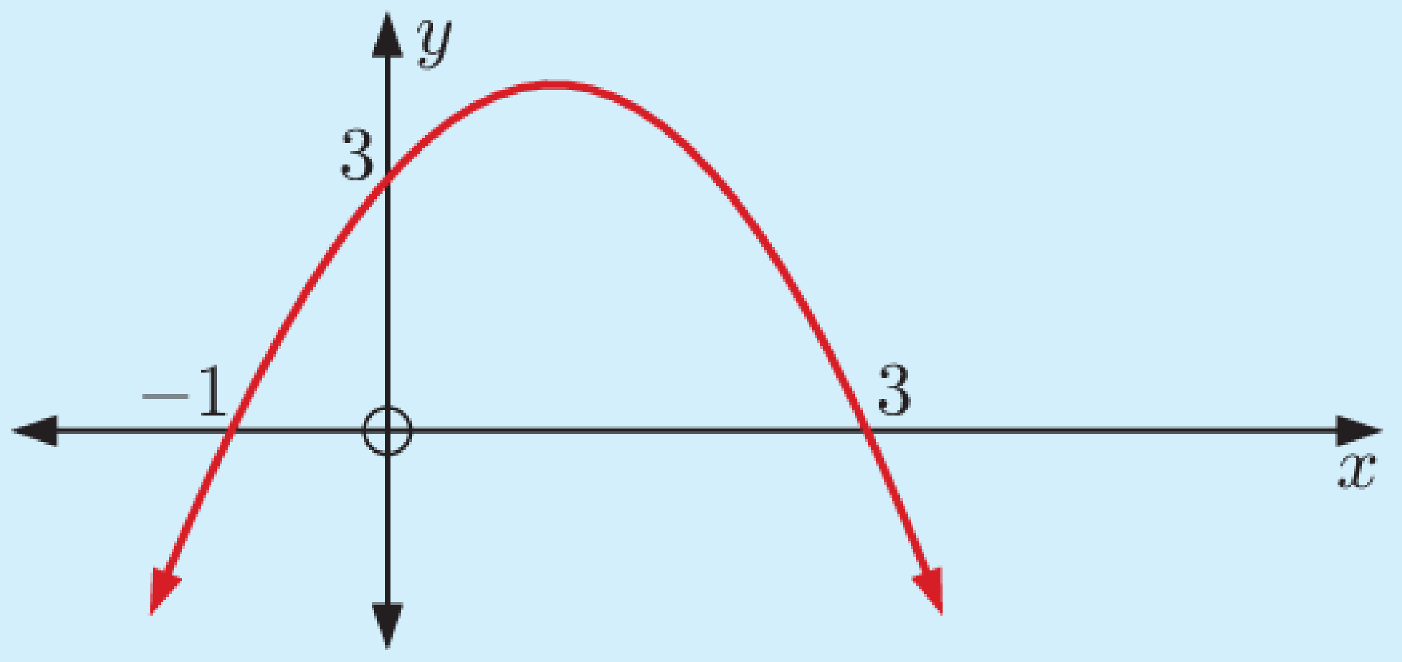
H FINDING A QUADRATIC FROM ITS GRAPH

If we are given sufficient information on or about a graph, we can determine the quadratic in whatever form is required.

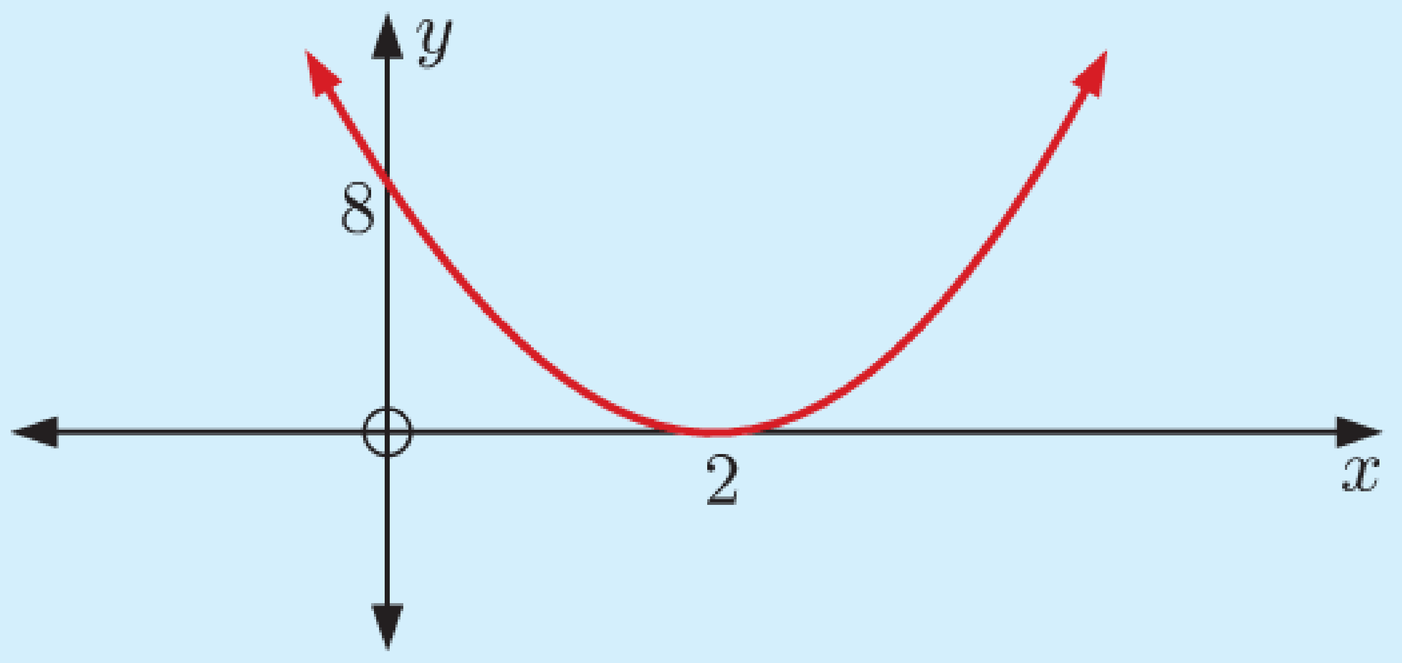
Example 15 **Self Tutor**

Find the equation of the quadratic with graph:

a



b



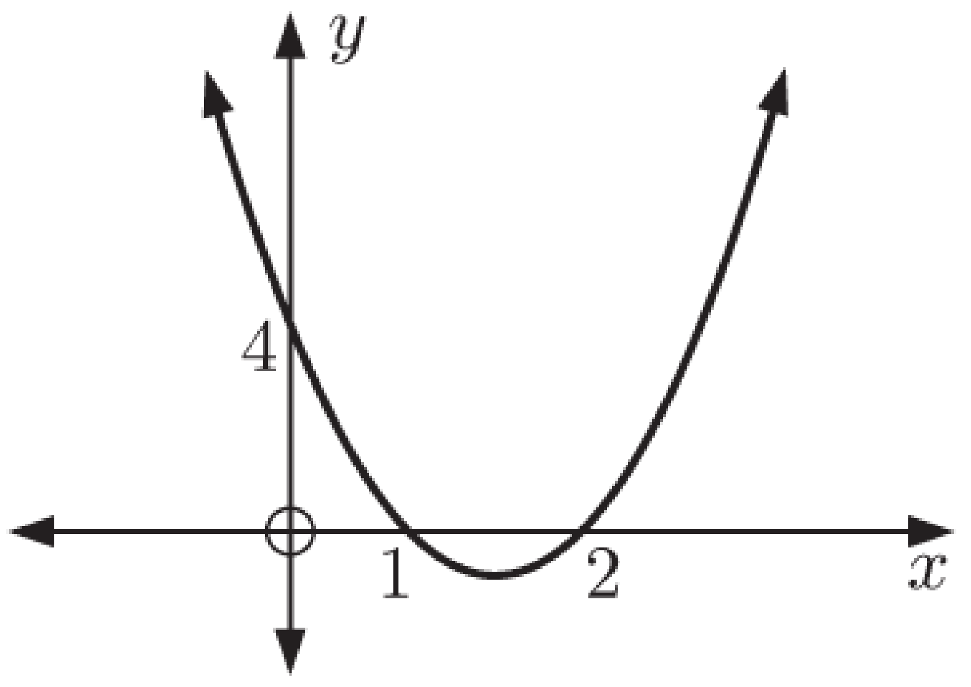
a Since the x-intercepts are -1 and 3,
$y = a(x+1)(x-3)$.
When $x = 0$, $y = 3$
$\therefore\ 3 = a(1)(-3)$
$\therefore\ a = -1$
The quadratic is $y = -(x+1)(x-3)$.

b The graph touches the x-axis at $x = 2$,
so $y = a(x-2)^2$.
When $x = 0$, $y = 8$
$\therefore\ 8 = a(-2)^2$
$\therefore\ a = 2$
The quadratic is $y = 2(x-2)^2$.

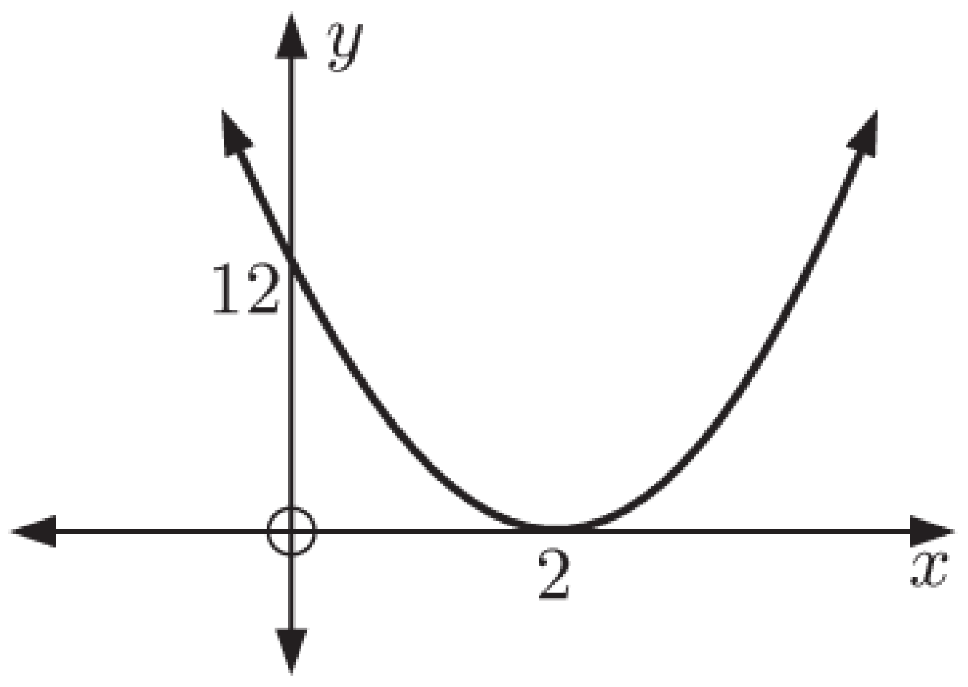
EXERCISE 6H

1 Find the equation of the quadratic with graph:

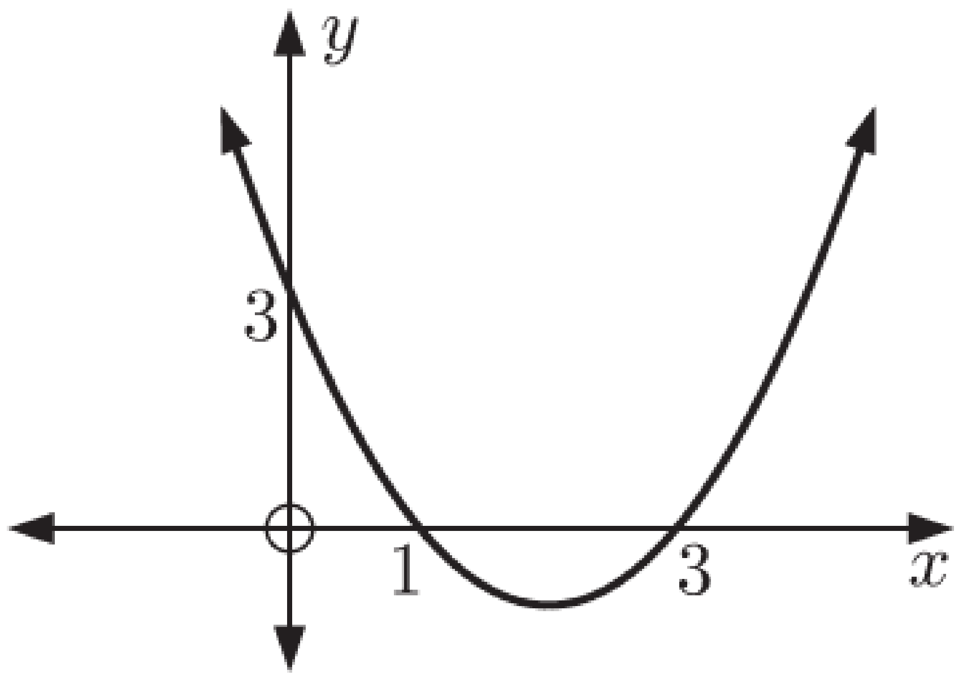
a



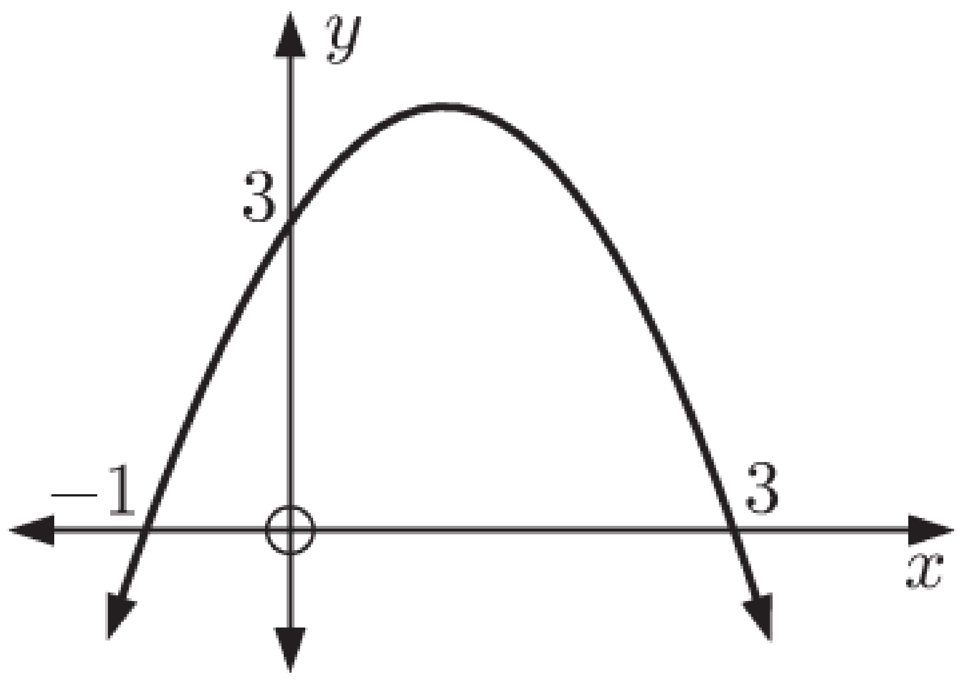
b



c



d



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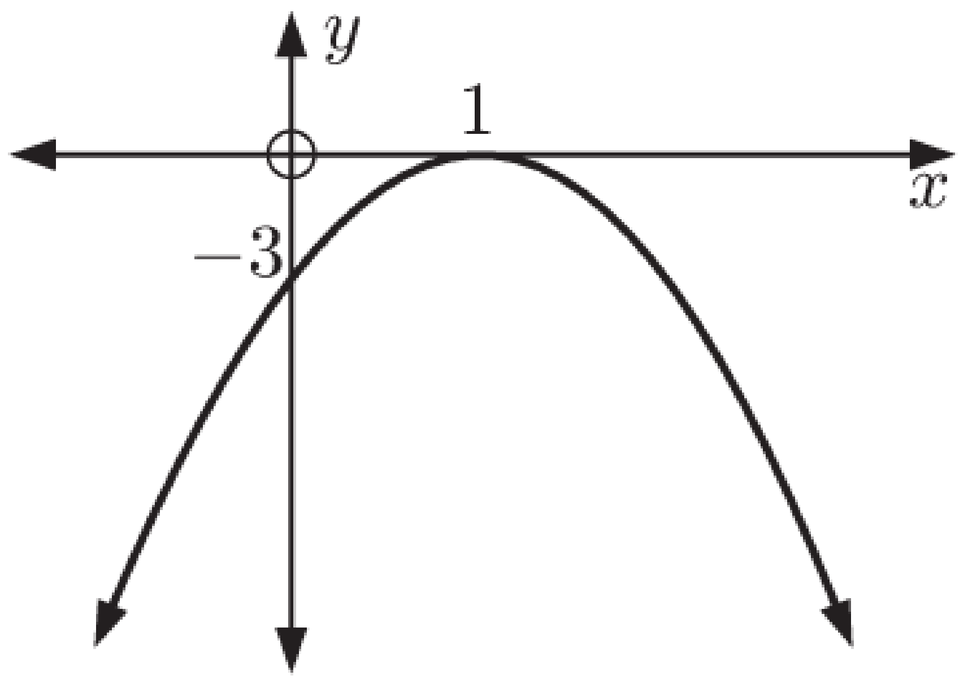

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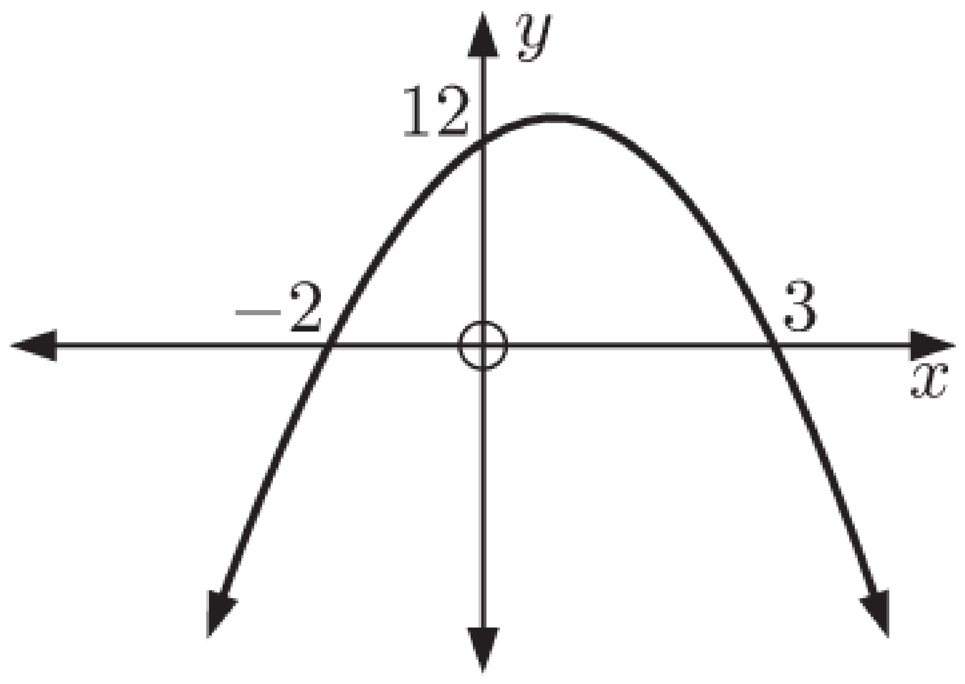
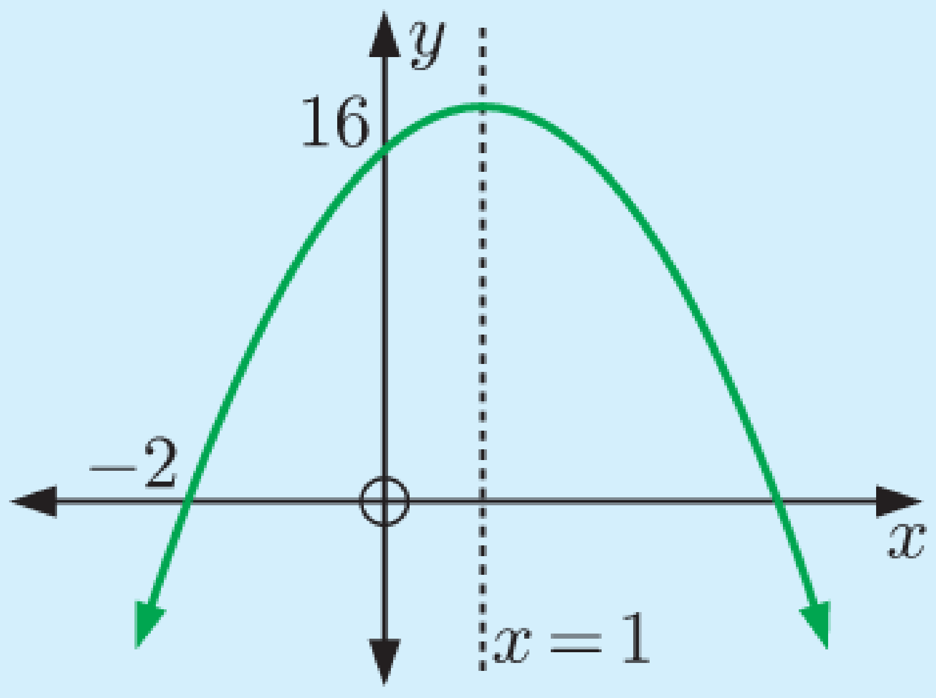
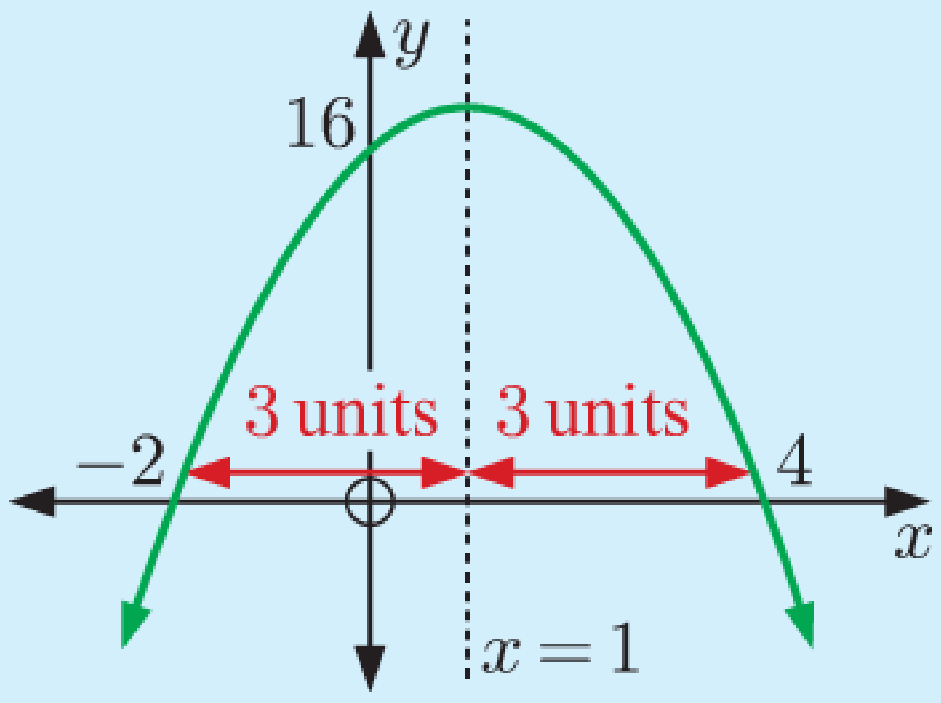


Example 16 **Self Tutor**

Find the equation of the quadratic with graph:



The axis of symmetry $x = 1$ lies midway between the x-intercepts.



$\therefore$ the other x-intercept is 4.

$\therefore$ the quadratic has the form

$$y = a(x+2)(x-4) \quad \text{where} \quad a < 0$$

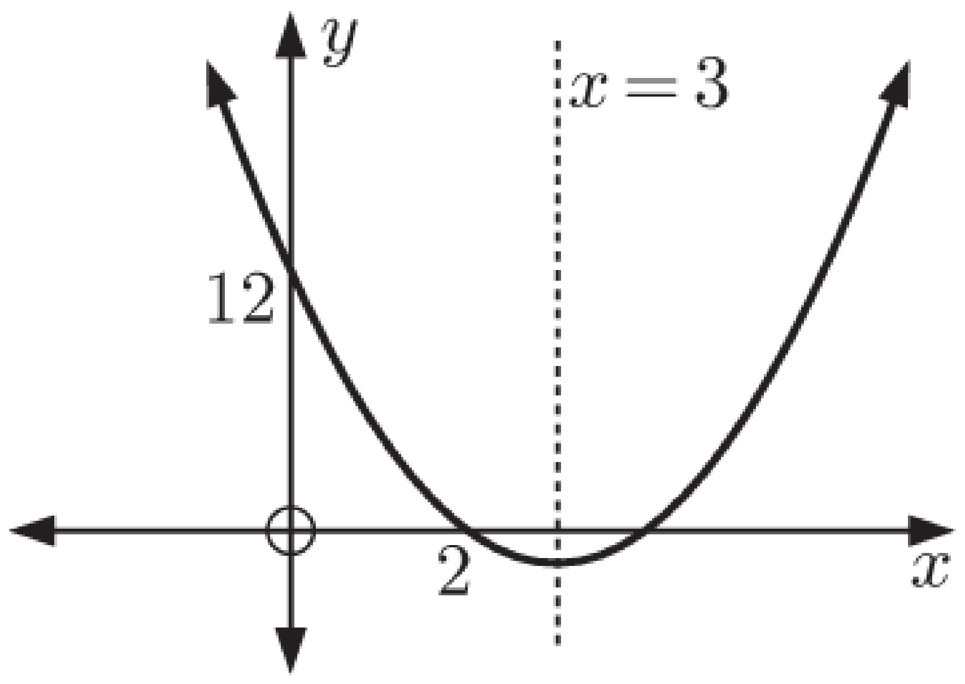
But when $x = 0$, $y = 16$

$$\therefore \quad 16 = a(2)(-4)$$
$$\therefore \quad a = -2$$

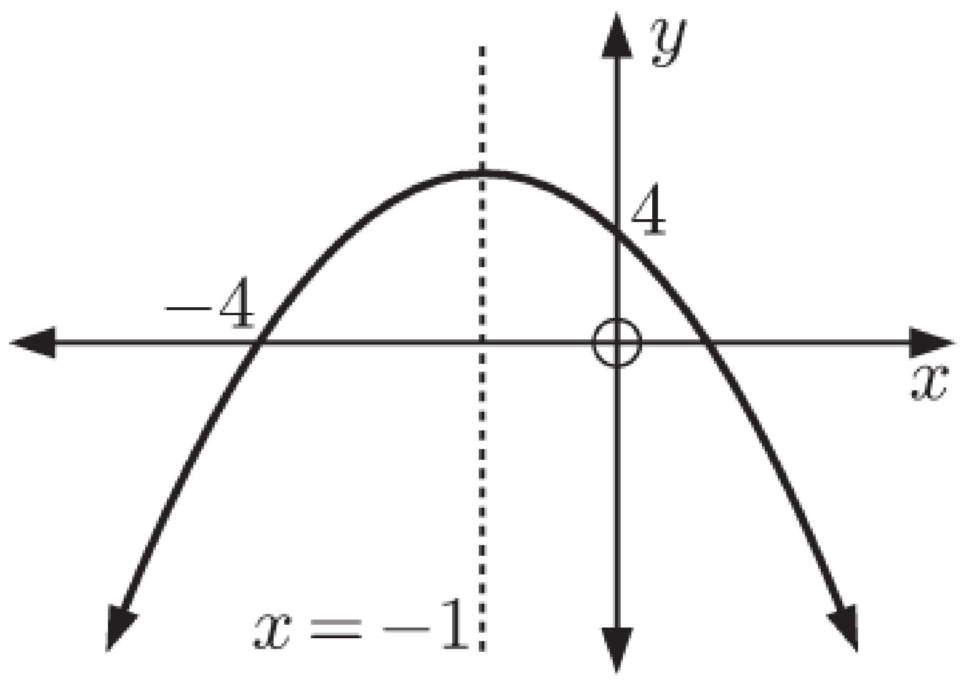
The quadratic is $y = -2(x+2)(x-4)$.

2 Find the equation of the quadratic with graph:

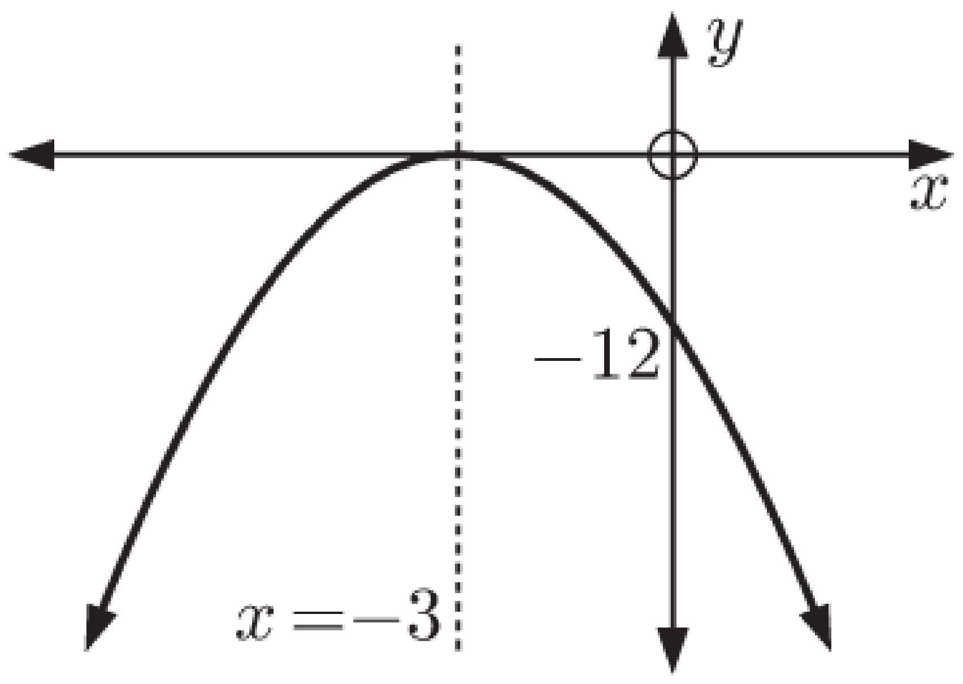
a



b



c



Example 17 **Self Tutor**

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x-intercepts are 4 and -3, the quadratic has the form $y = a(x-4)(x+3)$, $a \neq 0$.

When $x = 2$, $y = -20$

$$\therefore \quad -20 = a(2-4)(2+3)$$
$$\therefore \quad -20 = a(-2)(5)$$
$$\therefore \quad a = 2$$

The quadratic is

$$\begin{aligned} y &= 2(x-4)(x+3) \\ &= 2(x^2 - x - 12) \\ &= 2x^2 - 2x - 24 \end{aligned}$$

3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

- **a** cuts the x-axis at 5 and 1, and passes through $(2, -9)$
- **b** cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
- **c** touches the x-axis at 3 and passes through $(-2, -25)$
- **d** touches the x-axis at -2 and passes through $(-1, 4)$
- **e** cuts the x-axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$
- **f** cuts the x-axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

Example 18

Self Tutor

Find the equation of the quadratic function which passes through the points $(-2, 5)$, $(1, -4)$, and $(3, 10)$.

Let the quadratic function be $y = ax^2 + bx + c$.

When $x = -2$, $y = 5$ $\quad\therefore\ 5 = a(-2)^2 + b(-2) + c$ $\quad$ or $\quad 4a - 2b + c = 5$

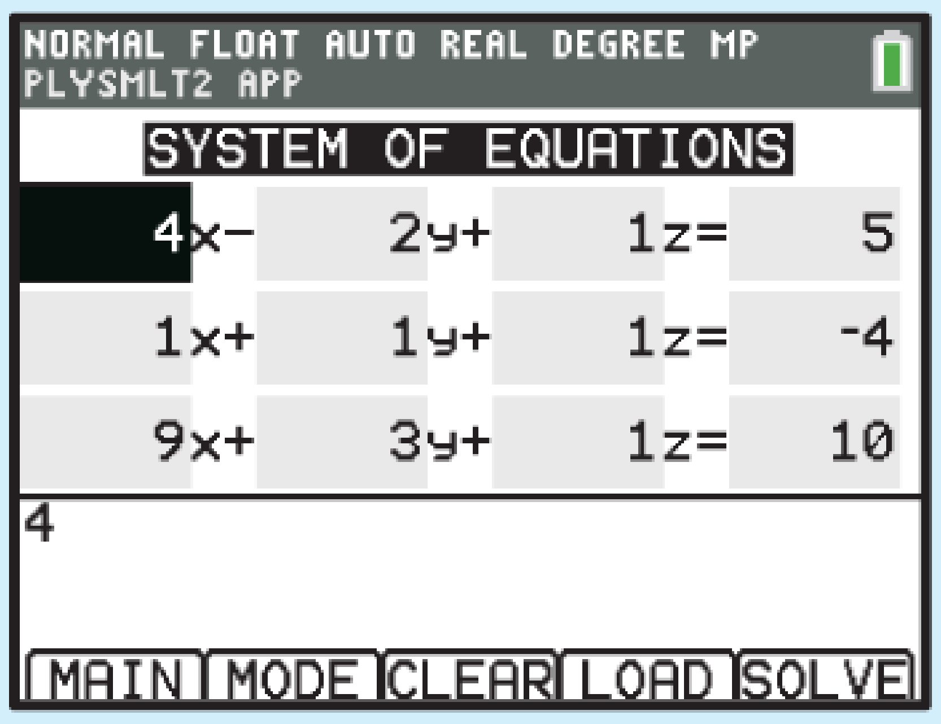
When $x = 1$, $y = -4$ $\quad\therefore\ -4 = a(1)^2 + b(1) + c$ $\quad$ or $\quad a + b + c = -4$

When $x = 3$, $y = 10$ $\quad\therefore\ 10 = a(3)^2 + b(3) + c$ $\quad$ or $\quad 9a + 3b + c = 10$

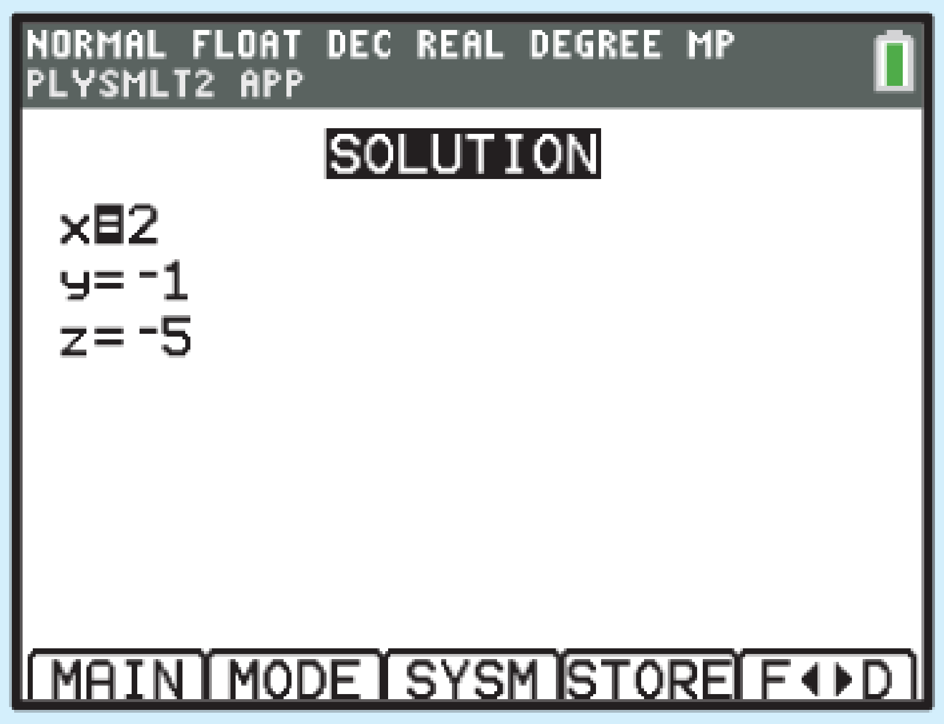
We solve the system of equations $\begin{cases} 4a - 2b + c = 5 \\ a + b + c = -4 \\ 9a + 3b + c = 10 \end{cases}$ simultaneously using technology.

We find that $a = 2$, $b = -1$, and $c = -5$.

So, the function is $y = 2x^2 - x - 5$.



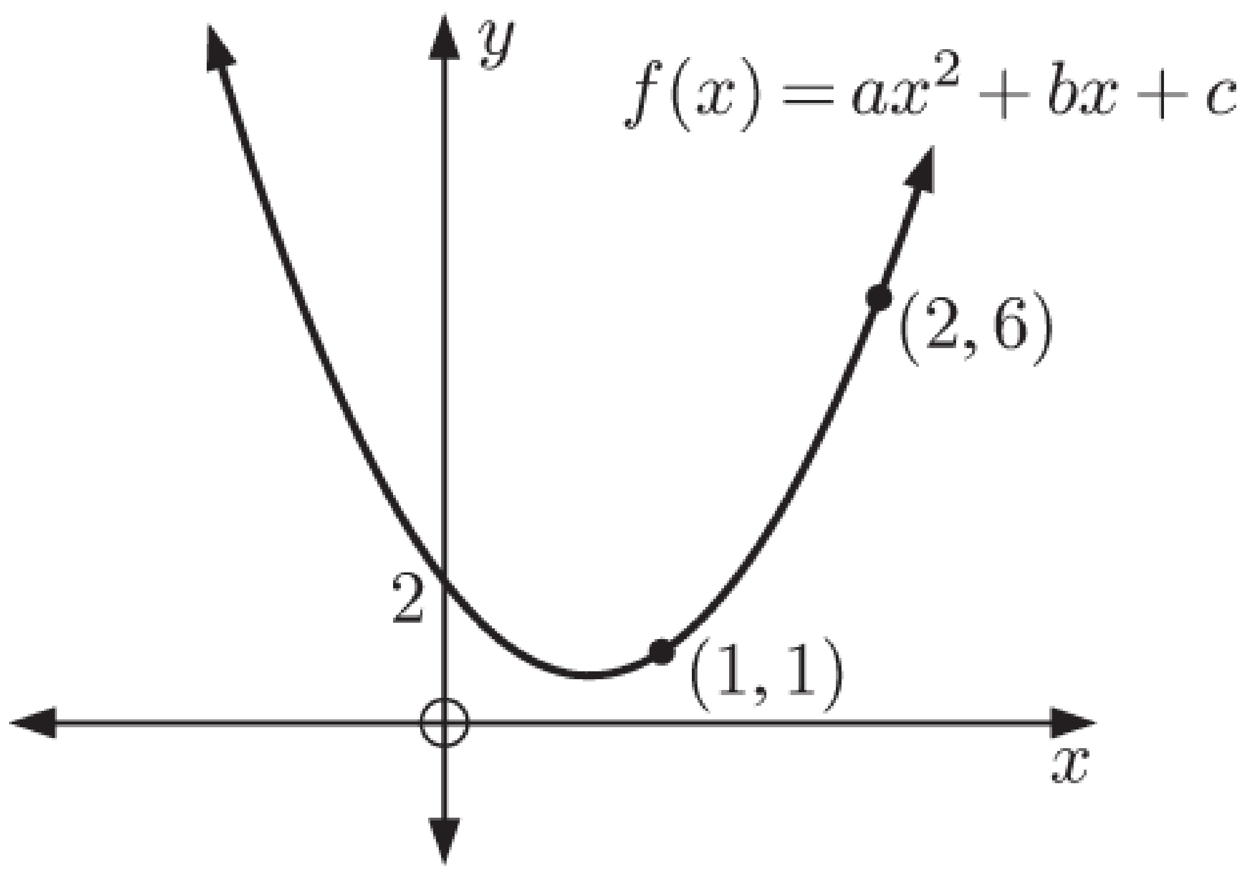
```
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
      SYSTEM OF EQUATIONS
  4x−     2y+     1z=     5
  1x+     1y+     1z=    -4
  9x+     3y+     1z=    10
4
```



```
NORMAL FLOAT DEC REAL DEGREE MP
PLYSMLT2 APP
           SOLUTION
 x=2
 y=-1
 z=-5
```

4 Consider the quadratic function $f(x) = ax^2 + bx + c$ alongside.

- **a** State the value of c.
- **b** Write two equations in terms of a and b.
- **c** Use technology to find a and b, and hence state the equation of the quadratic function.



5 Find the equation of the quadratic function which passes through the points:

- **a** $(1, -2)$, $(2, 4)$, and $(3, 12)$
- **b** $(-1, 3)$, $(2, 9)$, and $(4, -7)$
- **c** $(-8, 4)$, $(-4, -8)$, and $(6, 32)$.

6 Try to find the equation of the quadratic which passes through the points $(-1, 7)$, $(2, 1)$, and $(3, -1)$. Explain your answer.

7 A quadratic function $y = ax^2 + bx + c$ passes through the points $(1, -1)$, $(2, 1)$, and $(5, -5)$.

- **a** Plot the three points on a set of axes.
- **b** Explain why:
  - **i** a must be negative
  - **ii** c must be negative
  - **iii** b must be positive.
- **c** Use technology to find a, b, and c.
- **d** Find the coordinates of the vertex of the quadratic.

8 The quadratic function $f(x) = ax^2 + bx + c$ has vertex $(3, 7)$ and passes through the point $(5, 3)$.

- **a** Use the information given to write three linear equations in terms of a, b, and c.
- **b** Find the equation of the quadratic function.
- **c** Sketch the quadratic function.

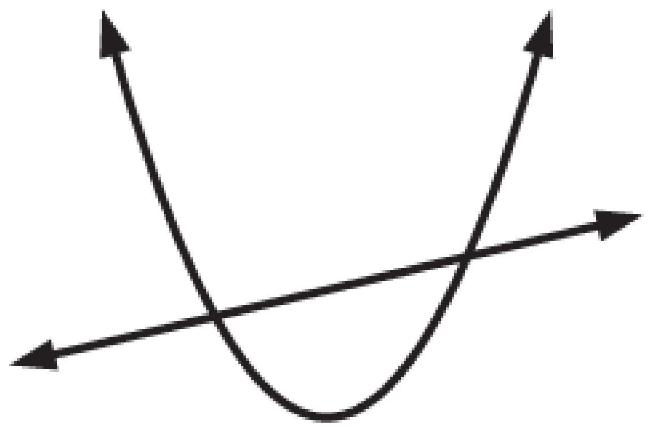
ACTIVITY 2

Click on the icon to run a card game for quadratic functions.

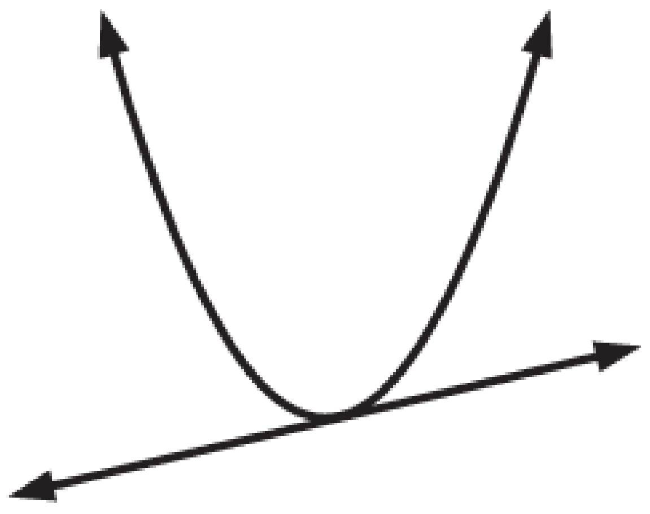
CARD GAME

I INTERSECTION OF GRAPHS

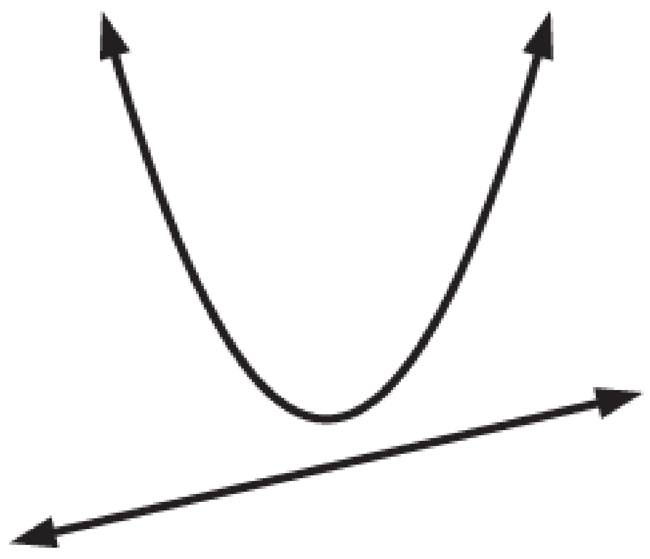
When a quadratic function and a linear function are graphed on the same set of axes, there could be 2, 1, or 0 points of intersection.



cutting
2 points of intersection



touching
1 point of intersection



missing
no points of intersection

We can use technology to find the intersection points of two functions.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

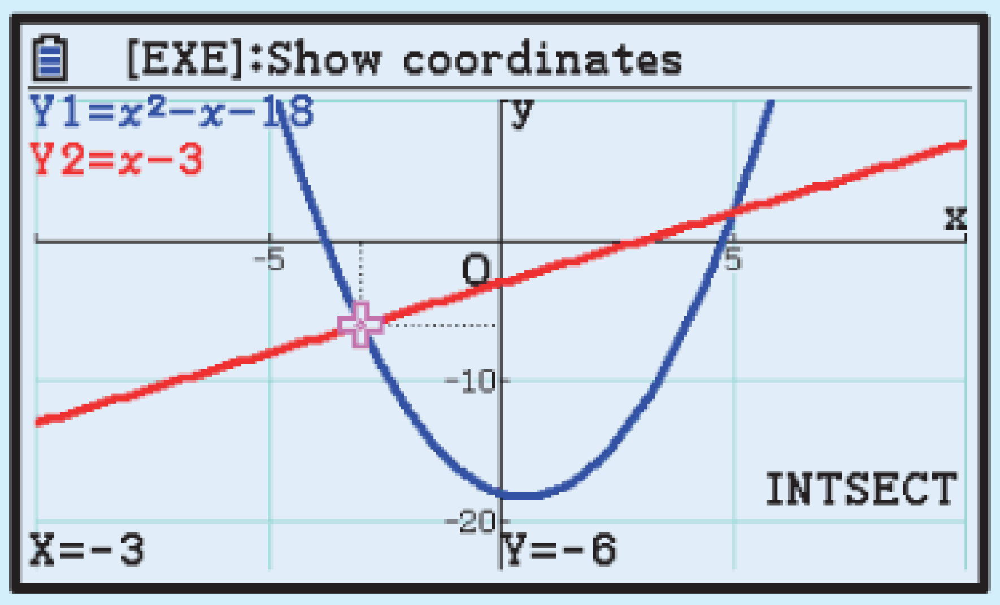
Example 19

Self Tutor

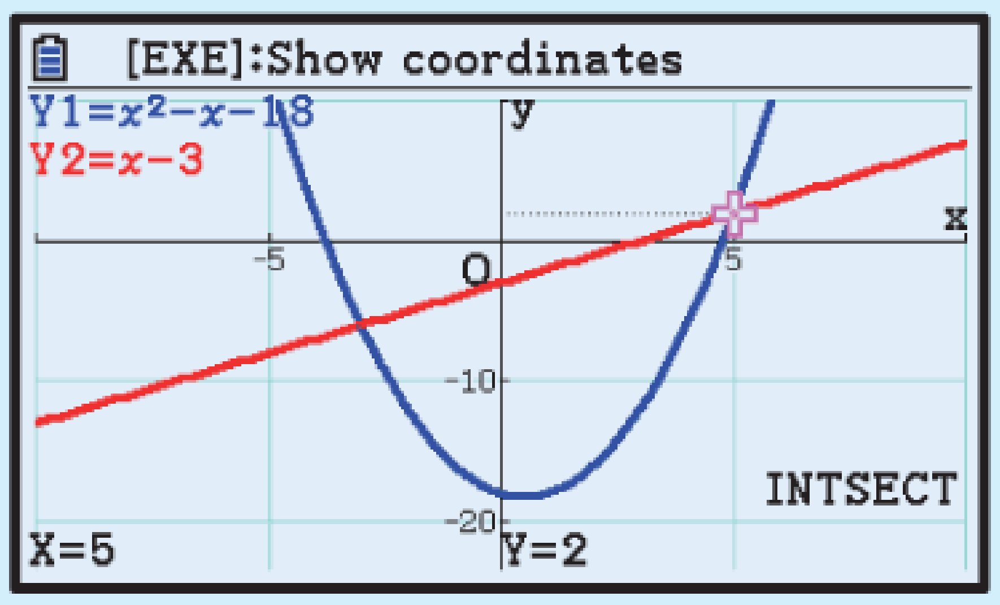
Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

We graph $Y_1 = X^2 - X - 18$ and $Y_2 = X - 3$ on the same set of axes.

The graphs intersect at $(-3, -6)$ and $(5, 2)$.



[EXE]:Show coordinates
Y1=x²−x−18
Y2=x−3
INTSECT
X=-3 Y=-6



[EXE]:Show coordinates
Y1=x²−x−18
Y2=x−3
INTSECT
X=5 Y=2

EXERCISE 6I

1 Find the coordinates of the point(s) of intersection of the graphs with equations:

a $y = x^2 + x - 1$ and $y = 2x - 1$

b $y = -x^2 + 2x + 3$ and $y = 2x - 1$

c $y = x^2 - 2x + 8$ and $y = x + 6$

d $y = -x^2 + 3x + 9$ and $y = 2x - 3$

e $y = x^2 - 4x + 3$ and $y = 2x - 6$

f $y = -x^2 + 4x - 7$ and $y = 5x - 4$.

2 Find, to 3 significant figures, the coordinates of the points of intersection of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$

b $y = x^2 - 5x + 2$ and $y = x - 7$

c $y = -x^2 - 2x + 4$ and $y = x + 8$

d $y = -x^2 + 4x - 2$ and $y = 5x - 6$

e $y = x^2 + 5x - 4$ and $y = -\frac{1}{2}x - 3$

f $y = -x^2 + 7x + 1$ and $y = -3x + 2$

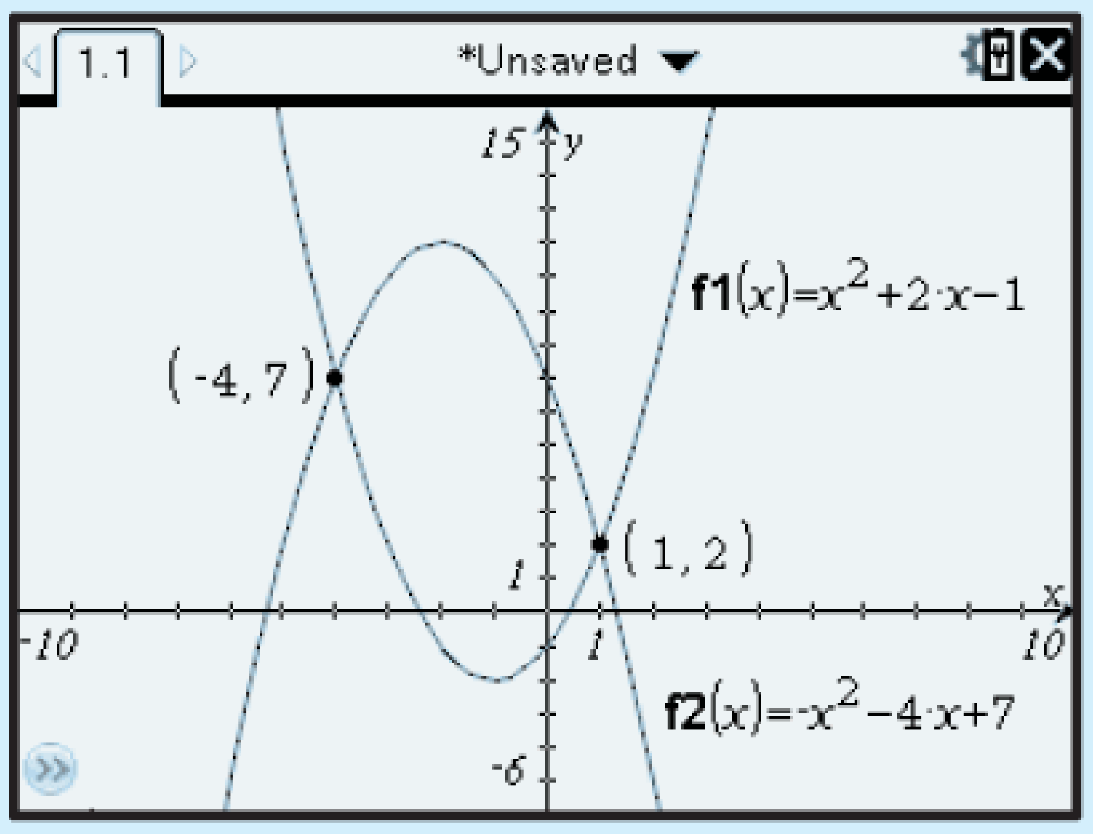
g $y = 3x^2 - x - 2$ and $y = 2x - \frac{11}{4}$.

Example 20 **Self Tutor**

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 + 2x - 1$ and $y = -x^2 - 4x + 7$.

We graph $Y_1 = X^2 + 2X - 1$ and $Y_2 = -X^2 - 4X + 7$ on the same set of axes.

The graphs intersect at $(-4, 7)$ and $(1, 2)$.



3 Find, to 3 significant figures, the coordinates of the points of intersection of the graphs with equations:

a $y = x^2 - 8x + 15$ and $y = -x^2 - 4x + 7$

b $y = x^2 - 3x + 4$ and $y = -x^2 + x + 2$

c $y = x^2 - 4x + 9$ and $y = -x^2 + 8x - 12$

d $y = -2x^2 - 3x + 15$ and $y = -x^2 + 5$

e $y = x^2 + 5x + 7$ and $y = 3x^2 - 5x + 6$

f $y = 7x^2 - 14x$ and $y = x^2 - 12x + 36$.

A quadratic may miss, touch, or intersect another quadratic.

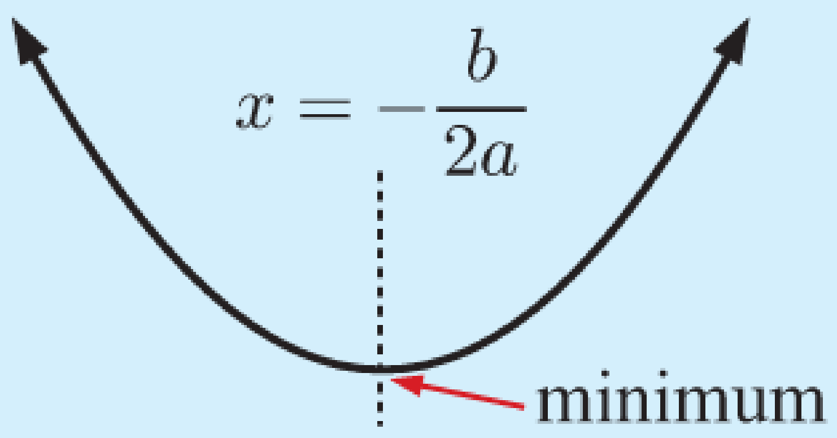
J QUADRATIC MODELS

There are many situations in the real world where there is a quadratic relationship between two variables.

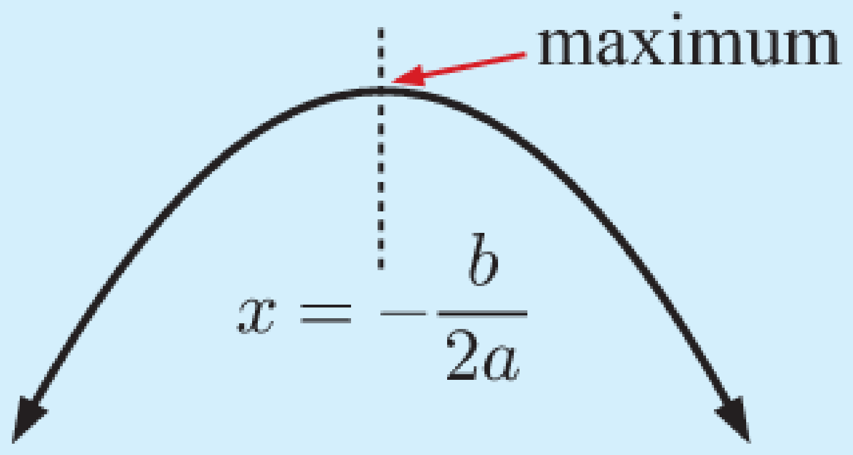
For example, when an object is subjected to a constant force, its resulting motion is parabolic. We observe this when objects are thrown or fall with gravity. We can use a quadratic model to answer questions about the situation. For example, the vertex of the function might tell us the maximum height reached by a ball, and the time at which this maximum height occurred.

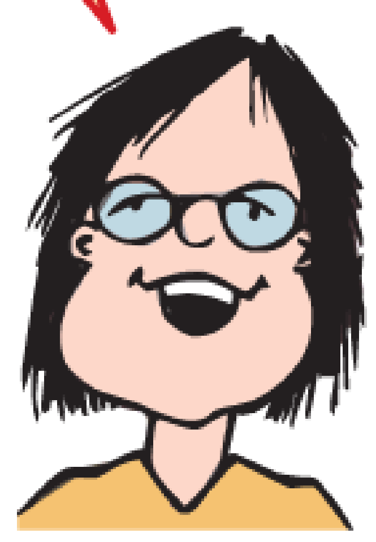
For the quadratic $y = ax^2 + bx + c$, we have seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.





The process of finding a maximum or minimum value is called **optimisation**.

Example 21

Self Tutor

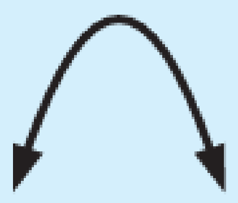
When a baseball is hit, its height above the ground after t seconds is given by $H(t) = -5t^2 + 30t + 1$ metres, $t \geqslant 0$.

a Find the ball's height above the ground after 2 seconds.
b How long does it take for the ball to reach its maximum height?
c Find the maximum height reached by the ball.
d How long does it take for the ball to hit the ground?

a $$\begin{aligned} H(2) &= -5(2)^2 + 30(2) + 1 \\ &= -20 + 60 + 1 \\ &= 41 \end{aligned}$$

After 2 seconds, the ball is 41 m above the ground.

b For the quadratic function $H(t)$, $a = -5$, $b = 30$, and $c = 1$.

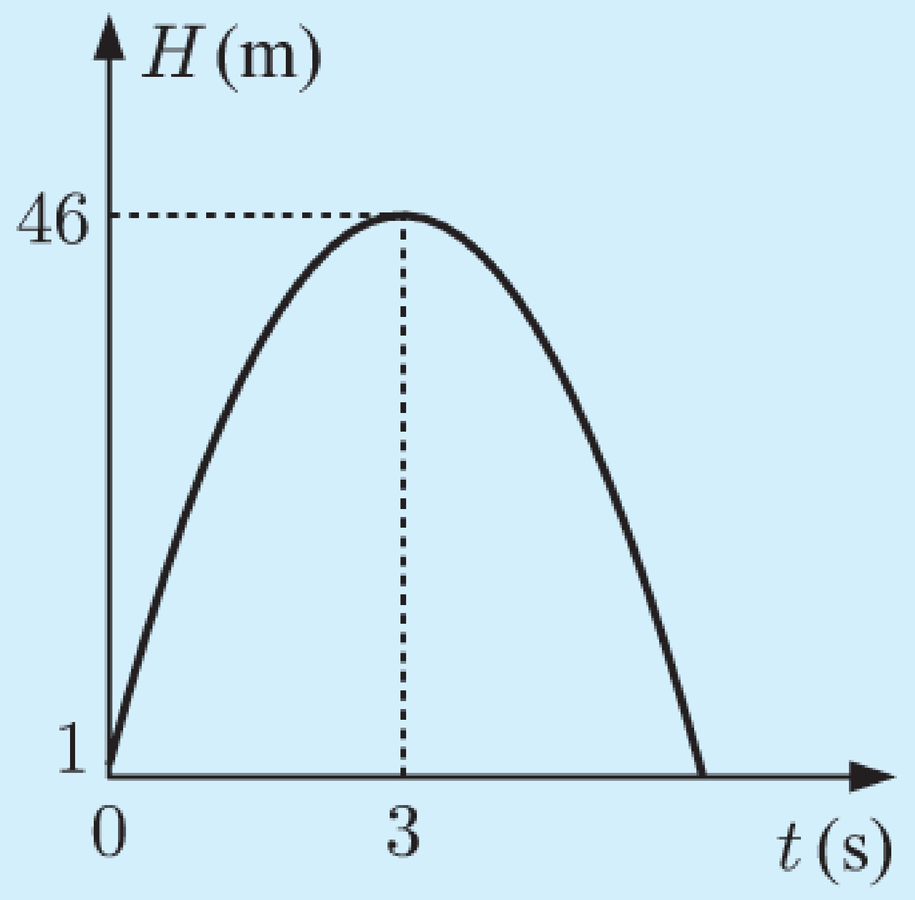
Since $a < 0$, the shape is  .

The maximum height occurs when $t = -\frac{b}{2a} = -\frac{30}{2 \times (-5)} = 3$

So, the maximum height is reached after 3 seconds.

c $$\begin{aligned} H(3) &= -5(3)^2 + 30(3) + 1 \\ &= -45 + 90 + 1 \\ &= 46 \end{aligned}$$

So, the maximum height reached is 46 metres.



d The ball hits the ground when

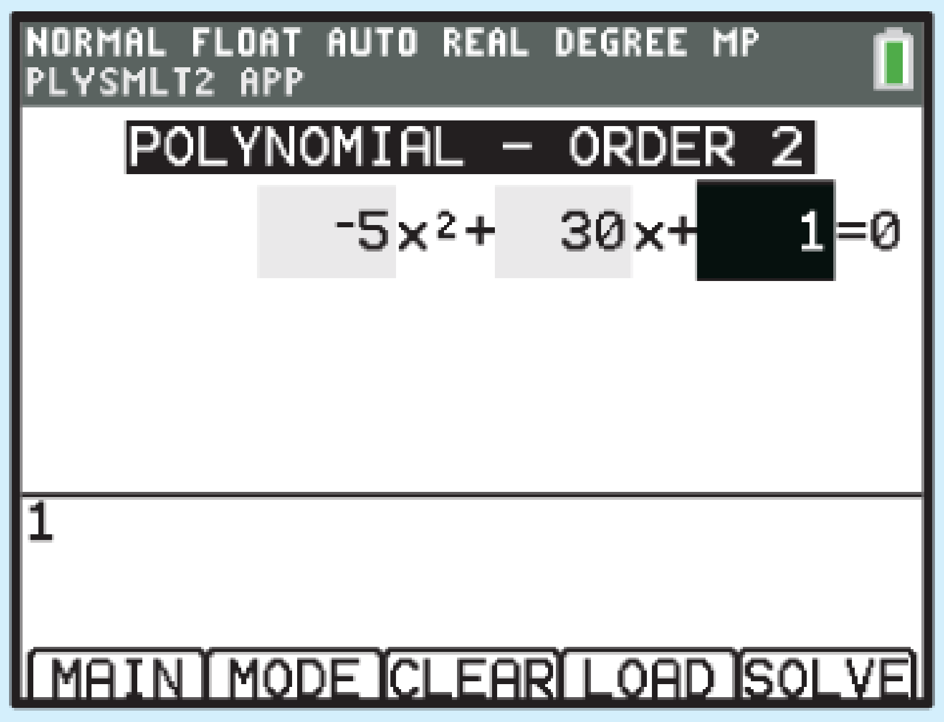
$$H(t) = 0$$

$$\therefore \quad -5t^2 + 30t + 1 = 0$$

$$\therefore \quad t \approx -0.0332 \text{ or } 6.0332$$

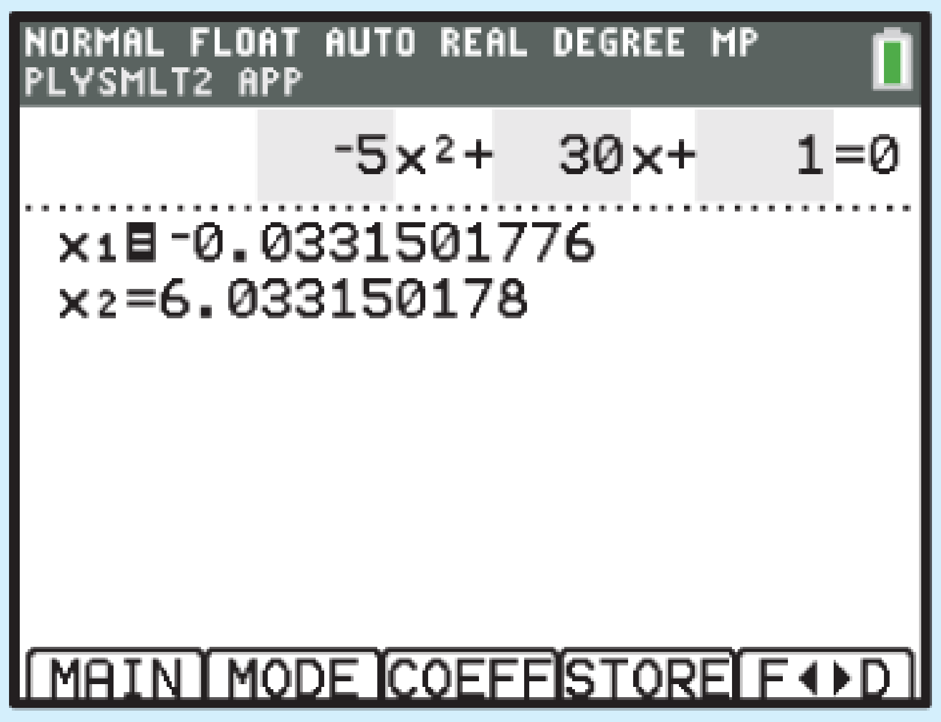
{using technology}

Since t must be positive, the ball hits the ground after approximately 6.03 seconds.



```
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
    POLYNOMIAL - ORDER 2
         -5x²+  30x+    1=0

1
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
         -5x²+  30x+    1=0
...........................
 x1=-0.0331501776
 x2=6.033150178
```

EXERCISE 6J

1 When Andrew dives off a pier, his height above the water after t seconds is given by $H(t) = -4t^2 + 4t + 3$ metres, $t \geqslant 0$.

- **a** How high above the water is the pier?
- **b** How long does it take for Andrew to reach the maximum height of his dive?
- **c** How far is Andrew above the water at his highest point?
- **d** How long does it take for Andrew to hit the water?

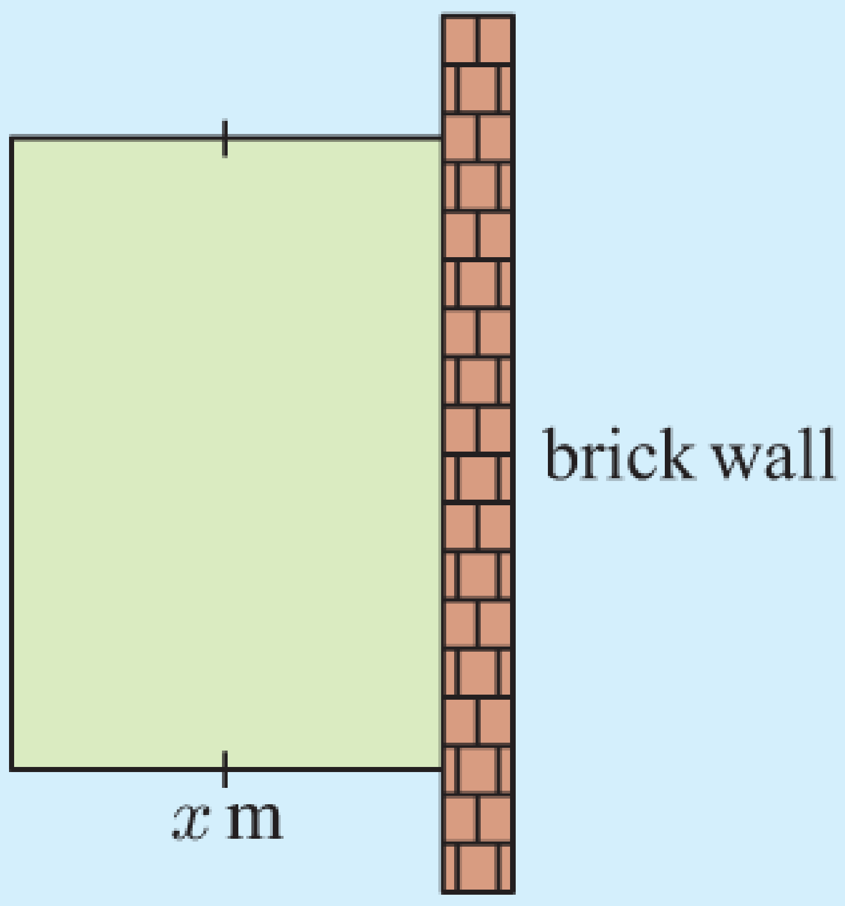
2 If Jasmine makes x necklaces each day for her market stall, her daily profit is given by $P(x) = -x^2 + 20x$ dollars, $0 \leqslant x \leqslant 25$.

- **a** How much profit does Jasmine make if she makes 7 necklaces per day?
- **b** How many necklaces should Jasmine make per day to maximise her profit?
- **c** Find the maximum daily profit that Jasmine can make.

Example 22 **Self Tutor**

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

- **a** Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.
- **b** Find the dimensions of the garden of maximum area.



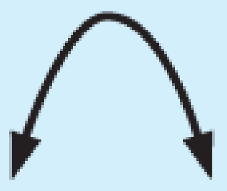
a Side [XY] has length $(40 - 2x)$ m.

Now, area = length × width

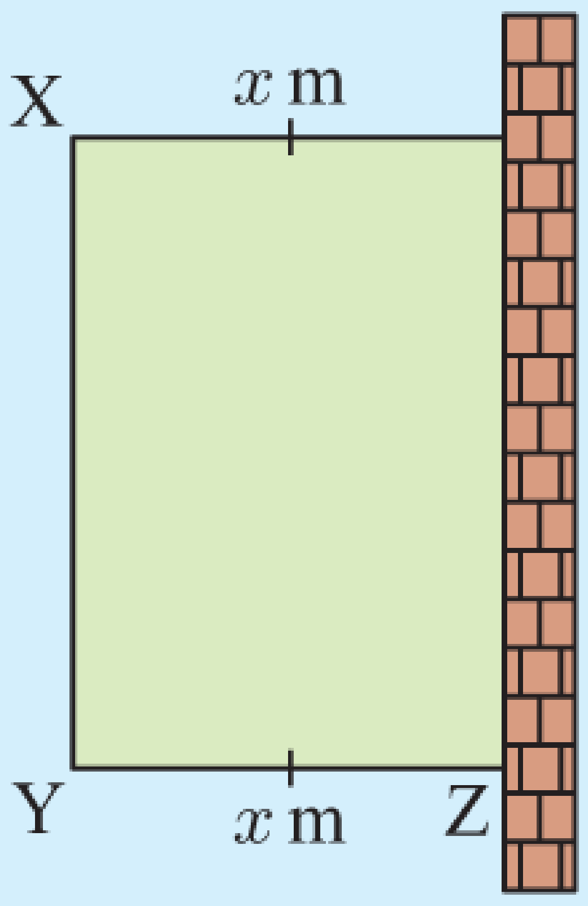
$$\therefore \quad A = x(40 - 2x) \text{ m}^2$$

b $A = 0$ when $x = 0$ or 20.

The vertex of the function lies midway between these values, so $x = 10$.

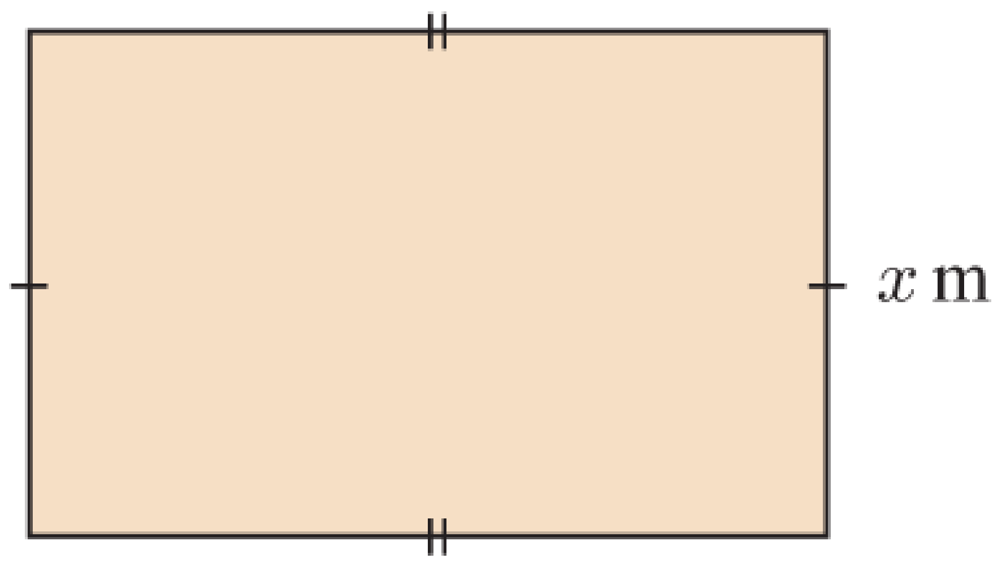
Since $a < 0$, the shape is  .

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.



3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

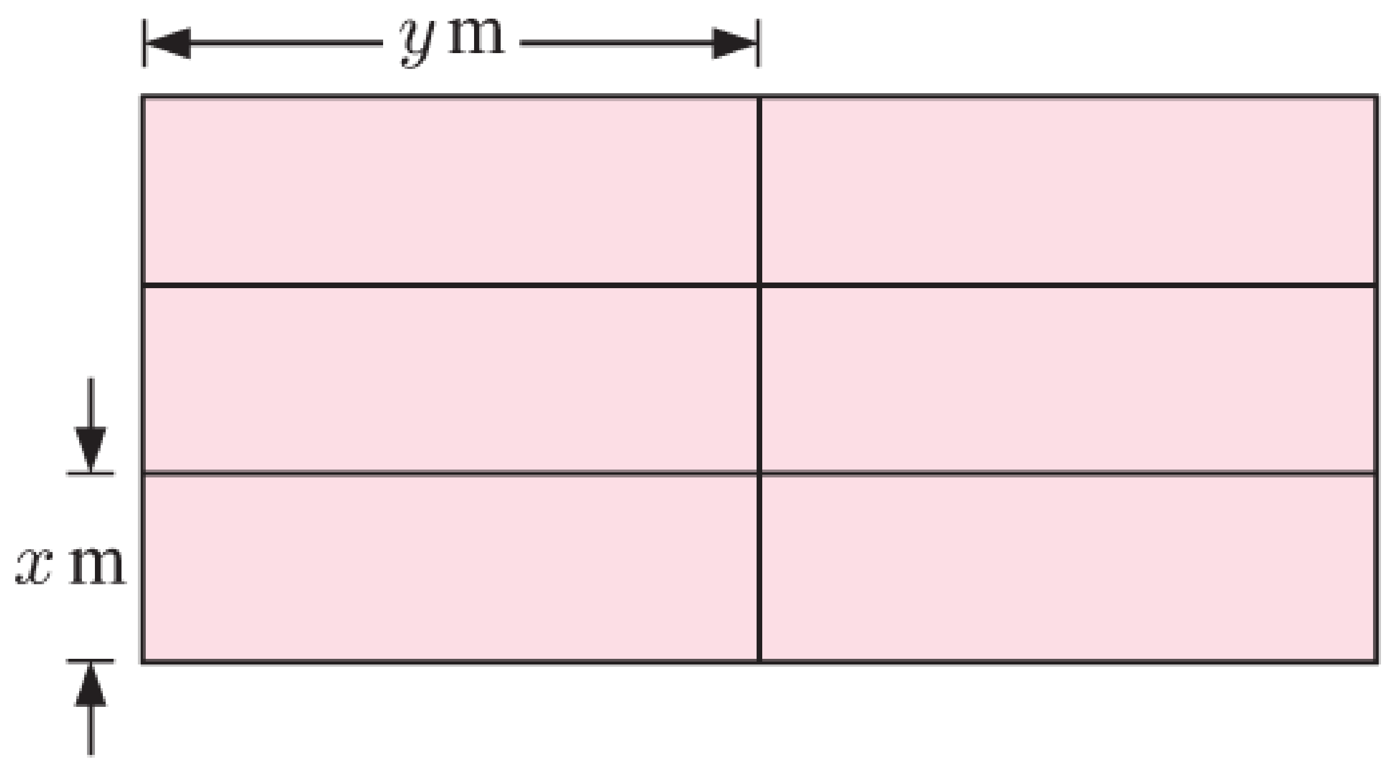
- **a** $A = 100x - x^2$ where x m is the length of one of its sides
- **b** the area is maximised if the rectangle is a square.



4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock to maximise its area?

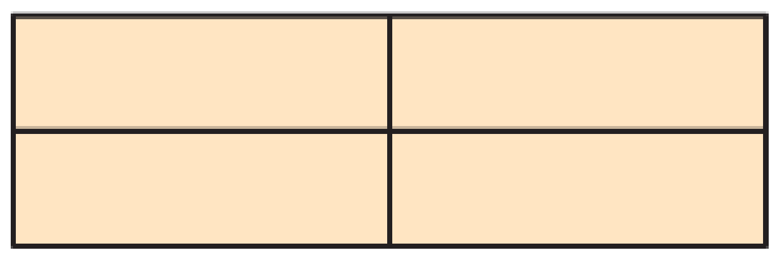
5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

- **a** Explain why $9x + 8y = 1800$.
- **b** Show that the area of each pen is given by $A = -\frac{9}{8}x^2 + 225x$ m^2.
- **c** If the area enclosed is to be maximised, what are the dimensions of each pen?

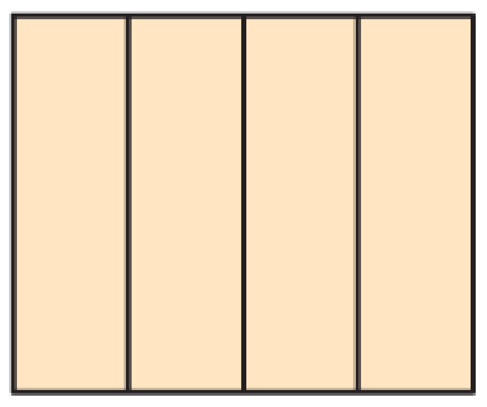


6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

a



b

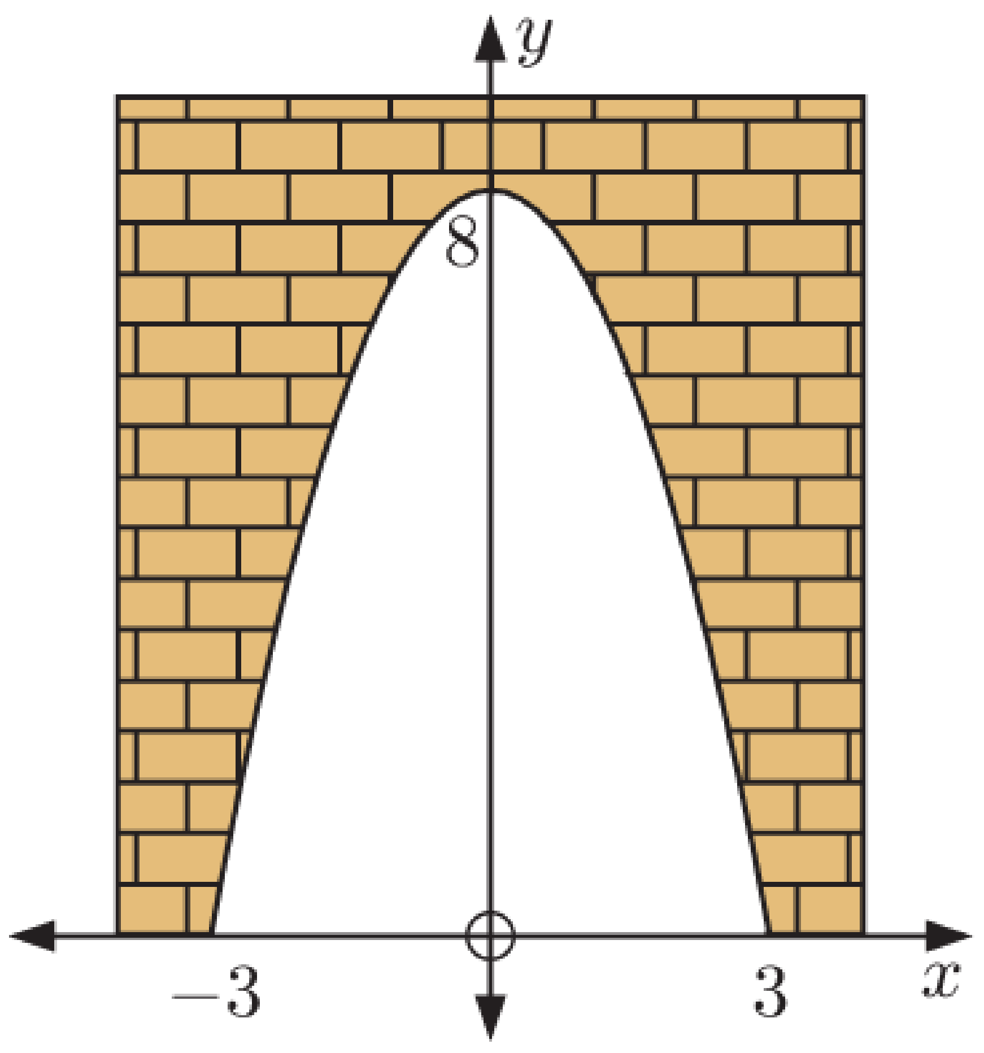


7 Answer the **Opening Problem** on page **134**.

8 The shape of this tunnel can be described by the quadratic model $y = ax^2 + bx + c$. The units are metres.

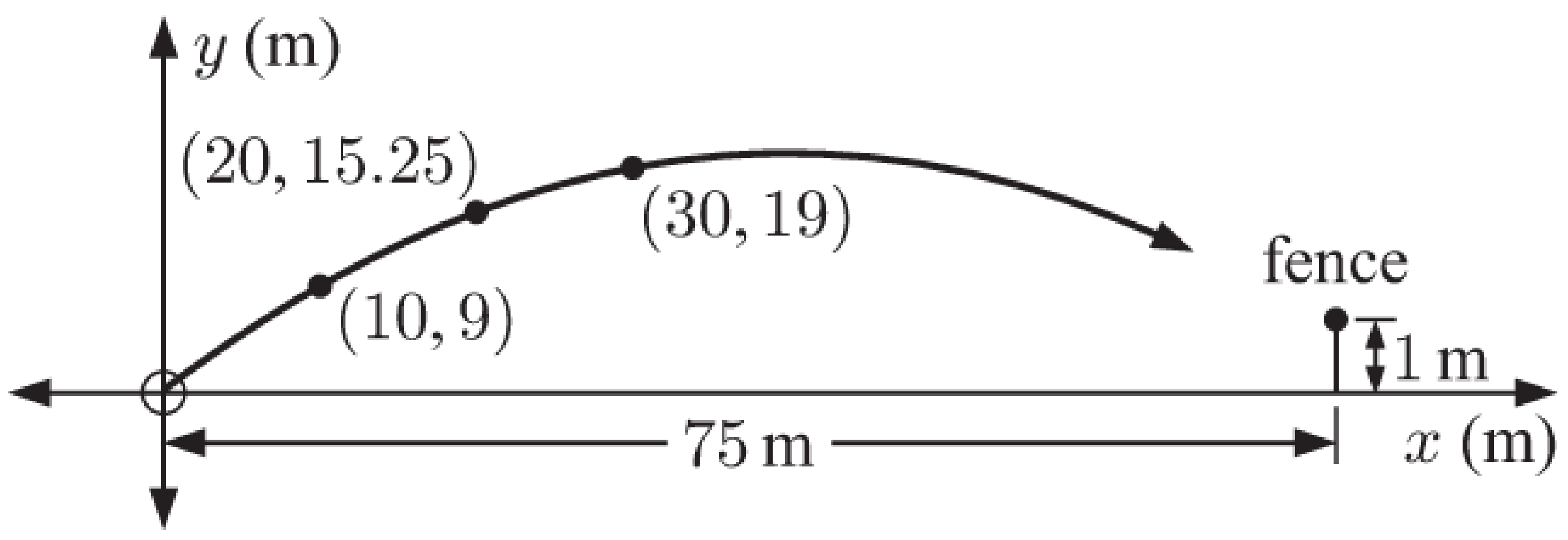
- **a** Find the value of:
  - **i** c
  - **ii** b
  - **iii** a

  Hence state the quadratic model which describes the shape of the tunnel.
- **b** Transportable site offices are being shifted on a semi-trailer. The combination is significantly larger than a standard vehicle, being 5 m high and 4 m wide. Will the semi-trailer be able to take its load through the tunnel?



9 A cricketer hits the ball in the air with the path shown alongside.

- **a** Use technology to find the quadratic model connecting y and x.
- **b** Find the maximum height reached by the ball.



- **c** The 1 m high boundary fence is 75 m away. Will the ball clear the boundary fence?

10 A stone is thrown into the air from the top of a cliff 60 m above sea level. The stone reaches its maximum height of 80 m above sea level after 2 seconds.

The stone's height above sea level after t seconds can be modelled by the quadratic $H(t) = at^2 + bt + c$ metres.

a State the value of c.

b Use the remaining information to write two linear equations involving a and b.

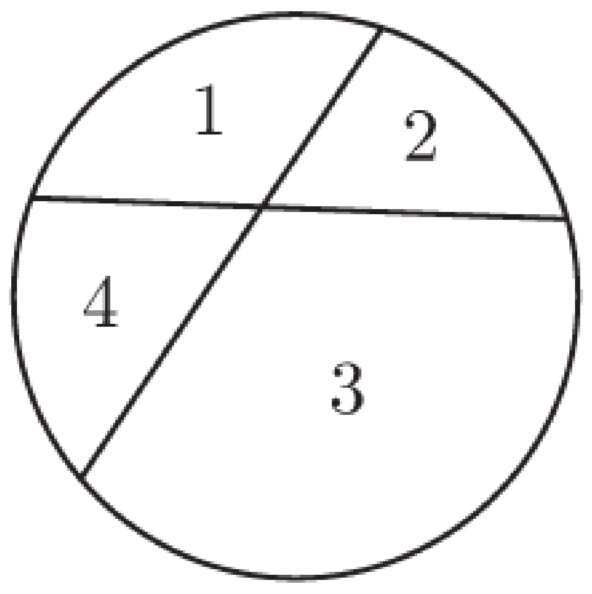
c Hence find a and b.

d Find the stone's height above sea level after 3 seconds.

e How long will it take for the stone to hit the water?

11 Let $P(n)$ be the maximum number of pieces into which a pizza can be cut using n straight cuts.

For example, with 2 cuts we can make a maximum of 4 pieces, so $P(2) = 4$.



a Find $P(3)$ and $P(4)$. Draw diagrams to illustrate your answers.

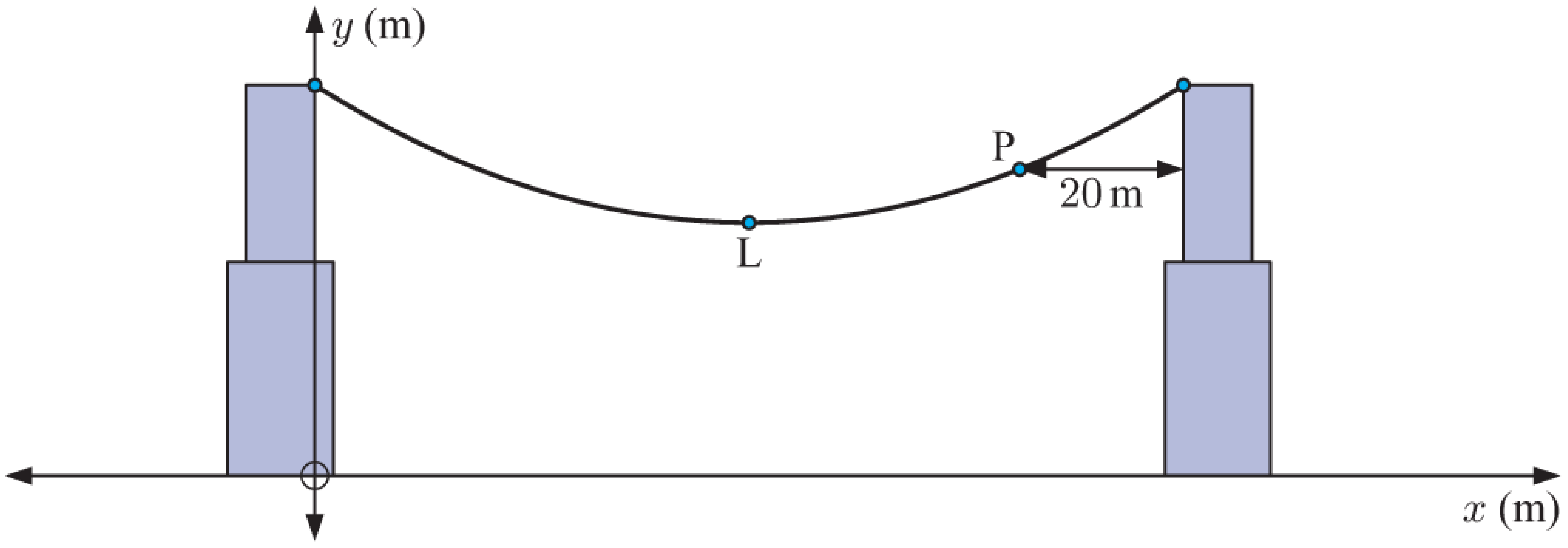
b $P(n)$ can be modelled by the quadratic function $P(n) = an^2 + bn + c$. Use technology to find a, b, and c.

c Use your model to find $P(5)$. Draw a diagram to illustrate your answer.

d Find the maximum number of pieces into which a pizza can be cut using 12 cuts.

e For what values of n can this model be used?

12



A tightrope connects two elevated platforms which are 50 m high and 100 m apart.

Julian wants to know the height of the tightrope above ground level as a function of the distance along it. He knows that the tightrope is 30 m high at its lowest point L, and he assumes the height follows a quadratic function.

a Find the quadratic model connecting y and x.

b Hence estimate the height of the tightrope above ground level at point P.

c For what values of x is the quadratic model valid?

RESEARCH

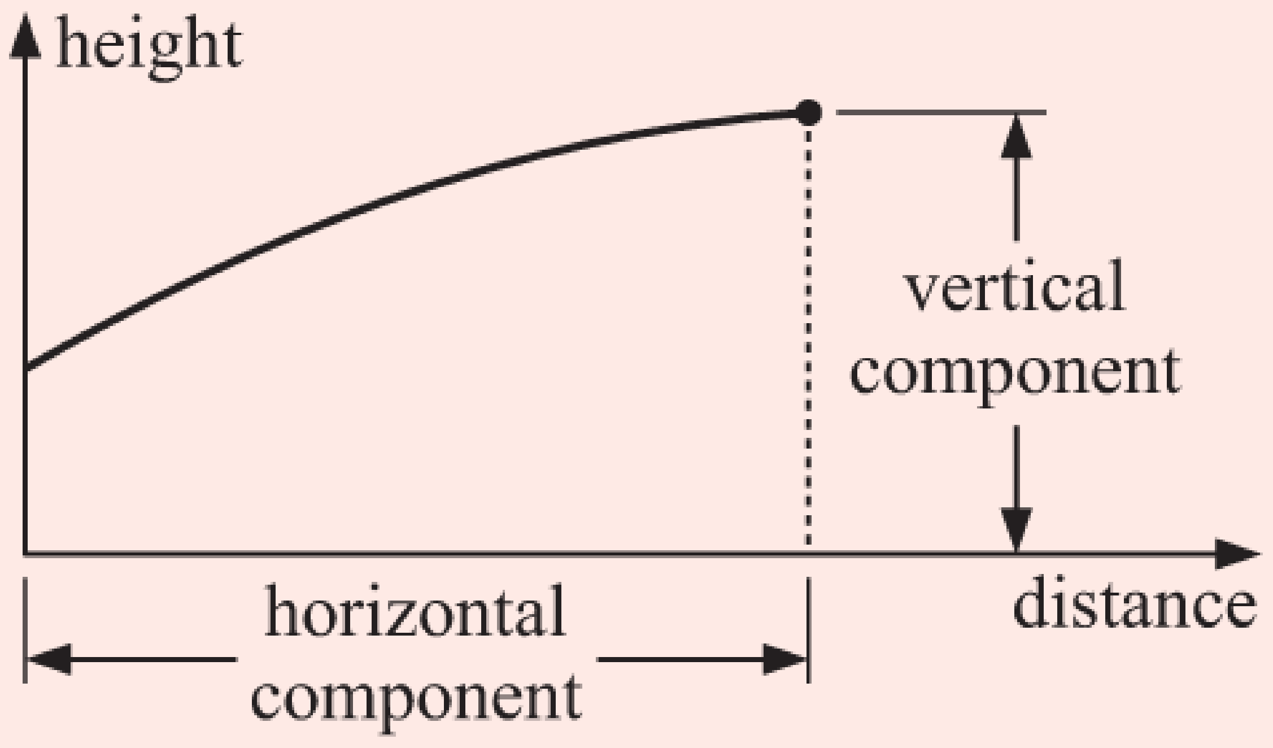
The height of a free-hanging rope, such as the tightrope in the previous Exercise, does not in fact follow a quadratic model.

- Research the correct model for the height of a free-hanging rope.
- Do you think the quadratic model gives a good approximation for the height of the tightrope? What advantages are there to using the quadratic model as opposed to the correct model?

ACTIVITY 3 PROJECTILE MOTION

A **projectile** is an object upon which the only force acting is gravity. If we ignore air resistance, balls and missiles travel through the air with **projectile motion**.

When an object moves with projectile motion, it has both a **horizontal component** and a **vertical component**. It is often useful to consider each component as a separate function of time. This allows us to locate the object at any instant.



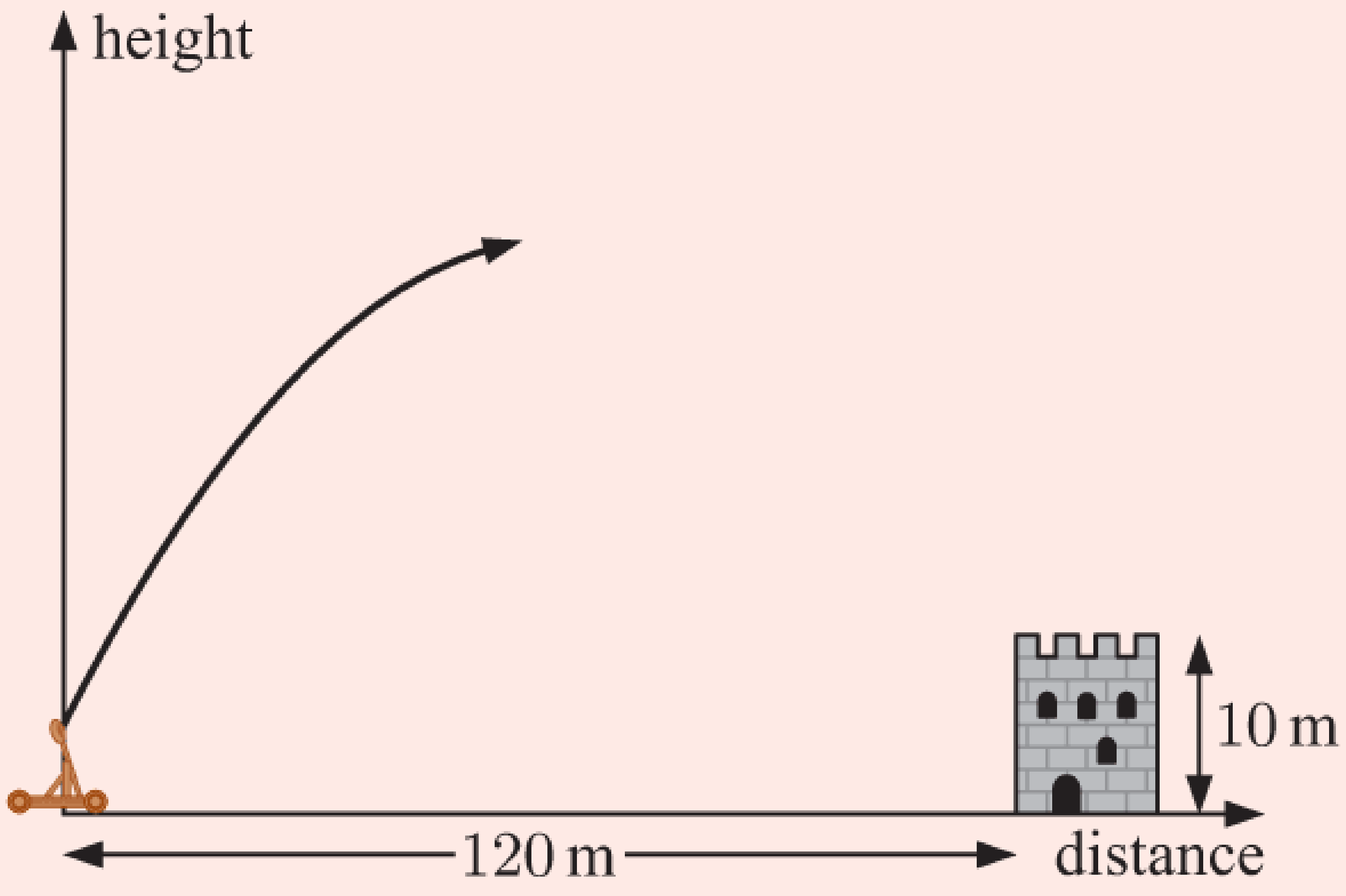
What to do:

1 Consider a ball thrown into the air. It moves with horizontal component $x = 20t$ metres and vertical component $y = -4.9t^2 + 14.7t + 1$ metres, where t is the time in seconds, $t \geqslant 0$.

- **a** Find the position of the ball after:
  - **i** 1 second
  - **ii** 2 seconds.
- **b**
  - **i** At what time was the ball at its highest point?
  - **ii** Find the maximum height reached by the ball.
  - **iii** How far had the ball travelled horizontally at this time?
- **c**
  - **i** At what time did the ball hit the ground?
  - **ii** How far had the ball travelled horizontally when it hit the ground?

2 A stone is released from a catapult towards a castle. The castle is 120 m away, and is 10 m high.

The stone moves with horizontal component $x = 30t$ metres and vertical component $y = -4.9t^2 + 20t + 5$ metres, where t is the time in seconds, $t \geqslant 0$.



- **a** Find the position of the stone after 1 second.
- **b** Find the maximum height reached by the stone.
- **c** Will the stone fly over the castle, hit the castle, or land in front of the castle? Explain your answer.

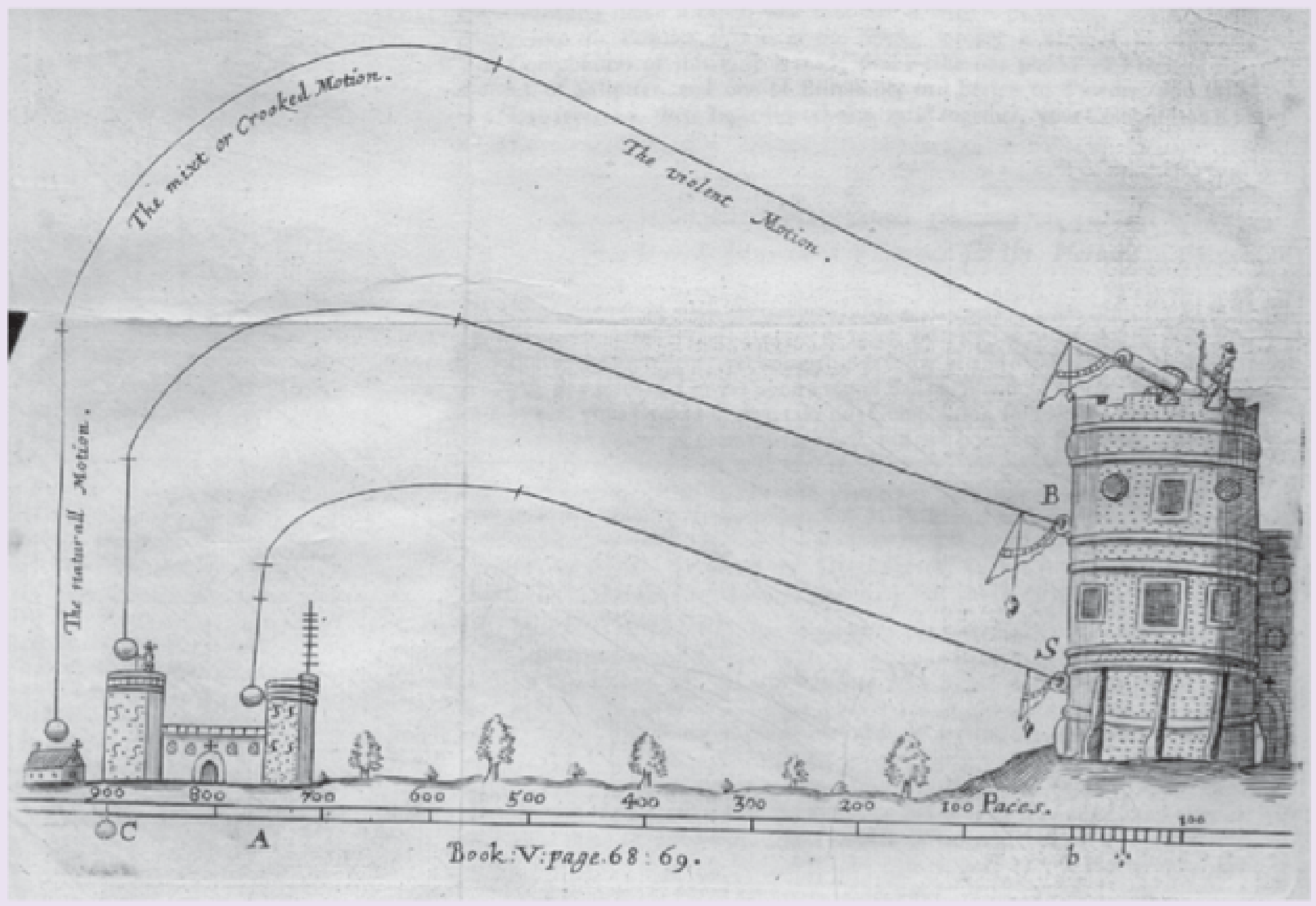
3 Click on the icon to run a cannon simulation.
Change the initial velocity and angle of trajectory, and observe the effect these have on the path of the cannonball.
What angle of trajectory gives the greatest range for the cannon?

SIMULATION

HISTORICAL NOTE

The theory of projectiles was developed in Europe in the 14th century, driven by the desire to improve guns and cannons. At that time, scientists were still using Aristotle's theory of motion which suggested that forces gave rise to momentum. This would mean that as soon as you stopped pushing something (even an object on wheels) it would stop moving.

Using this theory, authors separated the flight of the projectile into two or more sections, including a "violent motion" where the cannonball was forced upwards by an explosion, and a "natural motion" where it fell back to Earth. An example from **Samuel Sturmy** is shown in the picture alongside. It was published in *The Mariners Magazine* in 1669.



The word *random* was originally used to describe the range of a cannon. A gunner would have a *table of randoms* to specify how much gun powder to use, and at what angle to aim, in order to generate an expected range. However, early artillery was so unreliable that the word *random* started to be used to describe unpredictable events!

It was **Galileo Galilei** (1564 - 1642) who first suggested that in the absence of resistance, a projectile would move in a quadratic curve. This was later explained by Newtonian mechanics which suggested that forces give rise to acceleration.

REVIEW SET 6A

1 If $f(x) = x^2 - 3x - 15$ find:

a $f(0)$ **b** $f(1)$ **c** x such that $f(x) = 3$.

2 Consider the function $y = -x^2 - 4x + 7$.

a Construct a table of values for the function using $x = -3, -2, -1, 0, 1, 2, 3$.

b Use your table to graph the function.

3 Find the y-intercept for each function:

a $y = 3x^2 - x + 9$ **b** $f(x) = (x + 5)(x - 6)$ **c** $y = -(x - 4)^2$

4 Find the zeros of:

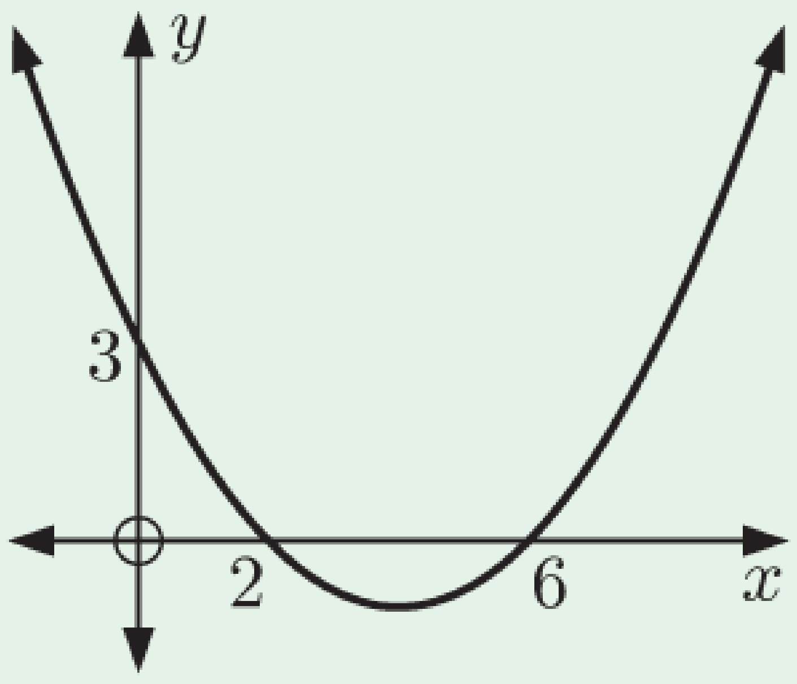
a $y = (3x - 5)(x + 2)$ **b** $y = -\frac{1}{2}(x - 7)^2$ **c** $f(x) = x^2 + 4x - 12$

5 On the same set of axes, sketch $y = x^2$ and:

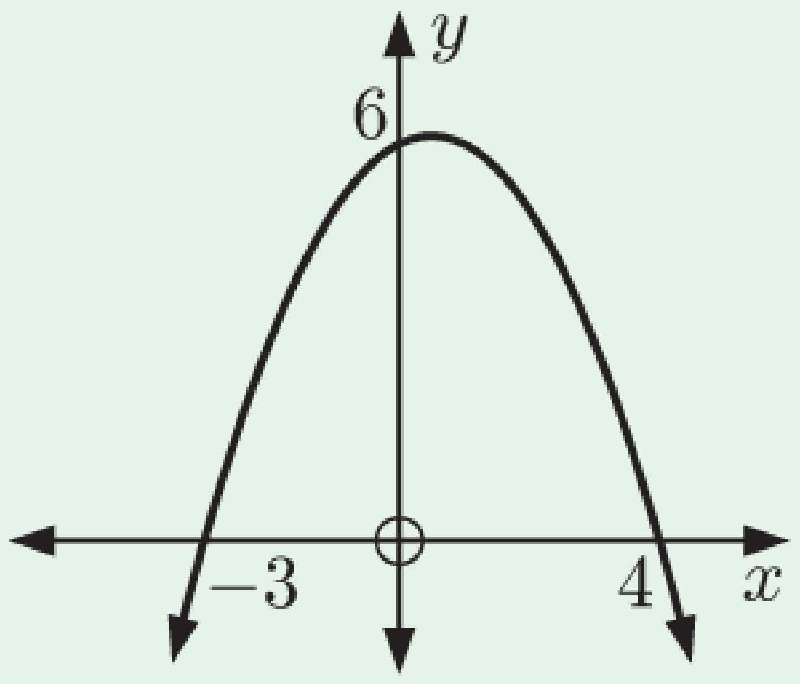
a $y = 3x^2$ **b** $y = -\frac{1}{2}x^2$

6 Find the axis of symmetry of each quadratic:

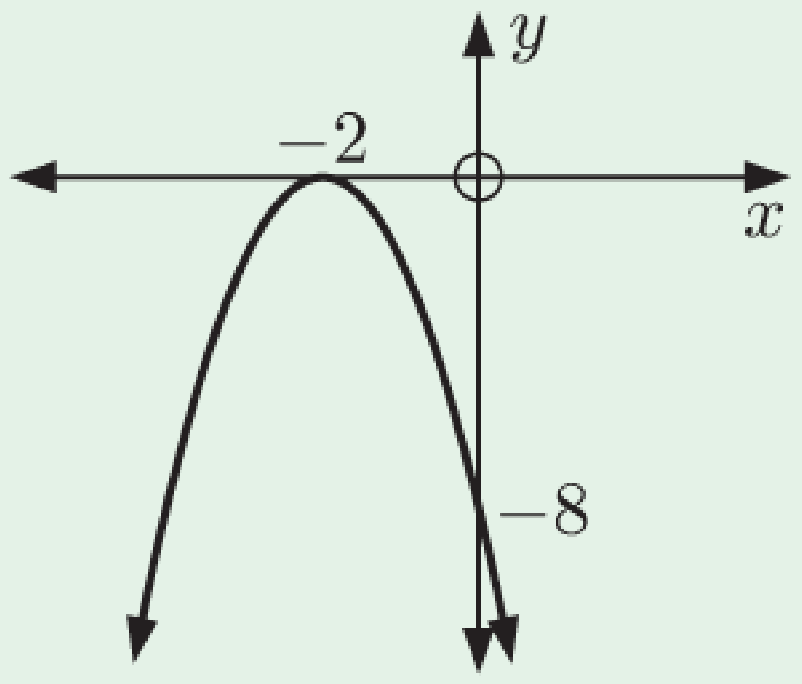
a



b



c



7 Consider the quadratic function $y = -2(x-1)(x+3)$.

a In which direction does the parabola open? Explain your answer.

b Find the y-intercept.

c Find the x-intercepts.

d Sketch the function showing the features you have found.

8 Find the axis of symmetry of the function:

a $y = (x+1)(x-4)$
b $y = x^2 - 7x - 3$
c $f(x) = -2x^2 + 3x - 5$

9 Find the coordinates of the vertex of each quadratic:

a $y = -x^2 + 8x + 5$
b $f(x) = (x-5)(x+4)$
c $f(x) = 3x^2 + 12x - 4$

10 Consider the function $y = x^2 - 2x - 15$.

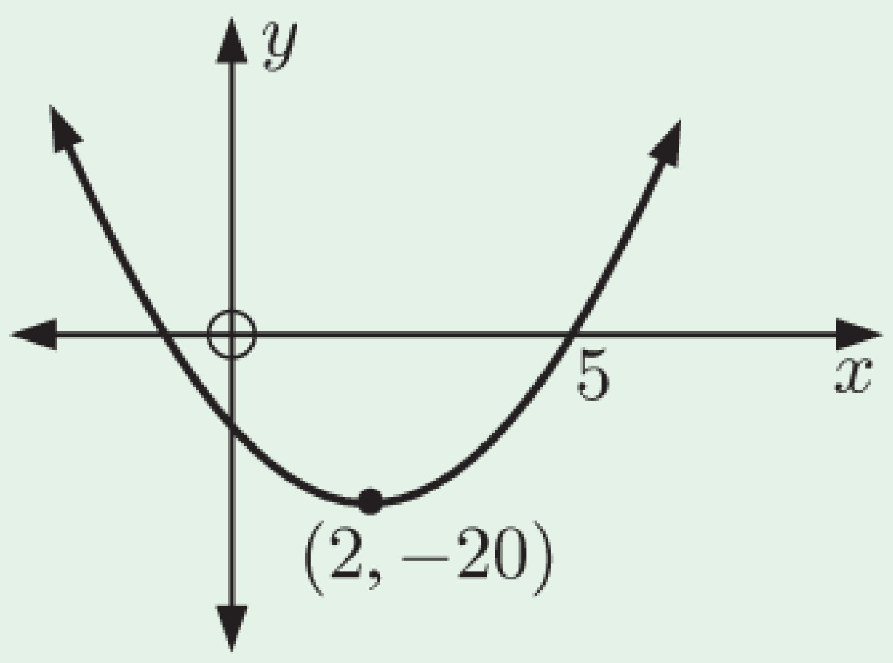
a Find the:

- **i** y-intercept
- **ii** x-intercepts
- **iii** axis of symmetry
- **iv** coordinates of the vertex.

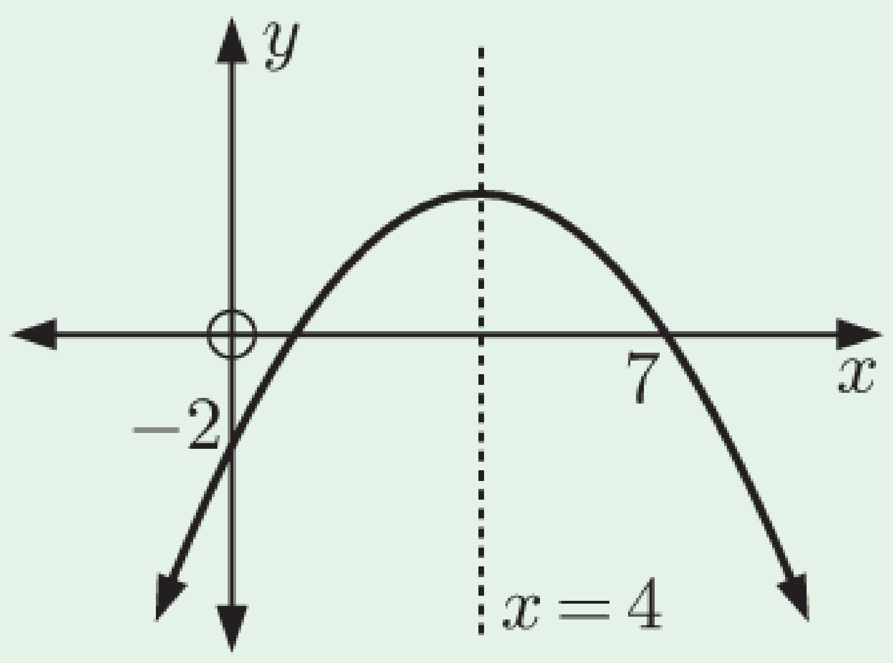
b Sketch the function showing the features you have found.

11 Find the equation of the quadratic with graph:

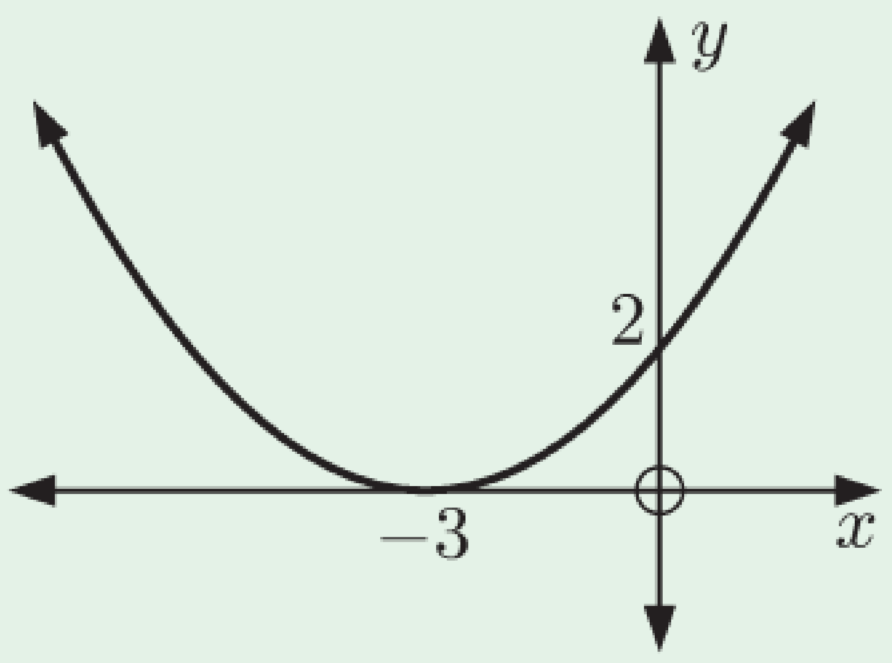
a



b



c



12 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at -1 and 2, and the y-axis at -6

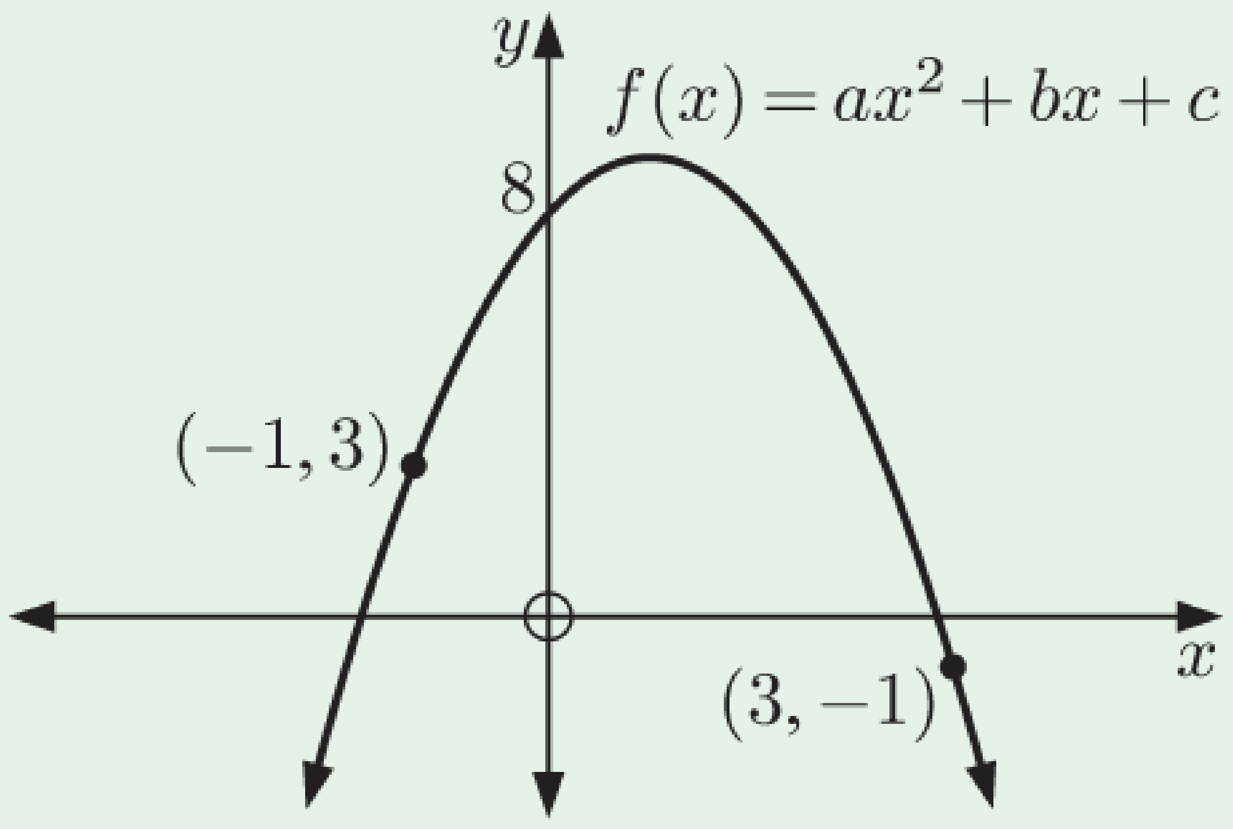
b touches the x-axis at 4 and passes through $(2, 12)$.

13 Consider the function $f(x) = ax^2 + bx + c$ shown.

a State the value of c.

b Use the other information to write two equations involving a and b.

c Use technology to find a and b, and hence state the equation of the quadratic.



14 Find the equation of the quadratic which passes through the points $(1, -2)$, $(2, 6)$, and $(3, 20)$.

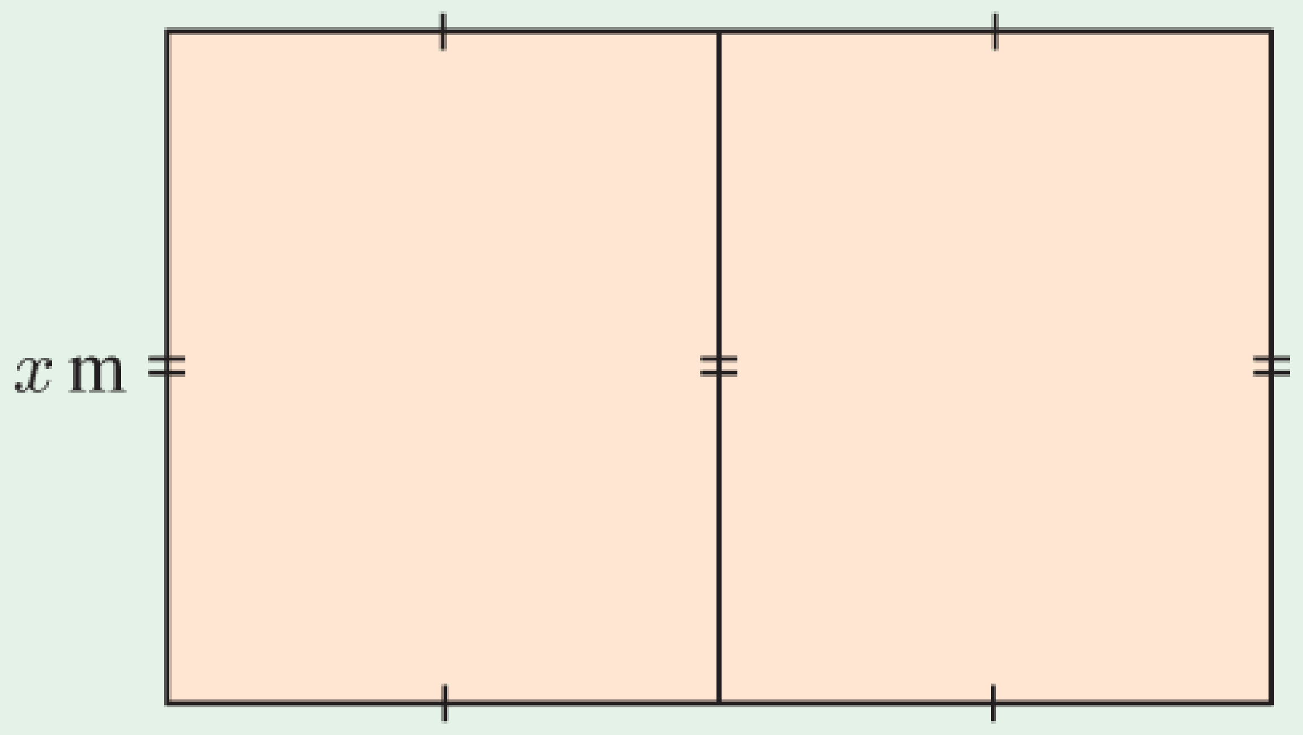
15 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

16 When Annie hits a softball, the height of the ball above the ground after t seconds is given by $h = -4.9t^2 + 19.6t + 1.4$ metres.

a Find the height of the ball after 1 second.

b Find the maximum height reached by the ball.

17 A farmer has 2000 m of fencing to enclose two identical adjacent fields.



- **a** Find an expression for the total area of the fields in terms of x.
- **b** Find the maximum possible total area of the two fields, and the dimensions of the fields which give this maximum total area.

18 Let $S(n)$ be the sum of the first n positive integers, so $S(1) = 1$ and $S(2) = 1 + 2 = 3$.

- **a** Find $S(3)$.
- **b** $S(n)$ can be modelled by the quadratic function $S(n) = an^2 + bn + c$. Use technology to find a, b, and c.
- **c** Check that your model gives the correct result for the sum of the first 6 positive integers.
- **d** Find the sum of the first 60 positive integers.
- **e** Find $S(2\frac{1}{2})$. Is this result meaningful?
- **f** For which values of n is it appropriate to use this model?

REVIEW SET 6B

1 Is $f(x) = -2x^2 + 13x - 4$ satisfied by the ordered pair $(1, 5)$?

2 Find the value(s) of x for which $g(x) = x^2 - 5x - 9$ has value 5.

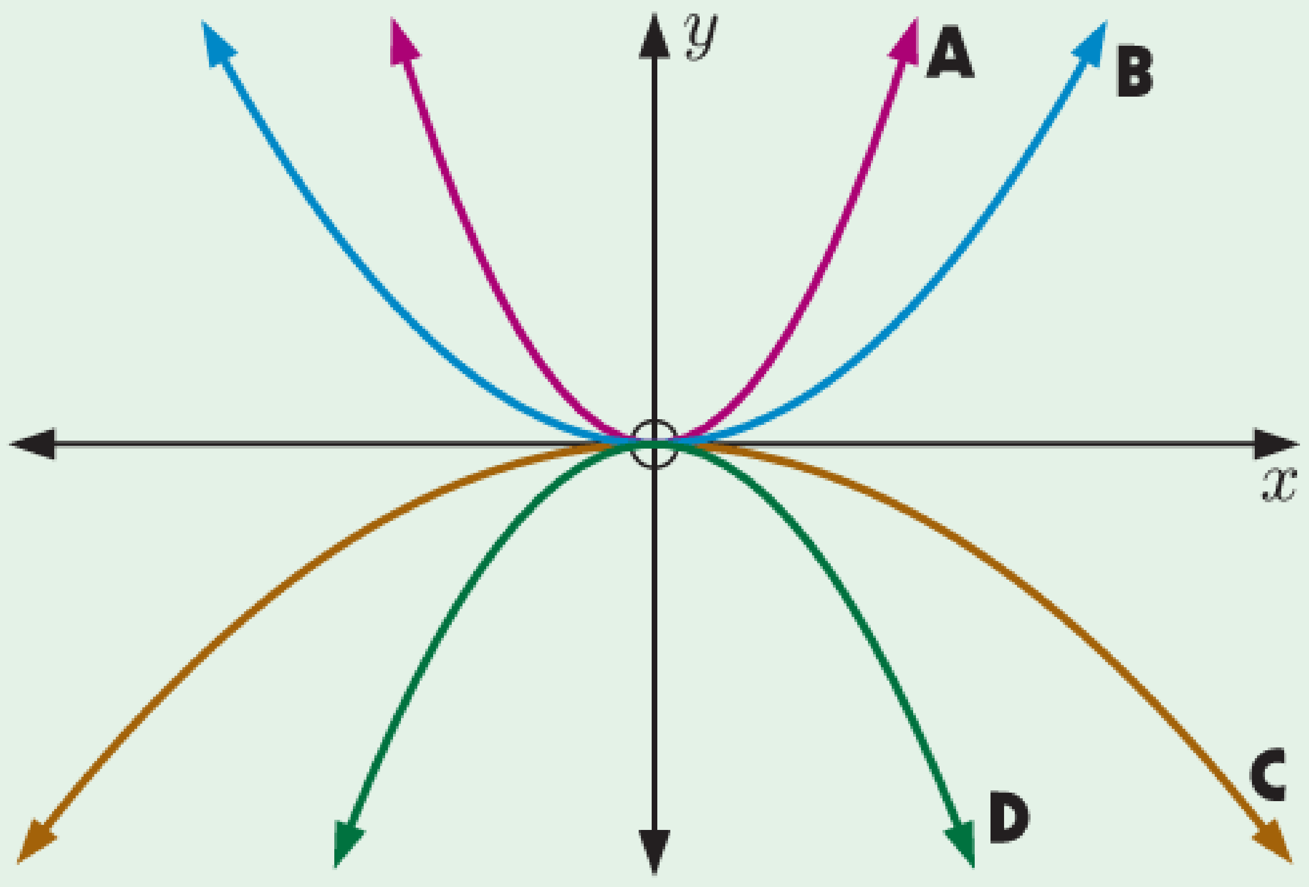
3 Use a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ to sketch the graph of $y = x^2 + 3x - 5$.

4 Find the axes intercepts of the function:

- **a** $y = (x + 5)(x - 1)$
- **b** $y = -2(2x - 3)(x + 4)$
- **c** $f(x) = 8x^2 - 2x - 3$

5 Match each function with its graph:

- **a** $y = x^2$
- **b** $y = -\frac{1}{2}x^2$
- **c** $y = 3x^2$
- **d** $y = -2x^2$



6 Consider the function $y = 3(x - 2)^2$.

- **a** In which direction does the parabola open? Explain your answer.
- **b** Find the y-intercept.
- **c** Find any x-intercept(s).
- **d** Sketch the function, showing the features you have found.

7 Find the axis of symmetry for the function:

- **a** $y = (x - 2)(x - 9)$
- **b** $y = -x^2 + 8x - 1$
- **c** $y = \frac{2}{3}x^2 - x + \frac{1}{3}$

8 A quadratic function has axis of symmetry $x = 6$, and one of its x-intercepts is -3. Find the other x-intercept.

9 Find the coordinates of the vertex of $y = -3x^2 + 8x + 7$. State whether the vertex is a maximum turning point or a minimum turning point.

10 Find the:

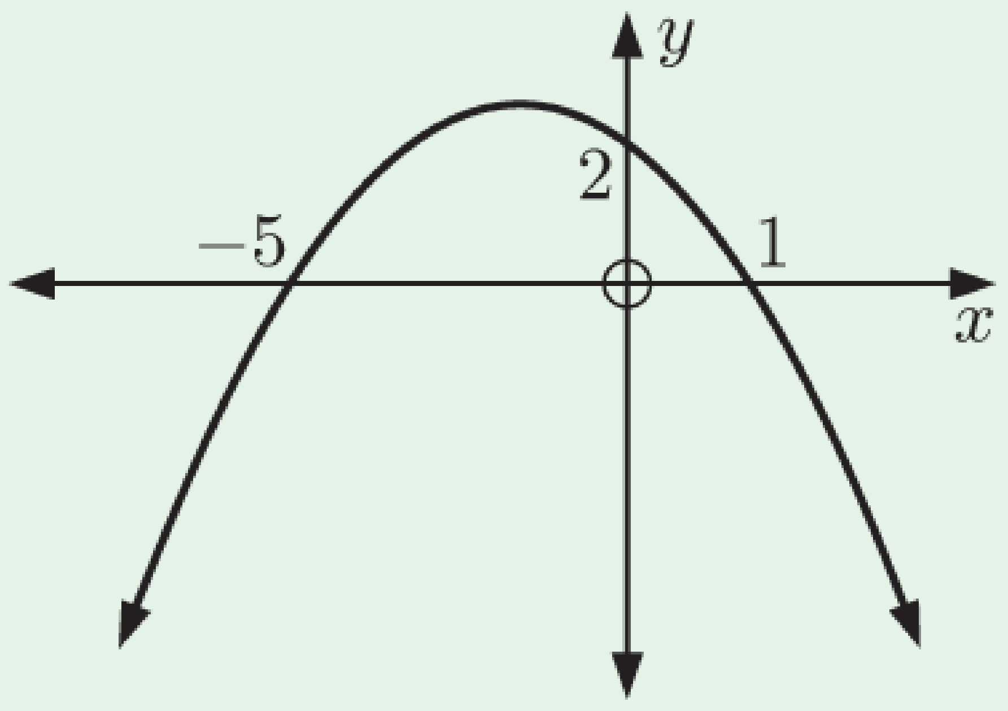
- **a** maximum value of $y = -x^2 + 10x - 9$
- **b** minimum value of $y = 2x^2 - 2x + 5$.

11 Consider the function $y = -x^2 + 7x - 10$.

- **a** Find the:
  - **i** y-intercept
  - **ii** x-intercepts
  - **iii** axis of symmetry
  - **iv** coordinates of the vertex.
- **b** Sketch a graph of the function showing all of the above features.
- **c** State the domain and range of the function.

12

- **a** Find the equation of the quadratic shown.
- **b** Hence find its vertex and axis of symmetry.



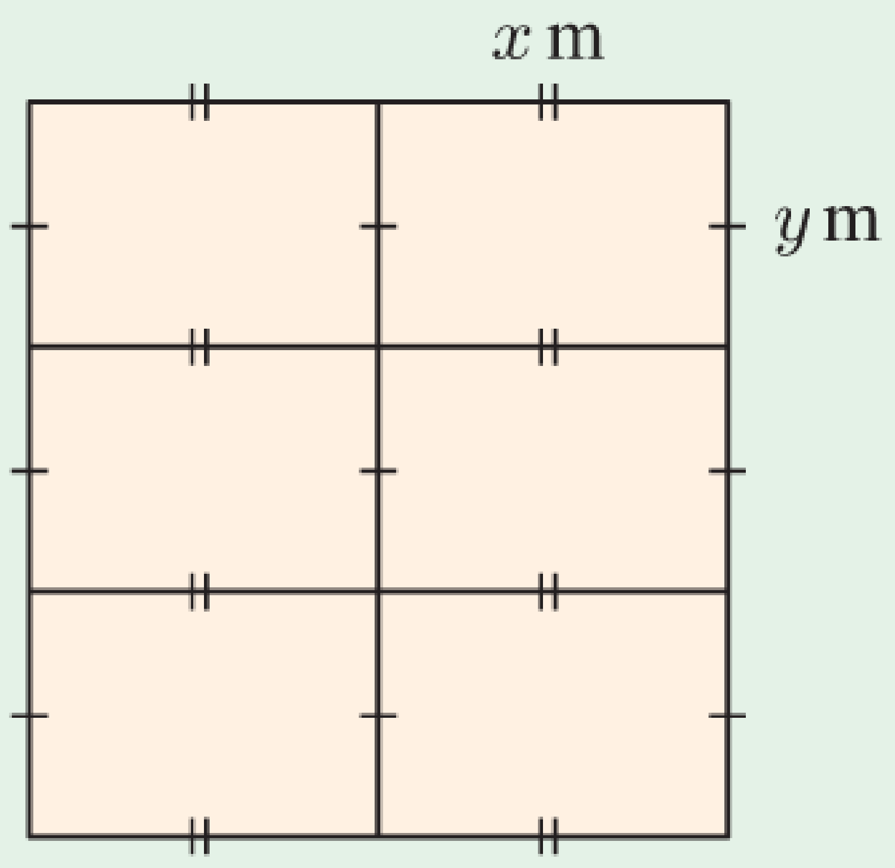
13 Find, in the form $y = ax^2 + bx + c$, the quadratic function whose graph:

- **a** touches the x-axis at 3 and passes through $(2, 2)$
- **b** has x-intercepts 3 and -2, and y-intercept 3
- **c** passes through $(-1, -9)$, $(1, 5)$, and $(2, 15)$
- **d** has vertex $(3, 15)$ and passes through the point $(1, 7)$.

14 Try to find the equation of the quadratic which passes through the points $(2, 5)$, $(6, -1)$, and $(2, -3)$. Explain your answer.

15 Find the coordinates of the point(s) of intersection of the graphs with equations $y = x^2 + 13x + 15$ and $y = 2x - 3$.

16 600 m of fencing is used to construct 6 rectangular animal pens as shown.



- **a** Explain why $8x + 9y = 600$.
- **b** Show that the area A of each pen is $A = -\frac{8}{9}x^2 + \frac{200}{3}x \text{ m}^2$.
- **c** Find the dimensions of each pen so that it has the maximum possible area. State the area of each pen in this case.

17 A retailer sells sunglasses for \$45, and has 50 customers per day. From market research, the retailer discovers that for every \$1.50 increase in the price of the sunglasses, he will lose a customer per day.

Let $\$x$ be the price increase of the sunglasses.

- **a** Show that the retailer collects $R = (45 + x)\left(50 - \frac{x}{1.5}\right)$ dollars in revenue each day.
- **b** Write the revenue function in the form $R = ax^2 + bx + c$.
- **c** Find the price the retailer should set for his sunglasses in order to maximise his daily revenue. How much revenue is made per day at this price?

Chapter 7

Direct and inverse variation

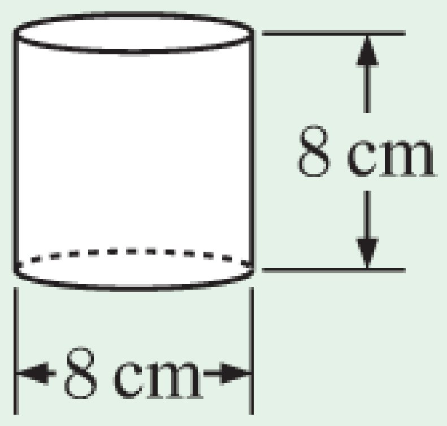
Contents:

- **A** Direct variation
- **B** Powers in direct variation
- **C** Inverse variation
- **D** Powers in inverse variation
- **E** Determining the variation model
- **F** Using technology to find variation models

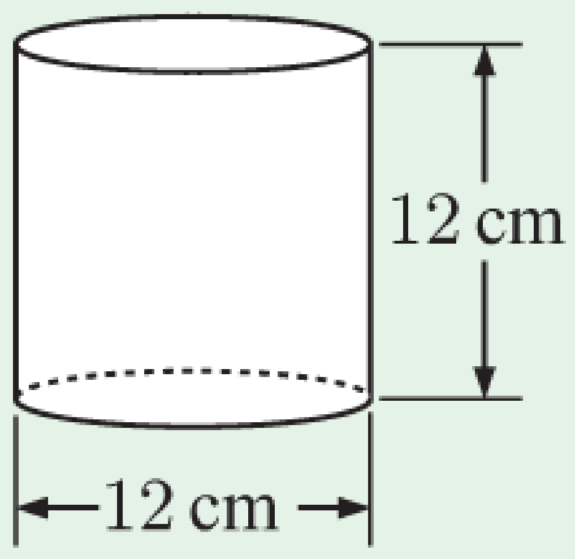
OPENING PROBLEM

Mrs Cornwall has set a challenge for the students in her Mathematics class. Each student must try to guess the number of jelly beans in a large cylindrical jar.

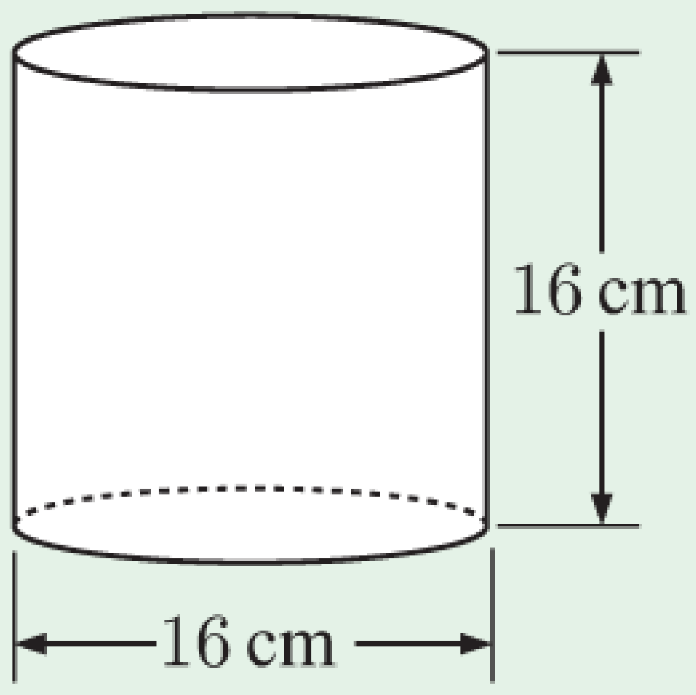
Jill measures the jar, and finds that its height and width are each 20 cm. She cannot find another jar that size, but she finds three other jars with equal height and width at home. She fills them with jelly beans and counts how many they can contain:



95 jelly beans



321 jelly beans



763 jelly beans

Things to think about:

- **a** What does the graph of the *number of jelly beans* against the *jar height* look like?
- **b** Does the *number of jelly beans* increase *in proportion* with the *jar height*?
- **c** Explain why we should expect the *number of jelly beans* to increase in proportion with the *cube* of the *jar height*.
- **d** Can you find an equation which connects the *number of jelly beans* and the *jar height*?
- **e** What guess do you think Jill should make in the challenge?

A DIRECT VARIATION

Suppose petrol costs \$3 per gallon.

The table alongside shows how the *cost* y of the petrol varies with the *amount* x of petrol bought.

× 2 × 3

Amount of petrol (x gallons)	0	1	2	3	4	5	6
Cost (\$$y$)	0	3	6	9	12	15	18

× 2 × 3

Notice that:

- If x is doubled from 1 to 2, y is also doubled (from 3 to 6).
- If x is tripled from 2 to 6, y is also tripled (from 6 to 18).

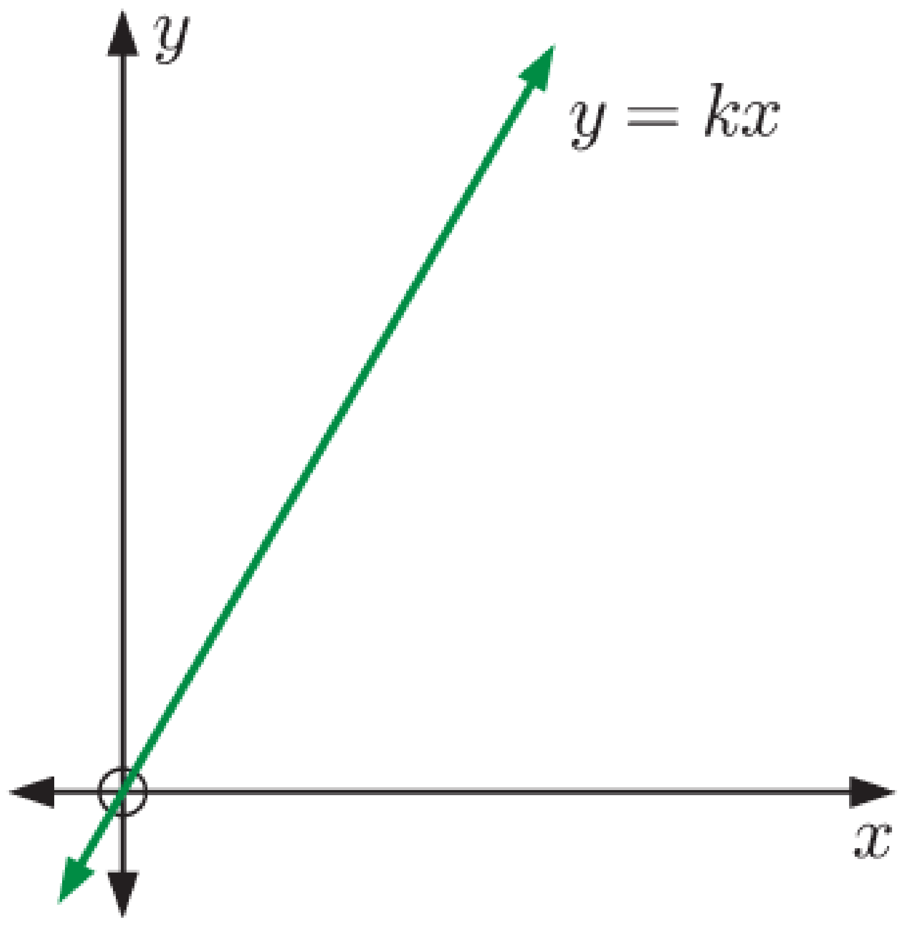
The variables x and y vary *in proportion* to each other. We say that x and y are **directly proportional**.

> Two variables are **directly proportional** or **vary directly** if multiplying one of them by a number results in the other one being multiplied by the same number.

If the variables x and y are directly proportional, we write $y \propto x$.

If $y \propto x$ then $y = kx$ where k is a constant called the **proportionality constant**.

When y is graphed against x, the graph is a straight line with **gradient** k, which passes through the **origin**.



$\propto$ reads "*is directly proportional to*", or "*varies directly as*".

In the petrol scenario on the previous page, the proportionality constant $k = 3$, and the variables are connected by the formula $y = 3x$.

Example 1 **Self Tutor**

The table alongside shows the relationship between the *side length* of a square and its *perimeter*.

Side length (x cm)	1	2	3	5
Perimeter (P cm)	4			

a Copy and complete the table.

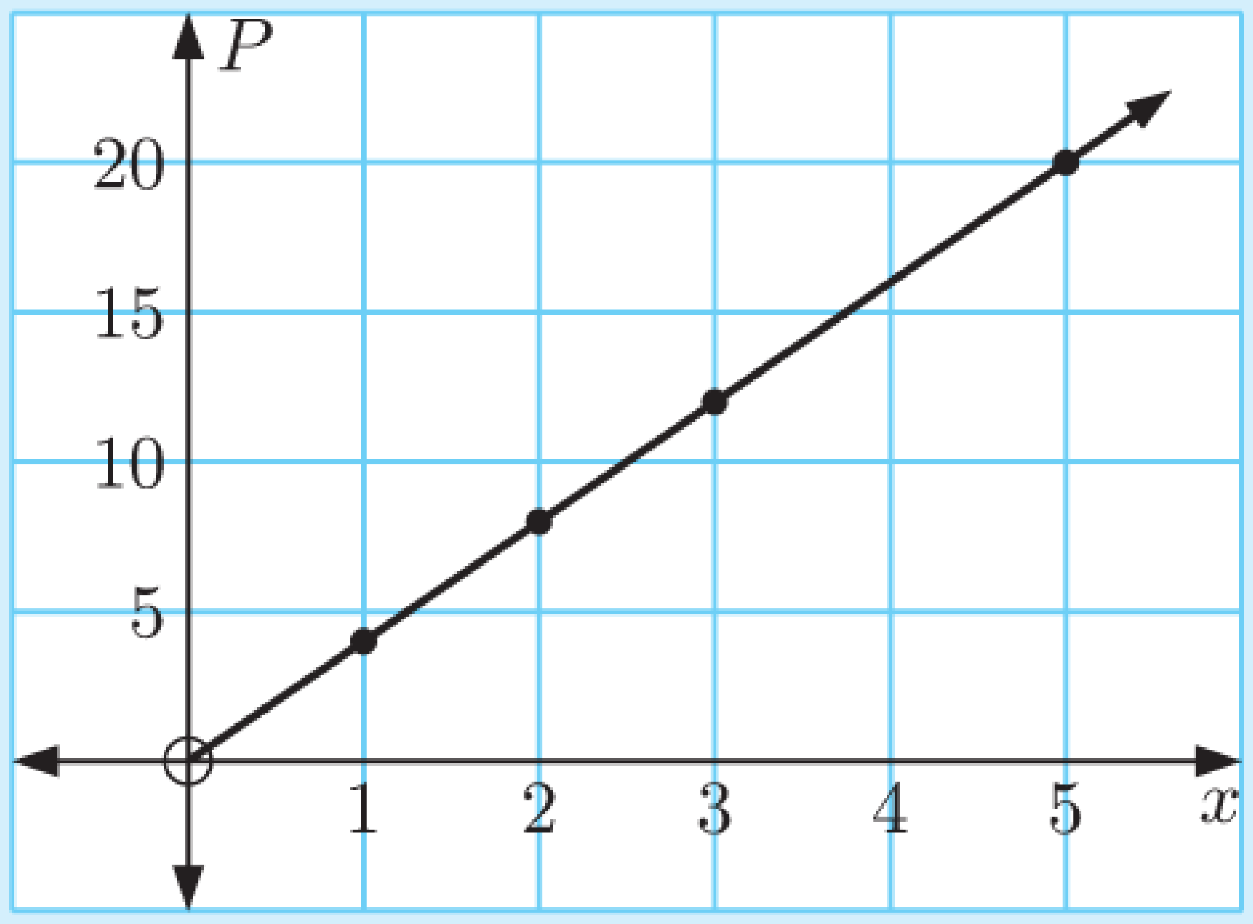
b Use a graph to show that P is directly proportional to x.

c Find a formula connecting P and x.

a

Side length (x cm)	1	2	3	5
Perimeter (P cm)	4	8	12	20

b The graph is a straight line which passes through the origin, so $P \propto x$.



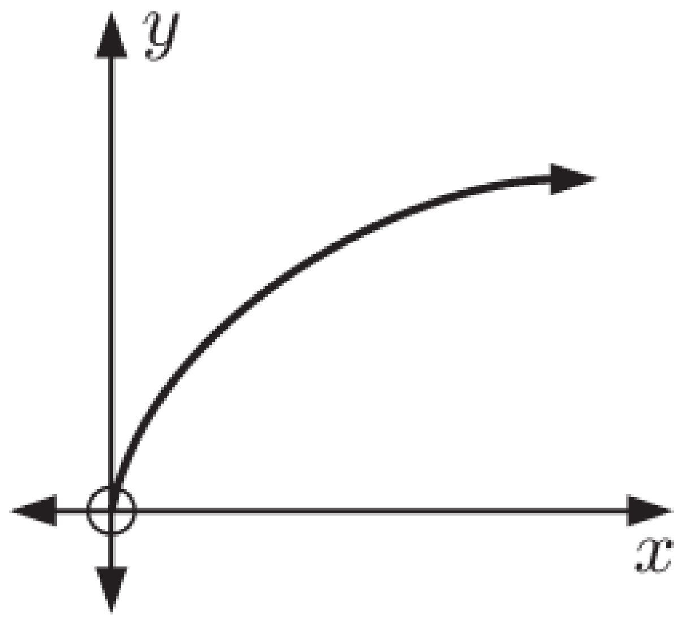
c The gradient of the line $= \dfrac{8-4}{2-1}$

$= 4$

$\therefore$ the proportionality constant $k = 4$, and the formula connecting P and x is $P = 4x$.

EXERCISE 7A

1 Which graph indicates that y is directly proportional to x?

A


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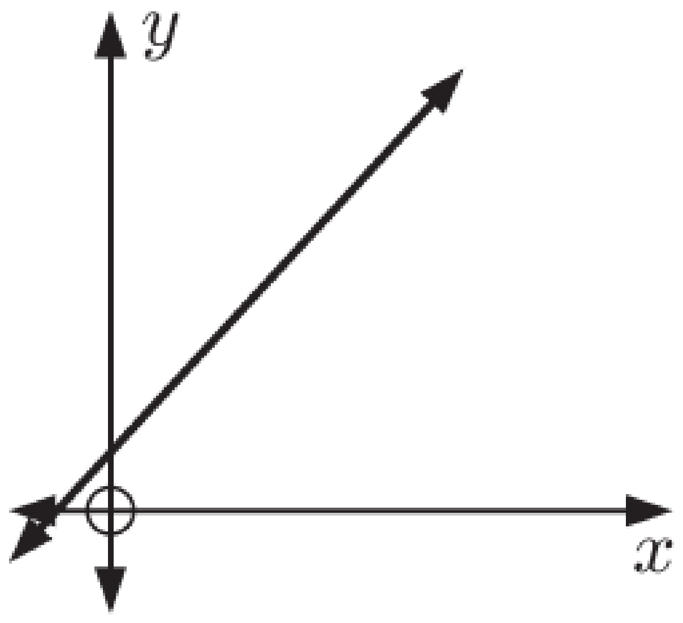

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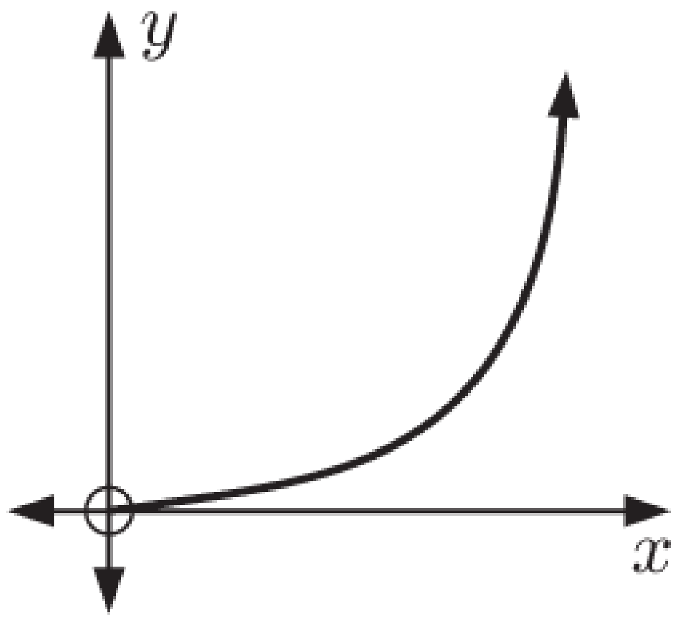
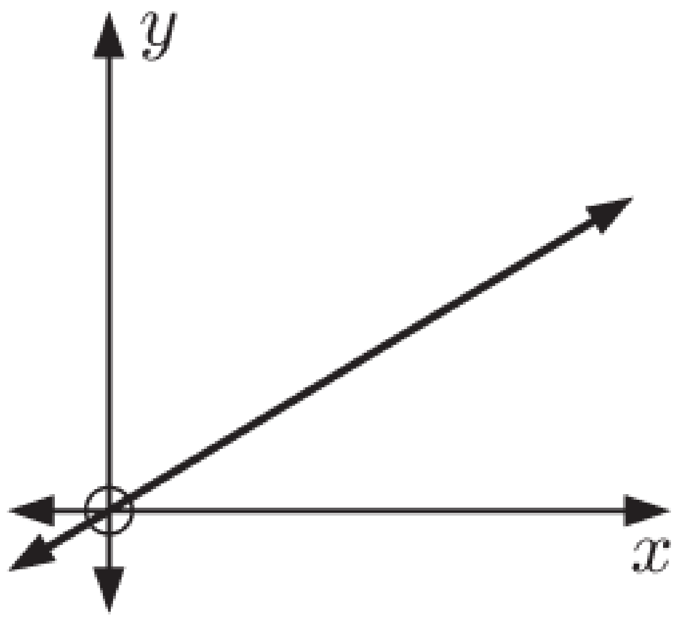


D


2 For each table of values:

i Plot the graph of y against x.

ii Determine whether y is directly proportional to x. If it is, find the proportionality constant.

a

x	0	1	2	3	4
y	0	5	10	15	20

b

x	0	1	2	3	4
y	2	4	6	8	10

c

x	2	3	4	6
y	3	4.5	6	7.5

d

x	2	4	5	7
y	10	20	30	35

3 The table below shows the cost of purchasing w kg of prawns.

Weight (w kg)	0	1	2	3	4
Cost (\$$C$)	0	15	30	45	60

a Draw the graph of C against w.

b Explain why C and w are directly proportional.

c Find a formula connecting C and w.

4 This table shows the cost of parking in a parking lot.

Time (t hours)	*Cost* (€C)
$0 < t \leqslant 1$	6
$1 < t \leqslant 2$	12
$2 < t \leqslant 3$	18
$3 < t \leqslant 4$	24
$4 < t \leqslant 5$	30

a Find the cost of parking for:

i 1 hour **ii** 2 hours

iii 2.5 hours **iv** 4 hours.

b Are C and t directly proportional? Explain your answer.

5 Suppose y is directly proportional to x. State what happens to:

a y if x is doubled

b x if y is multiplied by 6

c y if x is divided by 3

d x if y is increased by 20%

e y if x is decreased by 70%

f x if y is increased by 10.

6 Water freezes at 0°C, which is approximately 32°F. Explain why the variables *temperature in* °C and *temperature in* °F cannot be directly proportional.

7 Water is poured into this drinking glass at a constant rate. Determine whether the following pairs of variables are directly proportional:

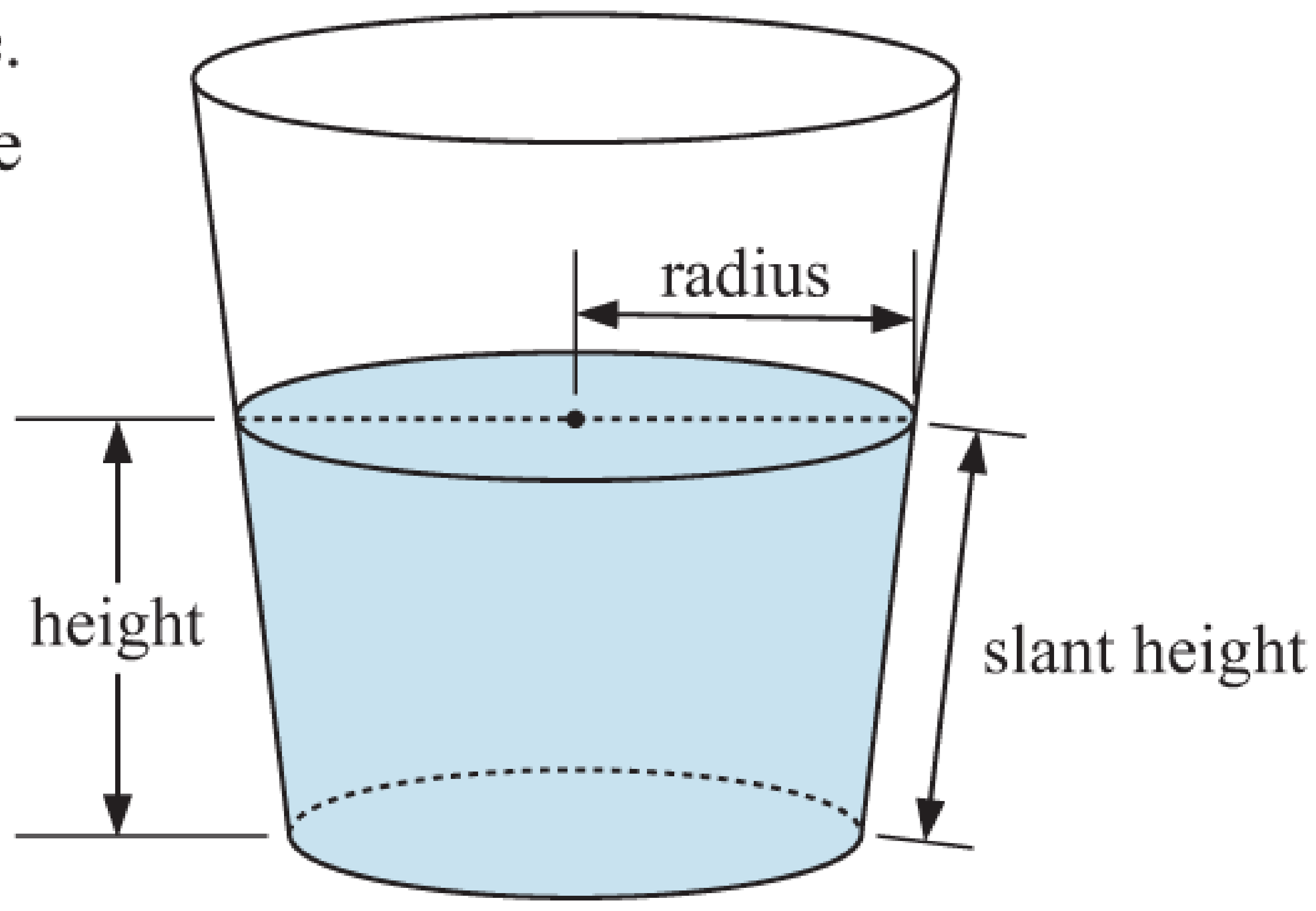
a *volume of water* and *time*

b *volume of water* and *height*

c *height* and *slant height*

d *height* and *radius*

e *slant height* and *weight of water*.



Example 2 **Self Tutor**

Suppose $p \propto t$ and that $p = 17$ when $t = 5$. Find t when $p = 60$.

t	5	?
p	17	60

$\times \frac{60}{17}$

To change p from 17 to 60, we multiply by $\frac{60}{17}$.

Since $p \propto t$, we must also multiply t by $\frac{60}{17}$

$\therefore \quad t = 5 \times \frac{60}{17} \approx 17.6$

8 Suppose $y \propto x$ and that $y = 24$ when $x = 7$. Find:

- **a** y when $x = 21$
- **b** x when $y = 30$.

9 Suppose $M \propto l$ and that $M = 40$ when $l = 9$. Find:

- **a** M when $l = 45$
- **b** l when $M = 8$.

10 The amount of energy used by a heater is directly proportional to the time it is on. When it is on for 2 hours, the heater uses 3.6 kWh of energy. Find:

- **a** the proportionality constant and explain what it means
- **b** the energy used by the heater if it is on for 1.5 hours
- **c** the time it would take for the heater to use 6 kWh of energy.

11 Isabella has drawn a scale diagram of her school. An actual length of 60 m is represented by 3 cm on her diagram.

- **a** Find the proportionality constant. Explain how it relates to the scale of the diagram.
- **b** The football pitch is 110 m long. How long is the pitch on Isabella's diagram?
- **c** The front fence is 15 cm long on Isabella's diagram. Find the actual length of the front fence.

12 The momentum of a gridiron player is directly proportional to his velocity. When he moves at 5 $\text{m}\,\text{s}^{-1}$, his momentum is 610 $\text{kg}\,\text{m}\,\text{s}^{-1}$. Find:

- **a** the player's momentum when he moves at 3 $\text{m}\,\text{s}^{-1}$
- **b** the velocity required for his momentum to be 420 $\text{kg}\,\text{m}\,\text{s}^{-1}$.

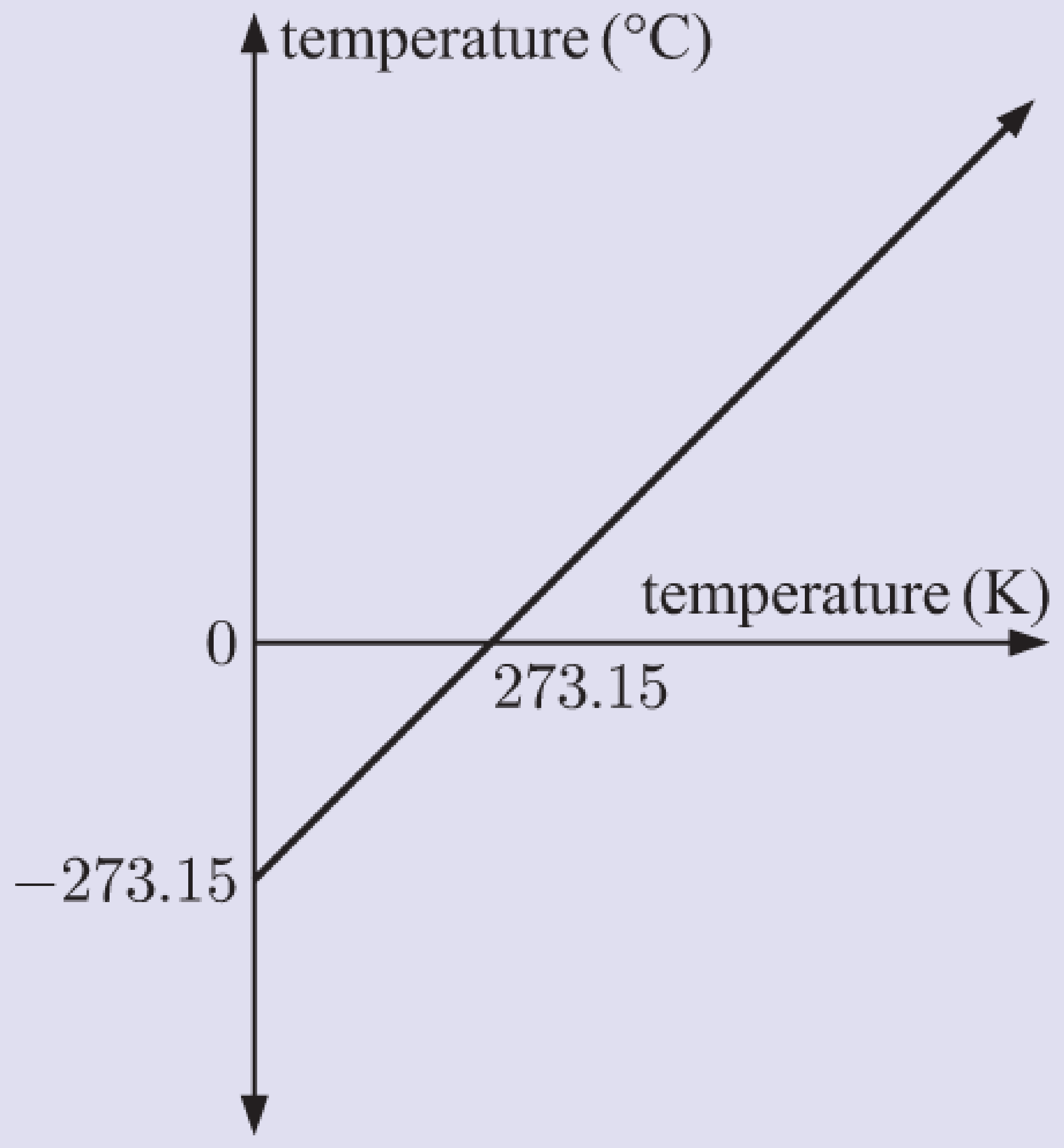
13 A plumber charges a £30 call-out fee plus a constant rate per hour of labour.
Suppose the total charge for a job requiring t hours labour is £C.

- **a** Explain why:
  - **i** C and t are not directly proportional
  - **ii** $(C - 30)$ and t are directly proportional.
- **b** The total charge for a 1.5 hour job is £120. Find the total charge for a 5 hour job.

THEORY OF KNOWLEDGE

In 1848, **William Thomson** (1824 - 1907), also known as **Lord Kelvin**, proposed the need for a temperature scale starting at *absolute zero*. His idea stemmed from research showing a proportional relationship between the kinetic energy of a system and its temperature. By extrapolating his results to a point where the kinetic energy of a system was zero, Thomson was able to predict *absolute zero* as about $-273°$C.

The SI unit for temperature is the kelvin (K), named in Thomson's honour. *Absolute zero* is regarded as 0 kelvin, and is defined as $-273.15°$C. An increase of 1 kelvin corresponds to an increase of 1°C, so 0°C is equivalent to 273.15 K, and 100°C is equivalent to 373.15 K.



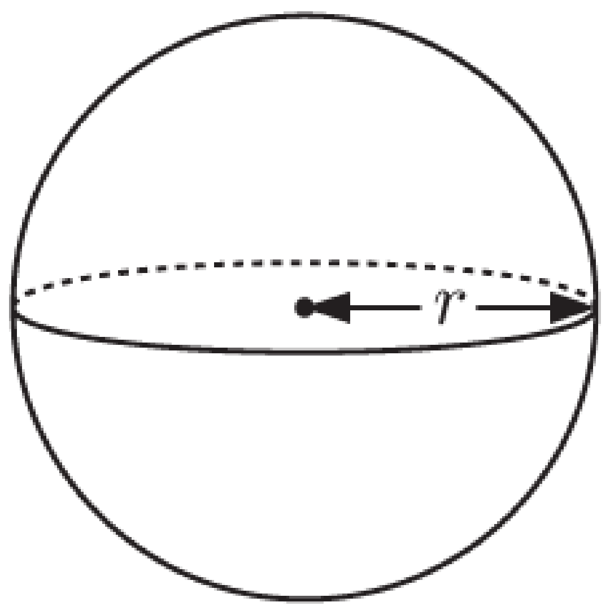
1. In what ways is it useful to use variables in direct proportion?
2. Which measure of temperature is most convenient?
3. What is the most *natural* measure of temperature?

B POWERS IN DIRECT VARIATION

In many circumstances, the variables we consider are not directly proportional, but there is direct variation between their *powers*.

For example, the surface area of a sphere with radius r is $A = 4\pi r^2$.

A is not directly proportional to r. However, $A \propto r^2$ with proportionality constant 4π.



Example 3 **Self Tutor**

Suppose $y = \dfrac{2x^3}{7}$. State two variables which are directly proportional, and determine the proportionality constant k.

$y = \frac{2}{7}(x^3)$, so $y \propto x^3$ and $k = \frac{2}{7}$.

y is directly proportional to the cube of x, and the proportionality constant $k = \frac{2}{7}$.

EXERCISE 7B

1 State which two variables are directly proportional, and determine the proportionality constant k:

a $A = \pi r^2$ **b** $V = \frac{4}{3}\pi r^3$ **c** $T = \dfrac{3n^4}{4}$

2 Suppose $y \propto x^3$. Describe what happens to:

a y if x is doubled

b y if x is divided by 10

c y if x is increased by 20%

d x if y is multiplied by 2.5 .

Example 4 Self Tutor

Suppose D is directly proportional to z^2, and that $D = 14$ when $z = 5$. Find:

a D when $z = 15$

b z when $D = 140$, given $z > 0$.

a

$\times 3$

z	5	15
D	14	

z is multiplied by 3

$\therefore$ z^2 is multiplied by $3^2 = 9$

$\therefore$ D is multiplied by 9 {as $D \propto z^2$}

$\therefore$ $D = 14 \times 9 = 126$

b

z	5	
D	14	140

$\times 10$

D is multiplied by 10

$\therefore$ z^2 is multiplied by 10 {as $D \propto z^2$}

$\therefore$ z is multiplied by $\sqrt{10}$ {as $z > 0$}

$\therefore$ $z = 5\sqrt{10} \approx 15.8$

3 Suppose M is directly proportional to t^2, where $t > 0$. When $t = 6$, $M = 40$. Find:

a M when $t = 9$

b t when $M = 120$.

4 Suppose V is directly proportional to the cube of y, and that when $y = 3$, $V = 30$. Find:

a V when $y = 12$

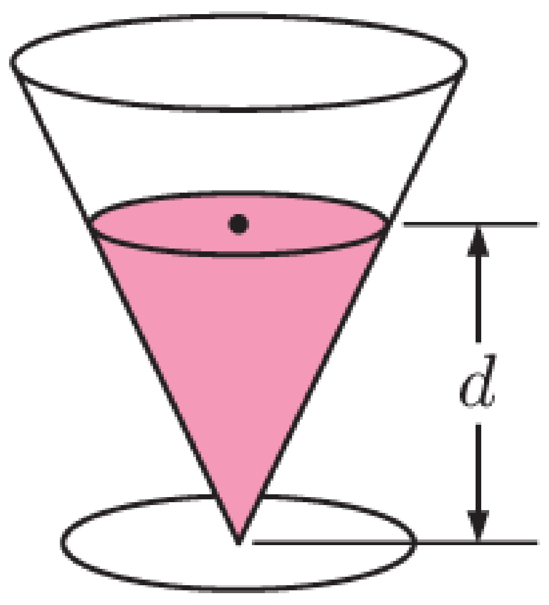
b y when $V = 180$.

5 The mass of a square sheet of glass is directly proportional to the square of its length. Given that a 30 cm square sheet has mass 900 g, find the mass of a 50 cm square sheet.

6 The amount of medicine in this glass is directly proportional to the cube of its depth d.
The glass is 6 cm high, and has capacity 40 mL.

a Find the amount of medicine needed to fill the glass to a depth of 4 cm.

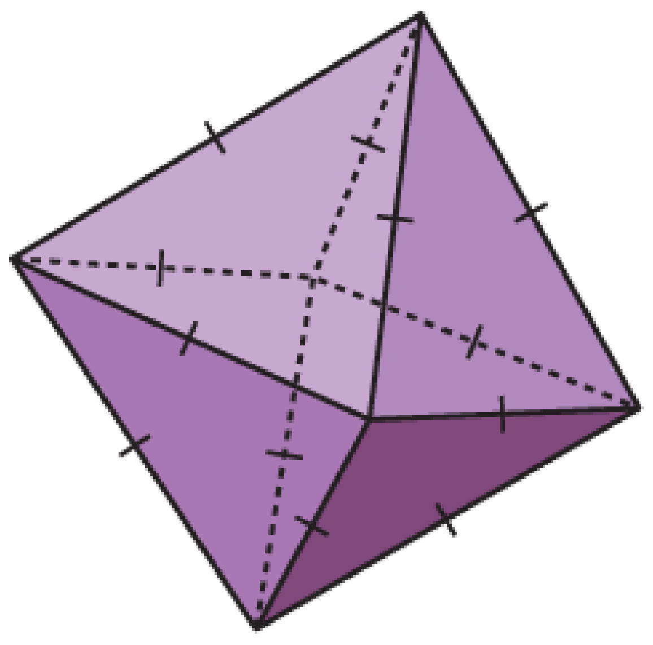
b Jimmy needs 30 mL of medicine. To what depth should the glass be filled?



7 The volume of a regular octahedron is directly proportional to the cube of its side lengths.

a If the side lengths increase by 5%, by what percentage does the volume change?

b What percentage change in side length is required to double the volume?



8 The *kinetic energy* of an object with mass m and speed v is given by $E = \frac{1}{2}mv^2$.

a For objects travelling at a particular speed, what proportionality exists between E and m?

b For an object with constant mass, what proportionality exists between E and v?

c What happens to the kinetic energy of an object if its speed decreases by 10%?

d The brakes of a car turn kinetic energy into heat at a constant rate. Explain why the stopping distance of a car is proportional to the square of the speed.

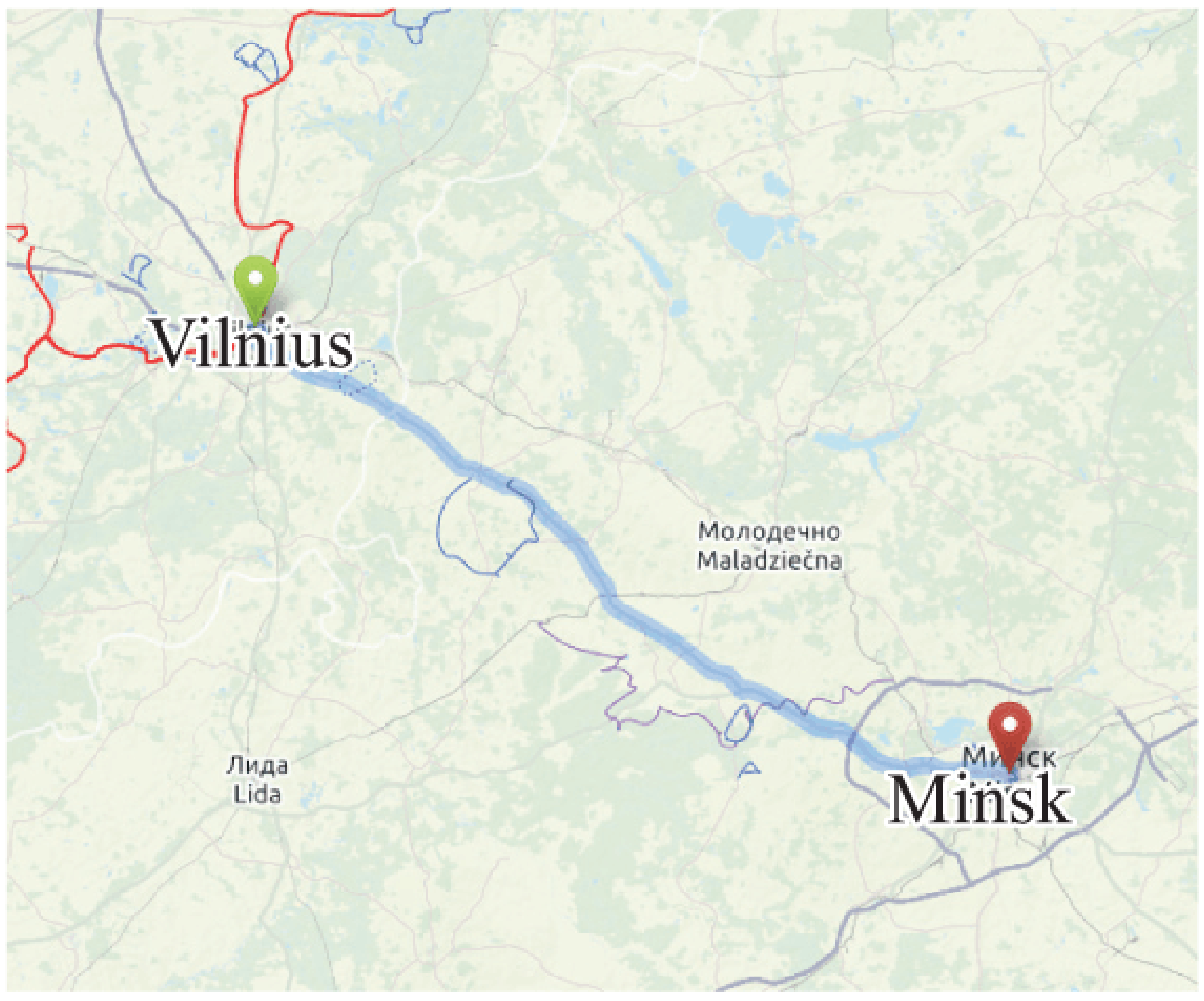
C INVERSE VARIATION

Emilija is driving 200 km from Vilnius to Minsk.

Travelling at an average speed of 50 km h^{-1}, the trip will take $\frac{200}{50} = 4$ hours. Travelling at an average speed of 100 km h^{-1}, the trip will only take $\frac{200}{100} = 2$ hours.

Notice that doubling the *average speed* will halve the *time taken*. In other words, when one variable was *multiplied* by 2, the other was *divided* by 2.

In a case like this, we have an **inverse proportion**.



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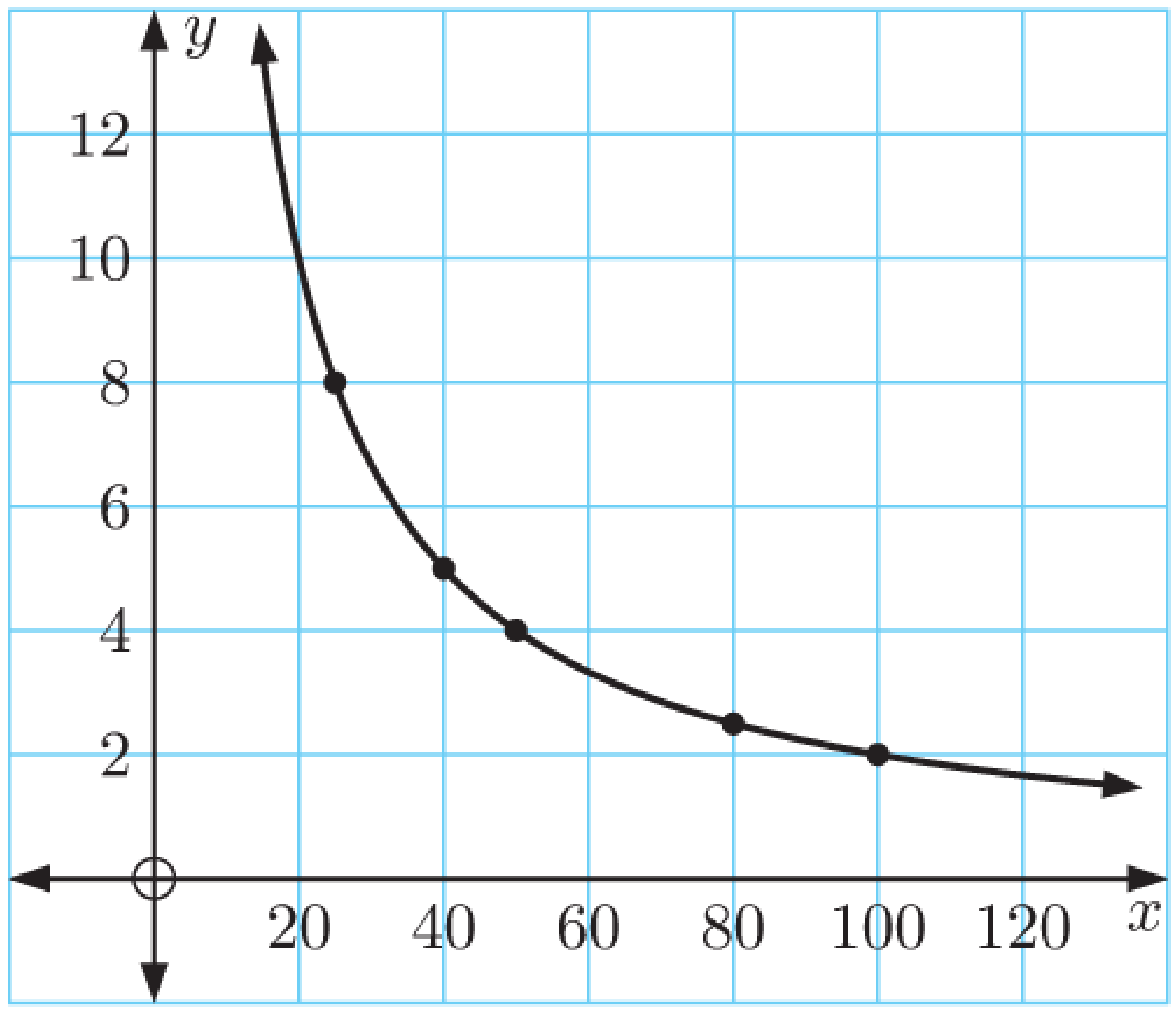
Two variables are **inversely proportional** or **vary inversely** if, when one is *multiplied* by a constant, the other is *divided* by the same constant.

Suppose Emilija travels with average speed x km h^{-1}, and the trip takes y hours.

The table below shows some possible combinations of x and y.

x	25	40	50	80	100
y	8	5	4	2.5	2

When these points are plotted, they form part of a **hyperbola**.



Notice that:

- As x gets closer and closer to 0, y gets infinitely large. The curve approaches but never reaches the vertical asymptote $x = 0$.
- As x gets infinitely large, y gets closer and closer to 0. The curve approaches but never reaches the horizontal asymptote $y = 0$.
- $xy = 200$ for all points in the table.

If y is **inversely proportional** to x, then y is **directly proportional** to $\dfrac{1}{x}$.

We write this as $y \propto \dfrac{1}{x}$.

Consequently, $\boldsymbol{y = \dfrac{k}{x}}$ or $\boldsymbol{xy = k}$.

Example 5 **Self Tutor**

Suppose y is inversely proportional to x, and that $y = 12$ when $x = 10$. Find:

a y when $x = 40$

b x when $y = 19$

Since y is inversely proportional to x, $y \propto \frac{1}{x}$.

a

$\times 4$

x	10	40
y	12	

x is multiplied by 4

$\therefore$ y is multiplied by $\frac{1}{4}$ $\{$as $y \propto \frac{1}{x}\}$

$\therefore$ $y = 12 \times \frac{1}{4} = 3$

b

x	10	
y	12	19

$\times \frac{19}{12}$

y is multiplied by $\frac{19}{12}$

$\therefore$ x is multiplied by $\frac{12}{19}$ $\{$as $y \propto \frac{1}{x}\}$

$\therefore$ $x = 10 \times \frac{12}{19} \approx 6.32$

EXERCISE 7C

1 For each of the following tables, calculate the value of xy for each point. Hence determine whether x and y are inversely proportional. If an inverse proportionality exists, determine the law connecting the variables, and draw the graph of y against x.

a

x	2	3	4	6
y	12	8	6	4

b

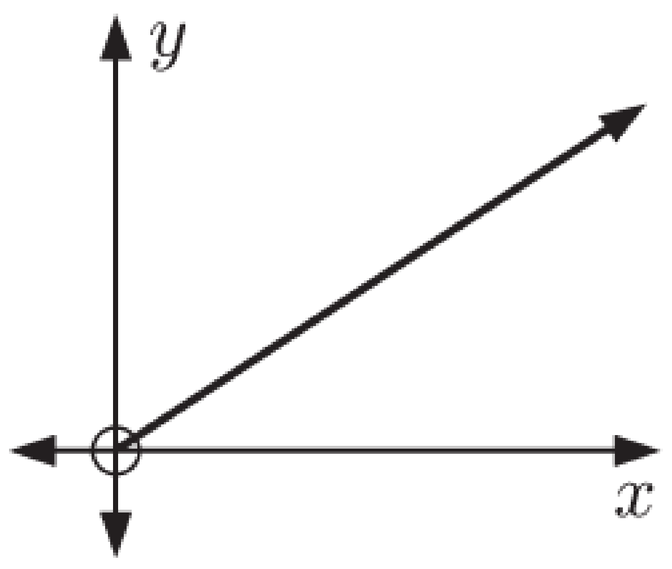
x	1	2	4	5
y	20	10	6	4

c

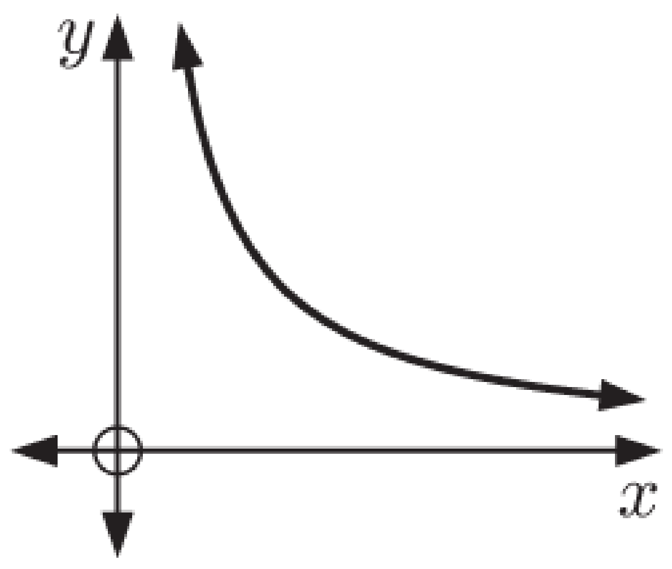
x	3	5	10	12
y	20	12	6	5

2 Which graph could indicate that y is inversely proportional to x?

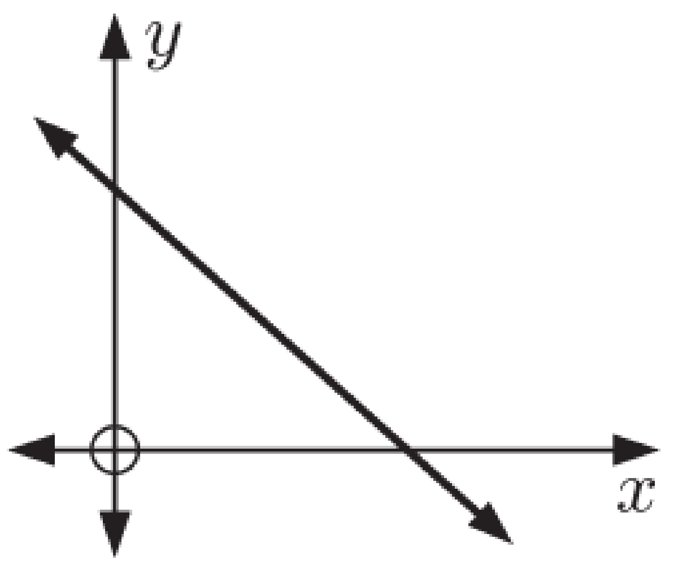
A



B



C



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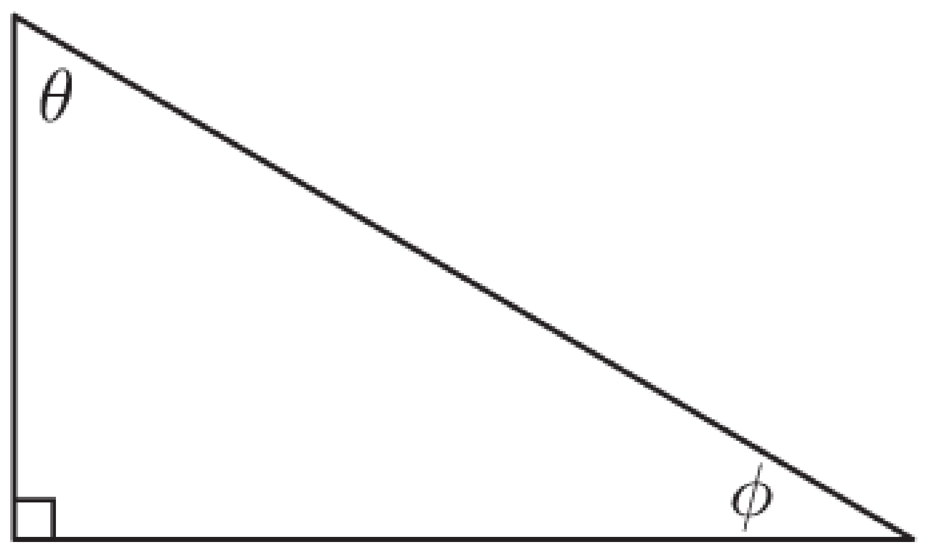


θ and ϕ are the two smaller angles of a right angled triangle.

a Are θ and ϕ inversely proportional?

b Are $\tan\theta$ and $\tan\phi$ inversely proportional?

Explain your answers.

4 Suppose y is inversely proportional to x. Explain what happens to y if:

a x is doubled

b x is divided by 7

c x is multiplied by $\frac{9}{5}$

d x is increased by 30%.

5 Suppose C is inversely proportional to t, and that $C = 15$ when $t = 6$. Find:

a C when $t = 18$

b t when $C = 20$.

6 The time taken to landscape a garden is inversely proportional to the number of gardeners working on the task. If 5 gardeners could do the task in 6 hours, how long would it take 3 gardeners to do the task?

7 When a constant force is applied to an object, the object's *acceleration* is inversely proportional to its *mass*.
When the force is applied to a 5 kg object, the resulting acceleration is 1.5 m s^{-2}.

- **a** Find the acceleration which would result if the same force acted on a 2 kg object.
- **b** Find the mass of an object whose acceleration under the force is 10 m s^{-2}.

8 Wendy wants to use the money in her savings account to buy shares in her favourite company.

- **a** Explain why the *number of shares* she can buy is inversely proportional to the *share price*.
- **b** The share price is currently £6.25, and Wendy can buy n shares at that price.
  - **i** If the share price drops by £0.15, how many shares can Wendy now buy?
  - **ii** By what percentage must the share price fall so that Wendy can buy $1.5n$ shares?

D POWERS IN INVERSE VARIATION

In this Section we consider the inverse variation between one variable and the power of another. We use the same techniques as for direct variation.

Example 6 **Self Tutor**

Suppose P is inversely proportional to the square of n, and that $P = 40$ when $n = 6$.

- **a** Find P when $n = 12$.
- **b** Given $n > 0$, find n when $P = 90$.

Since P is inversely proportional to n^2, $P \propto \dfrac{1}{n^2}$

a

$\times 2$

n	6	12
P	40	

n is multiplied by 2

$\therefore$ n^2 is multiplied by $2^2 = 4$

$\therefore$ $\dfrac{1}{n^2}$ is multiplied by $\frac{1}{4}$

$\therefore$ P is multiplied by $\frac{1}{4}$ $\quad \{$as $P \propto \dfrac{1}{n^2}\}$

$\therefore$ $P = 40 \times \frac{1}{4} = 10$

b

n	6	
P	40	90

$\times \frac{90}{40}$

P is multiplied by $\frac{90}{40} = \frac{9}{4}$

$\therefore$ $\dfrac{1}{n^2}$ is multiplied by $\frac{9}{4}$ $\quad \{$as $P \propto \dfrac{1}{n^2}\}$

$\therefore$ n^2 is multiplied by $\frac{4}{9}$

$\therefore$ n is multiplied by $\sqrt{\frac{4}{9}} = \frac{2}{3}$ $\quad \{$as $n > 0\}$

$\therefore$ $n = 6 \times \frac{2}{3} = 4$

EXERCISE 7D

1 Suppose $y \propto \dfrac{1}{x^3}$. Explain what happens to:

- **a** y if x is doubled
- **b** y if x is multiplied by $\frac{3}{5}$
- **c** x if y is multiplied by 64.

2 Suppose y is inversely proportional to the square of x, and that $y = 27$ when $x = 8$.

  a Find y when $x = 24$. **b** Given $x > 0$, find x when $y = 75$.

3 Suppose M is inversely proportional to the cube of c, and that $M = 64$ when $c = 12$.

  a Find M when $c = 8$. **b** Find c when $M = 1$.

4 A drink company wants to adjust the dimensions of their cylindrical soft drink cans, but keep the same volume. The current can is 12.9 cm high with radius 3.04 cm.

  a Explain why the height of the can is inversely proportional to the square of its radius.

  b Find the height of the can if the radius chosen is 3.39 cm.

  c Find the radius of the can if the height chosen is 15.3 cm.

  d The production manager restricts the possible radius of the can to values between 2.7 cm and 3.8 cm. Can you suggest why this was done?

5 The *tidal acceleration* acting between two bodies is inversely proportional to the cube of the *distance* between them.

  a Find the percentage change in the tidal acceleration if the distance between the objects increases by 10%.

  b Find the percentage change in the distance between two objects such that the tidal acceleration triples.

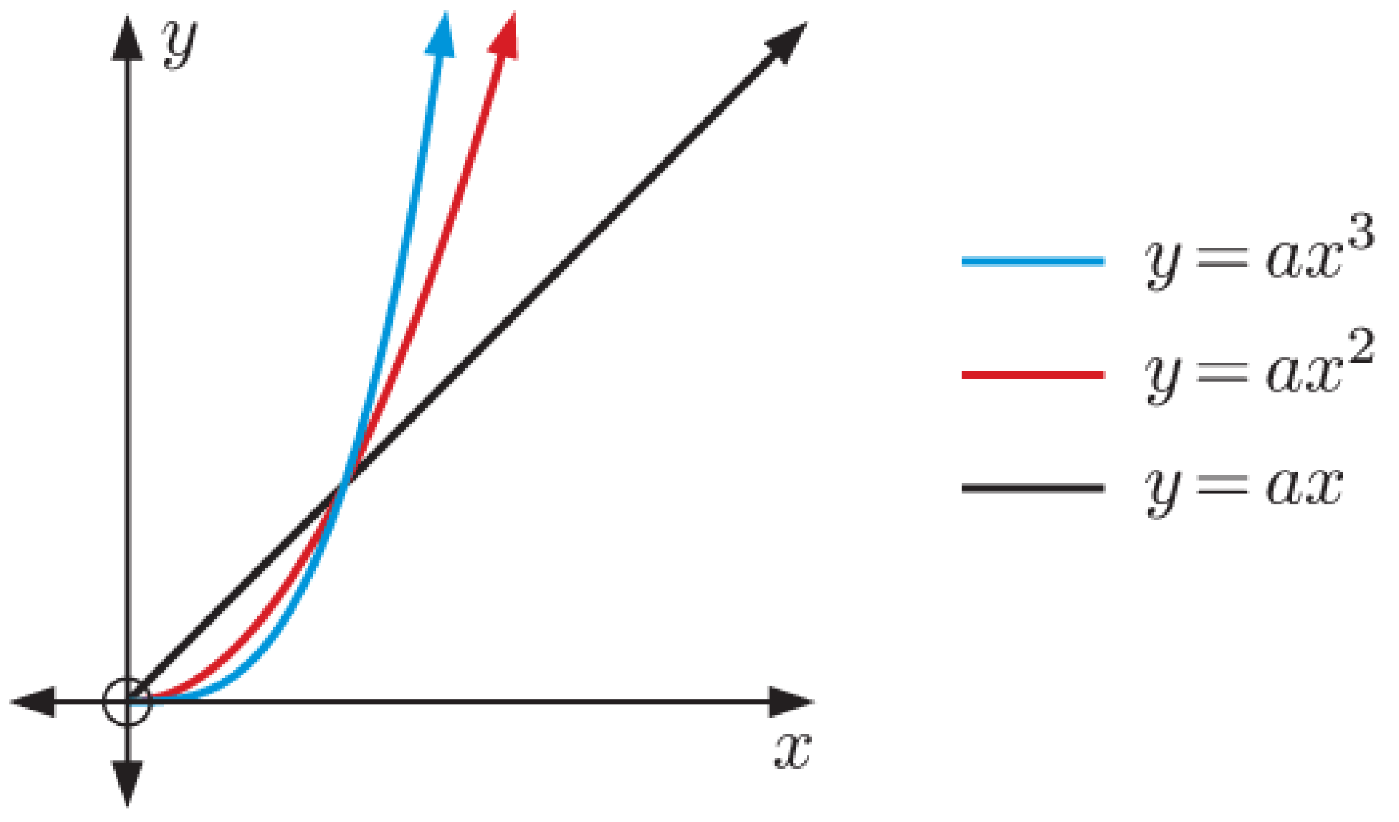
DISCUSSION

Suppose $y \propto \frac{1}{x}$, and $z \propto \frac{1}{y^2}$. What variation exists between x and z?

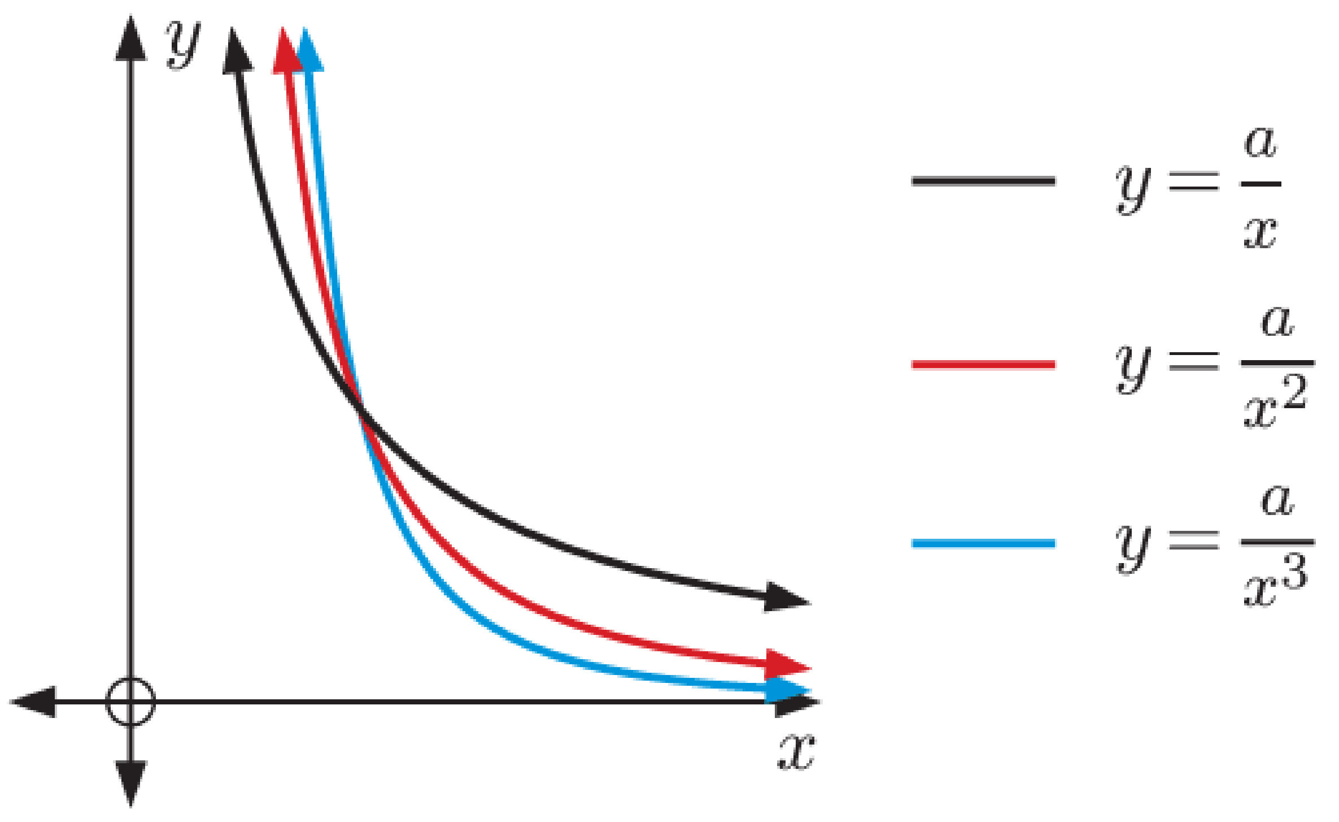
E DETERMINING THE VARIATION MODEL

The direct and inverse variations we have studied have equations of the form $y = ax^n$ where $n \in \mathbb{Z}$, $n \neq 0$. These equations are called **variation models**.

- If $n > 0$ we have **direct variation**.
- If $n < 0$ we have **inverse variation**.



The graph passes through the origin $(0, 0)$.

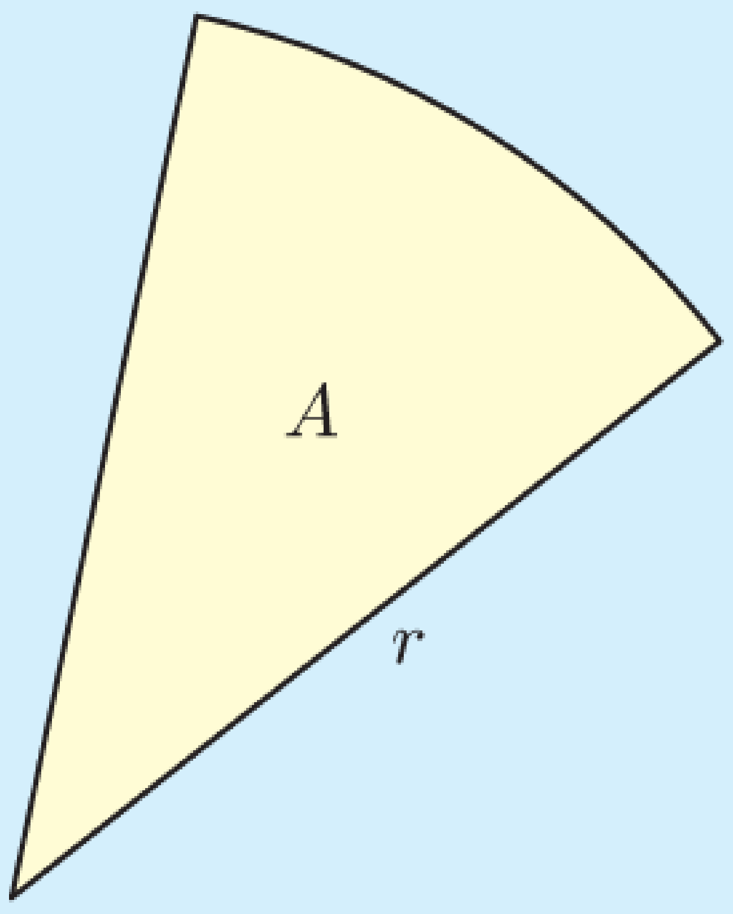


The graph is asymptotic to both the x and y axes.

In some cases, we know what type of variation exists between the variables. Given this knowledge, we can use a point which lies on the graph to find the exact equation of the variation model.

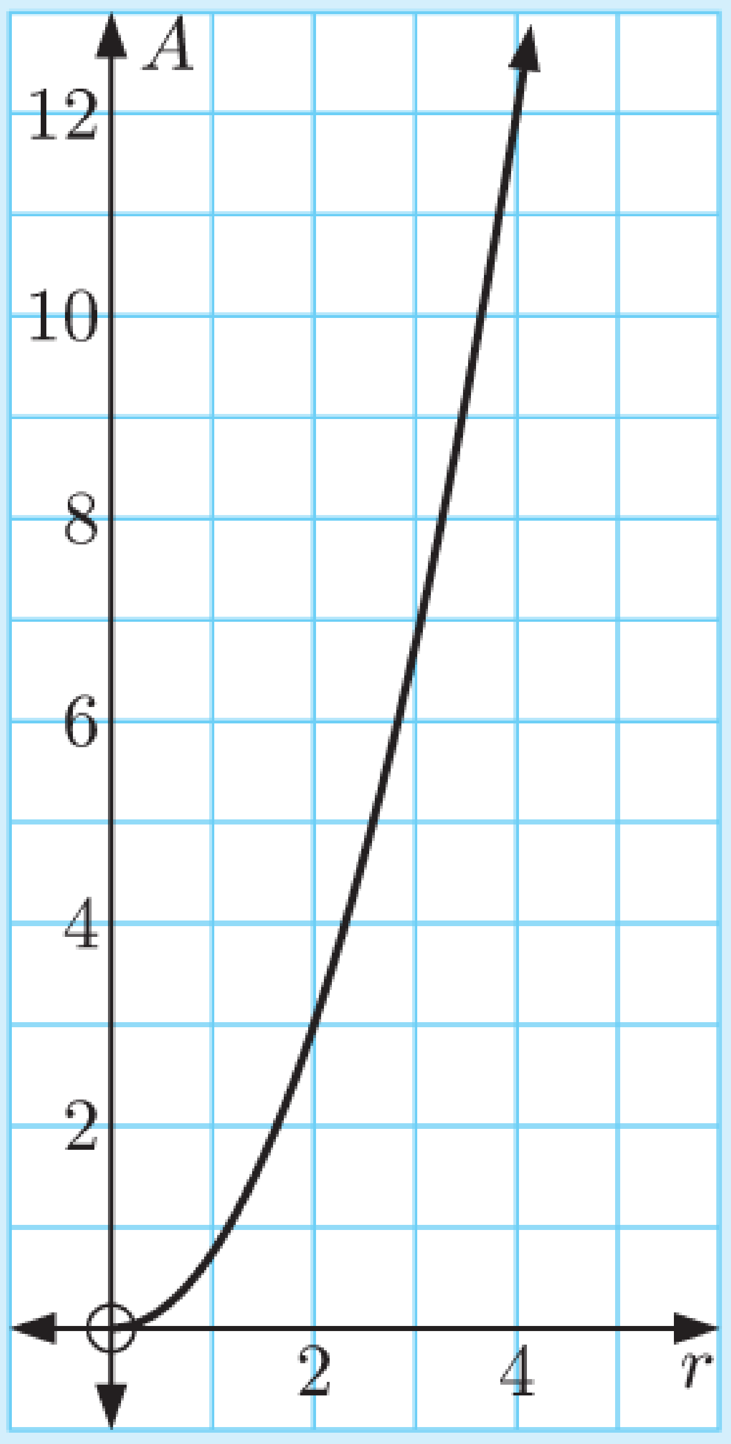
Example 7

Self Tutor



The area A of a sector of given angle is directly proportional to the square of its radius r. The graph of A against r is shown alongside.

Find the equation of the variation model.

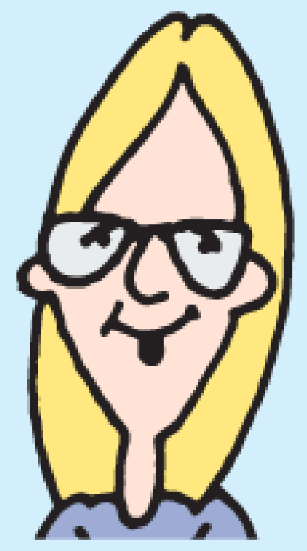


Since $A \propto r^2$, $A = ar^2$ where a is a constant.

From the graph we see that when $r = 2$, $A = 3$

$\therefore$ $3 = a \times 4$ and so $a = \frac{3}{4}$.

The equation of the variation model is $A = \frac{3}{4}r^2$.

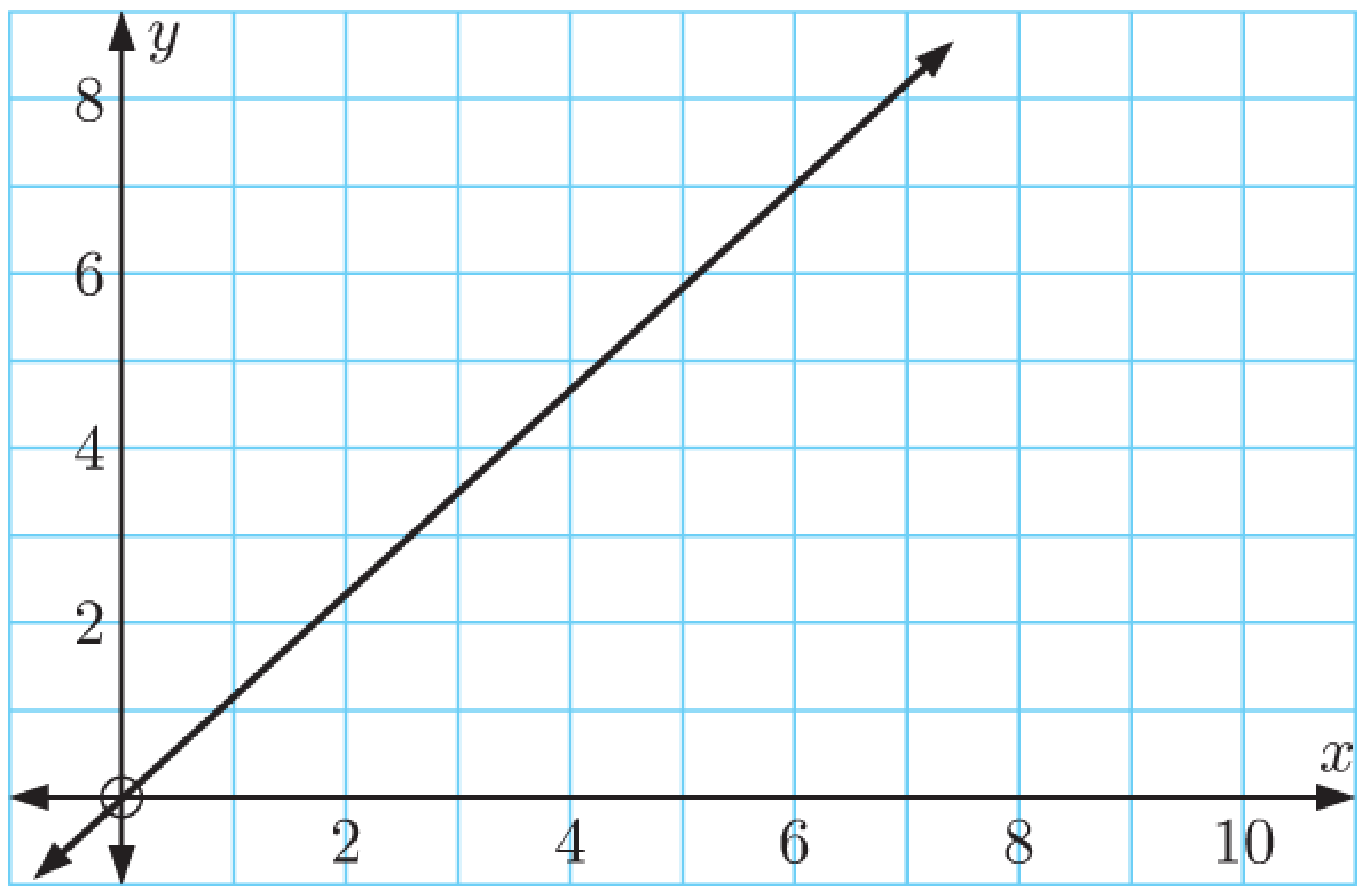


Check the other data points on the graph to make sure they obey this model.

EXERCISE 7E

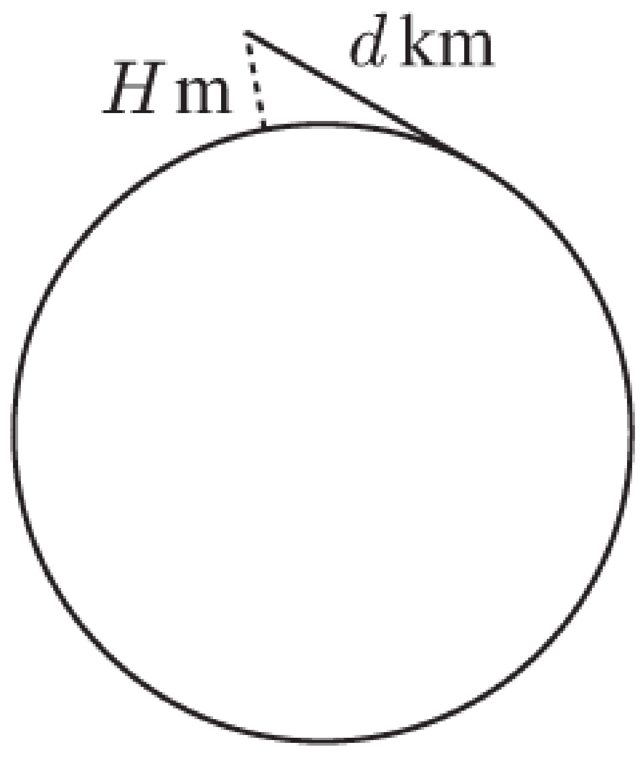
1 Consider the graph alongside.

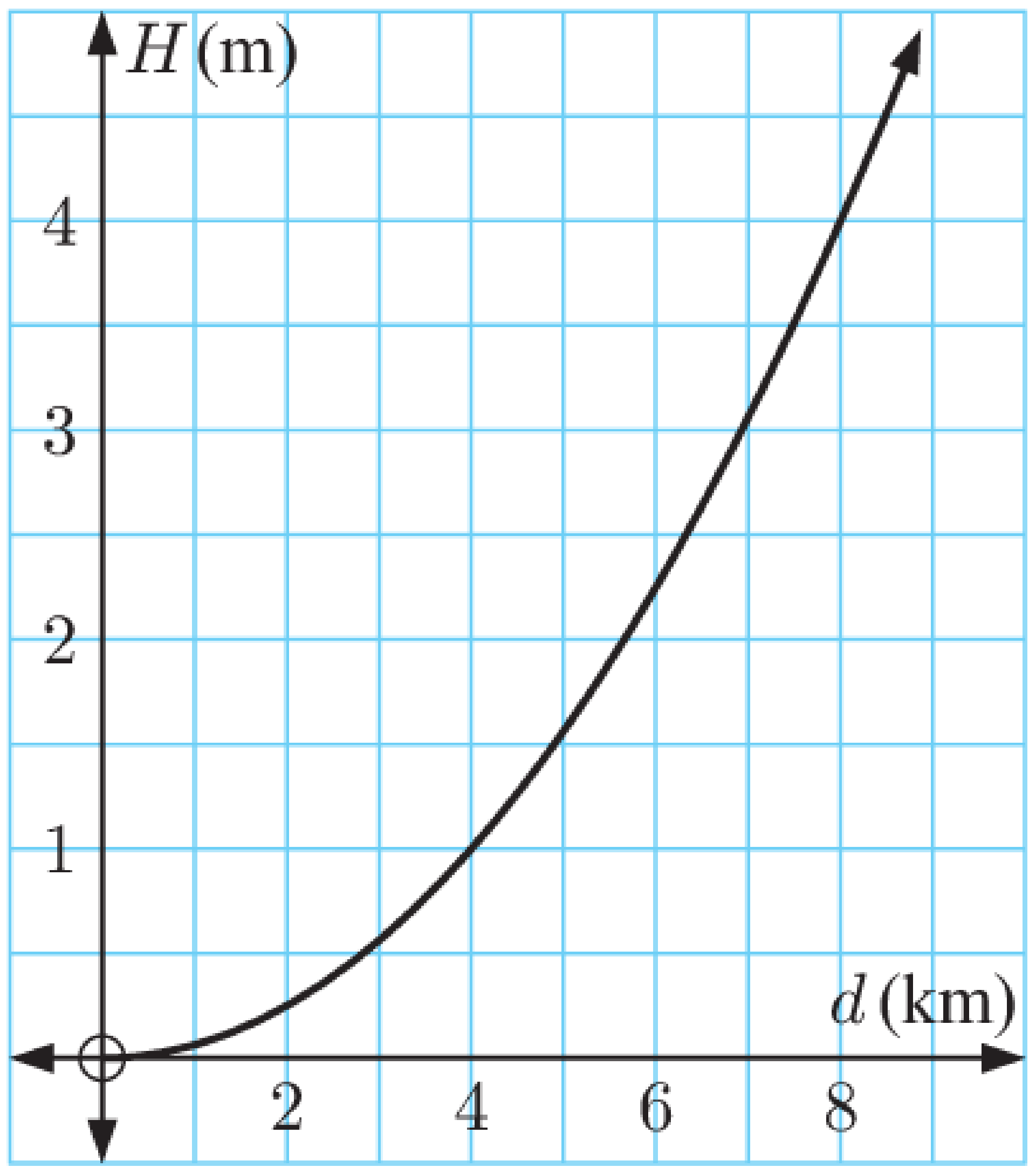
- **a** Explain why $y \propto x$.
- **b** Find the equation of the variation model.



2 The height (H m) of a person above sea level is directly proportional to the square of the distance (d km) to the horizon.
The graph of H against d is shown alongside.

- **a** Find the equation of the variation model.
- **b** Teresa is 16 km from the horizon. Find her height above sea level.





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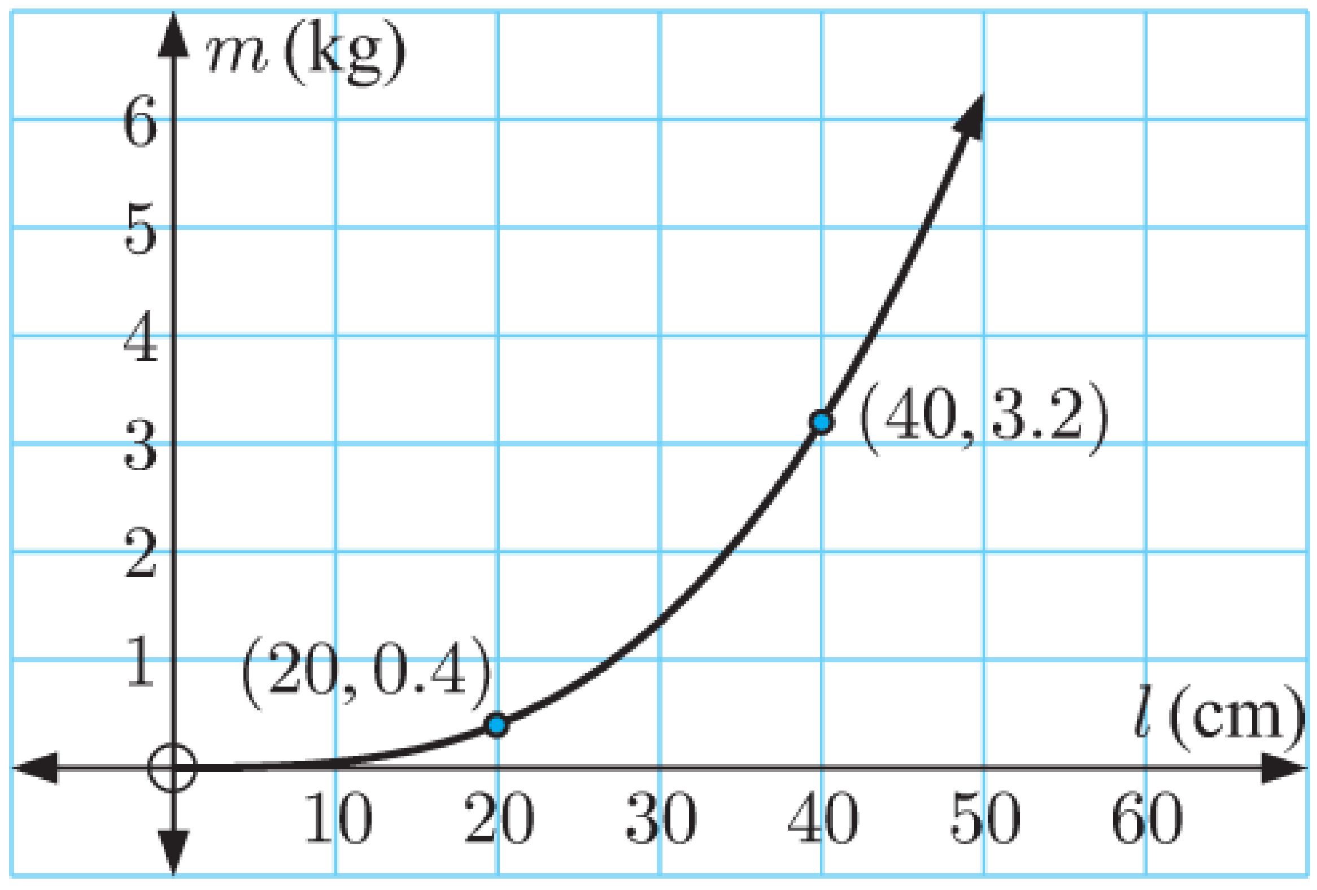


Scale models of a car are made in different sizes. The mass of a model car (m kg) is directly proportional to the cube of its length (l cm).

The graph of m against l is shown alongside.

a Find a model connecting m and l.

b Find the mass of a model car which is 50 cm long.

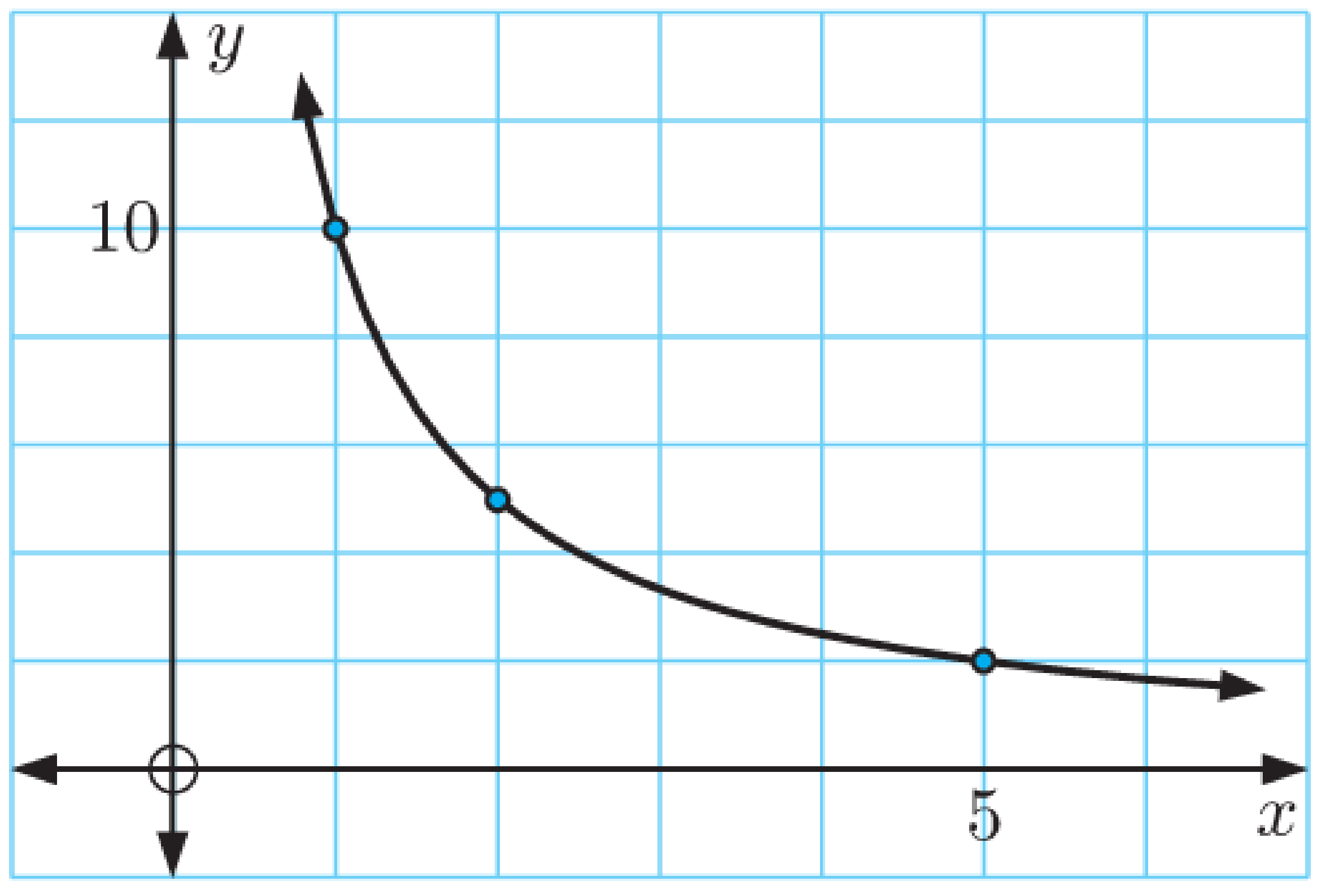
c Find the length of a model car with mass 1 kg.

4 It is suspected that variable y varies inversely with variable x.

a Do the three marked points confirm this relationship? Explain your answer.

b State the equation of the model connecting y and x.

c Find the value of y when $x = 8$.



5 The table opposite contains data from an experiment.

x	0.25	0.5	1	2
y	80	20	5	1.25

a Show that the model relating x to y has the form $y = \frac{k}{x^2}$ and find the value of k.

b Find the value of $x > 0$ when $y = 0.5$.

6 A car designer wants to find the relationship between the air resistance R and the velocity v km h^{-1} of the car. He performs a wind tunnel experiment and records the results in the table alongside:

v	10	20	30	40
R	0.5	4	13.5	32

a The designer initially suspects that $R \propto v^2$.

i Assuming this relationship is true, use the first data point to find the model for R in terms of v.

ii Use the other data points to show that this model is incorrect.

b Show that a model of the form $R = kv^3$ fits all data points.

c Hence find the air resistance when the car is travelling at 50 km h^{-1}.

F USING TECHNOLOGY TO FIND VARIATION MODELS

So far we have only considered data which a variation model fits exactly. This is unlikely to be the case when we collect data from real-world situations, due to experimental error, inaccuracies in measurement, and rounding. In these circumstances, we can use technology to find the variation model which best fits the data.

Variation models are often referred to as **power models**, because the variable is raised to a power. We use the **power regression** function on our calculator to find variation models, in the same way we found linear regression models in **Chapter 5**.

Example 8

Self Tutor

A company manufactures ball bearings in different sizes. The table alongside shows the radius and mass of each type of ball bearing.

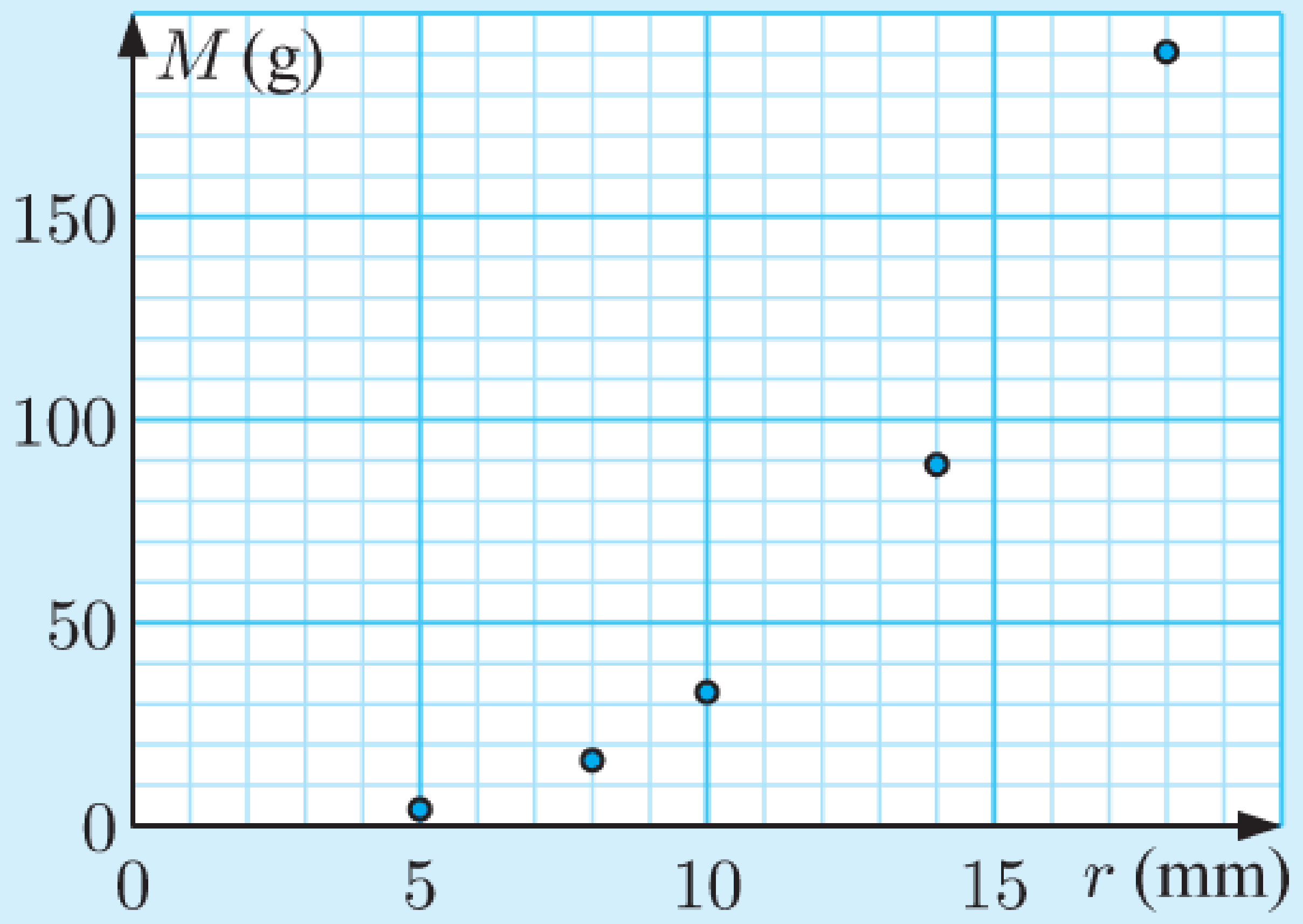
Radius (r mm)	5	8	10	14	18
Mass (M g)	4.08	16.73	32.67	89.65	190.54

a Draw a scatter diagram of the data. Discuss the shape of the graph.

b Obtain the power model which best fits the data.

c Estimate the mass of a ball bearing with radius 22 mm.

a

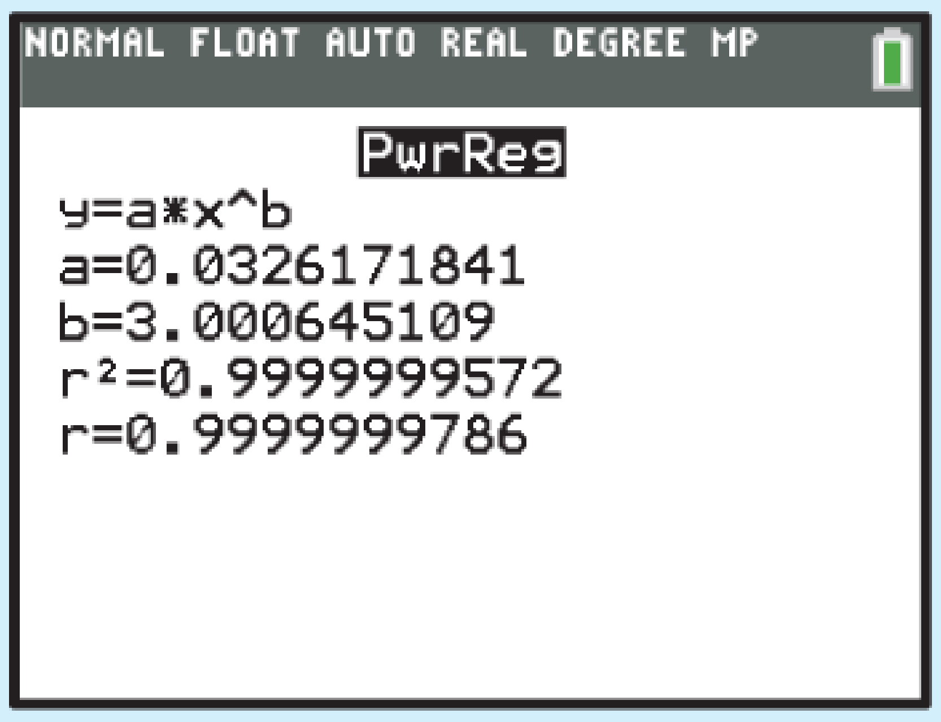


The data points appear to lie on a curve passing through the origin. This suggests that a variation model is appropriate.

b The correlation coefficient r is very close to 1, so the fit is excellent.

The power is very close to 3, so it is reasonable to conclude that M is directly proportional to r^3.

The model is $M \approx 0.0326r^3$.



```
NORMAL FLOAT AUTO REAL DEGREE MP
PwrReg
y=a*x^b
a=0.0326171841
b=3.000645109
r²=0.9999999572
r=0.9999999786
```

c When $r = 22$, $M \approx 0.0326 \times 22^3 \approx 347$

So, a ball bearing with radius 22 mm has mass ≈ 347 g.

GRAPHICS CALCULATOR INSTRUCTIONS

DISCUSSION

In the Example above, why should we expect that mass $\propto$ radius^3?

Discuss other situations where you might expect variation models.

EXERCISE 7F

1 For each data set, obtain the power model which best fits the data:

a

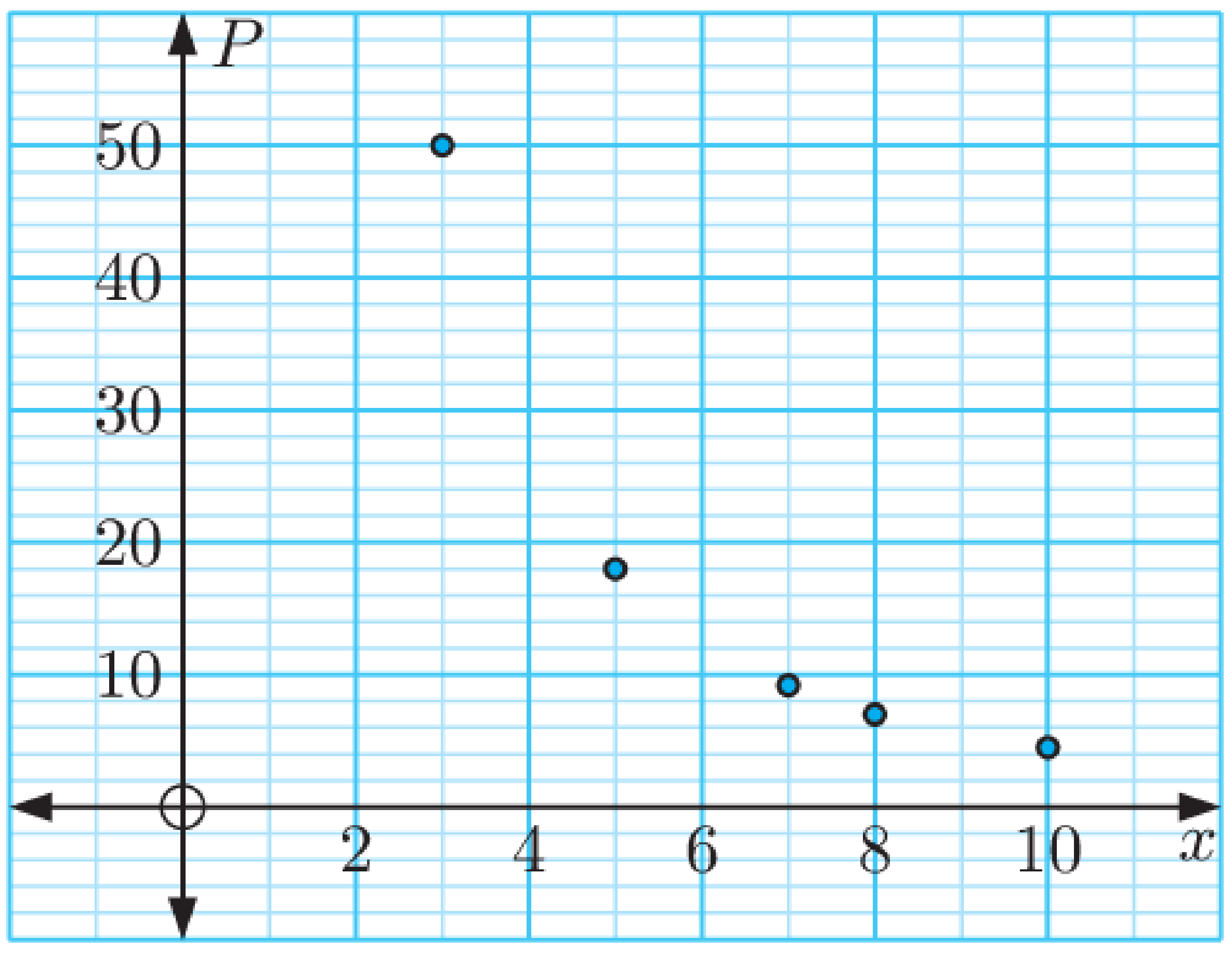
x	1	2	3	4
y	0.6	9.7	48.8	153.5

b

x	2	3	6	9
y	100	29.6	3.7	1.1

2 The data below is displayed on the scatter diagram.

x	3	5	7	8	10
P	50	18	9.2	7	4.5



a Do you think there is direct variation or inverse variation between the variables? Explain your answer.

b Obtain the variation model which best fits the data.

c Estimate the value of P when $x = 4$.

3 Zach never remembers to charge his phone until the battery has completely run out. This table shows the percentage charge his phone receives when it is charged for different times:

Time (t minutes)	8	25	32	45
Percentage charge ($C\%$)	10	31	40	56

a What type of variation would you expect between C and t? Explain your answer.

b Find the power model which best fits the data.

c Estimate the percentage charge of the phone after 56 minutes.

d For what values of t is it reasonable to apply this model? Explain your answer.

4 The land blocks in a new housing development have different dimensions, but they all have approximately the same area. Some of the land block dimensions are shown in the table below.

Width (w metres)	15	18	20	22	24
Length (l metres)	40	33	30	27	25

a What type of variation do you expect between l and w? Explain your answer.

b Draw a scatter diagram of the data. Is the diagram consistent with your answer to **a**?

c Obtain the power model which best fits the data.

d Estimate the length of a land block which is 23 m wide.

e For what range of values of w can this model sensibly be applied? Explain your answer.

5 When a car turns a corner, its *turning radius* depends on its *speed*.

Speed (s m s^{-1})	5.56	6.00	6.41	6.80	7.17	7.52
Turning radius (R m)	1.0	1.2	1.4	1.6	1.8	2.0

a Obtain the variation model which best fits the data.

b Estimate the turning radius for a car travelling at 10 m s^{-1}.

c Estimate the speed of a car if its turning radius is 4 m.

6 Answer the **Opening Problem** on page **164**.

7 The force between two positively charged spheres was measured at different distances. The results are recorded below.

Distance (d m)	0.1	0.25	0.5	0.75	1	1.5
Force (F N)	563	90.0	22.5	10.0	5.63	2.50

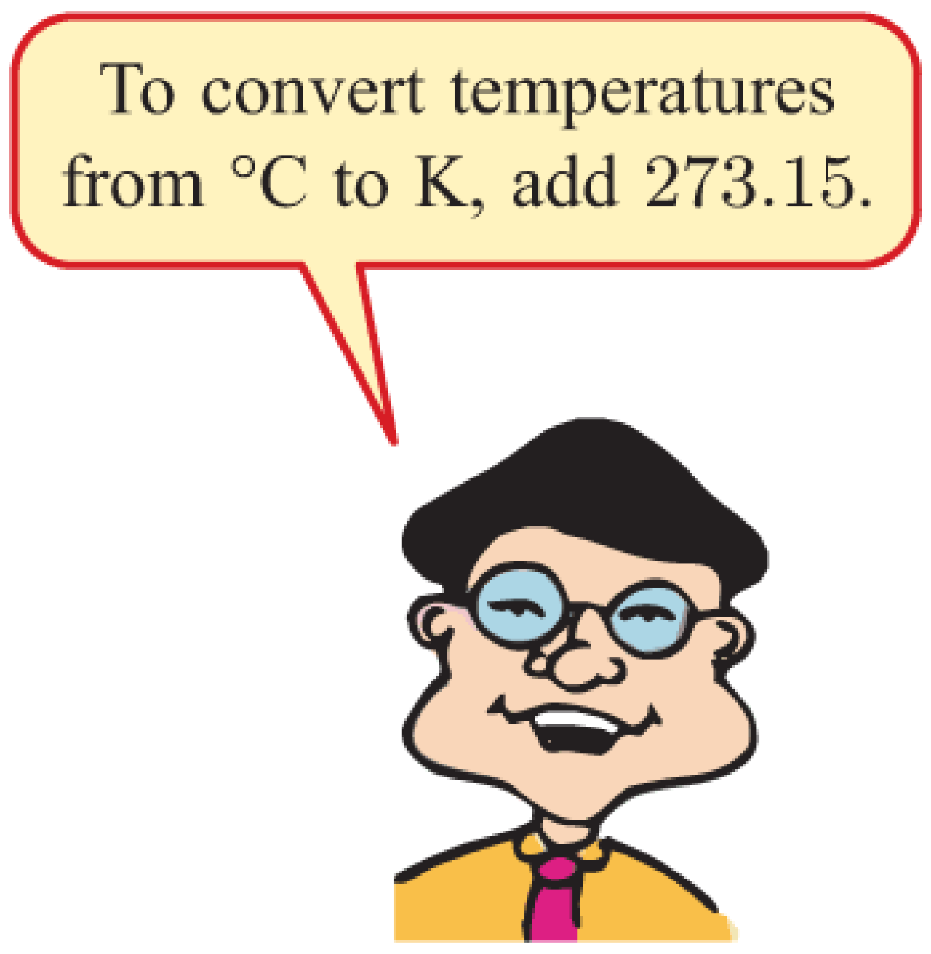
- **a** Obtain a power model which best fits the data.
- **b** Estimate the strength of the force when the spheres are 0.4 m apart.
- **c** Find the distance between the spheres if the force between them is 650 N.

8 Donna had heard that the pressure of a constant volume of gas is directly proportional to its *temperature*.

To test this for herself, Donna observed how the pressure in a tube changed as the gas inside was heated. Her results are shown in the table:

Temperature (T °C)	5	10	15	20	25	30	35
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

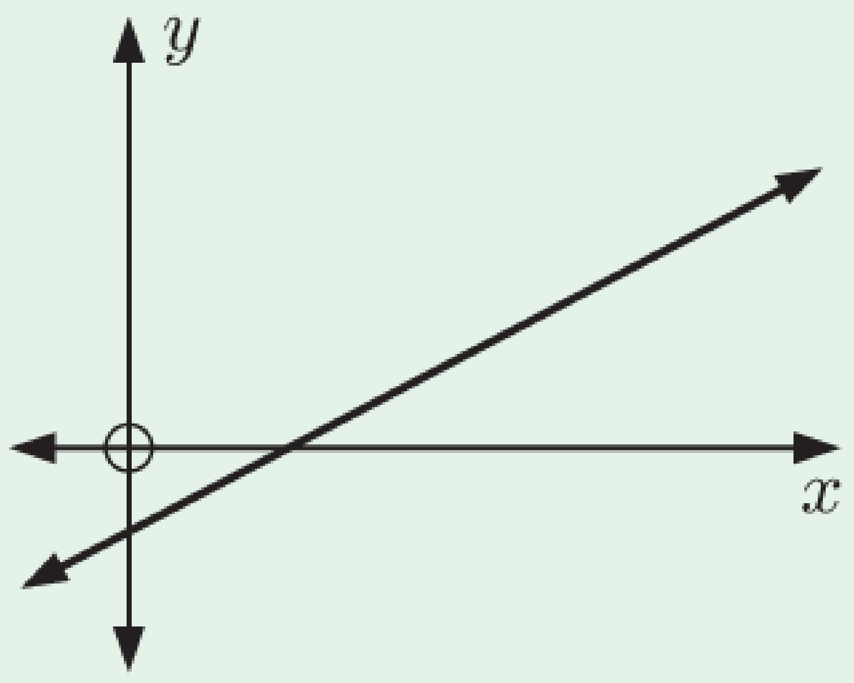
- **a** Obtain the power model which best fits the data. Explain why P is *not* directly proportional to T.
- **b** Donna's friend Penny suggests that the temperatures should be recorded in kelvin (K) rather than degrees Celsius.
  - **i** Redraw the table with the temperatures recorded in kelvin.
  - **ii** Obtain the power model which best fits the data. Use the correlation coefficient to explain why this power model is a much better fit for the data.
  - **iii** Is it now reasonable to conclude that pressure and temperature are directly proportional? Explain your answer.


To convert temperatures from °C to K, add 273.15.

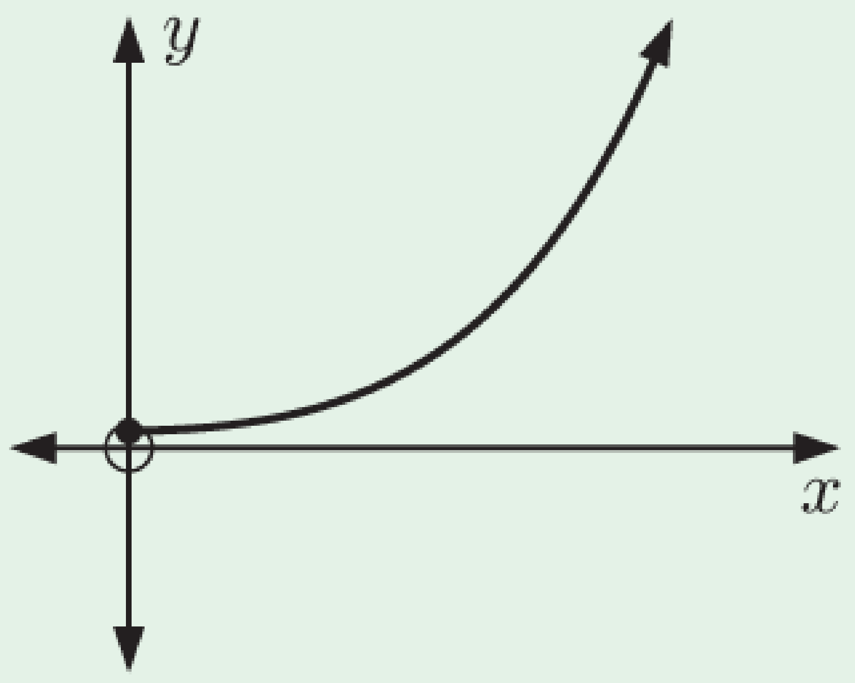
REVIEW SET 7A

1 Which graph indicates that y is directly proportional to x?

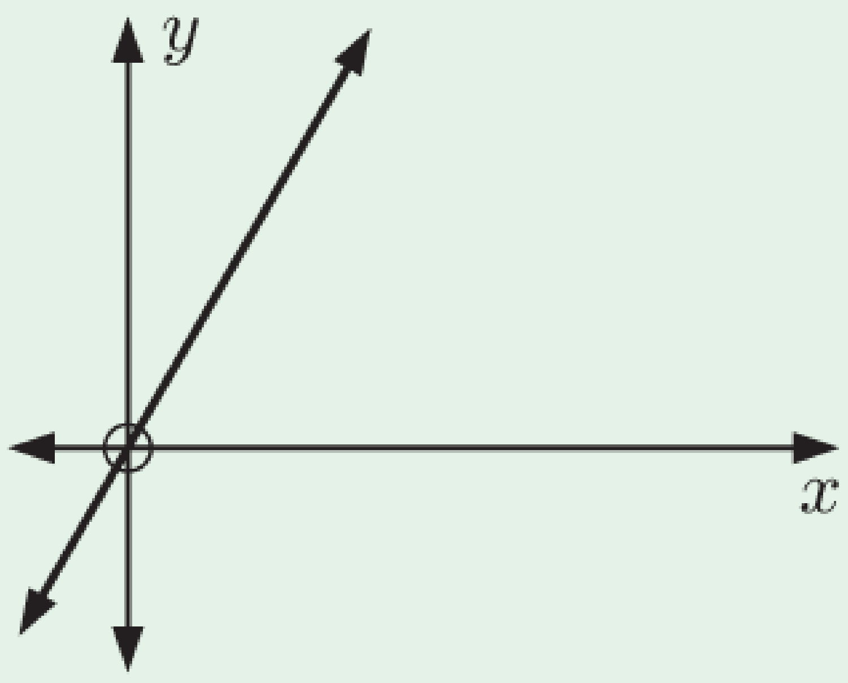
A



B



C



2 Suppose A is directly proportional to t. Explain what happens to:

- **a** A when t is multiplied by 4
- **b** t when A is increased by 5%.

3 A person's weight on the Moon is directly proportional to their weight on Earth. A person weighing 750 N on Earth weighs only 124 N on the Moon.

Given that John weighs 640 N on Earth, find his weight on the Moon.

4 State which two variables are directly proportional, and determine the proportionality constant k:

a $y = 5x^2$ **b** $P = \frac{2n^4}{3}$ **c** $V = \frac{\sqrt{5}}{4}a^3$

5 Suppose y is directly proportional to x^2, where $x > 0$. When $x = 8$, $y = 30$.

a Find a model connecting x and y.

b Hence find:

i y when $x = 4$ **ii** x when $y = 150$.

6 For each of the following data sets, use values of xy to determine whether x and y are inversely proportional. If an inverse proportionality exists, determine the law connecting the variables, and draw the graph of y against x.

a

x	2	4	5	8
y	20	10	8	5

b

x	3	5	8	10
y	20	12	8	6

7 The *frequency* of a light wave is inversely proportional to its *wavelength*.
Light waves that are seen as orange have wavelength 600 nm and frequency 500 THz.
Light waves that are seen as blue have wavelength 480 nm. Find the frequency of this blue light wave.

8 The resistance to the flow of electricity in a wire varies inversely to the square of the diameter of the wire. When the diameter is 0.44 cm, the resistance is 0.24 ohms. Find:

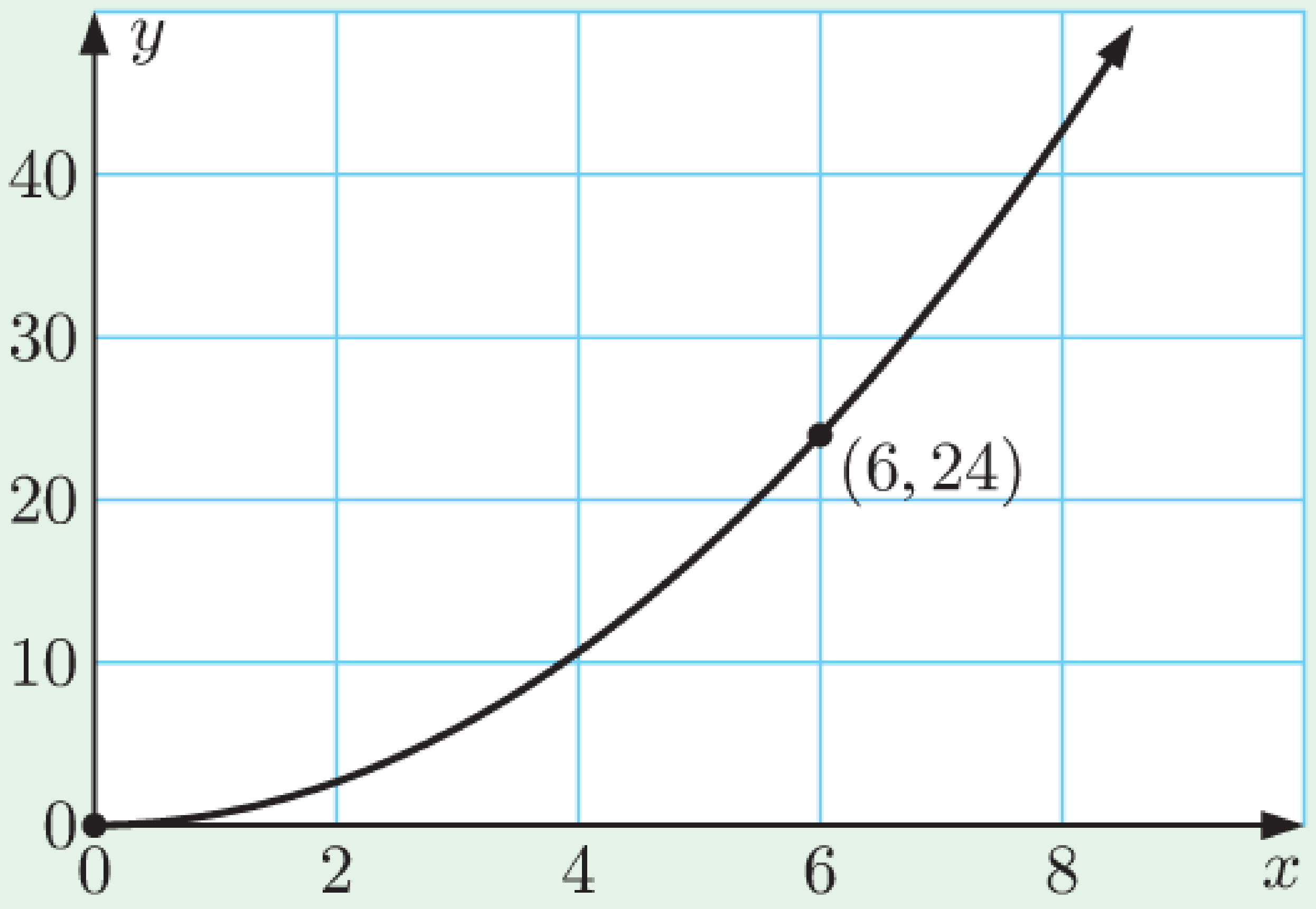
a the resistance when the diameter is 0.3 cm

b the diameter when the resistance is 0.45 ohms.

9 In the graph alongside, y varies directly with the square of x.

a Find the equation of the model connecting the variables.

b Find the value of y when $x = 11$.



10 It is suspected that two variables D and p are related by a law of the form $D = \frac{k}{p^2}$ where k is a constant. An experiment was conducted to find D for various values of p, and the results are given in the table.

p	1.5	2	2.5	3	4
D	40	22.5	14.4	10	5.625

a Graph D against p for these data values.

b What features of the graph suggest that $D = \frac{k}{p^2}$ may be an appropriate model for the data?

c Determine the value of k.

d Hence find the value of D when $p = 5$.

11 For each data set, use technology to find the power model which best fits the data.

a

x	2	4	5	7
y	12	96	188	514

b

x	0.9	1.4	2.2	2.5
y	762	130	21.3	12.8

12 The current was measured across a range of resistors for a constant potential difference:

Resistance (R ohms)	6.8	9.1	15	30	68	100
Current (I amperes)	17.6	13.2	8.00	4.00	1.76	1.20

a Draw a graph of I against R.

b Do you think there is a direct variation relationship or inverse variation between the variables? Explain your answer.

c Obtain the variation model which best fits the data.

d Estimate the current for a 250 ohm resistor given the same potential difference.

REVIEW SET 7B

1 Consider the table of values alongside.

t	2	5	8	10	15
P	5	12.5	20	25	37.5

a Draw a graph of P against t.

b Explain why P and t are directly proportional.

c Find a formula connecting P and t.

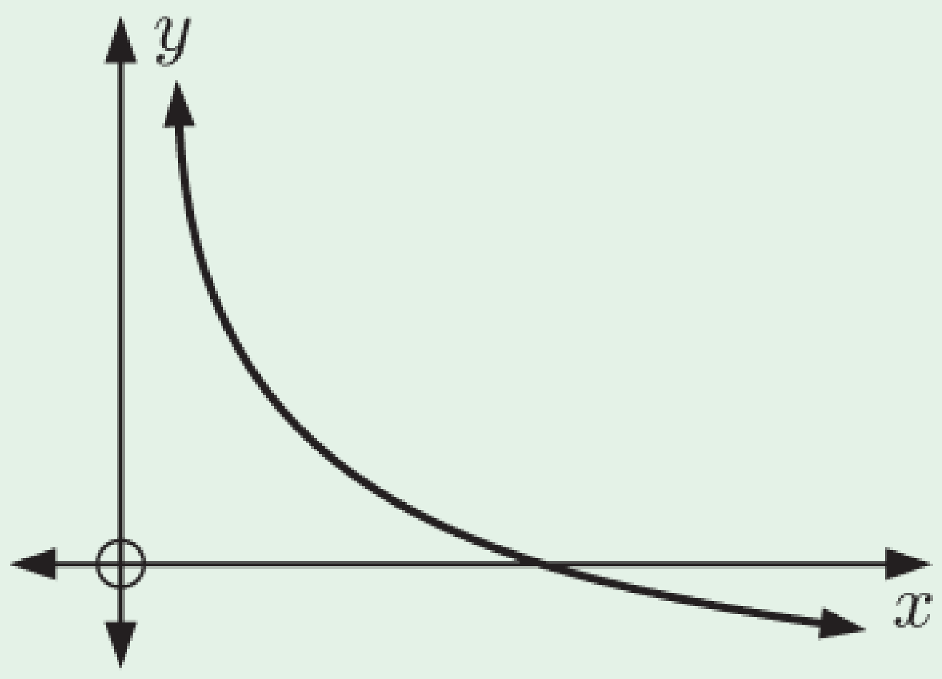
2 The force on a length of wire in a constant magnetic field is directly proportional to the current flowing through the wire. When the current is 1.09 A, the force on the wire is 2.18 N. Find:

a the force when the current is 1.45 A

b the current when the force is 3.6 N.

3 Which graph could indicate that y is inversely proportional to x?

A



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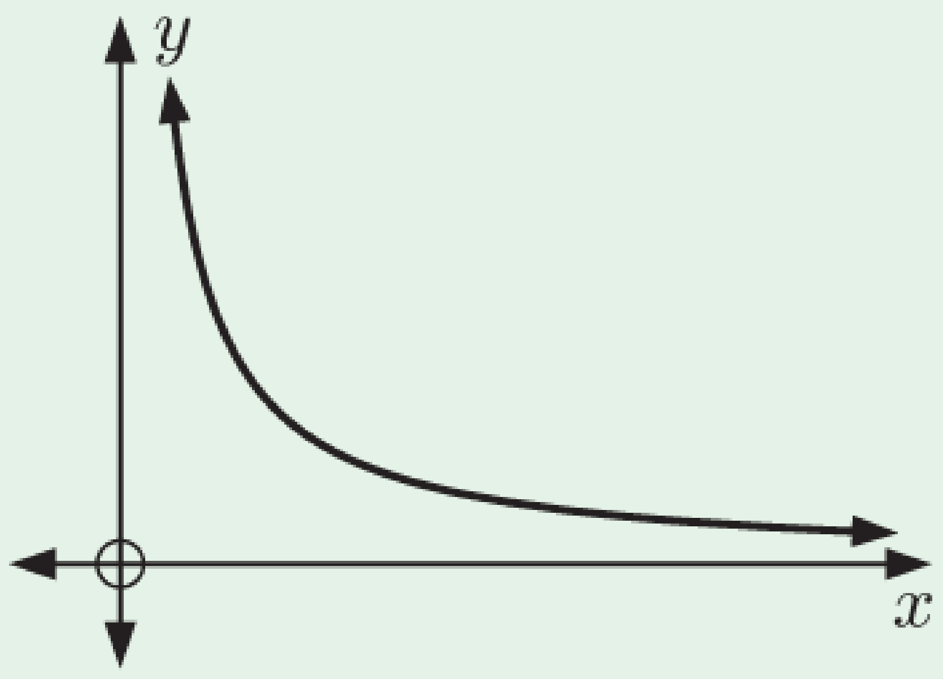

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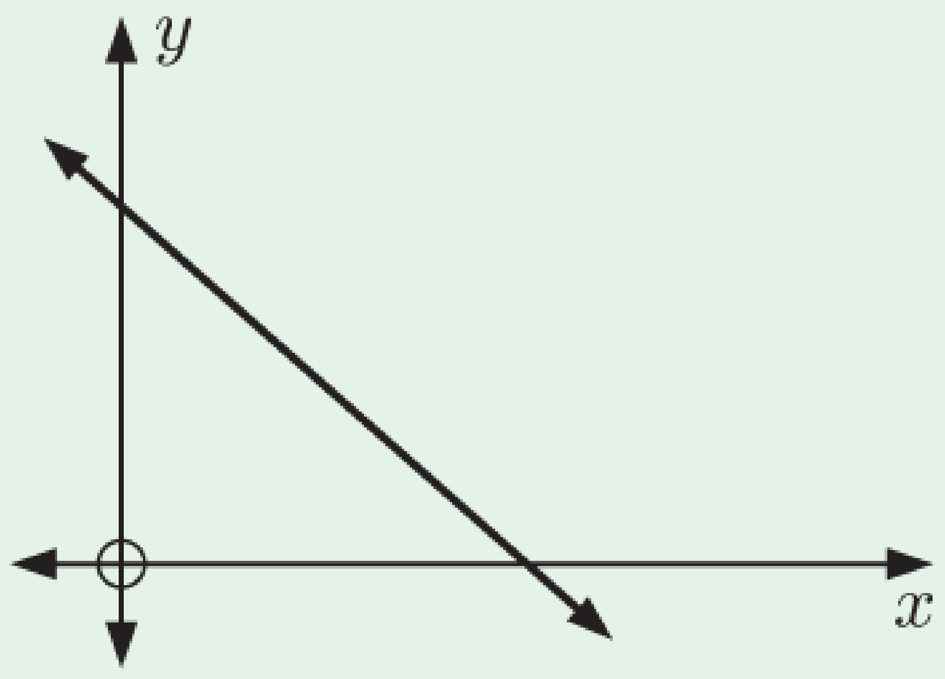
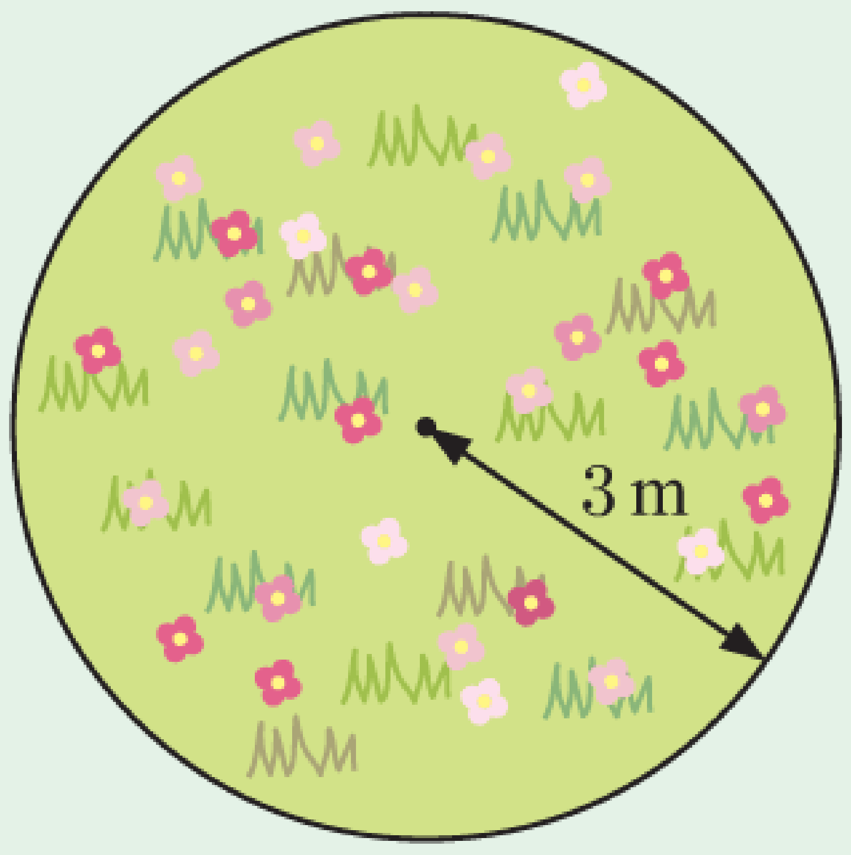


4 Tamzin works at the botanic gardens. She has been told by the head gardener that 250 kg of compost is needed to cover a circular garden bed of radius 3 m. However, she has new plants to add to the collection, so she has decided to increase the size of the garden bed at the same time.



a Explain why the amount of compost needed is directly proportional to the square of the radius of the garden bed.

b How much *extra* compost will Tamzin need to extend the radius of the garden bed by 15%?

c By how far can Tamzin extend the radius of the garden bed with an extra 40 kg of compost?

5 Suppose y is inversely proportional to the cube of x, and that $y = 16$ when $x = 6$.

- **a** Find a model connecting x and y.
- **b** Hence find:
  - **i** y when $x = 4$
  - **ii** x when $y = -2$.

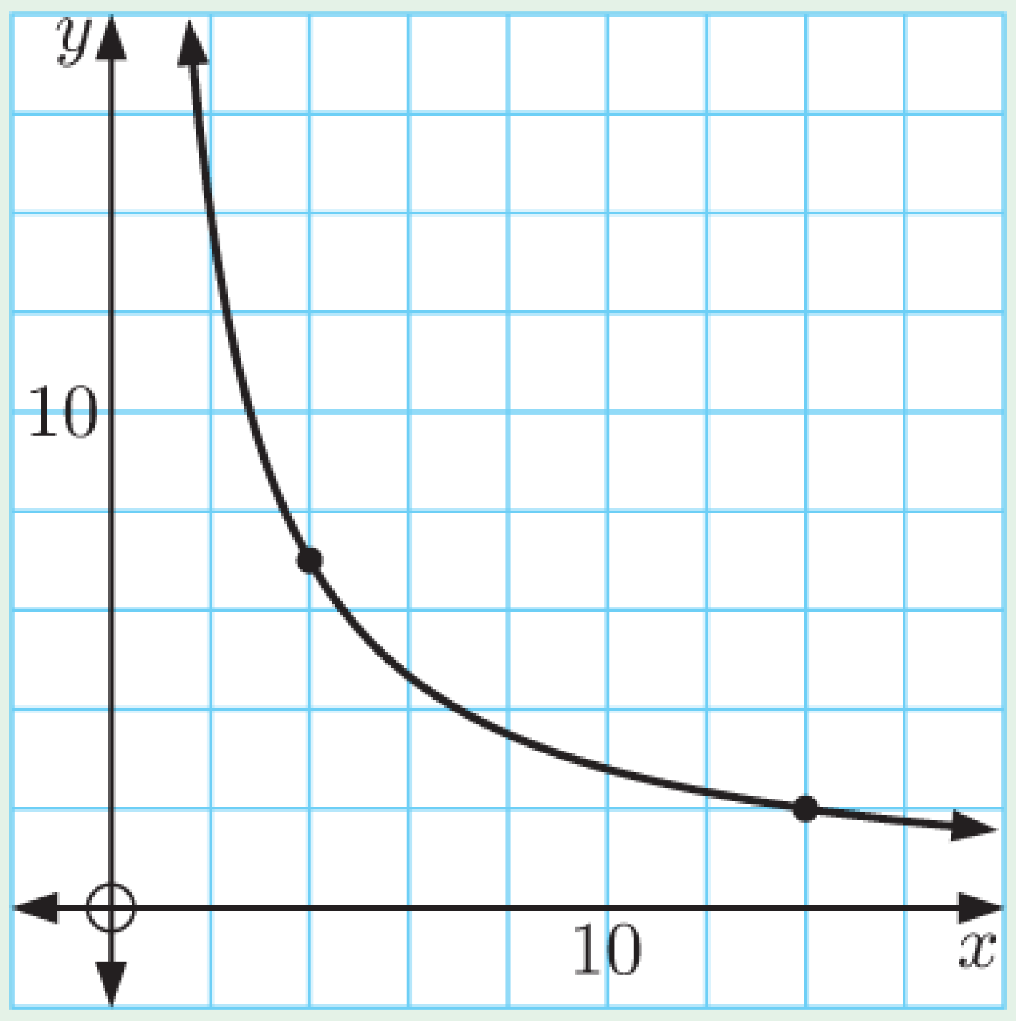
6 If 3 people could paint a grain silo in 18 days, how long would it take 8 people to paint the silo working at the same rate?

7 For an object moving in a circular orbit under gravity, the radius of the orbit is inversely proportional to the square of the object's orbital speed.

If the radius of the orbit increases by 20%, what happens to the orbital speed of the object?

8 In the graph alongside, y varies inversely with x.

- **a** Find the equation of the variation model.
- **b** Find the value of y when $x = 0.1$.



9 Kelly makes glass regular pyramids of height h cm. She suspects that the volume of glass V cm^3 she uses is directly proportional to a power of h, so $V \propto h^n$. A table of volumes for various heights is shown below.

h	2	4	6
V	3.2	25.6	86.4

- **a** Explain why we should expect that $n = 3$.
- **b** Use the first piece of data to find the variation model connecting V and h.
- **c** Check that the model you have found is satisfied by the remaining data points.
- **d** Find:
  - **i** V when $h = 8$
  - **ii** h when $V = 50$.

10 For the data set alongside:

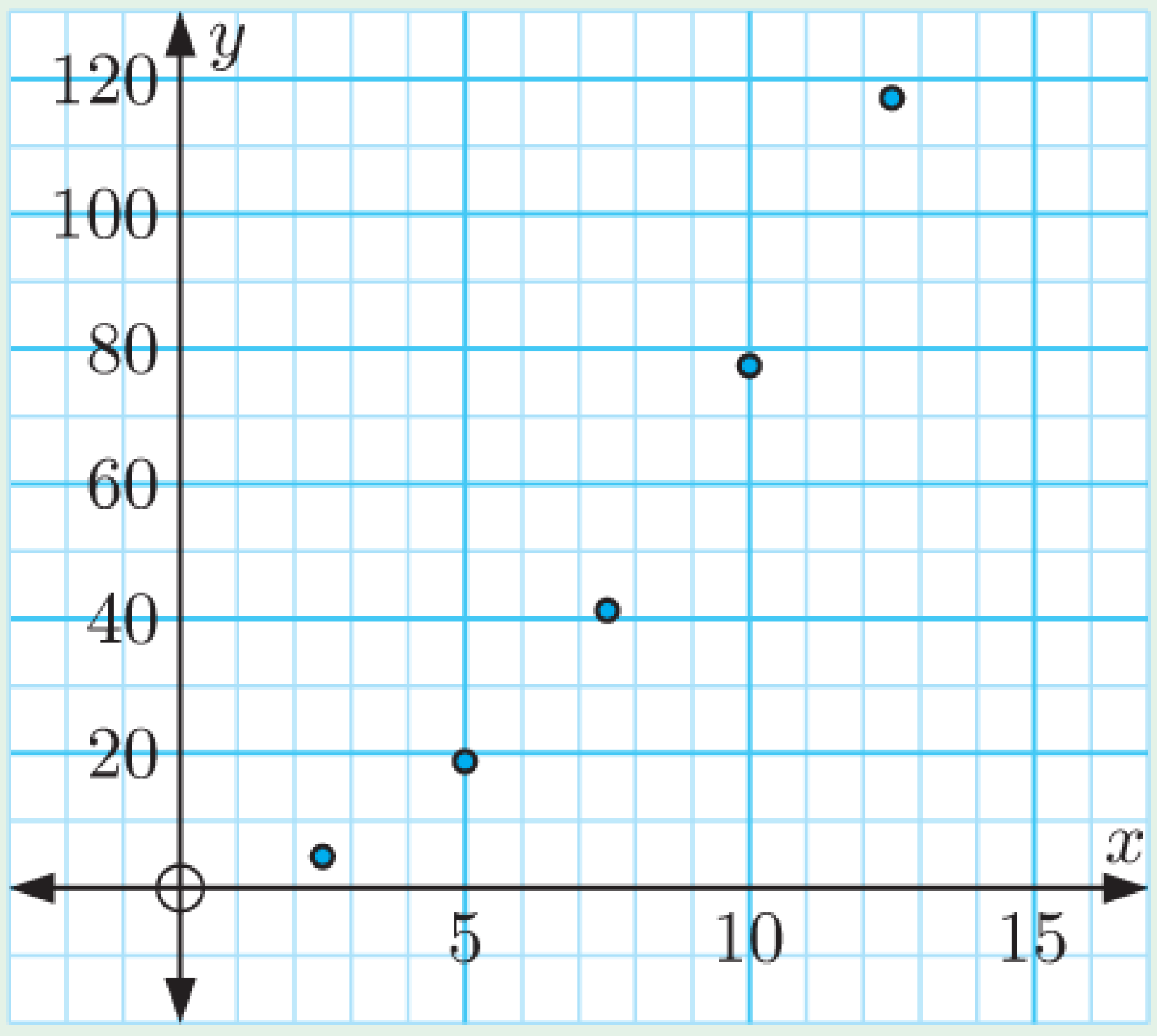
- **a** Find the power model which best fits the data.
- **b** Estimate the value of y when $x = 5$.

x	2	3	6	8	10
y	45	20	5	2.8	1.8

11 The data below is displayed on the scatter diagram.

x	2.5	5	7.5	10	12.5
y	4.7	18.8	41.2	75	117.2

- **a** Do you think there is a direct variation relationship or inverse variation between the variables? Explain your answer.
- **b** Obtain the variation model which best fits the data.



12 Abbas wanted to test the sound intensity of the speakers in the local hall. He set the output to a constant power, and took measurements at different distances. The results are given below.

Distance (d m)	1	5	10	15	20
Sound intensity (I W m^{-2})	63.7	2.55	0.637	0.283	0.159

a Abbas thinks that $I \propto \frac{1}{d}$. By calculating the value of $I \times d$ for each data point, show that he is incorrect.

b Find the power model which best fits the data.

c Hence find the percentage change in intensity if the distance is increased by 40%.

Chapter 8

Exponentials and logarithms

Contents:

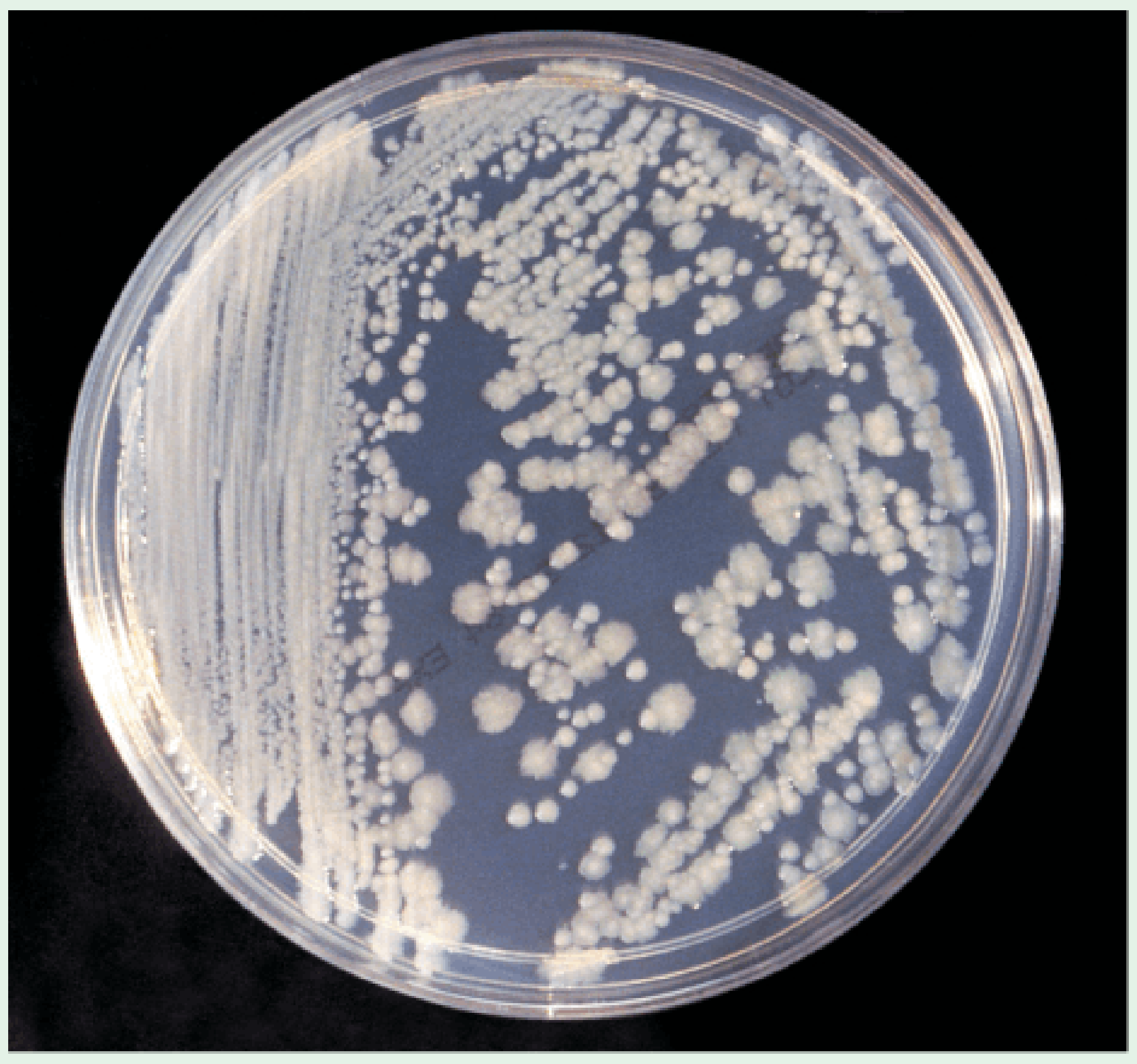
A Exponential functions
B Graphing exponential functions from a table of values
C Graphs of exponential functions
D Exponential equations
E Growth and decay
F The natural exponential
G Logarithms in base 10
H Natural logarithms

OPENING PROBLEM

A Petri dish contains a colony of 10 million bacteria. It takes one hour for each of these bacteria to divide into two bacteria, so every hour the total number of bacteria doubles.

This table shows the population b million bacteria after time t hours.

t (hours)	0	1	2	3	4
b (millions)	10	20	40	80	160



Things to think about:

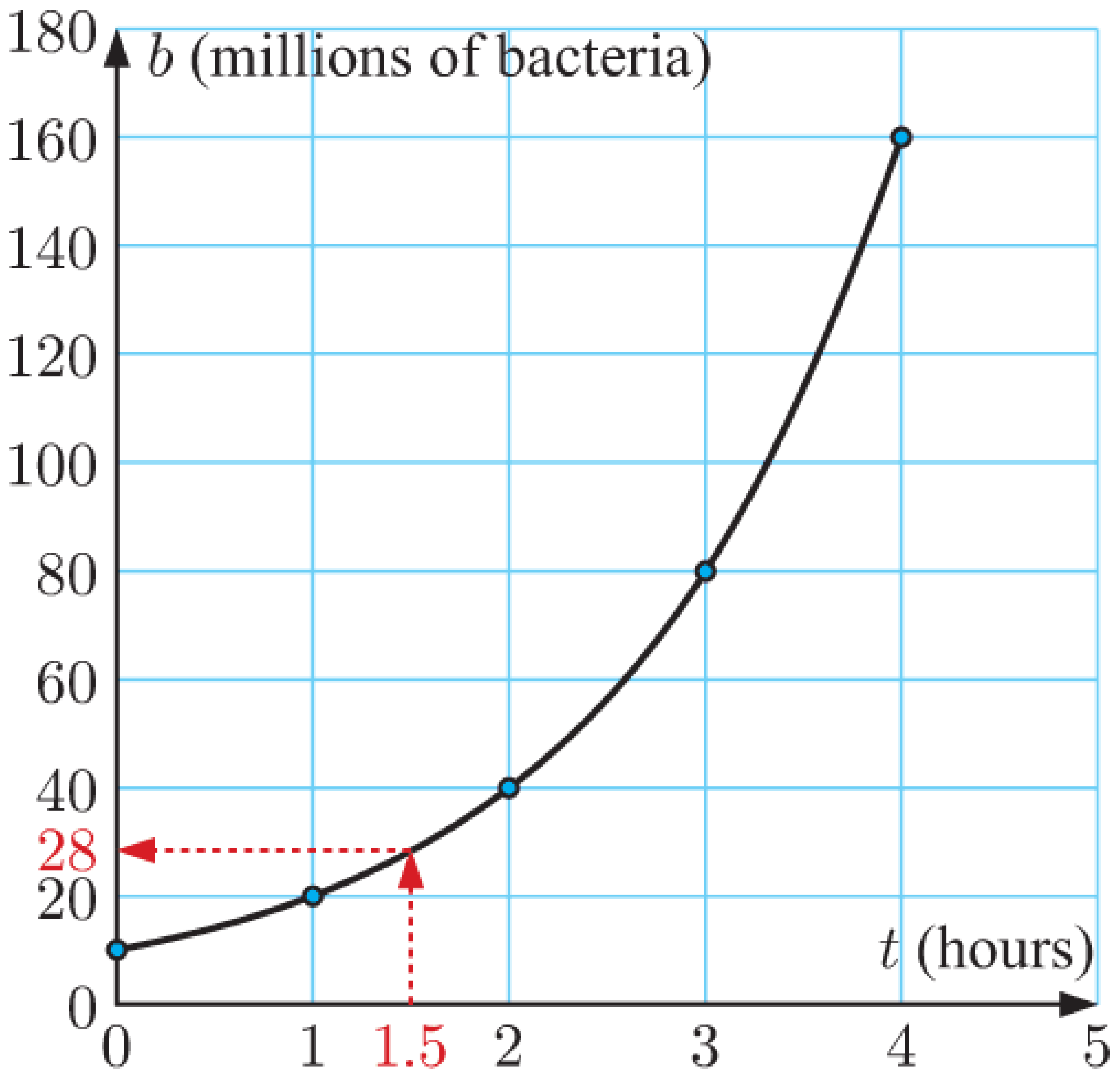
- **a** What type of sequence do these values form?
- **b** If b is plotted against t, what shape do these points form on a graph?
- **c** Is it reasonable to connect the points with a smooth curve?
- **d** What function can be used to model the bacteria population over time?

The growth of bacteria in the **Opening Problem** is shown on the graph alongside.

The bacteria do not all divide at the same time, but instead divide throughout the hour. The population is therefore continually growing, and it is reasonable to join the points with a smooth curve.

We can use the curve to estimate the population of bacteria at times other than on the hour. For example, after $1\frac{1}{2}$ hours there are about 28 million bacteria present.

We can also model the bacteria population using a function.



From our previous work on sequences, we observe that the population after each hour forms a geometric sequence with common ratio 2. The general term of the sequence is $b_t = 20 \times 2^{t-1}$, where $t \in \mathbb{N}$ is an integer number of hours.

However, we are also interested in what happens when t is not an integer.

We therefore use the function
$$\begin{aligned} b(t) &= 20 \times 2^{t-1} \\ &= 10 \times 2 \times 2^{t-1} \\ &= 10 \times 2^t \end{aligned}$$

We call this an **exponential function**.

In this Chapter we study exponential functions and their applications in growth and decay. We also study **logarithms** which are the inverse of exponential functions.

A EXPONENTIAL FUNCTIONS

An **exponential function** is a function in which the variable appears in the exponent.

For example, $f(x) = 5^x$ and $g(x) = 3 \times 2^{-x} + 1$ are exponential functions.

Example 1 **Self Tutor**

For the function $f(x) = 3^x + 5$, find:

a $f(6)$ **b** $f(0)$ **c** $f(-2)$

a $f(6) = 3^6 + 5$
$= 729 + 5$
$= 734$

b $f(0) = 3^0 + 5$
$= 1 + 5$
$= 6$

c $f(-2) = 3^{-2} + 5$
$= \frac{1}{3^2} + 5$
$= 5\frac{1}{9}$

EXERCISE 8A

1 Decide whether each function is an exponential function:

a $y = 3^x$ **b** $f(x) = 4^{2x} - 1$ **c** $y = x^3 - 2$

d $f(x) = 7 - 2^{-x}$ **e** $g(x) = \sqrt{x} + 5$ **f** $f(x) = -3 \times 5^{\frac{x}{2}}$

2 If $f(x) = 2^x - 3$, find:

a $f(2)$ **b** $f(1)$ **c** $f(0)$ **d** $f(-1)$ **e** $f(-2)$

3 If $f(x) = 5 \times 3^x$, find:

a $f(1)$ **b** $f(3)$ **c** $f(0)$ **d** $f(-4)$ **e** $f(-1)$

4 If $f(x) = 2 \times 2^x$, find:

a $f(4)$ **b** $f(0)$ **c** $f(1)$ **d** $f(-1)$ **e** $f(-5)$

5 If $g(x) = 5^{-x}$, find:

a $g(1)$ **b** $g(3)$ **c** $g(0)$ **d** $g(-2)$ **e** $g(-3)$

6 If $h(x) = 3 \times (1.1)^x$, use your calculator to evaluate:

a $h(0)$ **b** $h(1)$ **c** $h(5)$ **d** $h(-2)$ **e** $h(3.8)$

7 Determine whether the given point satisfies the exponential function:

a $y = 2^x + 1$ $(3, 9)$ **b** $f(x) = 5^{2x}$ $(1, 5)$

c $f(x) = 3^{-x} - 2$ $(0, -1)$ **d** $y = 6 \times 2^x$ $(-1, 3)$

e $f(x) = -4 \times 3^x + 1$ $(2, -13)$ **f** $y = 4^{-3x} + 2$ $(-1, 66)$

8 For the function $f(x) = 3^x - 1$, show that the axes intercepts are both zero.

9 For the function $f(x) = 2^{-x} - 8$, show that:

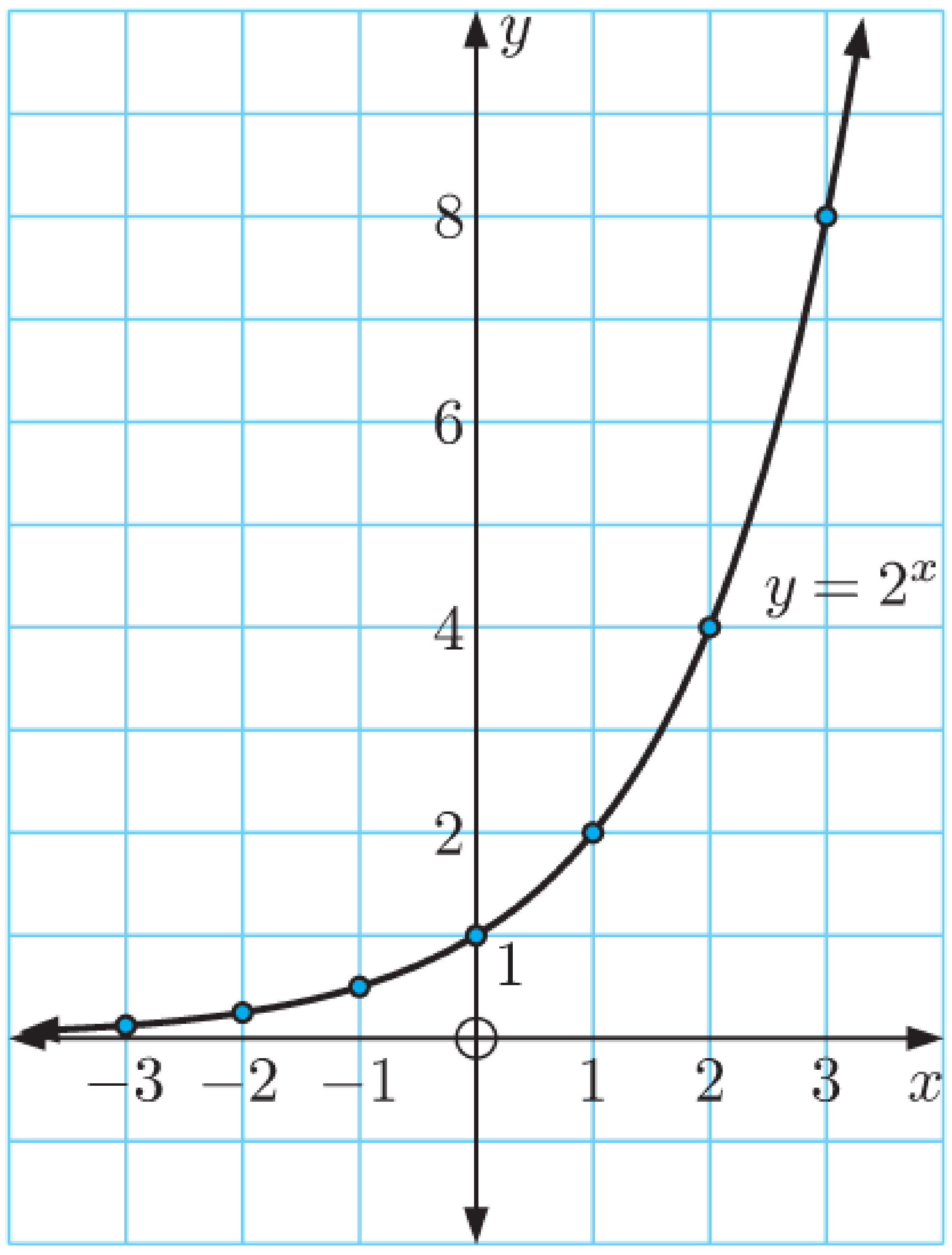
a the y-intercept is -7 **b** the x-intercept is -3.

B GRAPHING EXPONENTIAL FUNCTIONS FROM A TABLE OF VALUES

We can construct a table of values to help graph exponential functions, just as we did for quadratics.

For example, the table below gives values of $y = 2^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Drawing a smooth curve through the points, we see that as x becomes large and negative, the graph approaches the x-axis.

For example: when $x = -10$, $y = 2^{-10} \approx 0.001$

when $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

We conclude that $y = 0$ is a **horizontal asymptote** of the function.

We write "as $x \to -\infty$, $y \to 0$" to mean "as x approaches minus infinity, y approaches zero".

We also notice that as x becomes large and positive, y also becomes large and positive. We write "as $x \to \infty$, $y \to \infty$".

EXERCISE 8B

1 Consider the exponential function $y = 4^x$.

a Copy and complete the table of values.

x	-3	-2	-1	0	1	2	3
y			$\frac{1}{4}$	1	4		

b Complete the following statements:

i If x is increased by 1, the value of y is ......

ii If x is decreased by 1, the value of y is ......

c Use your table of values to draw the graph of $y = 4^x$.

d Copy and complete:

i As $x \to \infty$, $y \to$ ...... **ii** As $x \to -\infty$, $y \to$ ......

e Find the horizontal asymptote of $y = 4^x$.

2 Consider the exponential function $y = \left(\frac{1}{3}\right)^x$.

a Copy and complete the table of values.

x	-3	-2	-1	0	1	2	3
y							

b Use your table of values to draw the graph of $y = \left(\frac{1}{3}\right)^x$.

c Is the graph of $y = \left(\frac{1}{3}\right)^x$ increasing or decreasing?

d Copy and complete:

i As $x \to \infty$, $y \to$ ...... **ii** As $x \to -\infty$, $y \to$ ......

e Find the horizontal asymptote of $y = \left(\frac{1}{3}\right)^x$.

3 Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3,$ then use your table to graph each function:

a $y = 2^x + 3$ **b** $y = 3^x - 4$

c $y = 4^{-x}$ **d** $y = 5 \times 2^x$

GRAPHING PACKAGE

Use the **graphing package** or your **graphics calculator** to check your answers.

4 **a** Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ for the function:

i $y = 2^{-x}$ **ii** $y = \left(\frac{1}{2}\right)^x$

Comment on your answers.

b Explain why $2^{-x} = \left(\frac{1}{2}\right)^x$

c Sketch the graph of $f(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$.

C GRAPHS OF EXPONENTIAL FUNCTIONS

INVESTIGATION 1 EXPONENTIAL GRAPHS

In this Investigation we consider the graphs of exponential functions of the form $y = ka^x + c$ and $y = ka^{-x} + c$, where $a > 0$, $a \neq 1$.

GRAPHING PACKAGE

What to do:

1 **a** On the same set of axes, graph the functions $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.

b For the family $y = a^x$ where $a > 1$:

i Is the function increasing or decreasing?

ii What is the horizontal asymptote?

iii What effect does changing a have on the shape of the graph?

2 **a** On the same set of axes, graph the functions $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{5}\right)^x$, and $y = \left(\frac{1}{10}\right)^x$.

b For the family $y = a^x$ where $0 < a < 1$:

i Is the function increasing or decreasing?

ii What is the horizontal asymptote?

iii What effect does changing a have on the shape of the graph?

3 **a** On the same set of axes, graph the functions $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.

b For the family $y = 2^x + c$ where c is a constant:

i What effect does c have on the position of the graph?

ii What effect does c have on the shape of the graph?

iii What is the horizontal asymptote of $y = 2^x + c$?

4 **a** On the same set of axes, graph the functions:

i $y = 2^x$ and $y = 2^{-x}$ **ii** $y = 5^x$ and $y = 5^{-x}$

iii $y = \left(\frac{1}{3}\right)^x$ and $y = \left(\frac{1}{3}\right)^{-x}$.

b For any graph of the form $y = a^x$ or $y = a^{-x}$ where $a > 0$, $a \neq 1$:

- **i** What is the y-intercept of each graph?
- **ii** What is the horizontal asymptote of each graph?
- **iii** Describe how the graphs of $y = a^x$ and $y = a^{-x}$ are related.

c Explain why $a^{-x} = \left(\frac{1}{a}\right)^x$ for all $a > 0$, $a \neq 1$. Hence describe how the graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are related.

5 **a** On the same set of axes, graph the functions:

- **i** $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$
- **ii** $y = -1 \times 2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$

b For the family $y = k \times 2^x$ where k is a constant:

- **i** What effect does the *sign* of k have?
- **ii** What effect does the *size* of k have?
- **iii** What is the horizontal asymptote of each graph?

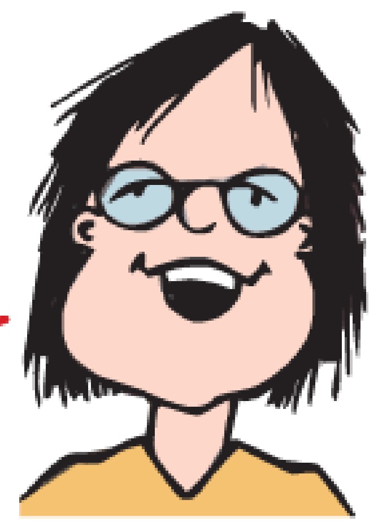
From the **Investigation** you should have discovered that:

For the general exponential functions $\boldsymbol{y = ka^x + c}$ and $\boldsymbol{y = ka^{-x} + c}$:

- c controls the vertical position of the graph.
- The horizontal asymptote is $y = c$.
- a and k both affect how steeply the graph increases or decreases.
- The sign of k determines whether the graph lies above or below the asymptote.

We can sketch the graph of an exponential function using:

- the horizontal asymptote
- the y-intercept
- two other points.



All exponential graphs have a horizontal asymptote.

Example 2 **Self Tutor**

Sketch the graph of $y = 2^{-x} - 3$. Hence state the domain and range of $f(x) = 2^{-x} - 3$.

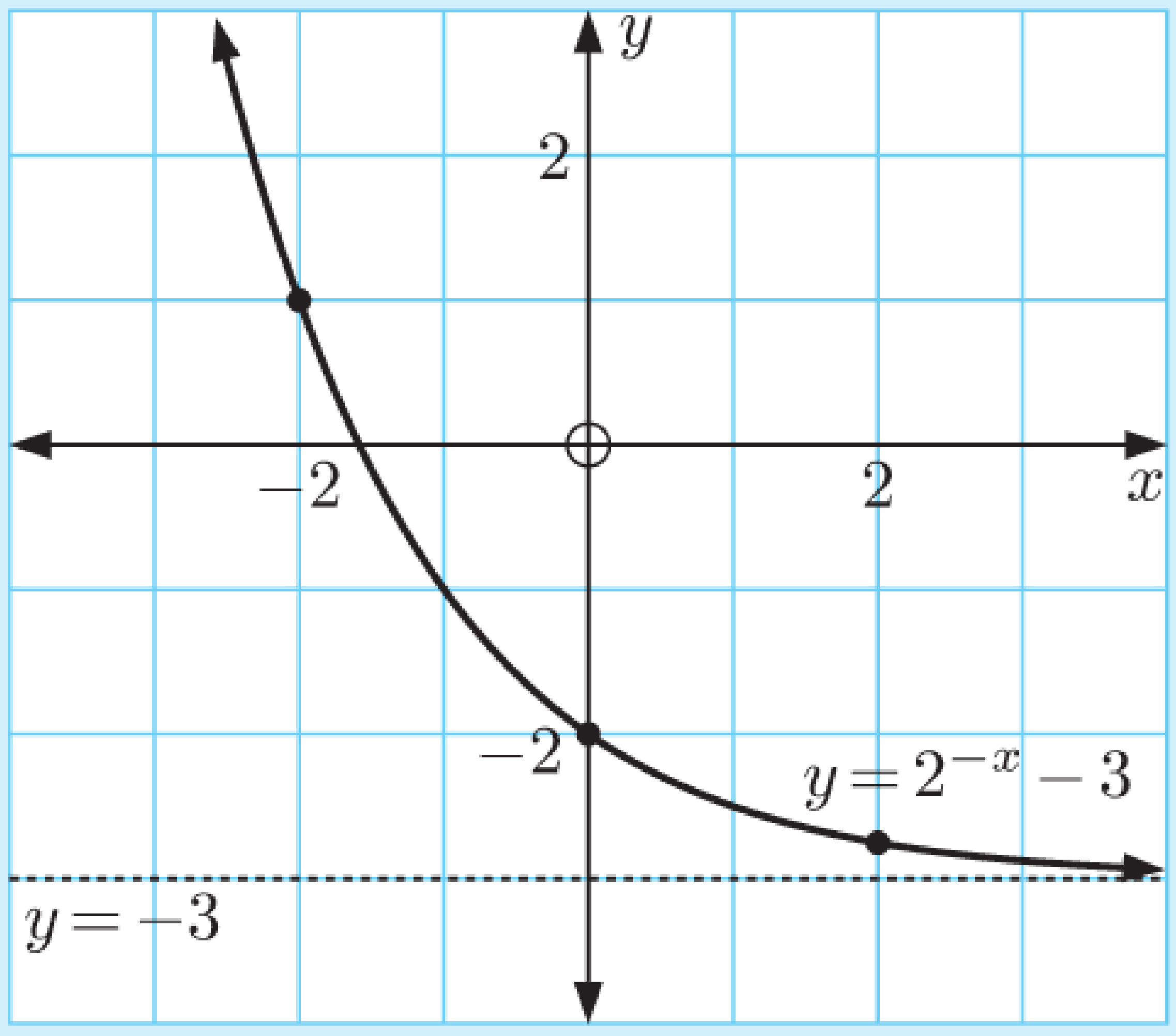
For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3 = 1 - 3 = -2$
$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3 = \frac{1}{4} - 3 = -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$

The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > -3\}$.



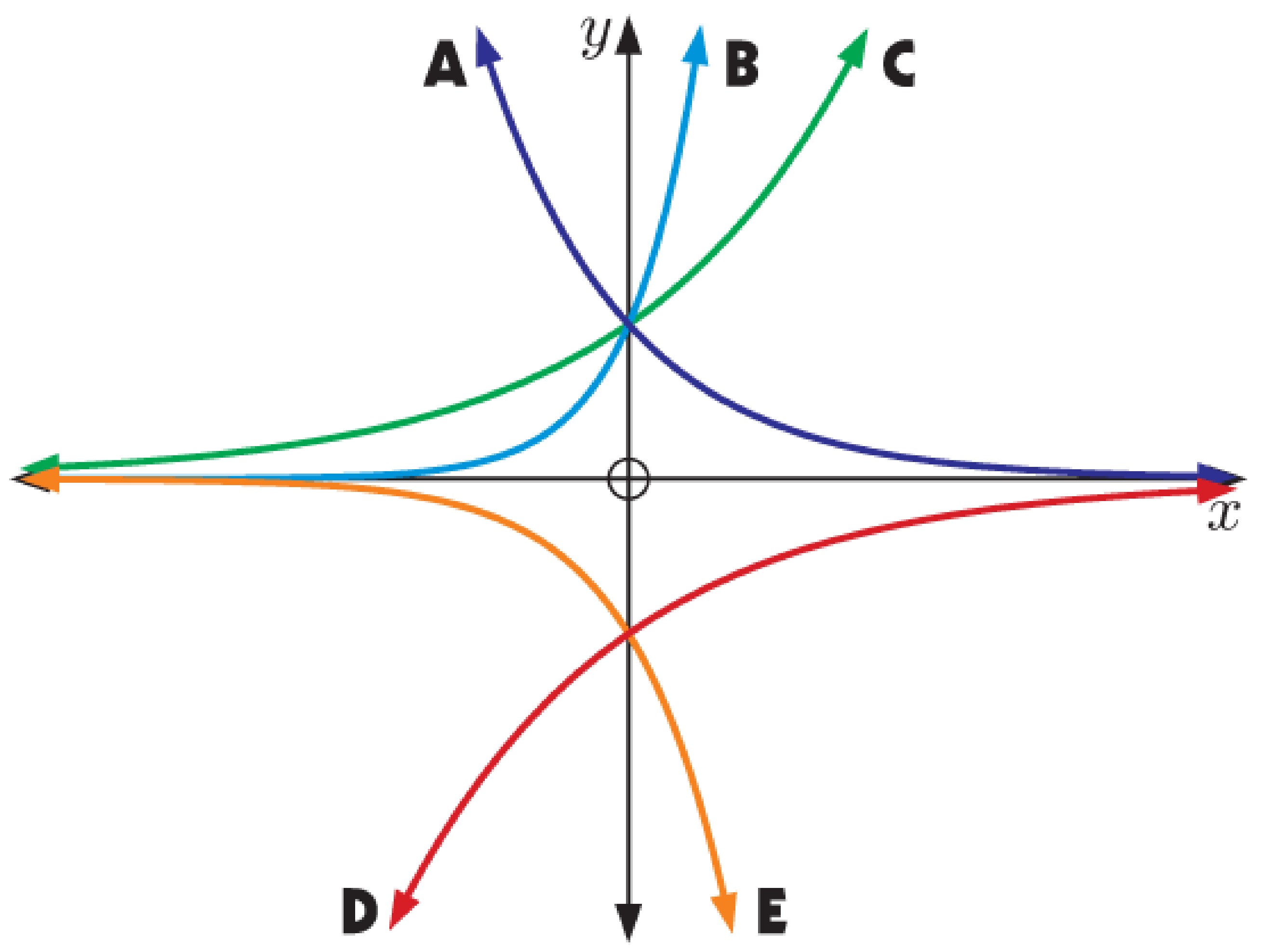
EXERCISE 8C

1 Match each function with its graph:

a $y = 2^x$ **b** $y = 10^x$

c $y = -5^x$ **d** $y = \left(\frac{1}{3}\right)^x$

e $y = -\left(\frac{1}{2}\right)^x$



2 Sketch each pair of functions on the same set of axes:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x + 3$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 5^x$ **d** $y = 2^x$ and $y = 2 \times 2^x$

3 Sketch each pair of functions on the same set of axes:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = \frac{1}{2} \times 3^x$

4 State the equation of the horizontal asymptote of:

a $y = 5^x - 1$ **b** $y = 2^{-x} + 4$ **c** $y = 3 \times 4^x + 1$ **d** $y = -\left(\frac{1}{2}\right)^x - 5$

5 Find the y-intercept of:

a $f(x) = 3^x + 4$ **b** $f(x) = 6^{-x} - 2$

c $f(x) = 3 \times 2^x + 7$ **d** $f(x) = -\frac{1}{2} \times 5^x + 6$

6 Consider the exponential function $f(x) = 3^x - 2$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

7 Consider the function $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$.

a Find: **i** $g(0)$ **ii** $g(2)$ **iii** $g(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

8 For each of the following exponential functions:

i Calculate the y-intercept. **ii** State the equation of the horizontal asymptote.

iii Find the values of y when $x = 2$ and $x = -2$.

iv Sketch the function using the information from **i** to **iii**.

v State the domain and range of the function.

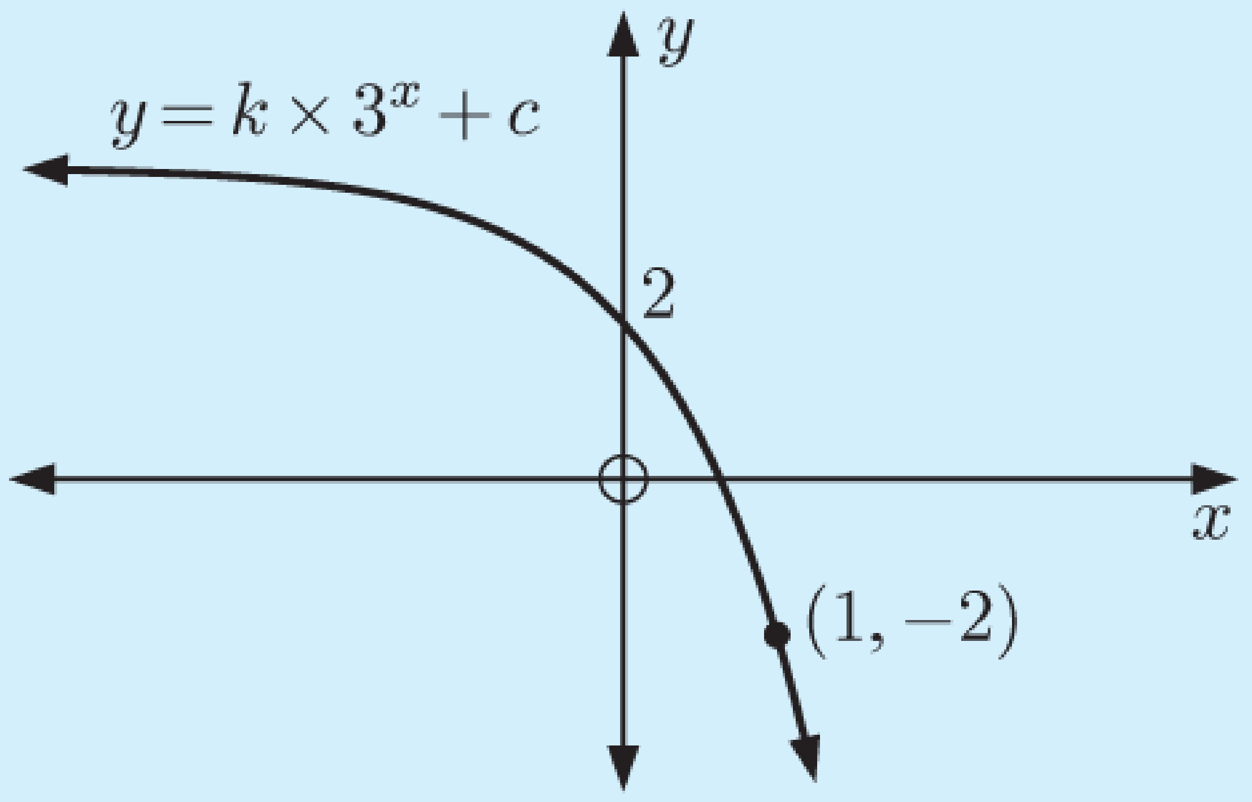
a $y = 2^x + 1$ **b** $y = 3^{-x} + 4$ **c** $y = \left(\frac{2}{5}\right)^x$ **d** $y = \left(\frac{1}{2}\right)^x - 3$

e $y = 2 - 2^x$ **f** $y = 4^{-x} + 3$ **g** $y = 3 - 2^{-x}$ **h** $y = -\frac{1}{2} \times 3^{-x} + 1$

Example 3 **Self Tutor**

This graph shows the curve $y = k \times 3^x + c$, where k and c are constants.

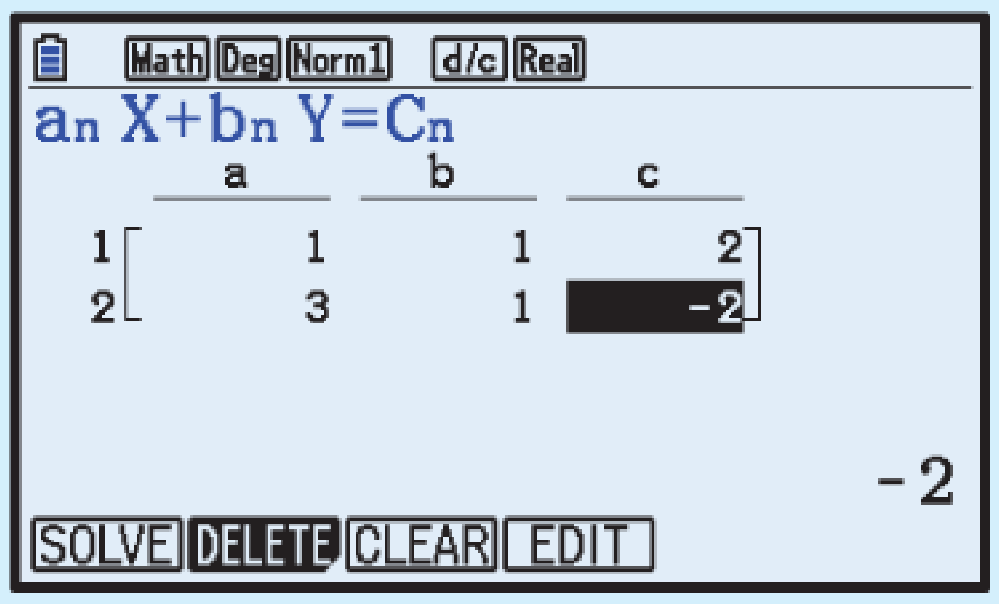
Find the values of k and c.



Substituting $(0, 2)$ into the equation gives $2 = k \times 3^0 + c$ $\quad \therefore \quad k + c = 2$

Substituting $(1, -2)$ into the equation gives $-2 = k \times 3^1 + c$ $\quad \therefore \quad 3k + c = -2$

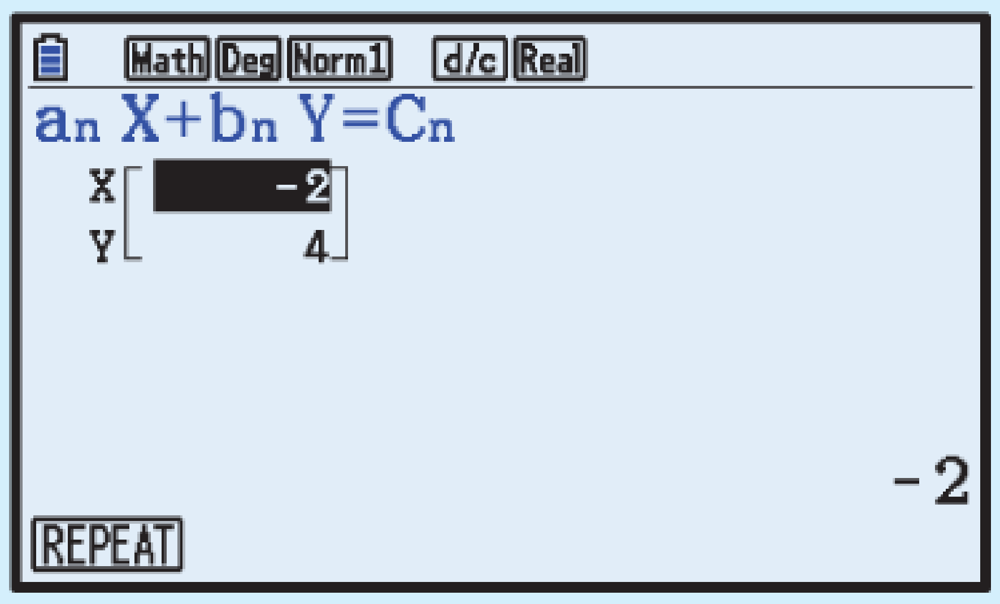
Solving the system of equations $\begin{cases} k + c = 2 \\ 3k + c = -2 \end{cases}$ simultaneously gives $k = -2$, $c = 4$.



$a_n X + b_n Y = C_n$

$$\begin{array}{c|ccc} & a & b & c \\ \hline 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & -2 \end{array}$$

-2



$a_n X + b_n Y = C_n$

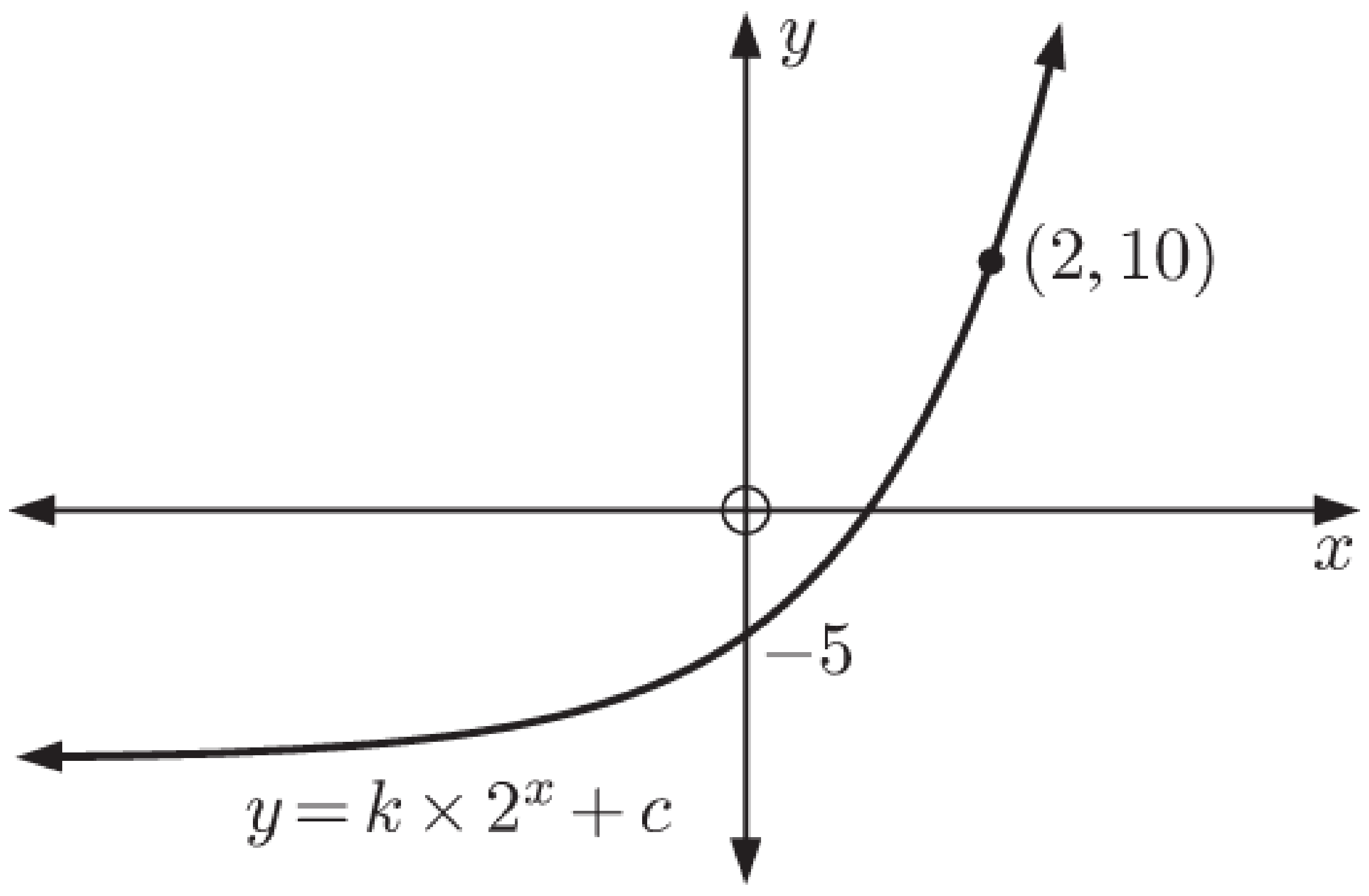
X -2

Y 4

-2

9 The graph alongside shows the curve $y = k \times 2^x + c$, where k and c are constants.

- **a** Find the values of k and c.
- **b** Find y when $x = 6$.



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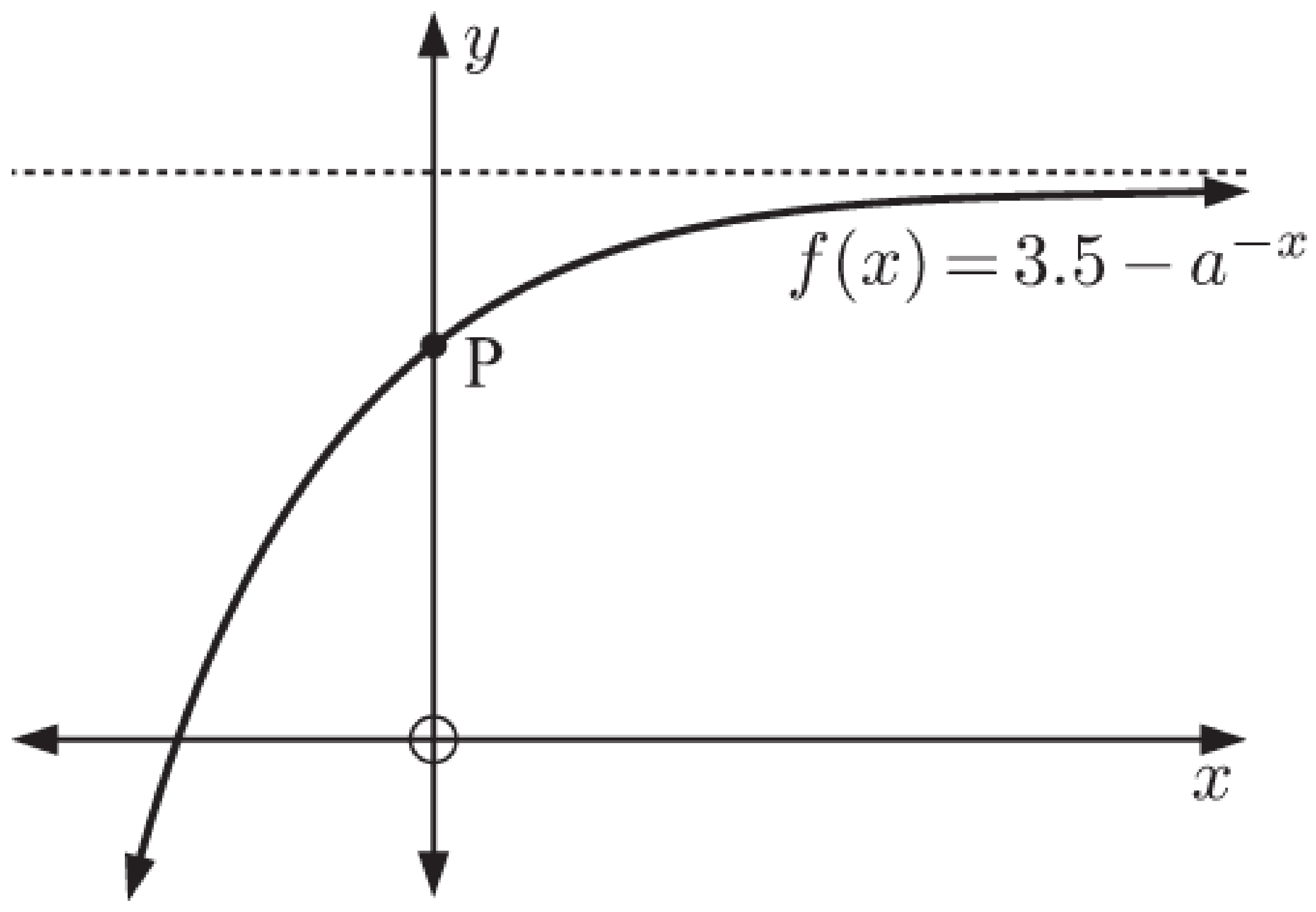


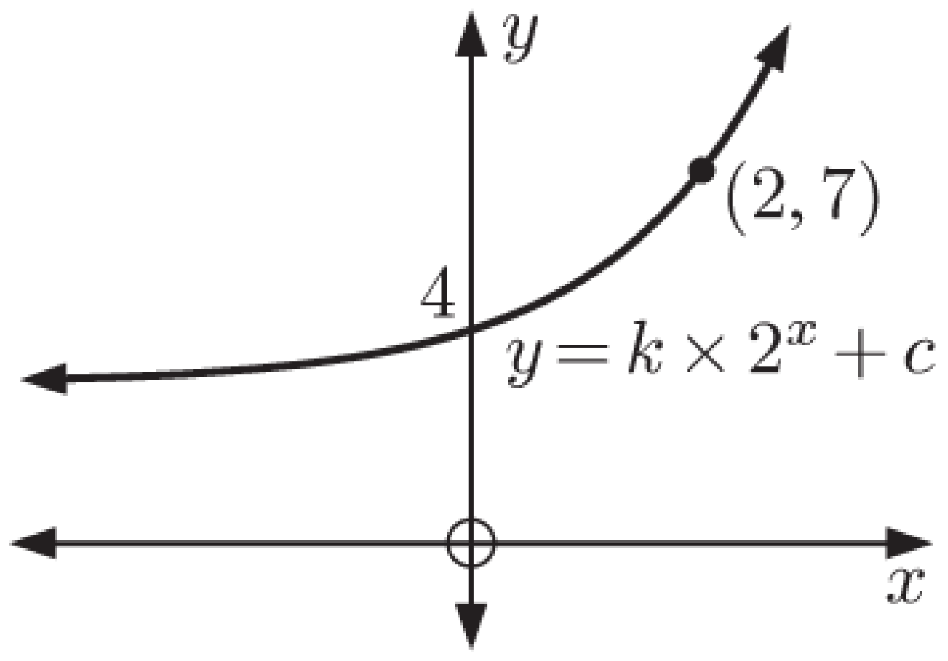
This graph shows the function $f(x) = 3.5 - a^{-x}$, where a is a positive constant.

The point $(-1, 2)$ lies on the graph.

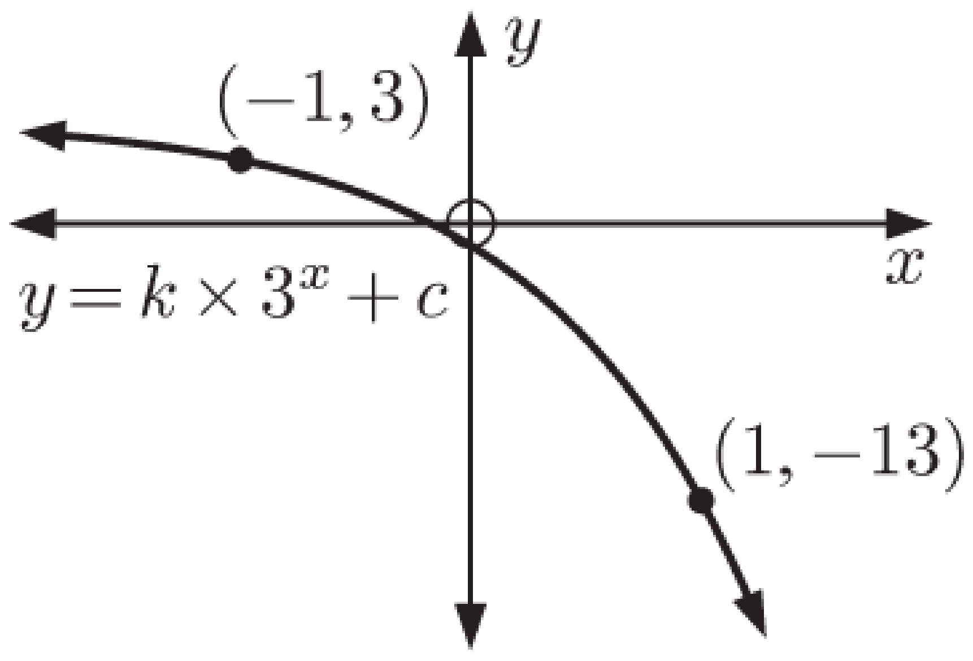
- **a** Write down the coordinates of the y-intercept, P.
- **b** Find the value of a.
- **c** Find the equation of the horizontal asymptote.

11 Find the exponential model in each graph:

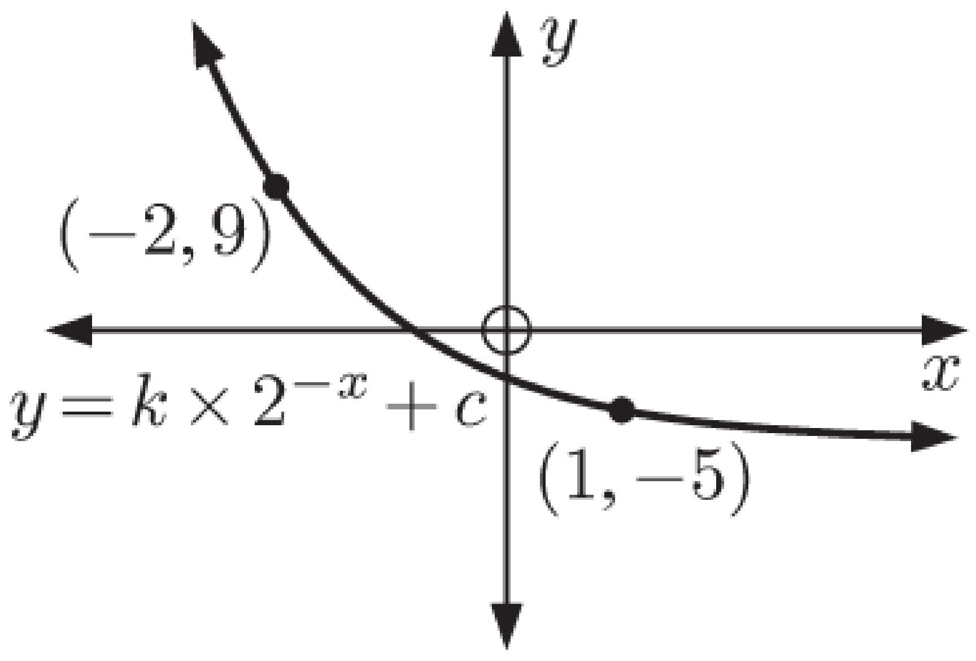
a

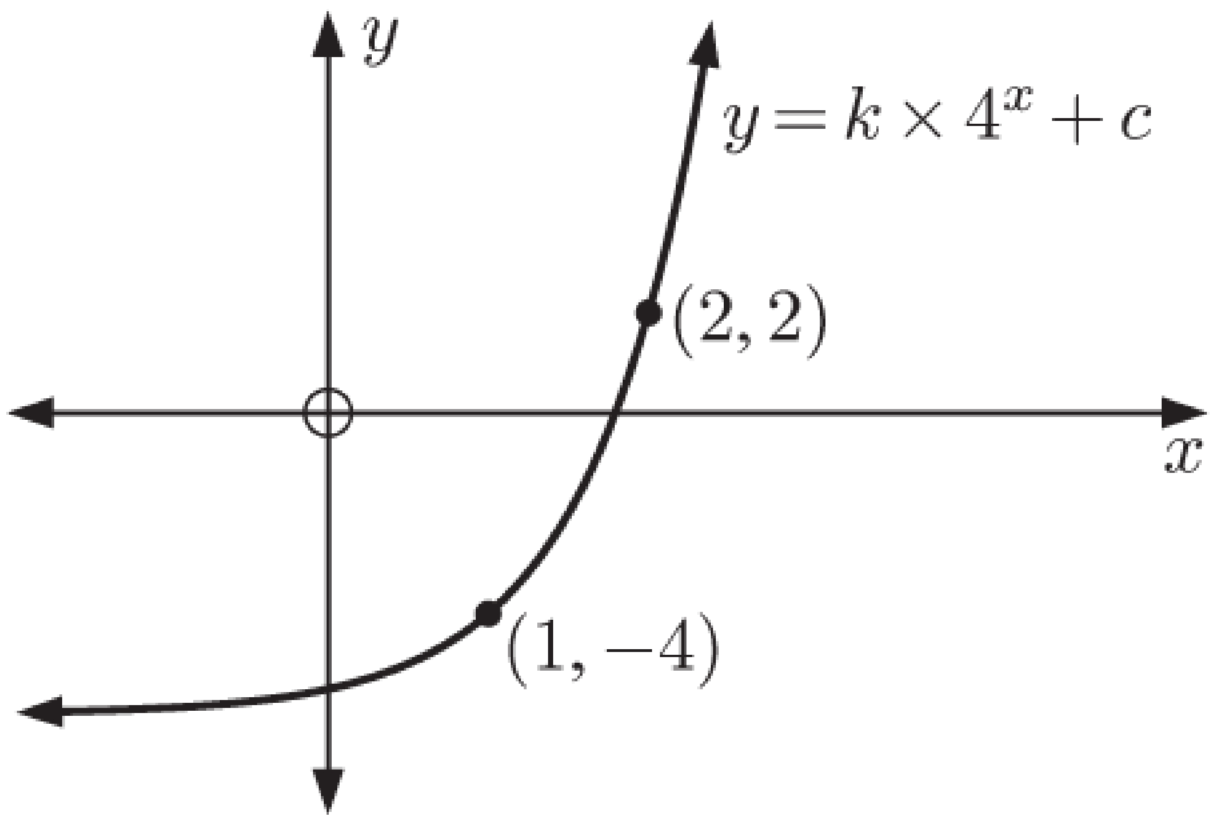


b



c





12 The graph of $y = k \times 4^x + c$ is shown alongside.

- **a** Explain why:
  - **i** k must be positive
  - **ii** c must be negative.
- **b** Find the equation of the exponential function.
- **c** Find the y-intercept.
- **d** State the horizontal asymptote.

13 An exponential function of the form $f(x) = k \times a^x + c$ has y-intercept 10, horizontal asymptote $y = 2$, and passes through the point $(5, 258)$. Find the exponential function.

14 An exponential function of the form $f(x) = k \times a^{-x} + c$ has y-intercept 1, horizontal asymptote $y = 4$, and passes through the point $(1, 2)$. Find the exponential function.

DISCUSSION

- For the exponential function $y = a^x$, why do we choose to specify:
  - **a** $a \neq 1$
  - **b** $a > 0$?
- What does the graph of $y = (-2)^x$ look like? What is its domain and range?

D EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the exponent or index.

For example, $2^x = 50$ and $7^{1-x} = 40$ are both exponential equations.

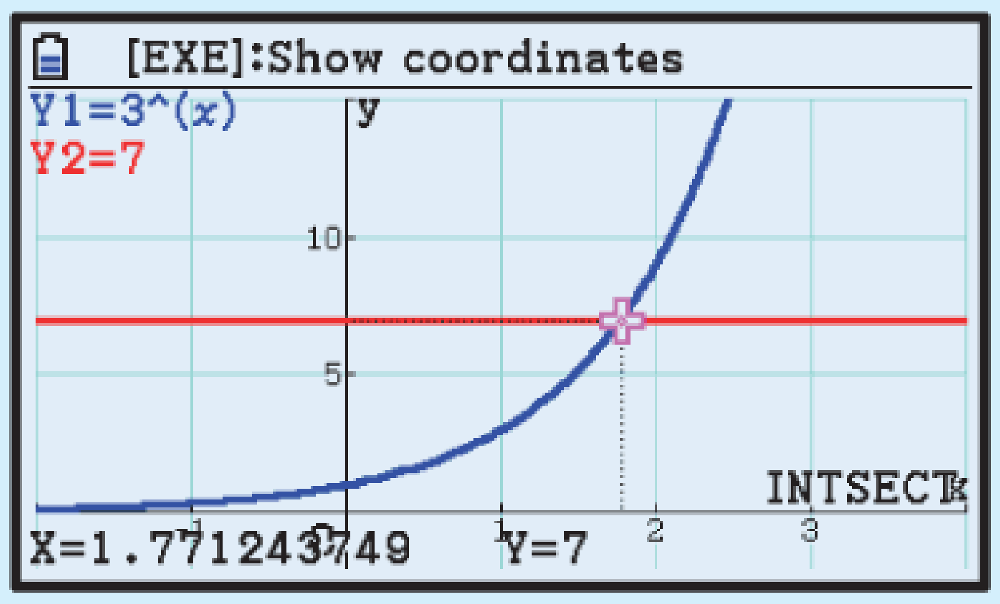
To solve an exponential equation, we graph each side on the same set of axes. The x-coordinate of the intersection point is the solution to the equation.

Example 4 **Self Tutor**

Use technology to solve $3^x = 7$.

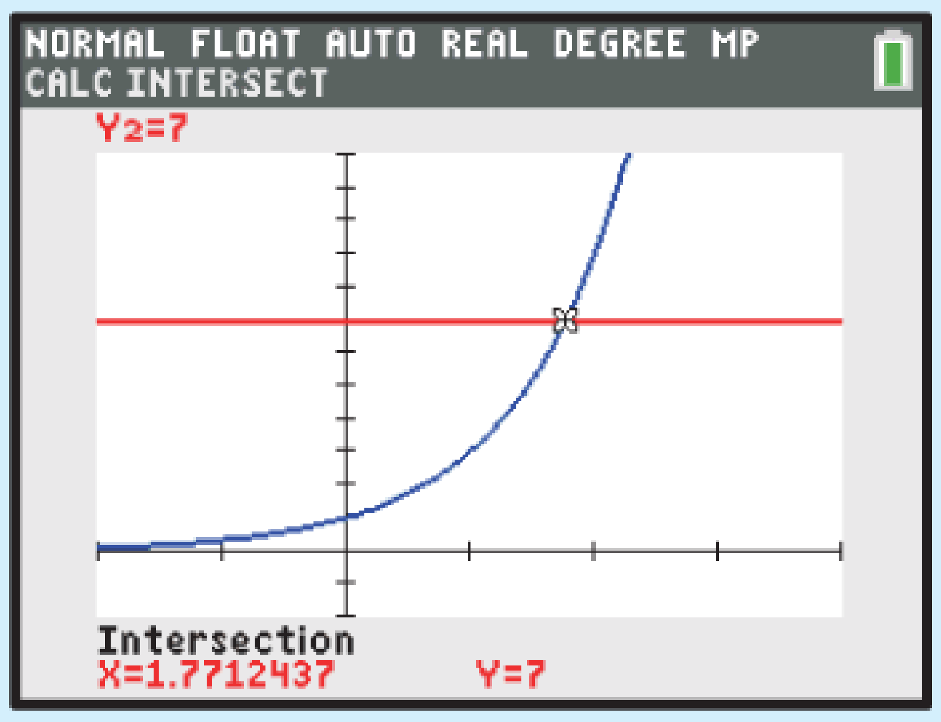
We graph $\text{Y}_1 = 3^\text{X}$ and $\text{Y}_2 = 7$ on the same set of axes, and find their point of intersection.

Casio fx-CG50



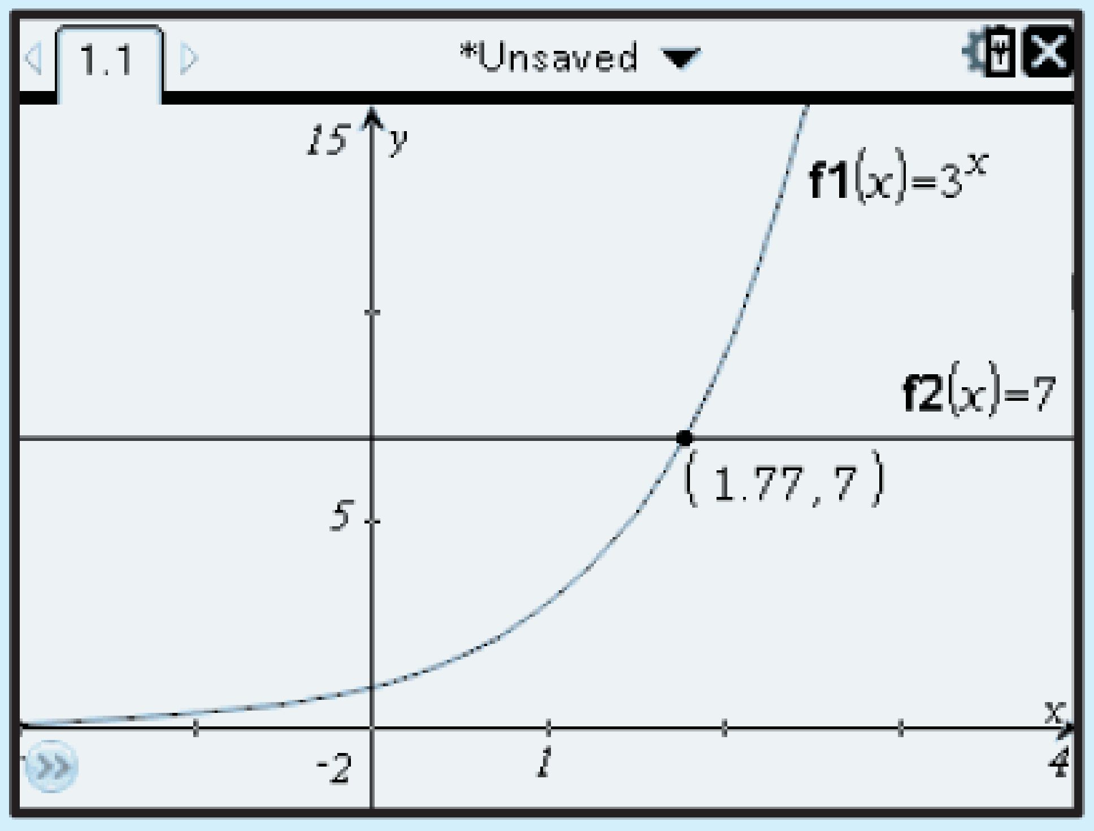
```
[EXE]:Show coordinates
Y1=3^(x)
Y2=7
INTSECT
X=1.771243749   Y=7
```

TI-84 Plus CE



```
NORMAL FLOAT AUTO REAL DEGREE MP
CALC INTERSECT
Y2=7
Intersection
X=1.7712437      Y=7
```

TI-*n*spire



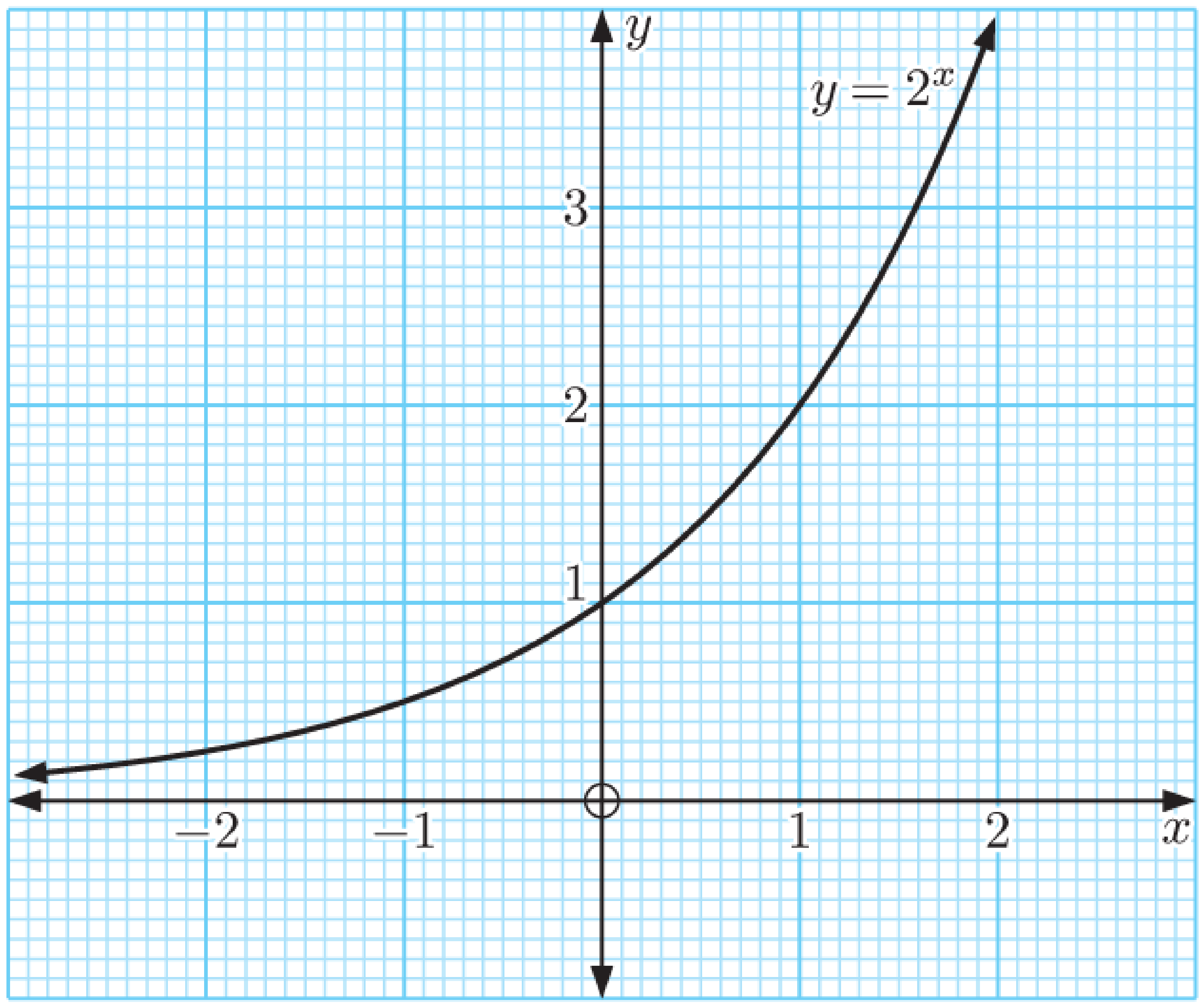
The solution is $x \approx 1.77$.

EXERCISE 8D

1 **a** Use the given graph of $y = 2^x$ to estimate the solution to:

  i $2^x = 3$ **ii** $2^x = 0.6$

b Use technology to check your answer.



2 Solve using technology:

a $2^x = 20$ **b** $4^x = 100$ **c** $3^x = 30$

d $(1.2)^x = 3$ **e** $(1.04)^x = 4.238$ **f** $(0.9)^x = 0.5$

3 Solve using technology:

a $3 \times 2^x = 93$ **b** $40 \times (0.8)^x = 10$ **c** $8 \times 3^x = 120$

d $21 \times (1.05)^x = 34$ **e** $500 \times (0.95)^x = 350$ **f** $250 \times (1.125)^x = 470$

g $3^x + 5 = 50$ **h** $60 + 10 \times (1.5)^x = 80$ **i** $20 + 80 \times (0.75)^x = 30$

4 For what values of k does the equation $10 - 8 \times (0.5)^x = k$ have:

a 1 solution **b** no solutions?

E GROWTH AND DECAY

In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.

Populations of animals, people, and bacteria usually grow exponentially until they become limited by resources. We also see exponential growth in the value of items which *appreciate* over time.

We observe exponential *decay* in radioactive substances, in the temperature of objects as they cool, and in the value of items which *depreciate* over time.

For the exponential function $y = k \times a^x + c$ where $a, k > 0$, $a \neq 1$, we see:

- growth if $a > 1$
- decay if $0 < a < 1$.

GROWTH

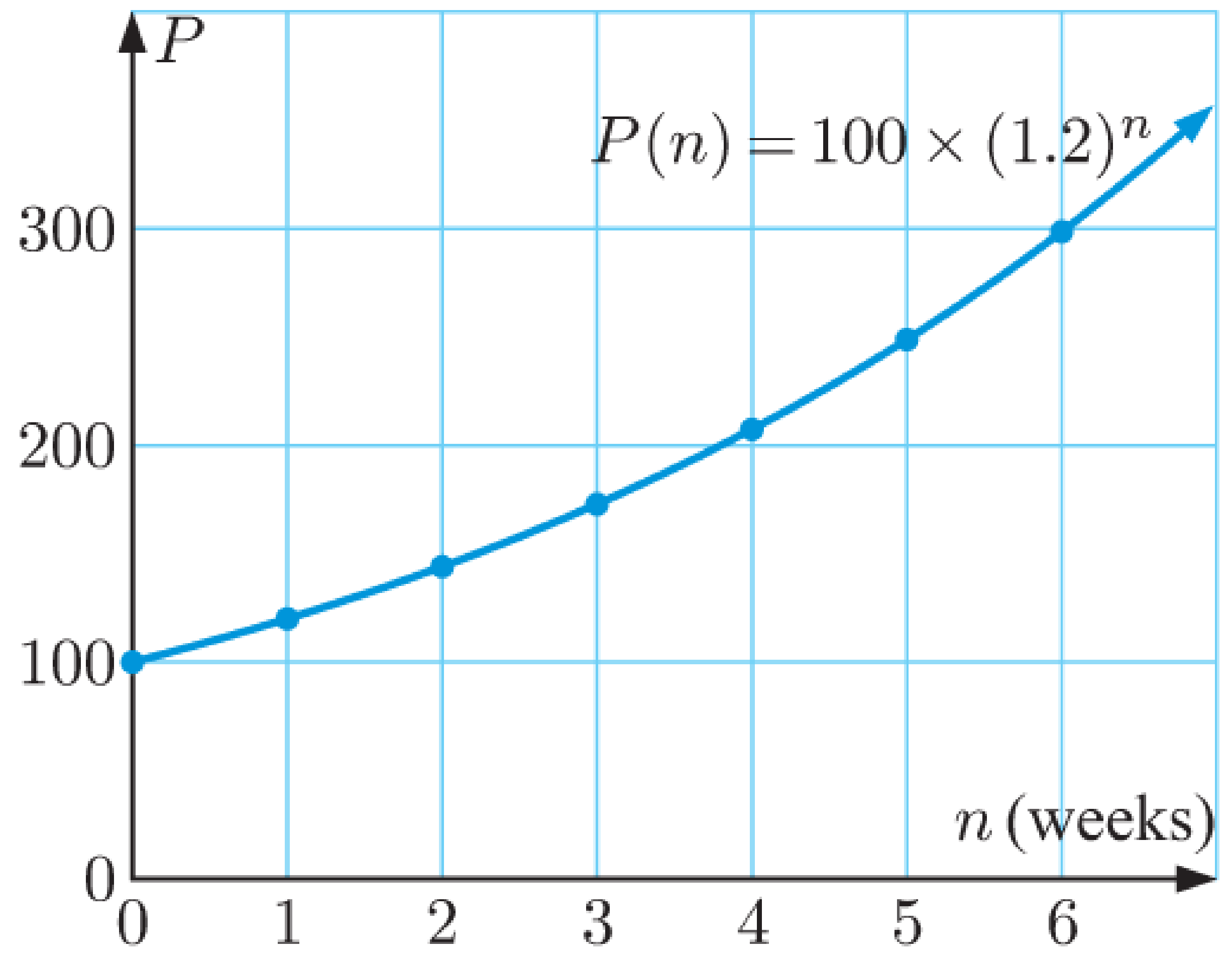
Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}
$P_1 = P_0 \times 1.2 = 100 \times 1.2$
$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$
$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$, $n \in \mathbb{Z}$, which is a geometric sequence.



However, while the population of mice must always be an integer, we expect that the population will grow continuously throughout the year, rather than in big, discrete jumps. We therefore expect it will be well approximated by the corresponding exponential function $P(n) = 100 \times (1.2)^n$, $n \in \mathbb{R}$.

Example 5 **Self Tutor**

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after: **i** 5 weeks **ii** 10 weeks.

c Draw the graph of the affected area over time.

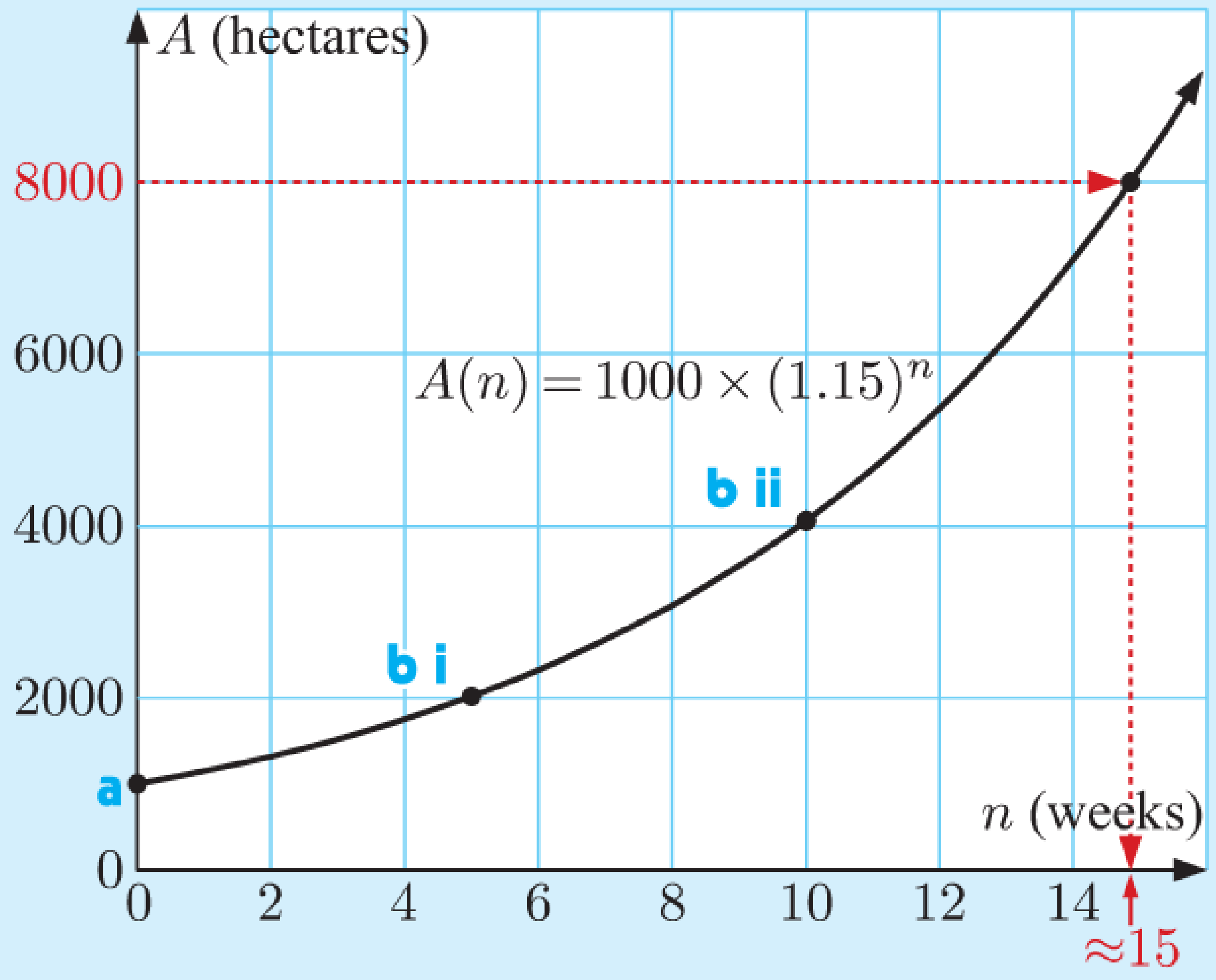
d Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

a $A(0) = 1000 \times 1.15^0 = 1000$
$\therefore$ the original affected area was 1000 hectares.

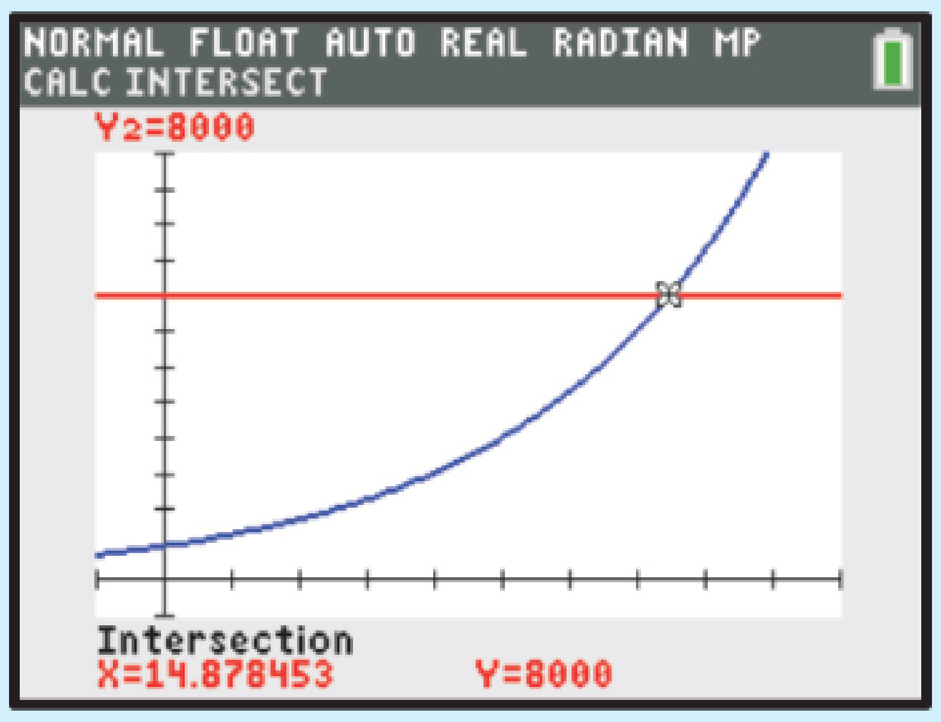
b **i** $A(5) = 1000 \times 1.15^5 \approx 2010$
The affected area is about 2010 hectares.

ii $A(10) = 1000 \times 1.15^{10} \approx 4050$
The affected area is about 4050 hectares.

c



d From the graph in **c**, it appears that it would take about 15 weeks for the affected area to reach 8000 hectares.
or Using technology, the solution is ≈ 14.9 weeks.


```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y2=8000
Intersection
X=14.878453        Y=8000
```

EXERCISE 8E.1

1 A weed in a field covers an area of $A(t) = 3 \times (1.08)^t$ square metres after t days.

a Find the initial area covered by the weed.

b By what percentage does the area increase each day?

c Find the area covered after:

i 2 days **ii** 10 days **iii** 30 days.

d Sketch the graph of $A(t)$ using the results of **a** and **c** only.

Use technology to graph $Y_1 = 3 \times (1.08)^X$ and hence check your answers.

GRAPHING PACKAGE

$a > 1$ indicates growth.

2 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times (1.07)^t$ grams.

a Find the initial weight.

b Interpret the value 1.07 in the model.

c Find the weight after:

i 4 hours **ii** 10 hours **iii** 24 hours.

d Sketch the graph of the bacteria weight over time using the results of **a** and **c** only.

Use technology to graph $Y_1 = 100 \times (1.07)^X$ and hence check your answers.

3 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times (1.23)^n$.

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of the population over time using **a** and **b** only.

d How long will it take for the population to reach 500?

Use technology to graph $Y_1 = 50 \times (1.23)^X$ and hence check your answers.

4 The speed of a chemical reaction is given by $V(t) = 5 \times (1.03)^t$ units, where t is the temperature in °C.

a Find the speed of the reaction at:

i 0°C **ii** 20°C.

b Find the percentage increase in speed from 0°C to 20°C.

c Draw the graph of $V(t)$ against t.

d At what temperature will the speed of the reaction reach 15 units?

5 A flu virus spreads in a school. The number of people N infected after t days is given by $N = 4 \times 1.332^t$, $t \geqslant 0$.

- **a** Find the number of people initially infected.
- **b** Calculate the number of people infected after 16 days.
- **c** There are 1200 people in the school. Estimate the time it will take for everybody in the school to catch the flu.
- **d** For what values of t is it reasonable to use this model?

6 In 1998, 200 bears were introduced to a large island off Alaska where previously there were no bears. The population increased exponentially according to $B(t) = B_0 \times a^t$, where $a > 0$ is a constant and t is the time in years since the introduction.

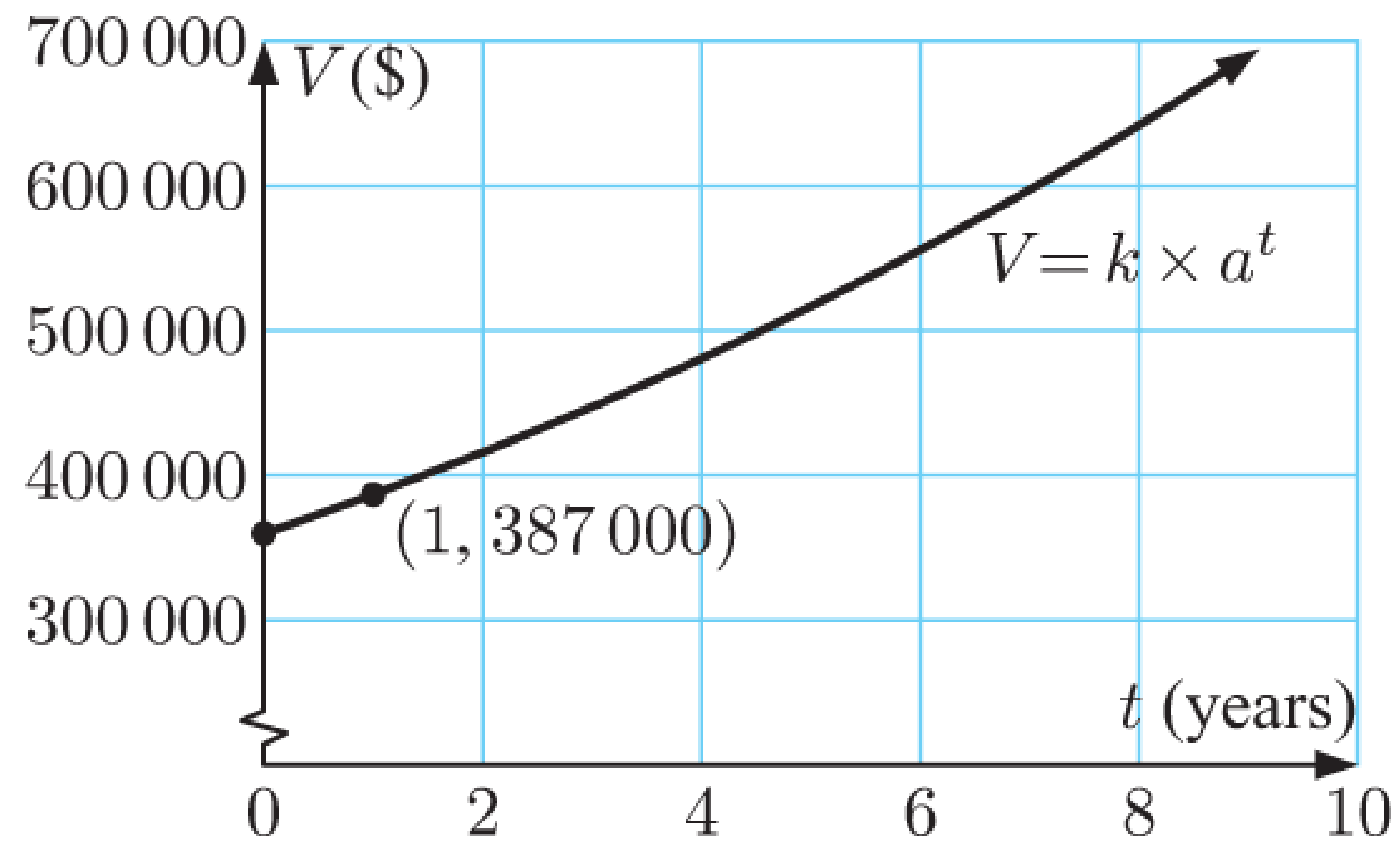
- **a** Find B_0.
- **b** In 2000 there were 242 bears. Find a, and interpret your answer.
- **c** Find the expected bear population in 2018.
- **d** Find the expected percentage increase in population from 2008 to 2018.
- **e** How long will it take for the population to reach 2000?

7 Kayla deposited €5000 into an account. The amount in the account increases by 8% each year.

- **a** Write a formula for the amount $A(t)$ in the account after t years.
- **b** Find the amount in the account after:
  - **i** 2 years
  - **ii** 5 years.
- **c** Sketch the graph of $A(t)$.

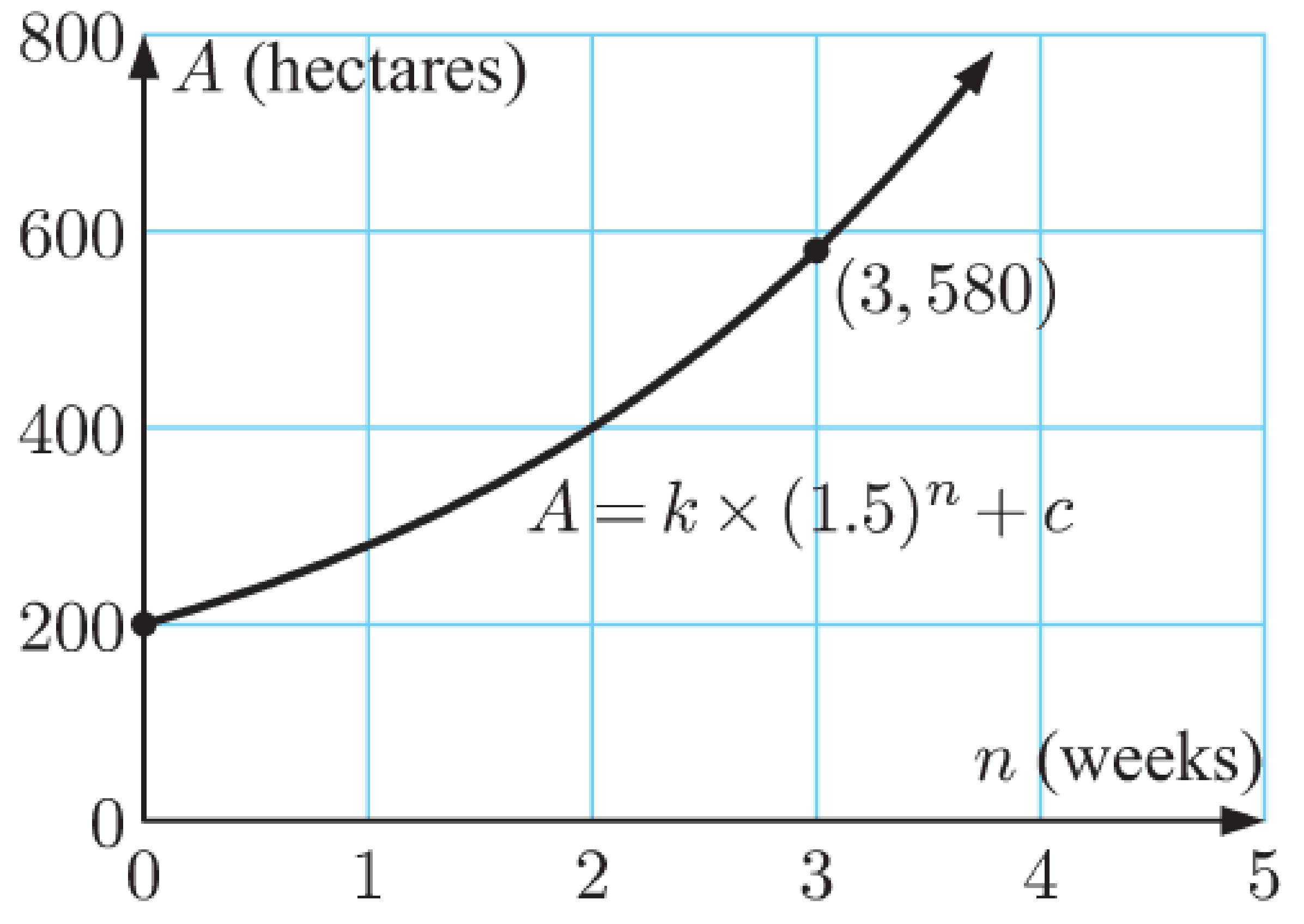
8 A house is expected to increase in value by 7.5% per year. Its expected value in t years' time is given by the exponential model $V = k \times a^t$ dollars, where $t \geqslant 0$. The model is graphed alongside.

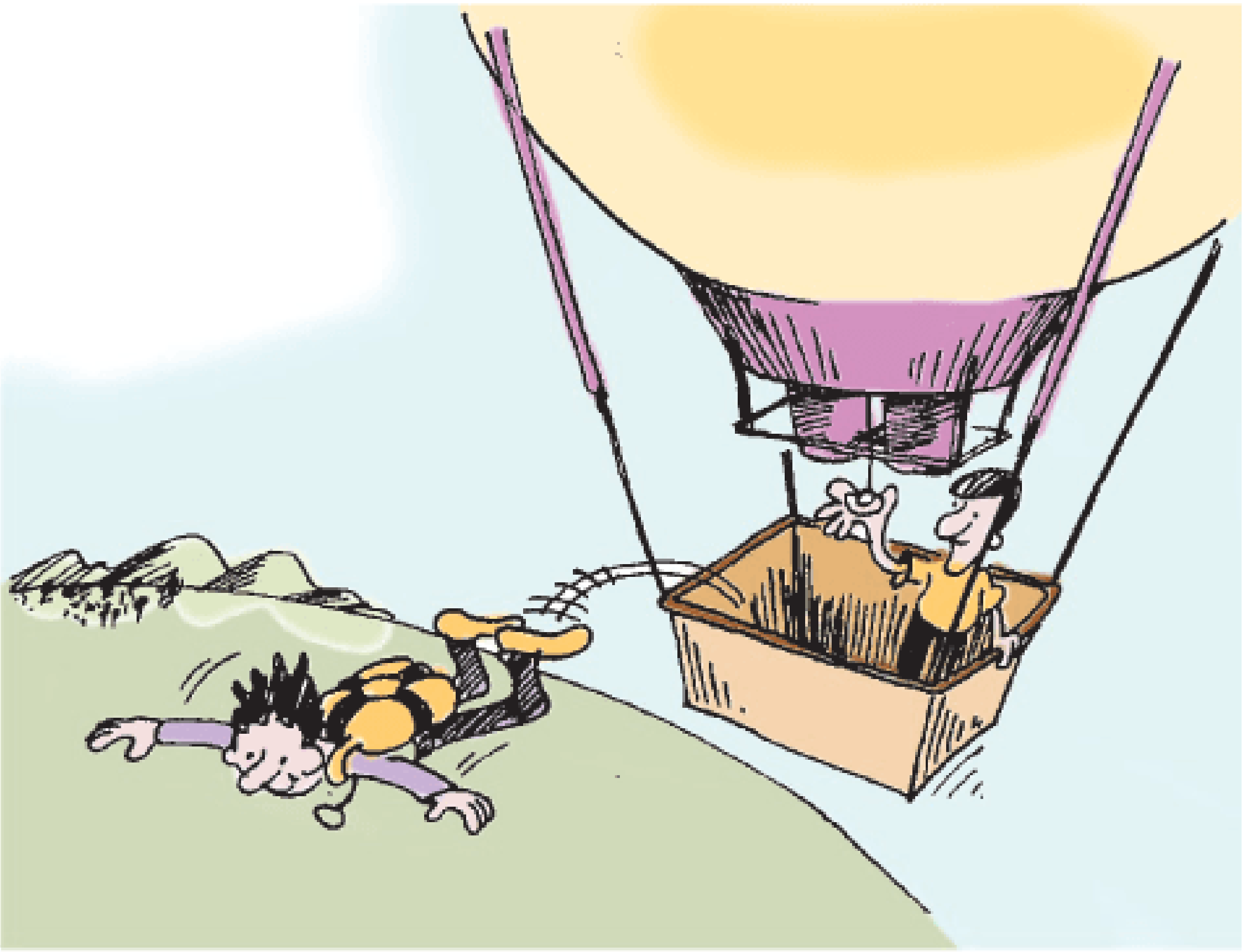
- **a** State the value of a.
- **b** Find the value of k, and interpret your answer.
- **c** How long will it take for the house's value to reach $550 000?



9 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A(n) = k \times (1.5)^n + c$ hectares, where n is the number of weeks after the initial observation, and k and c are constants.
The model is graphed alongside.

- **a** Use technology to find k and c.
- **b** Estimate the time taken for the infested area to reach 1000 hectares.





10 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = c - k \times (0.8)^t$ m s^{-1} where c and k are constants, and t is the time in seconds.

- **a** Explain why $c = k$.
- **b** After 2 seconds, the parachutist has speed 21.6 m s^{-1}. Find the exponential model connecting V and t.
- **c** Find the speed of the parachutist after 4 seconds.
- **d** Sketch the graph of V against t.
- **e** Describe how the speed of the parachutist varies over time.

PUZZLE

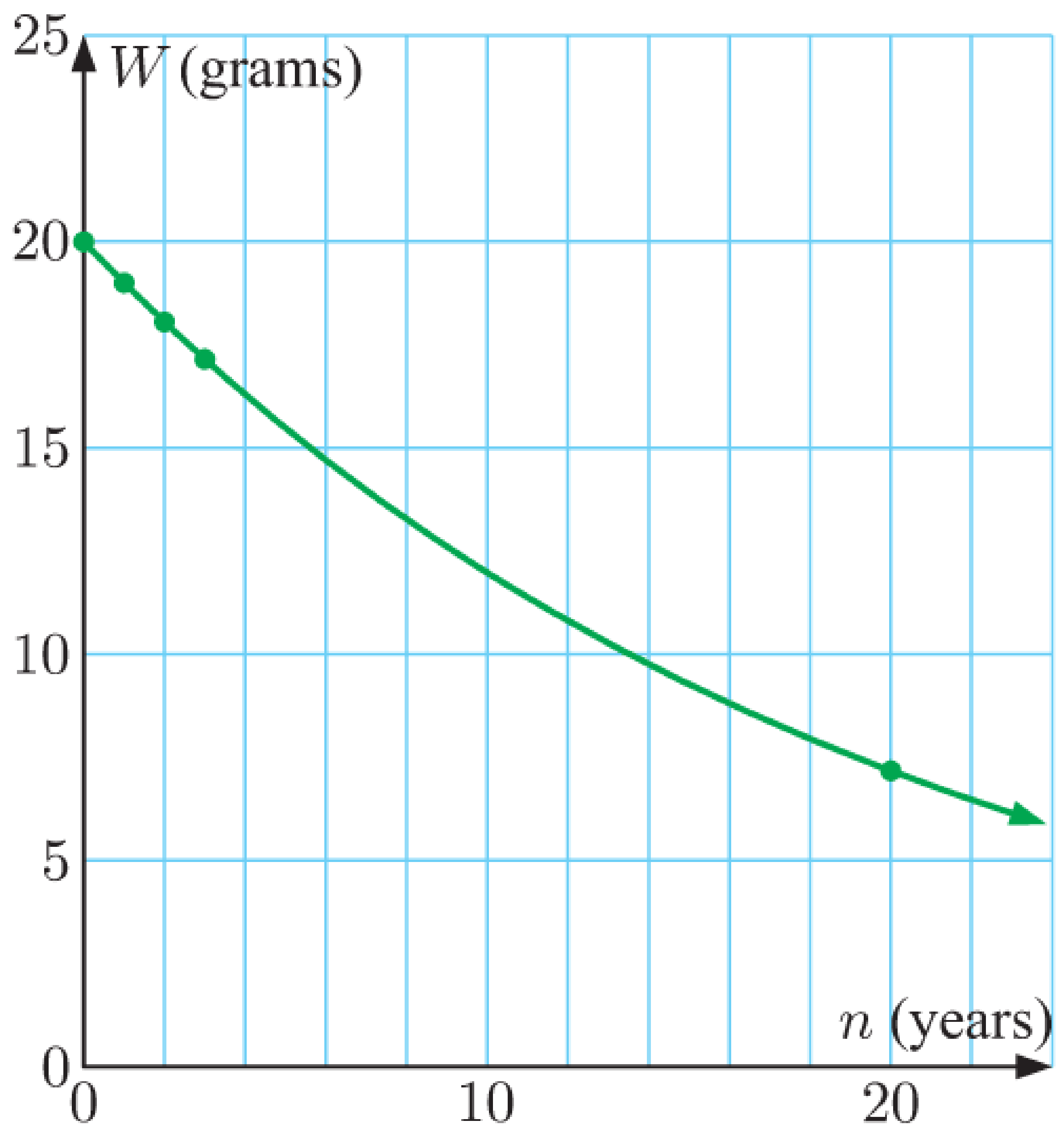
A population of insects doubles in size each week, and exhausts the food supply after 6 months. At what time does $\frac{3}{4}$ of the food remain?

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95.

If W_n is the weight after n years, then:

$$\begin{aligned}
W_0 &= 20 \text{ grams}\\
W_1 &= W_0 \times 0.95 = 20 \times 0.95 \text{ grams}\\
W_2 &= W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams}\\
W_3 &= W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams}\\
&\vdots\\
W_{20} &= 20 \times (0.95)^{20} \approx 7.2 \text{ grams}\\
&\vdots\\
W_{100} &= 20 \times (0.95)^{100} \approx 0.1 \text{ grams.}
\end{aligned}$$



From this pattern we see that $W_n = 20 \times (0.95)^n$, $n \in \mathbb{Z}$, which is again a geometric sequence.

However, we know that radioactive decay is a continuous process, so the weight remaining will actually be given by the smooth exponential curve $W(n) = 20 \times (0.95)^n$, $n \in \mathbb{R}$.

Example 6 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

- **a** Find $I(t)$ when $t = 0$, 1, 2, and 3.
- **b** What current flowed in the generator at the instant it was switched off?
- **c** Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.
- **d** Use your graph or technology to find how long it takes for the current to reach 4 amps.

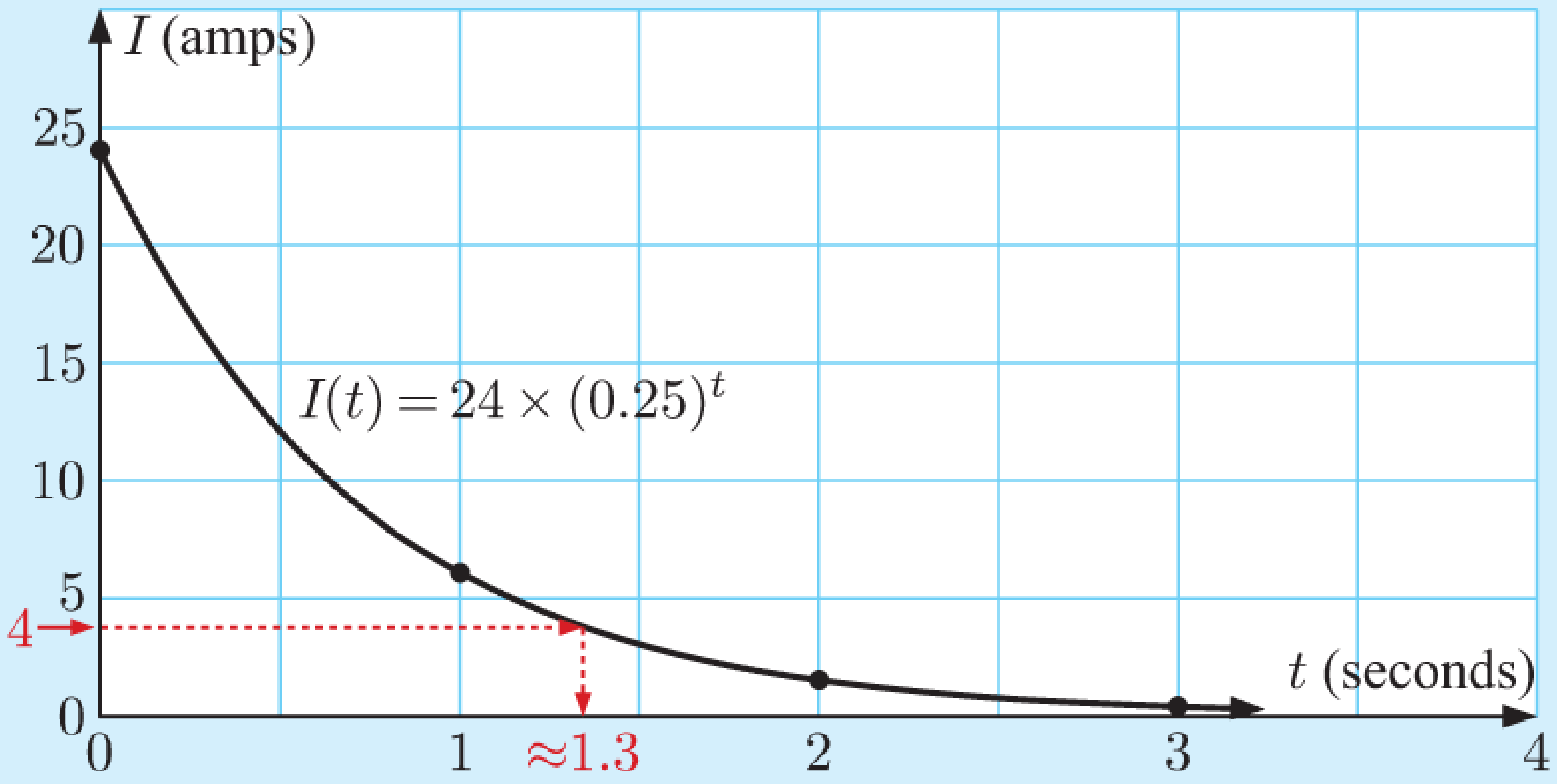
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b $I(0) = 24$

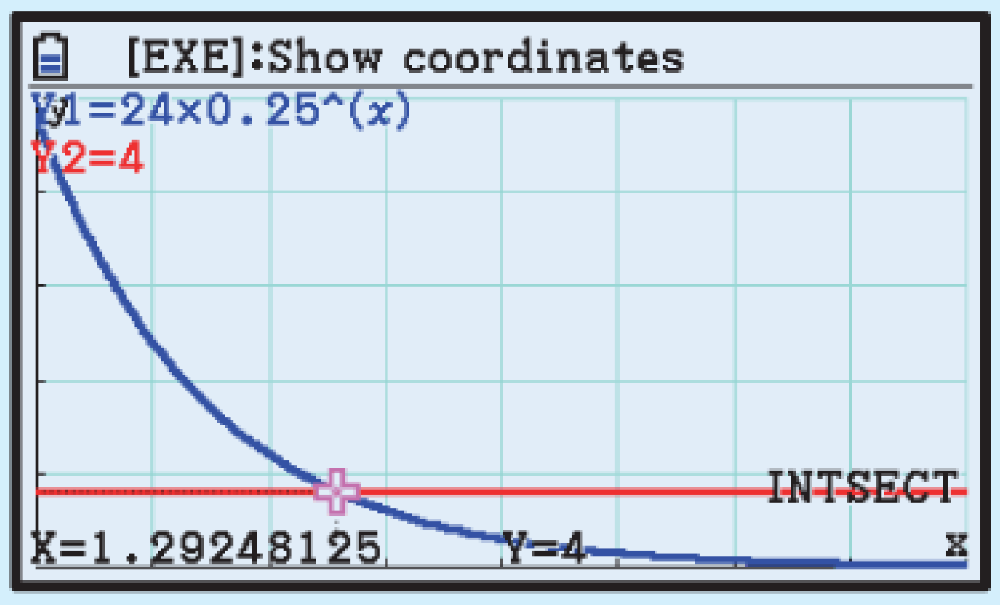
When the generator was switched off, 24 amps of current flowed in the circuit.

c



d From the graph above, the time to reach 4 amps is about 1.3 seconds.

or Using technology, the solution is ≈ 1.29 seconds.



EXERCISE 8E.2

1 When medication is taken by a patient, it is slowly used by their body. After t hours, the amount of drug remaining is given by $D(t) = 120 \times (0.9)^t$ mg.

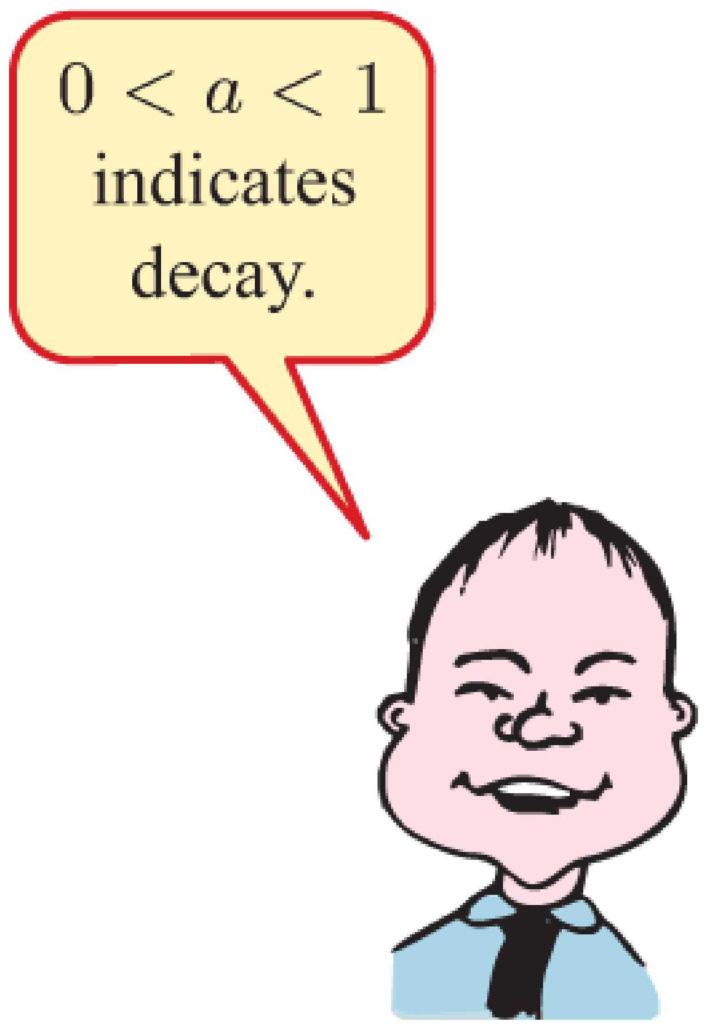
a Interpret the value 0.9 in the model.

b Find $D(t)$ when $t = 0$, 4, 12, and 24 hours.

c What was the original drug dose?

d Graph $D(t)$ against t for $t \geqslant 0$ using the information from **b**.

e Use your graph or technology to find the time when there is only 25 mg of the drug left in the body.


$0 < a < 1$ indicates decay.

2 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.

3 The current in a radio t seconds after it is switched off is given by $I(t) = 0.6 \times (0.03)^t$ amps.

a Find the initial current.

b Find the current after:

i 0.1 seconds **ii** 0.5 seconds **iii** 1 second.

c Graph $I(t)$ against t using **a** and **b** only.

4 The temperature of a liquid t minutes after it has been placed in a refrigerator, is given by $T(t) = 100 \times (1.02)^{-t}$ °C.

a Find the initial temperature.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

d How long will it take for the temperature to fall to 15°C?

5 An initial count of orangutans in a forest found that the forest contained 400 orangutans. Since then, the destruction of their habitat has caused the population to fall by 8% each year.

a Write a formula for the population P of orangutans t years after the initial count.

b Find the population of orangutans after:

i 1 year **ii** 5 years.

c Sketch the graph of the population over time.

d How long will it take for the population to fall to 200?

6 The intensity of light L diminishes below the surface of the sea according to the formula $L = 10 \times a^d$ units, where $a > 0$ is a constant and d is the depth in metres measured from the surface of the sea.

a Would you expect that $0 < a < 1$ or $a > 1$? Explain your answer.

b The intensity of light 1 m below the surface is 9.5 units. Find the value of a.

c Find the intensity of light 25 m below the surface.

d A light intensity of 4 units is considered adequate for divers to be able to see clearly. Calculate the depth corresponding to this intensity of light.

e Calculate the range of depths for which the light intensity is between 1 and 3 units.

7 A group of turtles was observed in a lake in 2005. The population of turtles has decreased by 5% each year since.

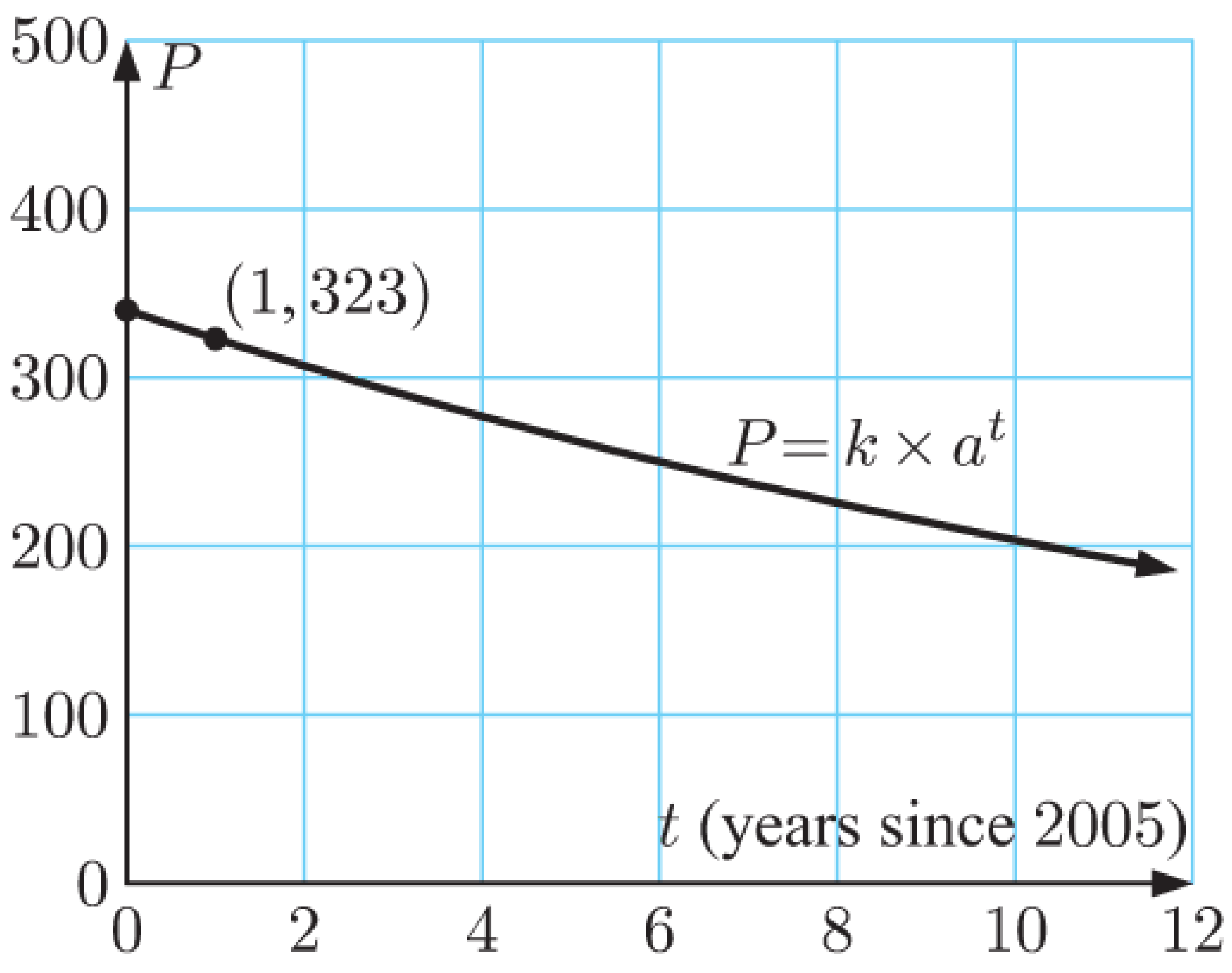
The graph alongside shows the population of turtles since 2005.

a State the value of a.

b Find the value of k, and interpret this value.

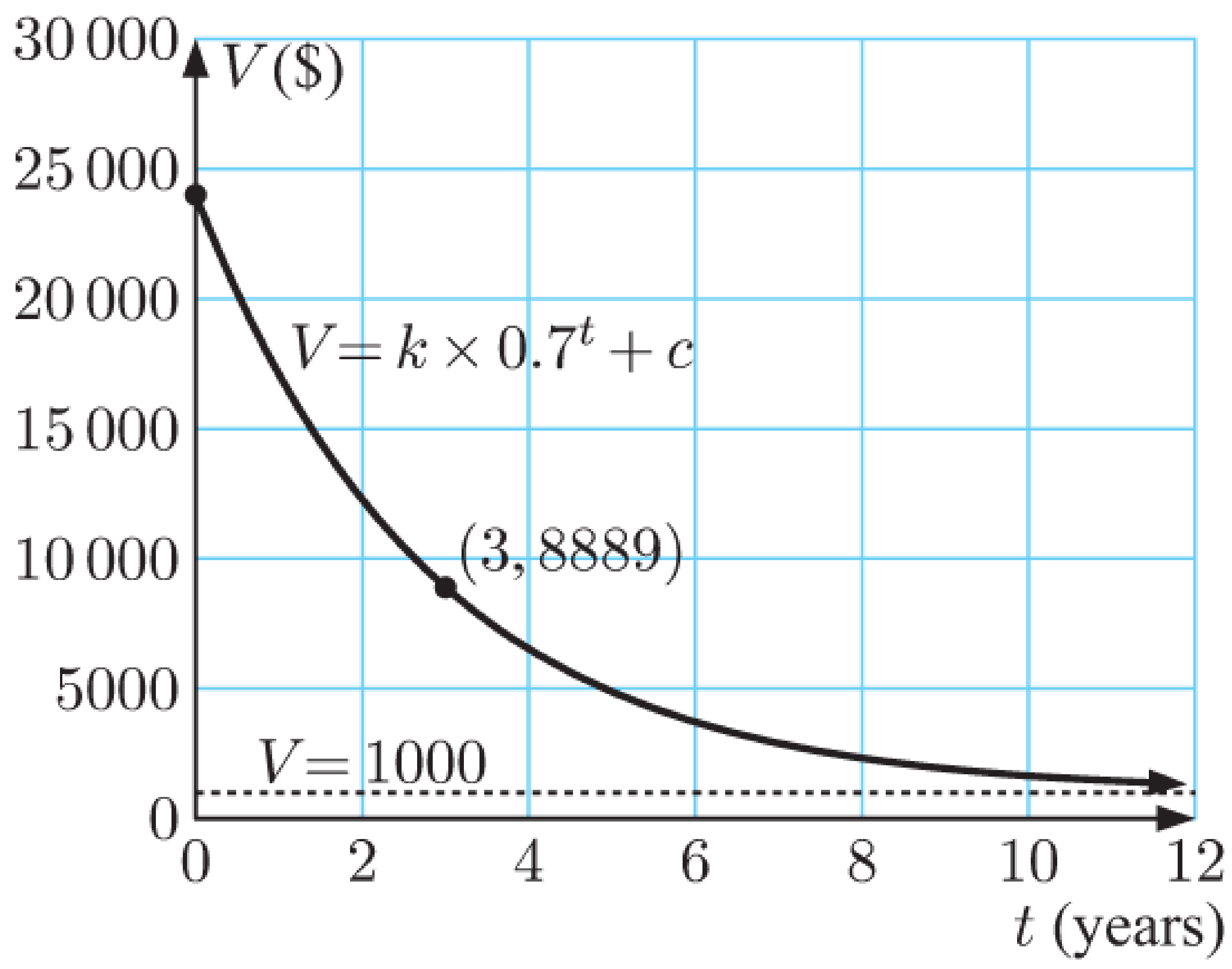
c Find the population of turtles in 2015.

d Do you think it reasonable to apply this model for negative values of t? Explain your answer.



8 The value of a car after t years is given by the exponential model $V = k \times 0.7^t + c$ dollars, where k and c are constants, and $t \geqslant 0$.

The model is graphed alongside.



- **a** State the value of c. Interpret this value in the context of the situation.
- **b** Find the value of k.
- **c** Find the initial value of the car.
- **d** How long will it take for the value of the car to reduce to \$5000? Give your answer to the nearest year.
- **e** Does the car depreciate in value by the same percentage each year? Explain your answer.

9 When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = c + k \times 0.875^t$ °C, where c and k are constants.

The temperature of the peas is 18°C after 1 minute, and 14.5°C after 2 minutes.

- **a** Use technology to find c and k.
- **b** What do you think is the temperature of the interior of the freezer? Explain your answer.
- **c** What was the temperature of the packet of peas:
  - **i** when first placed in the freezer
  - **ii** after 5 minutes
  - **iii** after 10 minutes?
- **d** Sketch the graph of $T(t)$.
- **e** How long does it take for the temperature of the packet of peas to fall to 0°C?

10 The **half-life** of a radioactive substance is the time it takes for the substance's weight to fall to half of its original value.

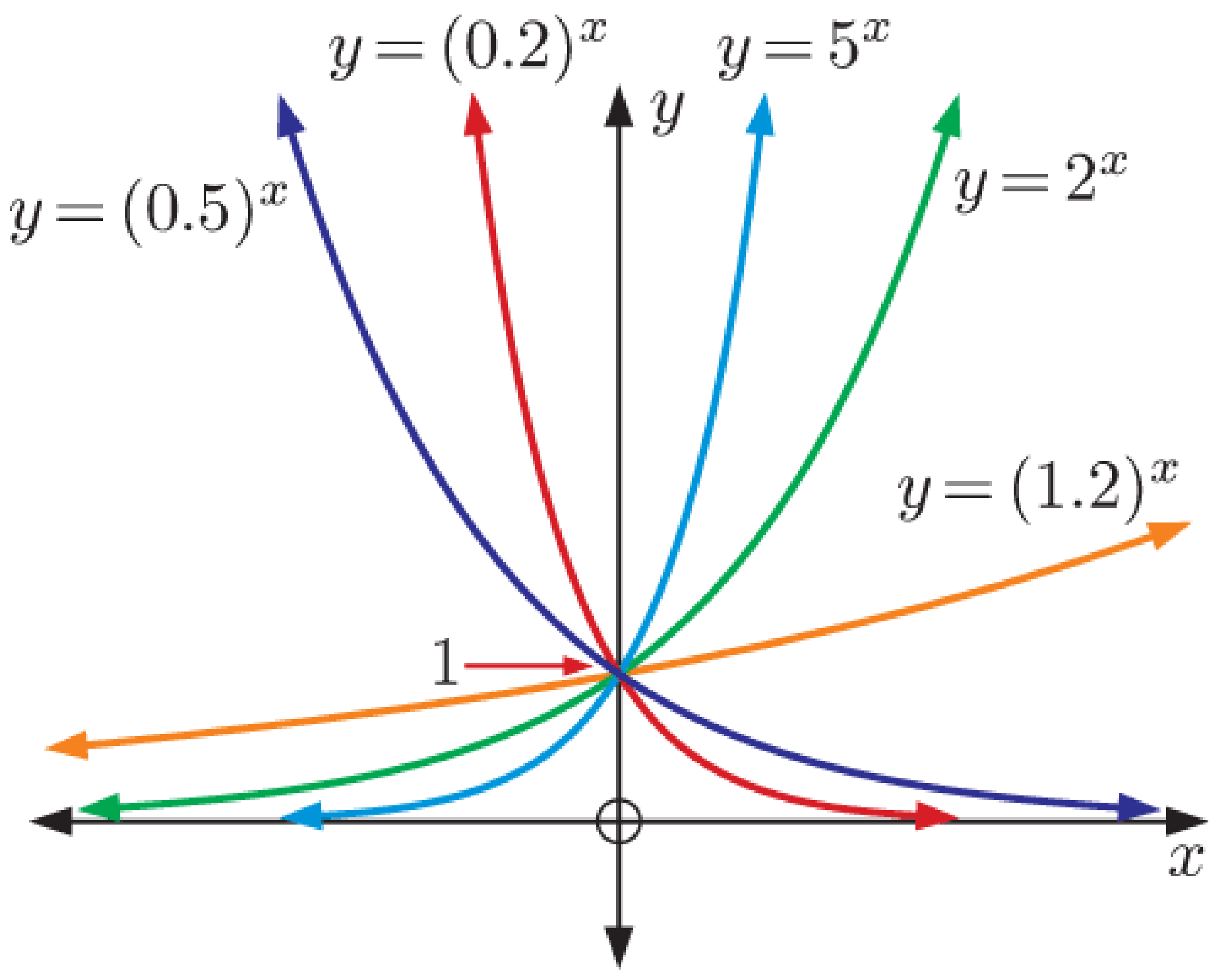
The radioactive isotope fermium-253 has a half-life of 3 days. The weight of fermium-253 detected t days after an explosion is $W(t) = 10 \times a^t$ mg.

- **a** Interpret the value 10 in this model.
- **b** Calculate the value of a, correct to 4 decimal places, and interpret this value.
- **c** Find the weight of fermium-253 after 2 days.
- **d** How long will it take for the weight of fermium-253 to fall to: **i** 3 mg **ii** 1.25 mg?

F THE NATURAL EXPONENTIAL

We have seen that the simplest exponential functions have the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base a, the graph is always positive.

Hence $a^x > 0$ for all $a > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Investigation**.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A discrete formula for calculating the amount to which an investment grows under compound interest is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,
i is the interest rate per compounding period,
n is the number of periods, or times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to become extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$
b quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$
c monthly
d daily
e by the second
f by the millisecond.

Comment on your answers.

2 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum compounded N times during the year.
Explain why the value of the investment:

a after the first period is $u_1 = \$1000\left(1+\frac{0.06}{N}\right)$

b at the end of the year is $u_N = \$1000\left(1+\frac{0.06}{N}\right)^N$.

3 Suppose u_0 is invested for t years at a fixed rate of $r\%$ per annum compounded N times per year.

a Explain why the value of the investment after:

i the first period is $u_1 = u_0\left(1+\frac{r}{N}\right)$

ii the first year is $u_N = u_0\left(1+\frac{r}{N}\right)^N$

iii t years is $u_n = u_0\left(1+\frac{r}{N}\right)^{Nt}$.

b Let $a = \frac{N}{r}$. Show that $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

4 Now suppose the number of interest payments per year N gets very large.

- **a** Explain why a gets very large as N gets very large.
- **b** Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

5 You should have found that for very large values of a, $\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$

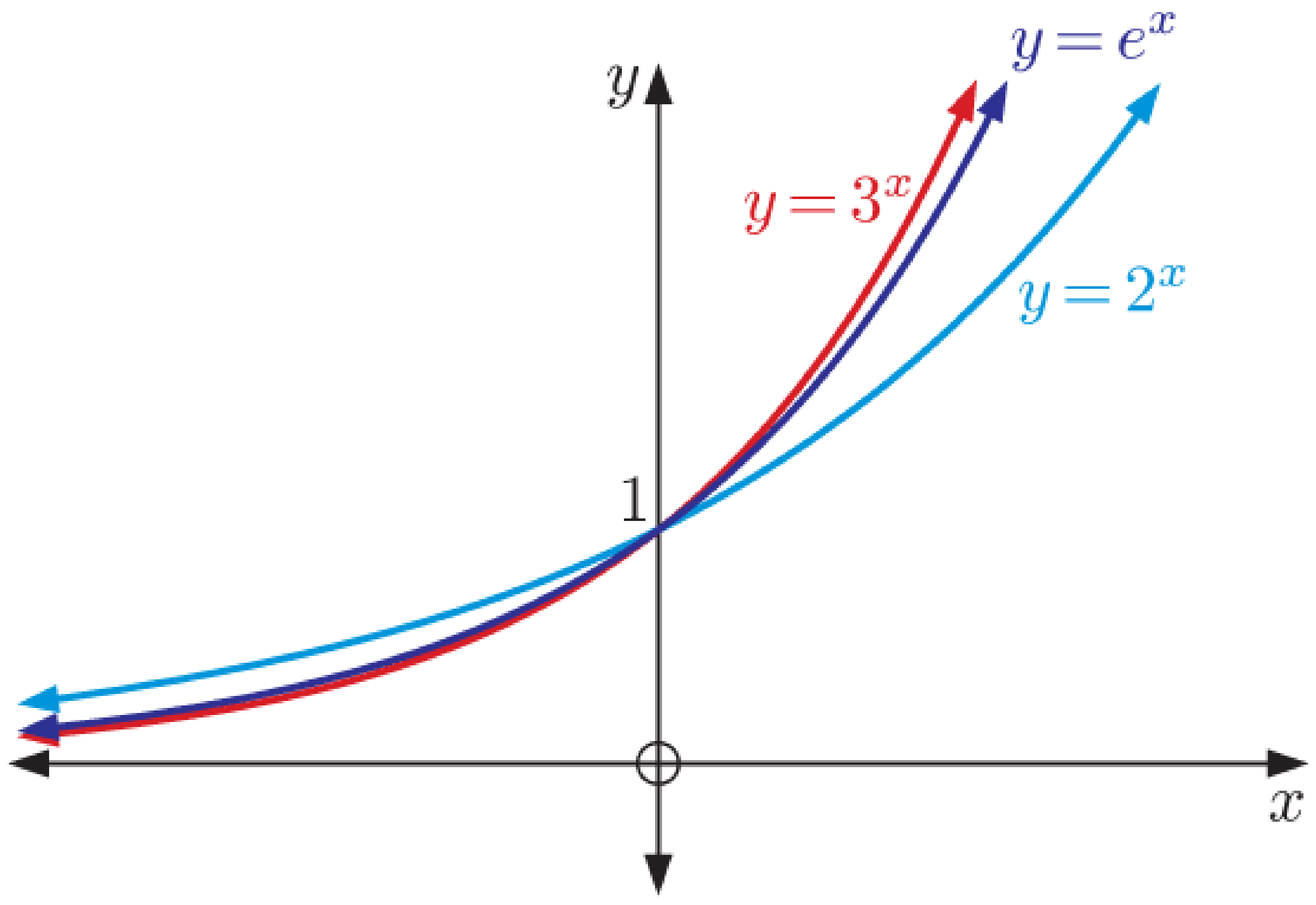
Use the $\boxed{e^x}$ key of your calculator to find the value of e^1. What do you notice?

6 For continuous growth, $u_n = u_0 e^{rt}$ where u_0 is the initial amount, r is the annual percentage rate, and t is the number of years.
Use this formula to find the final amount if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is paid continuously. Compare this value with your calculations in **1**.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

Since $e \approx 2.718\,28$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



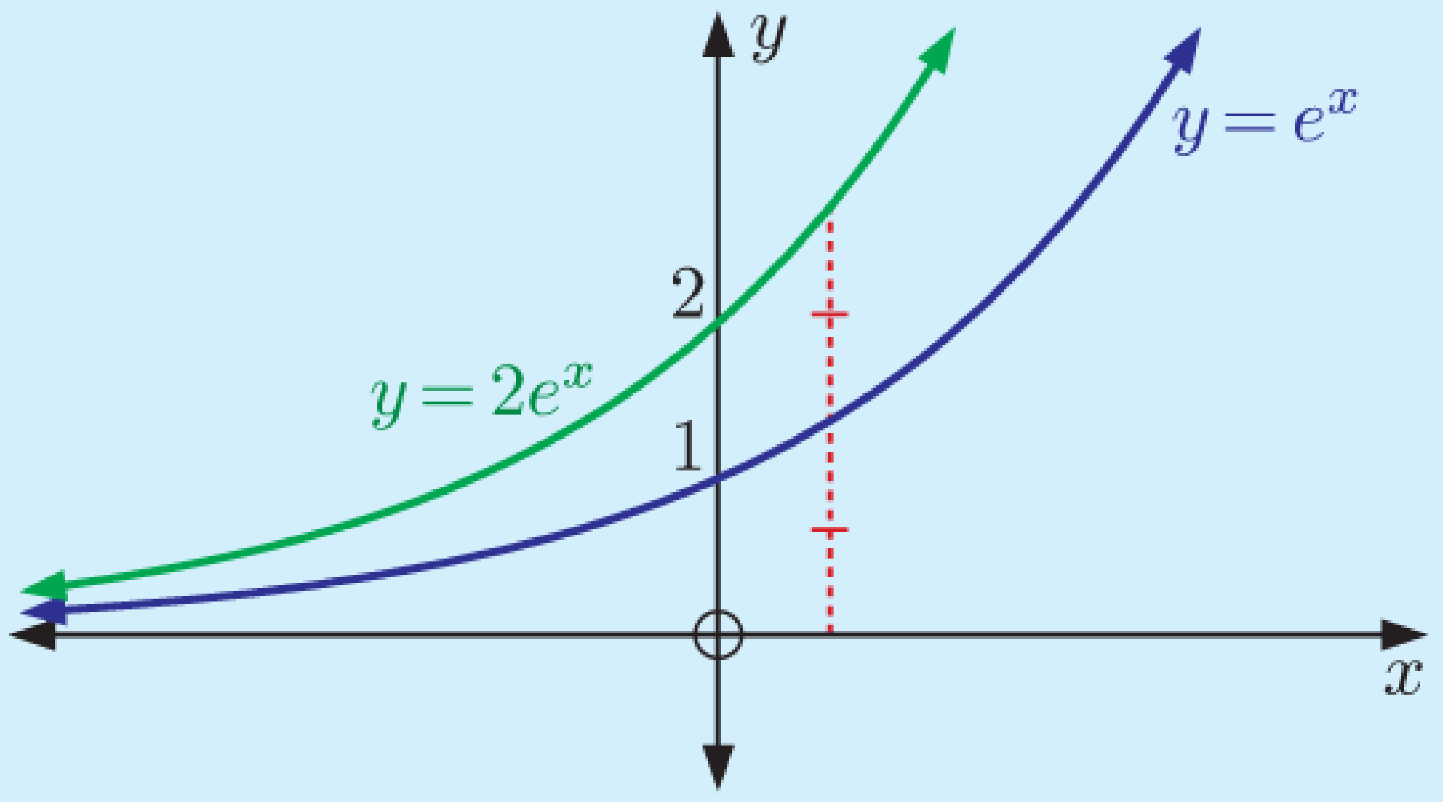
Example 7 🔊 **Self Tutor**

Sketch, on the same set of axes, the graphs of:

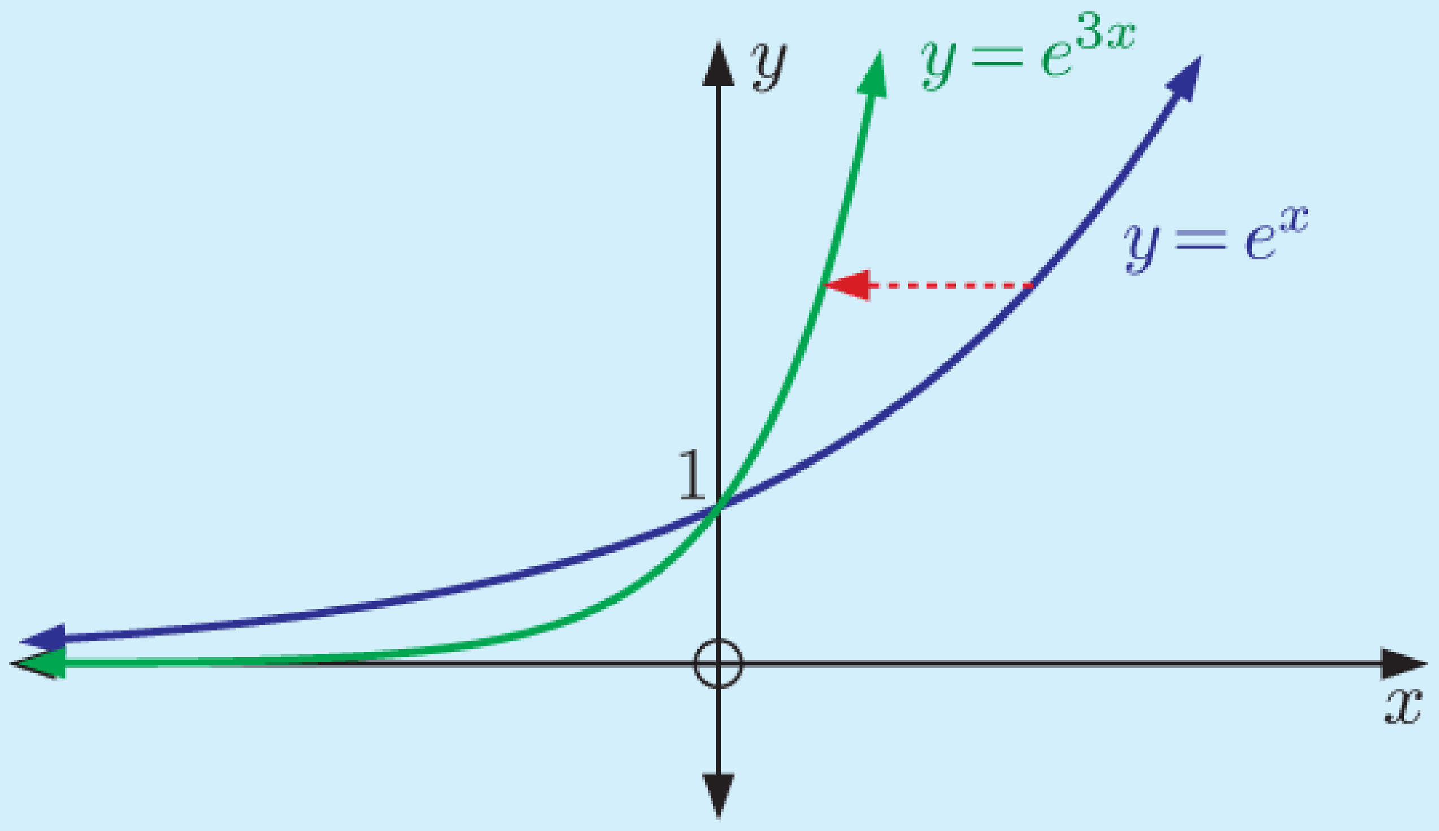
a $y = e^x$ and $y = 2e^x$

b $y = e^x$ and $y = e^{3x}$

a $y = 2e^x$ is a vertical dilation of $y = e^x$ with scale factor 2.



b $y = e^{3x}$ is a horizontal dilation of $y = e^x$ with scale factor $\frac{1}{3}$.



EXERCISE 8F

GRAPHING PACKAGE

1 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = -e^x$. What is the geometric connection between these two graphs?

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 Sketch, on the same set of axes as the graph of $y = e^x$:

a $y = e^x + 2$ **b** $y = e^x - 3$ **c** $y = 3e^x$

d $y = \frac{1}{2}e^x$ **e** $y = e^{2x}$ **f** $y = e^{\frac{x}{2}}$

For each graph, state the y-intercept and equation of the horizontal asymptote.

4 Evaluate, to 5 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.5}$ **d** e^{-1}

e $e^{2.31}$ **f** $e^{-2.31}$ **g** $e^{4.829}$ **h** $e^{-4.829}$

i $50e^{-0.1764}$ **j** $80e^{-0.6342}$ **k** $1000e^{1.2642}$ **l** $0.25e^{-3.6742}$

5 Use your calculator to solve these exponential equations:

a $e^x = 10$ **b** $e^{2x} = 15$ **c** $5e^{-x} = 0.3$ **d** $e^{3x} - 4 = -1$

6 **a** Use technology to help sketch the graph of $f(x) = 10 - e^x$.

b State the domain and range of $f(x)$.

c Describe the behaviour of $y = f(x)$ as $x \to \pm\infty$.

7 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture:

i initially **ii** after 30 minutes **iii** after $1\frac{1}{2}$ hours **iv** after 6 hours.

b Hence sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

8 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.

a What current is still flowing in the circuit after:

i 1 second **ii** 10 seconds?

b Use technology to help sketch the graph of $I(t) = 75e^{-0.15t}$.

c How long will it take for the current to fall to 1 amp?

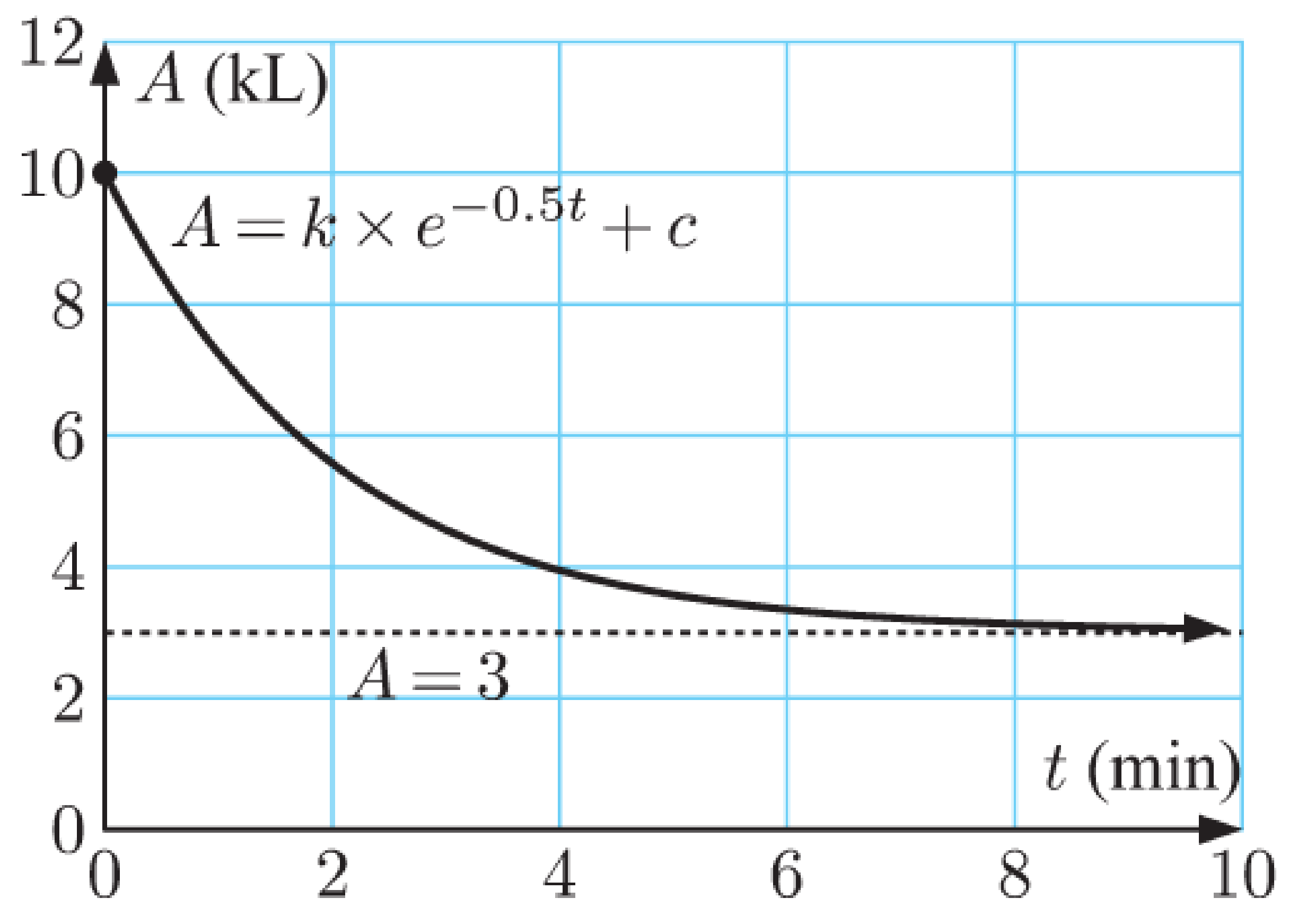
9 This graph shows the amount of water in a tank t minutes after it is punctured by a drill.

a Find the exponential model connecting A and t.

b Do you think the hole is in the bottom or the side of the tank? Explain your answer.

c Find the amount of water in the tank after 2 minutes.

d How long will it take for the tank to lose 6 kL of water?



10 A meteor hurtling through the atmosphere has speed of descent given by

$$V(t) = 650(4 + 2 \times e^{-0.1t}) \ \ \text{m s}^{-1}$$

where t is the time in seconds after the meteor is sighted.

- **a** Is the meteor's speed increasing or decreasing?
- **b** Find the speed of the meteor:
  - **i** when it was first sighted
  - **ii** after 2 minutes.
- **c** How long will it take for the meteor's speed to reach 3000 m s^{-1}?

ACTIVITY

Click on the icon to run a card game for exponential functions.

CARD GAME

HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748, Euler evaluated e correct to 18 decimal places.

Leonhard Euler

Euler also discovered some patterns in **continued fraction** expansions of e. He wrote that

$$\frac{e-1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \dots}}}}} \quad \text{and} \quad e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}}}$$

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

G LOGARITHMS IN BASE 10

The **logarithm in base a of b** is the power that a must be raised to in order to obtain b.

For example, to find $\log_2 8$, we ask "What power must 2 be raised to in order to obtain 8?". We know that $2^3 = 8$, so $\log_2 8 = 3$.

$a^x = b$ and $\log_a b = x$ are *equivalent* statements.

For any $b > 0$, $a^x = b \Leftrightarrow \log_a b = x$

The logarithm in base a of b is written $\log_a b$.

In this Section we will only consider logarithms in base 10.

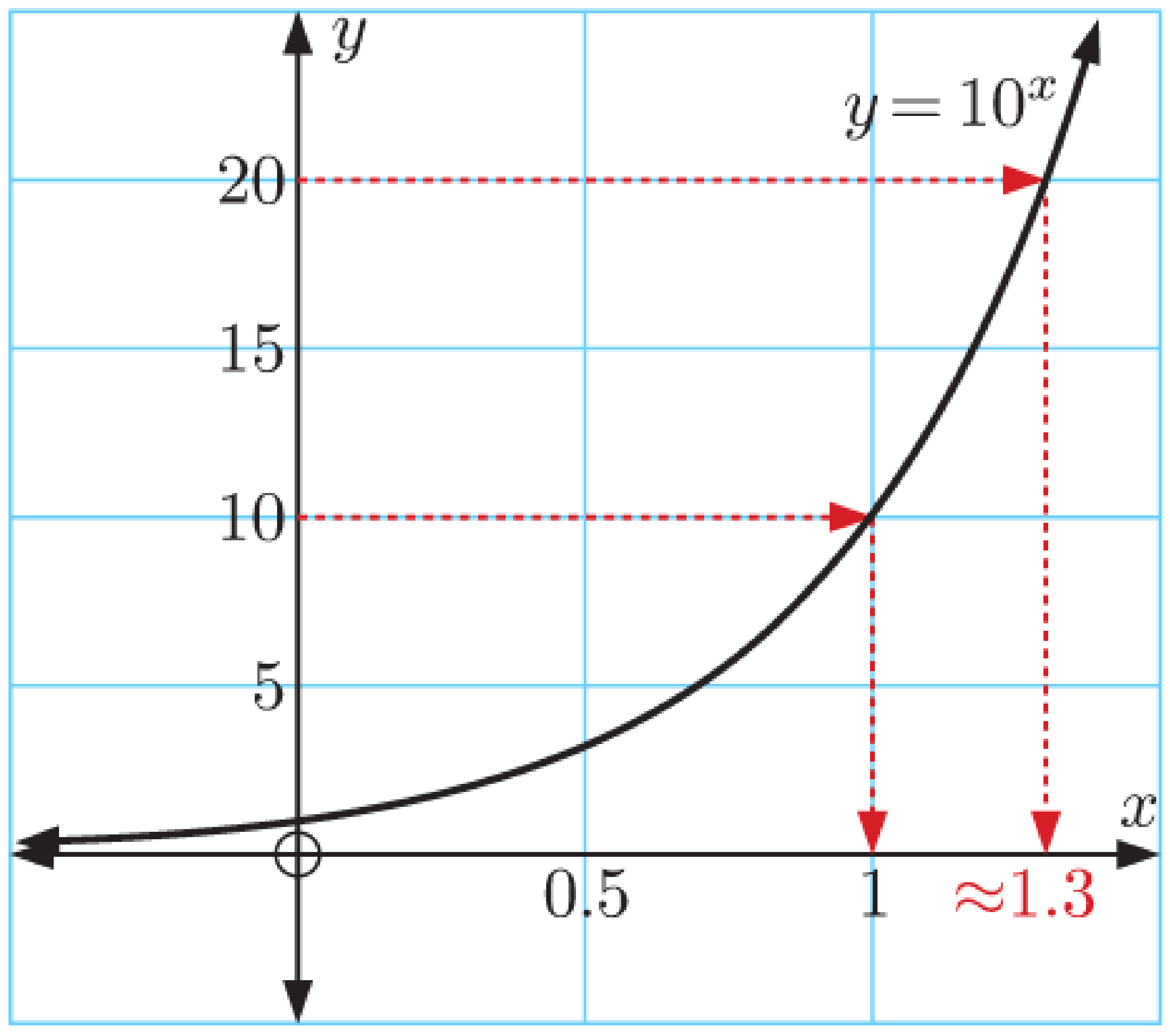
LOGARITHMS IN BASE 10

Consider the graph of $y = 10^x$ shown.

Notice that the range of the function is $\{y \mid y > 0\}$. This means that every positive number y can be written in the form 10^x.

For example:

- When $y = 10$, $x = 1$, so $10 = 10^1$.
- When $y = 20$, $x \approx 1.3$, so $20 \approx 10^{1.3}$.



When we write a positive number y in the form 10^x, we say that x is the **logarithm in base 10**, of y.

The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

For any $b > 0$, $10^x = b \Leftrightarrow \log_{10} b = x$

For example:

- The logarithm in base 10 of 1000 is 3, since $1000 = 10^3$.
  We write $\log_{10} 1000 = 3$ or simply $\log 1000 = 3$.
- $\log(0.01) = -2$ since $0.01 = 10^{-2}$.

If no base is indicated we assume it means base 10. $\log b$ means $\log_{10} b$.

By observing that $\log 1000 = \log(10^3) = 3$ and $\log(0.01) = \log(10^{-2}) = -2$, we conclude that $\boldsymbol{\log 10^x = x}$ for any $x \in \mathbb{R}$.

Example 8 **Self Tutor**

Find:

a $\log 100$ **b** $\log(0.1)$

a $\log 100 = \log(10^2)$
$= 2$

b $\log(0.1) = \log(10^{-1})$
$= -1$

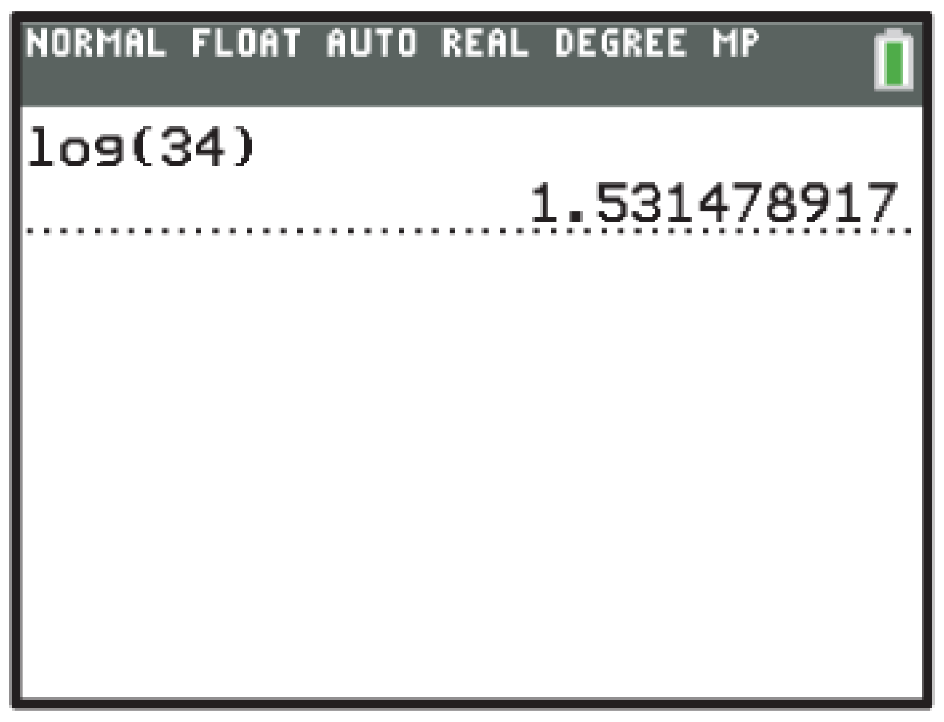
The logarithms in **Example 8** can be found by hand because it is easy to write 100 and 0.1 as powers of 10. The logarithms of most values, however, can only be found using a calculator.

For example, $\log 34 \approx 1.53$
so $34 \approx 10^{1.53}$

Logarithms allow us to write any number as a power of 10. In particular:

$$\boldsymbol{x = 10^{\log x}} \quad \text{for any} \quad x > 0.$$

GRAPHICS CALCULATOR INSTRUCTIONS



```
log(34)
                    1.531478917
```

Example 9

Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

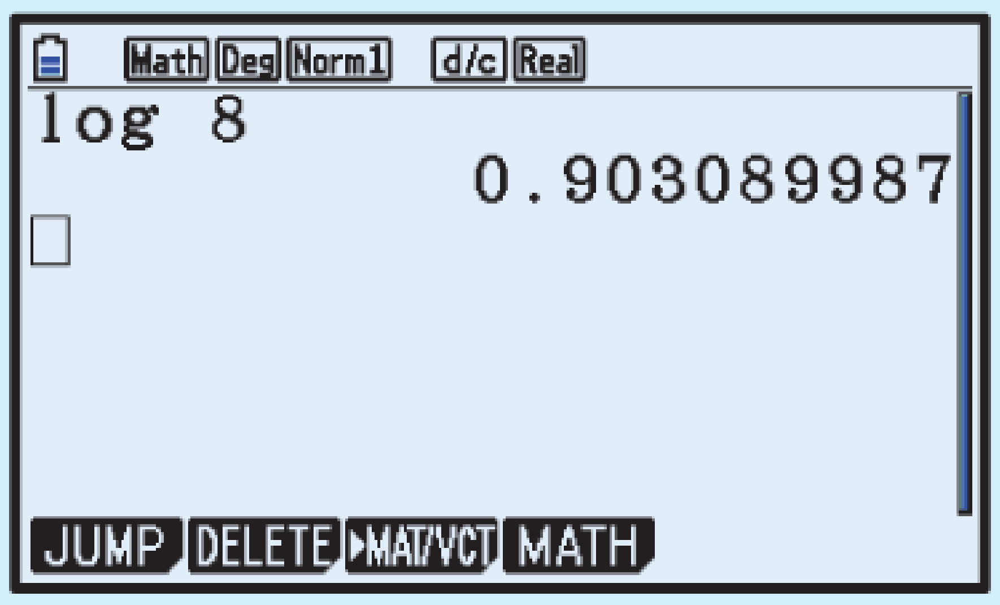
a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8}$
$\approx 10^{0.9031}$

b $800 = 10^{\log 800}$
$\approx 10^{2.9031}$

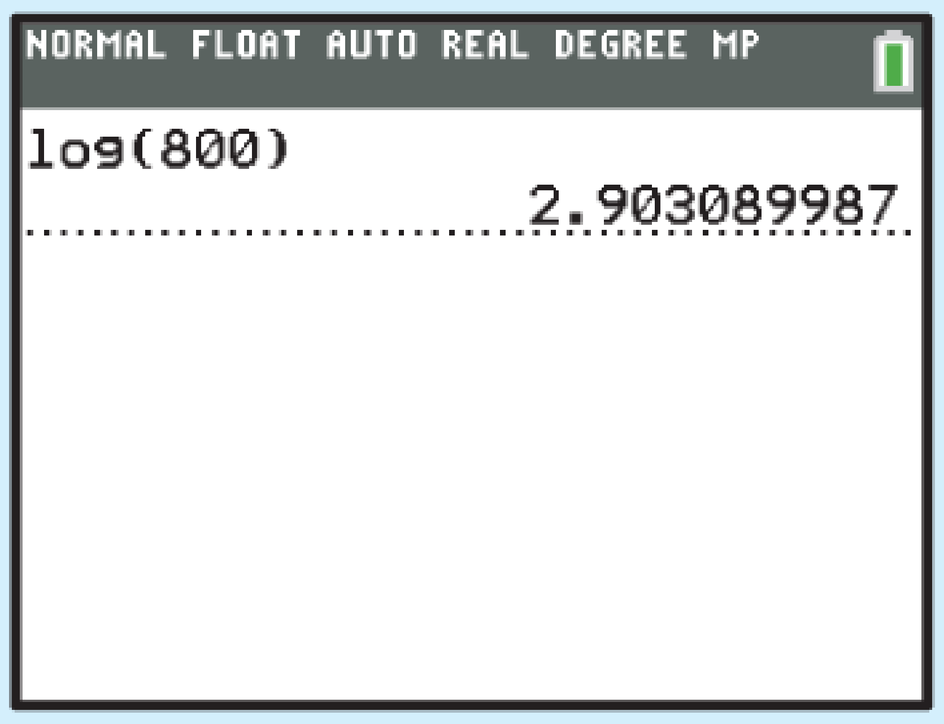
c $0.08 = 10^{\log 0.08}$
$\approx 10^{-1.0969}$

Casio fx-CG50



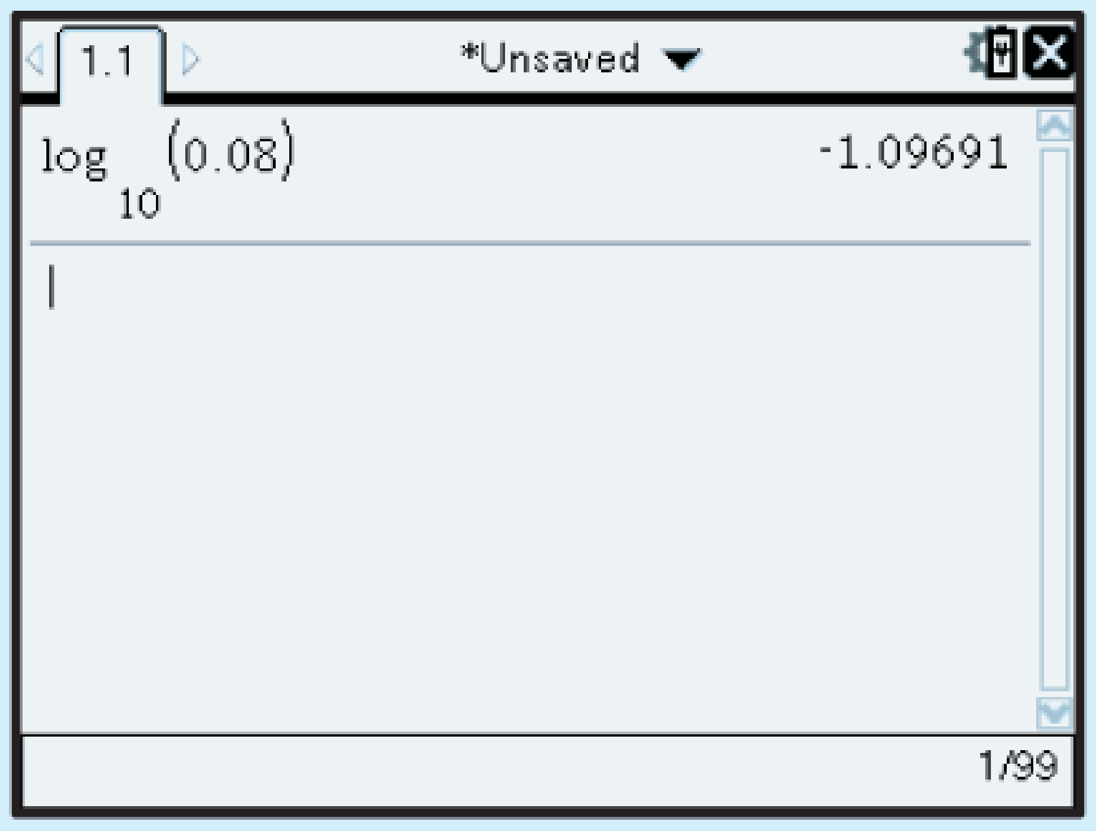
```
log 8
          0.903089987
```

TI-84 Plus CE



```
log(800)
          2.903089987
```

TI-*n*spire



```
log10(0.08)          -1.09691
```

EXERCISE 8G

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log(0.001)$ **c** $\log 10$ **d** $\log 1$

Check your answers using your calculator.

2 Simplify:

a $\log(10^n)$ **b** $\log(10^a \times 100)$ **c** $\log\left(\dfrac{10}{10^m}\right)$ **d** $\log\left(\dfrac{10^a}{10^b}\right)$

3 **a** Explain why $\log 237$ must lie between 2 and 3.

b Use your calculator to evaluate $\log 237$ correct to 2 decimal places.

4 **a** Between which two consecutive whole numbers does $\log(0.6)$ lie?

b Check your answer by evaluating $\log(0.6)$ correct to 2 decimal places.

5 Use your calculator to evaluate, correct to 2 decimal places:

a $\log 76$ **b** $\log 114$ **c** $\log 3$ **d** $\log 831$

e $\log(0.4)$ **f** $\log 3247$ **g** $\log(0.008)$ **h** $\log(-7)$

6 For what values of x is $\log x$:

a positive **b** zero **c** negative **d** undefined?

7 **a** Use your calculator to find $\log 83$.

b Hence write 83 as a power of 10, where the power is rounded to 4 decimal places.

8 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

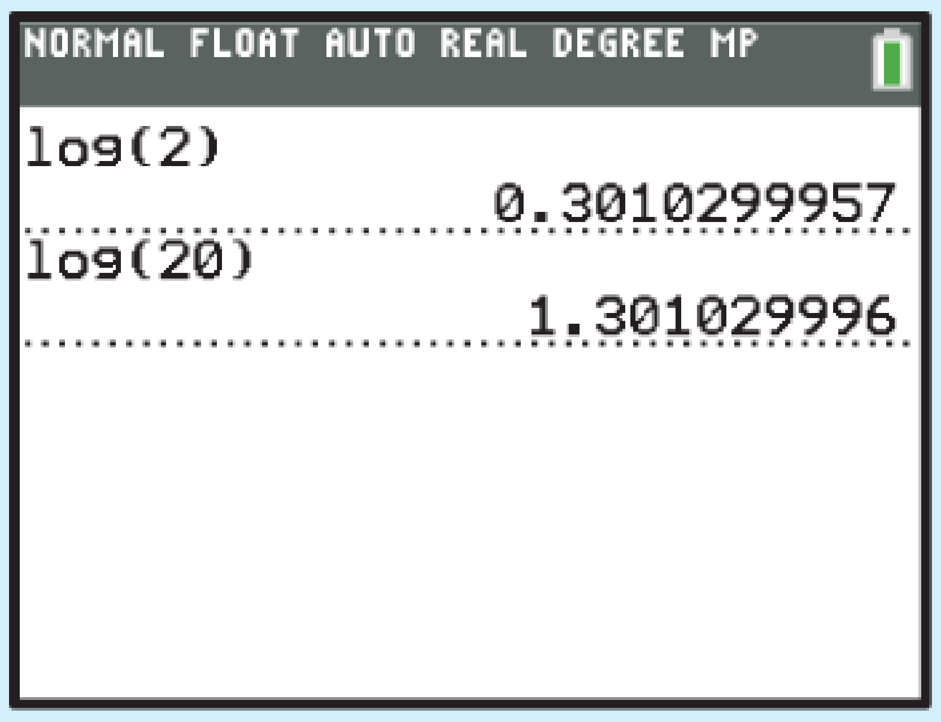
Example 10 **Self Tutor**

a Use your calculator to find:

i $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a



```
NORMAL FLOAT AUTO REAL DEGREE MP
log(2)
                    0.3010299957
log(20)
                     1.301029996
```

i $\log 2 \approx 0.3010$

ii $\log 20 \approx 1.3010$

b
$$\begin{aligned}\log 20 &= \log(2\times 10)\\ &= \log(10^{\log 2}\times 10^1) && \{x = 10^{\log x}\}\\ &= \log(10^{\log 2+1}) && \{\text{adding indices}\}\\ &= \log 2 + 1\end{aligned}$$

9 **a** Use your calculator to find:

i $\log 3$ **ii** $\log 300$

b Explain why $\log 300 = \log 3 + 2$.

10 **a** Use your calculator to find:

i $\log 5$ **ii** $\log(0.05)$

b Explain why $\log(0.05) = \log 5 - 2$.

11 Suppose the point (a, b) lies on the graph of $y = 10^x$.

a Write an equation connecting a and b.

b Hence show that $\log b = a$.

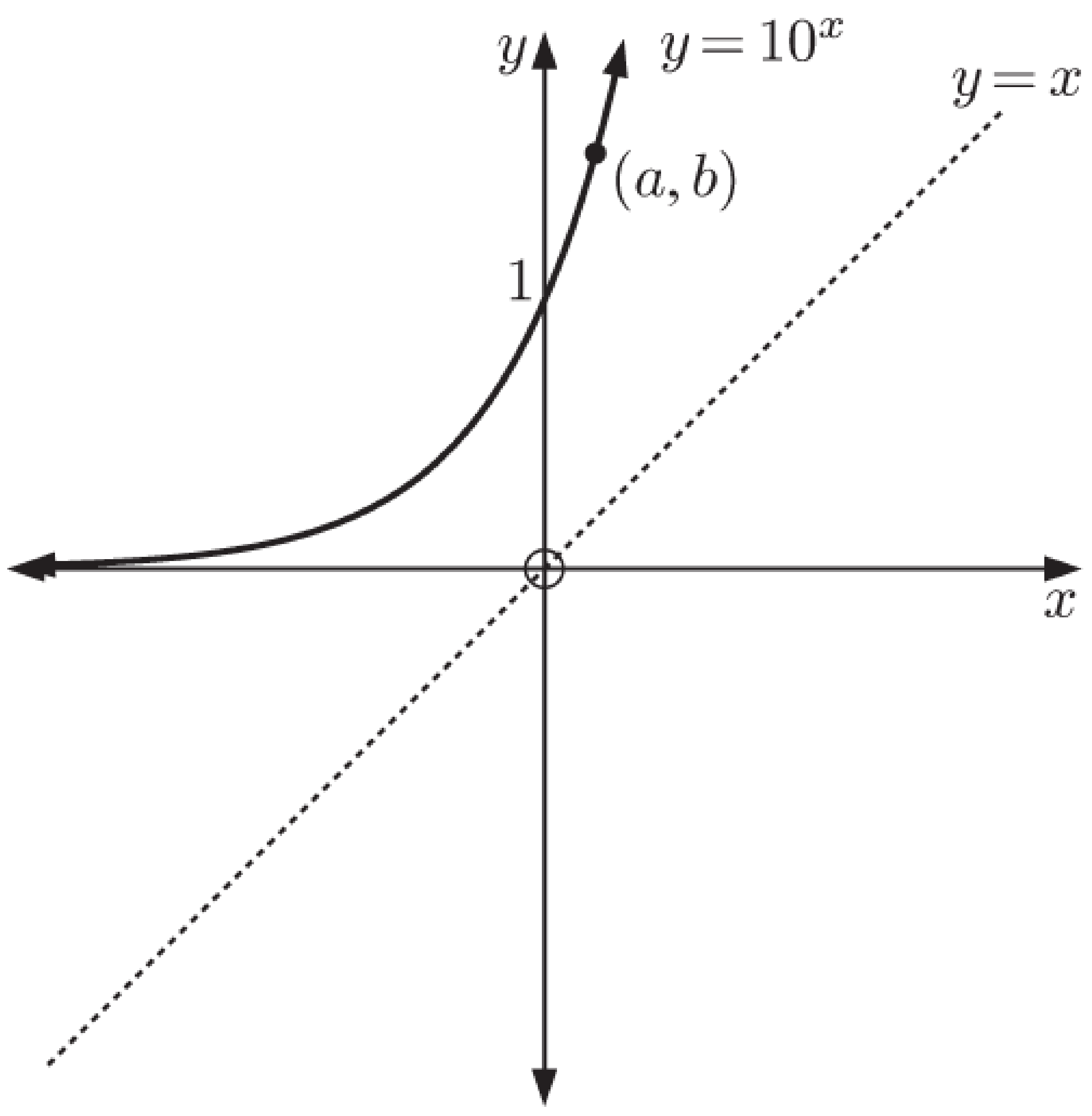
c Explain why the point (b, a) lies on the graph of $y = \log x$.

d What is the relationship between the functions $y = 10^x$ and $y = \log x$?

e Copy the graph of $y = 10^x$, and sketch the graph of $y = \log x$ on the same set of axes.

f Find the x-intercept of $y = \log x$.

g State the domain and range of $y = \log x$.



Example 11 **Self Tutor**

The magnitude of an earthquake which releases E joules of energy is given by $M = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$.

a Find the magnitude of an earthquake which releases 5.1×10^{10} joules of energy.

b How much energy is released by a magnitude 4.3 earthquake?

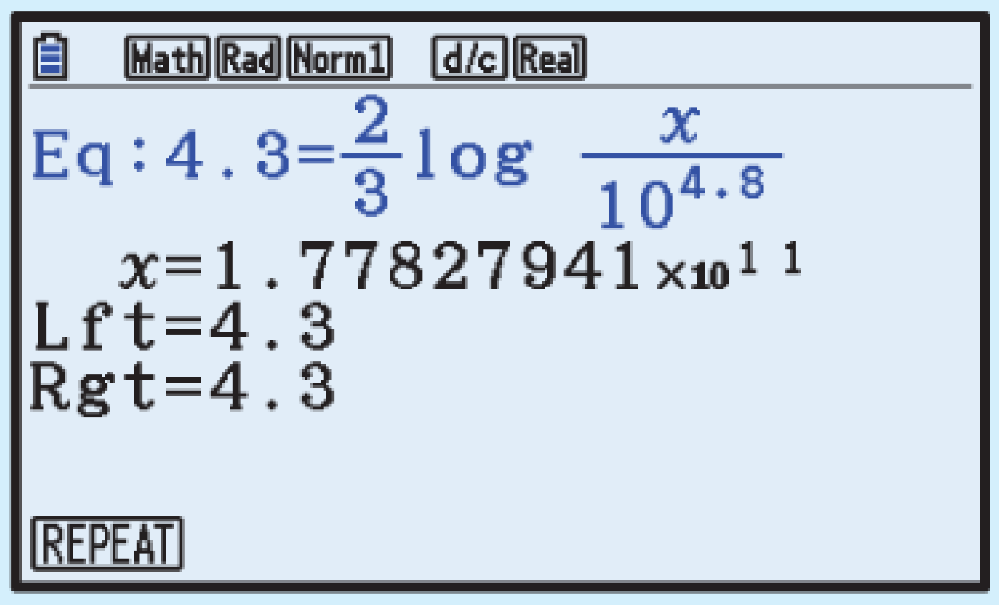
---

a When $E = 5.1 \times 10^{10}$, $M = \frac{2}{3}\log\left(\frac{5.1 \times 10^{10}}{10^{4.8}}\right)$

≈ 3.94

b When $M = 4.3$, $4.3 = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$

$\therefore$ using technology, $E \approx 1.78 \times 10^{11}$ joules.



$\text{Eq}:4.3=\frac{2}{3}\log\frac{x}{10^{4.8}}$

$x=1.77827941\times10^{11}$

Lft=4.3

Rgt=4.3

12 Use the formula in the Example above to find:

a the magnitude of an earthquake which releases 6.2×10^{13} joules of energy

b the energy released by a magnitude 5.1 earthquake.

13 In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by pH $= -\log C$, where C is the concentration of H_3O^+. Find:

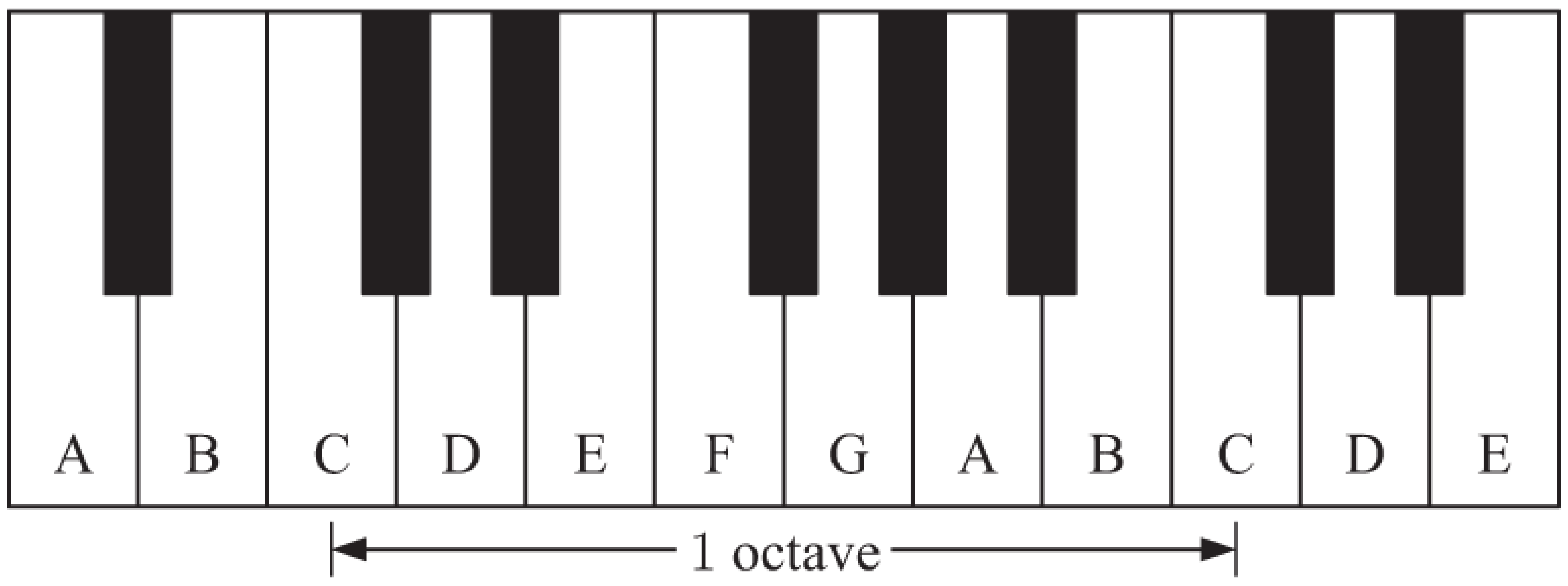
a the pH of a solution with H_3O^+ concentration 0.000 234 mol L^{-1}

b the H_3O^+ concentration in a solution with pH 6.2.

14 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet.

A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.

A note n octaves above "Middle C" has frequency f Hz. The variables n and f are related by the equation $n \approx 3.322\log\left(\frac{f}{261.6}\right)$.



a Find the frequency of "Middle C".

b How many octaves above "Middle C" is a note with frequency 784 Hz?

c Find the frequency of the note:

  i 3 octaves above "Middle C"   **ii** 1 octave below "Middle C".

d There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.

RESEARCH

1 Research the use of **decibels** in acoustics as a unit of measurement for loudness of sound. You may wish to use this as the basis for a Mathematical Exploration.

2 What other logarithmic scales are commonly used?

H NATURAL LOGARITHMS

In the previous Section we considered the logarithm in base 10.

The logarithm in base e is called the **natural logarithm**.

> The **natural logarithm** of a positive number is the power that e must be raised to in order to obtain that number.
>
> The natural logarithm is represented as $\log_e x$, or more commonly as $\ln x$.
>
> For all $x > 0$: $\boldsymbol{\ln e^x = x}$ and $\boldsymbol{e^{\ln x} = x}$.

Example 12 **Self Tutor**

Find:

a $\ln(e^3)$

b $\ln\left(\frac{1}{e^4}\right)$

c $e^{2\ln 5}$

a $\ln(e^3)$
$= 3$

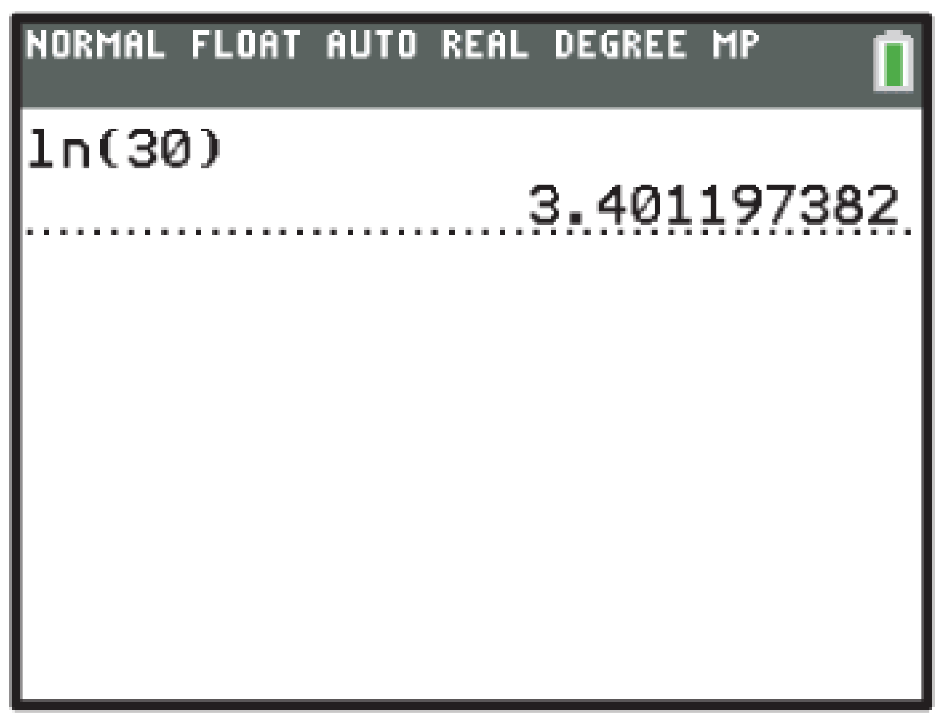
b $\ln\left(\frac{1}{e^4}\right)$
$= \ln(e^{-4})$
$= -4$

c $e^{2\ln 5}$
$= (e^{\ln 5})^2$
$= 5^2$
$= 25$

As with base 10 logarithms, we can use our calculator to find natural logarithms.

For example, $\ln 30 \approx 3.40$, which means that $30 \approx e^{3.40}$.

GRAPHICS CALCULATOR INSTRUCTIONS



```
NORMAL FLOAT AUTO REAL DEGREE MP
ln(30)
                    3.401197382
```

EXERCISE 8H

1 Without using a calculator find:

a $\ln(e^2)$ **b** $\ln(e^4)$ **c** $\ln 1$ **d** $\ln\left(\frac{1}{e^2}\right)$

Check your answers using a calculator.

2 Simplify:

a $\ln(e^a)$ **b** $\ln(e \times e^a)$ **c** $\ln(e^a \times e^b)$ **d** $\ln\left((e^a)^b\right)$

e $e^{\ln 3}$ **f** $e^{2\ln 3}$ **g** $e^{-\ln 5}$ **h** $e^{-2\ln 2}$

3 **a** Use your calculator to evaluate e and e^2 correct to 3 decimal places. Hence explain why $\ln 5$ lies between 1 and 2.
  b Use your calculator to find $\ln 5$ correct to 3 decimal places.

4 **a** Use your calculator to evaluate e^3 and e^4 correct to 4 decimal places.
  b Between which two consecutive whole numbers does $\ln 40$ lie?
  c Find $\ln 40$ correct to 3 decimal places.

5 Use your calculator to find, correct to 3 decimal places:

a $\ln 12$	**b** $\ln 68$	**c** $\ln(1.4)$	**d** $\ln(0.7)$	**e** $\ln 500$
f $\ln 850$	**g** $\ln(0.02)$	**h** $\ln(10^5)$	**i** $\ln(0.006)$	**j** $\ln 7500$

6 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

Example 13 **Self Tutor**

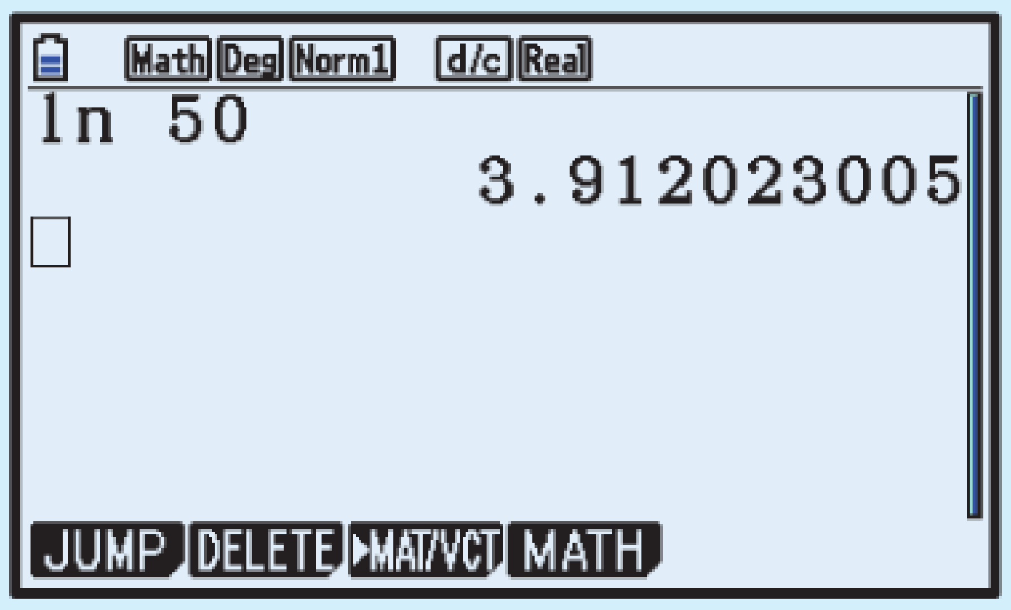
Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

a 50
$= e^{\ln 50}$ $\{x = e^{\ln x}\}$
$\approx e^{3.9120}$

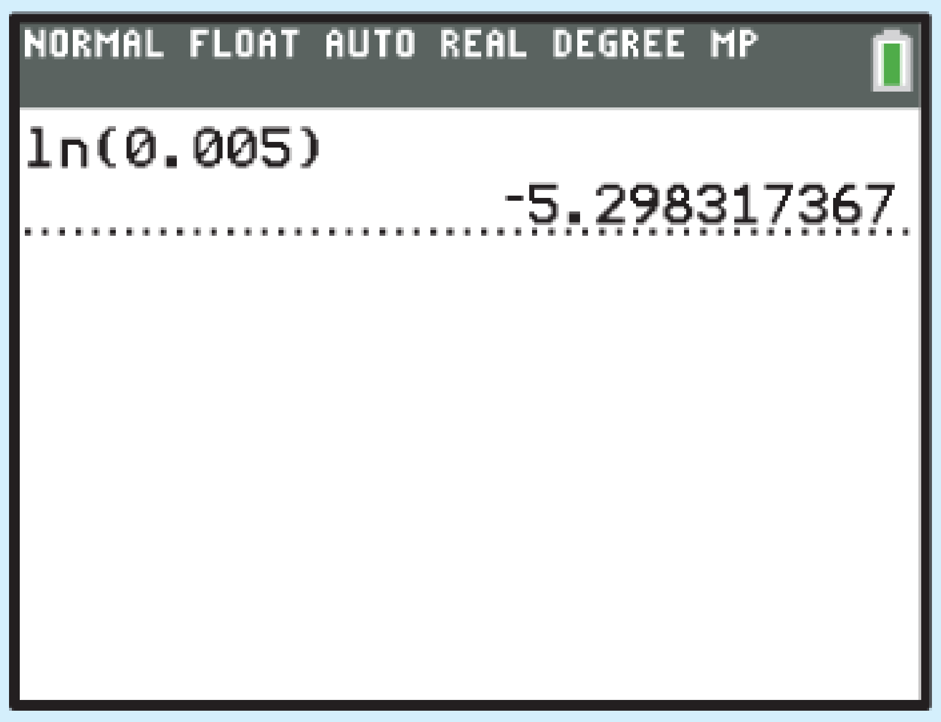
b 0.005
$= e^{\ln 0.005}$
$\approx e^{-5.2983}$

Casio fx-CG50



```
ln 50
                    3.912023005
```

TI-84 Plus CE



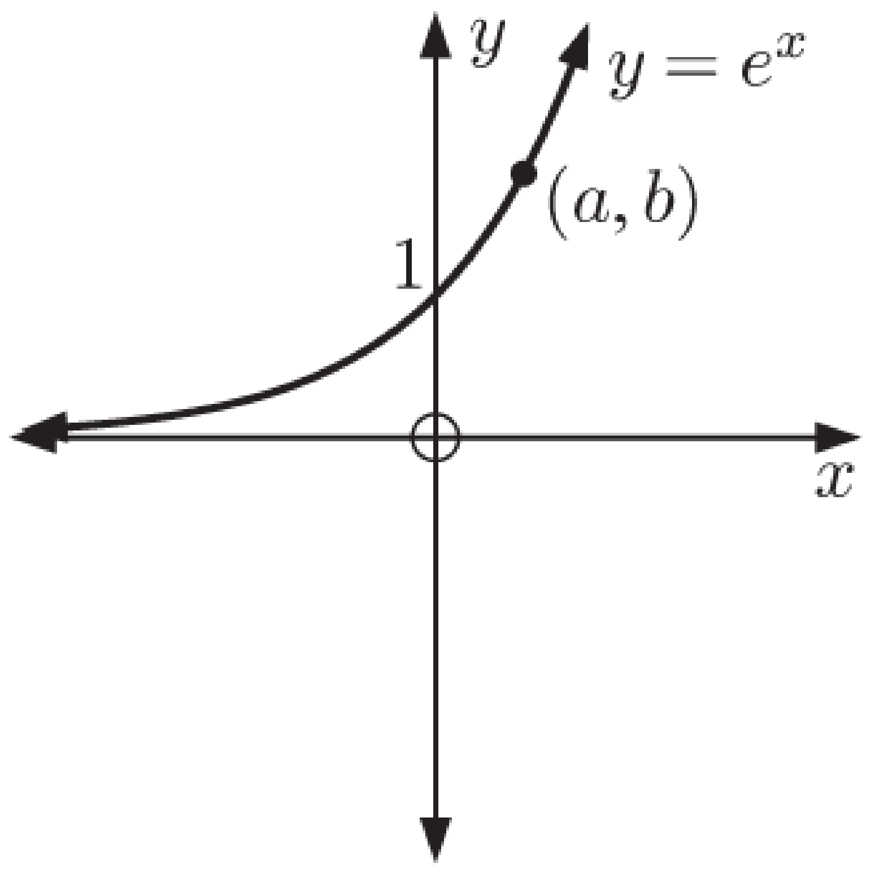
```
NORMAL FLOAT AUTO REAL DEGREE MP
ln(0.005)
                   -5.298317367
```

7 **a** Use your calculator to find $\ln 24$. Give your answer correct to 4 decimal places.
  b Hence write 24 in the form e^k.

8 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6	**b** 60	**c** 6000	**d** 0.6	**e** 0.006
f 15	**g** 1500	**h** 1.5	**i** 0.15	**j** 0.000 15

9 **a** Show that if (a, b) lies on the graph of $y = e^x$, then (b, a) must lie on the graph of $y = \ln x$.
  b What does this tell us about the functions $y = e^x$ and $y = \ln x$?
  c Sketch the graphs of $y = e^x$ and $y = \ln x$ on the same set of axes.
  d State the domain and range of $y = \ln x$.



10 *Hick's law* models the time taken for a person to make a selection from a number of possible options. For a particular person, Hick's law determines that the time taken to choose between n equally probable choices is $T = 2\ln(n+1)$ seconds.

a How long will it take this person to choose between:

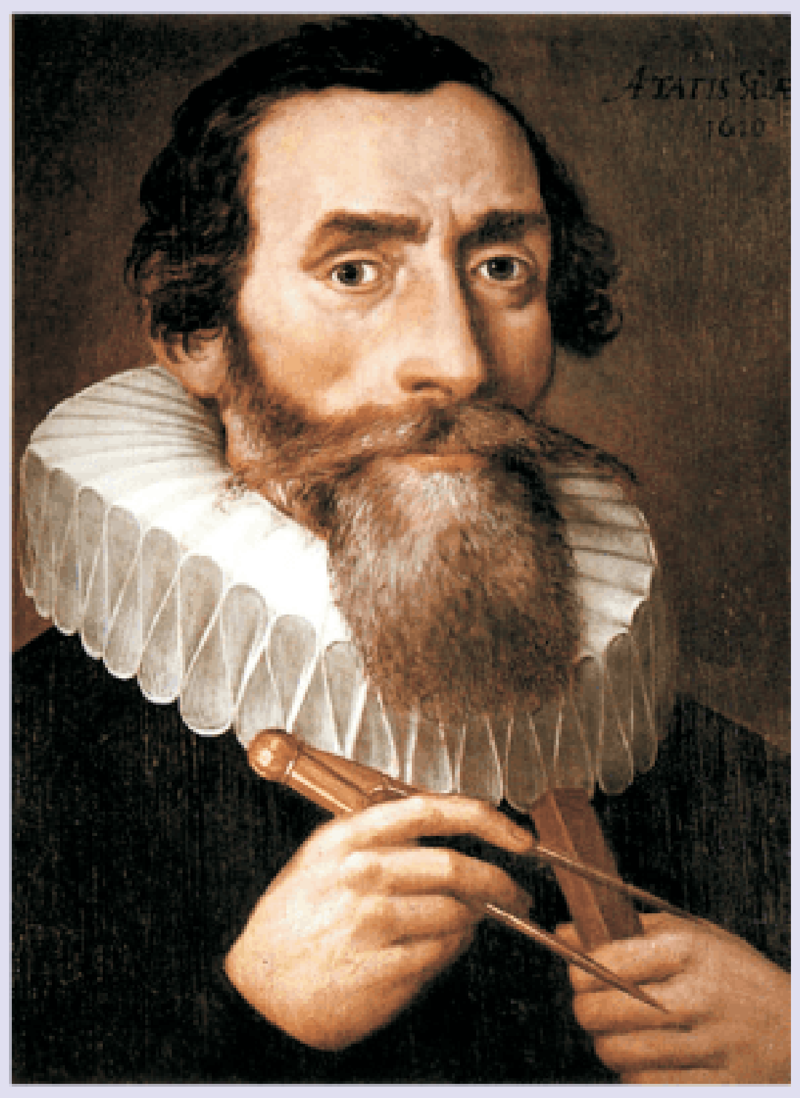
i 5 possible choices **ii** 15 possible choices?

b If the number of possible choices increases from 20 to 40, how much longer will the person take to make a selection?

THEORY OF KNOWLEDGE

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and techniques such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.


Johannes Kepler

Logarithms were an extremely important development, and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used logarithms to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

1. Can anyone claim to have *invented* logarithms?
2. Can we consider the process of mathematical discovery as an *evolution* of ideas?
3. Has modern computing effectively doubled the life of a mathematician?

Many areas of mathematics have been developed over centuries as several mathematicians have worked in a particular area, or taken the knowledge from one area and applied it to another field. Sometimes the process is held up because a method for solving a particular class of problem has not yet been found. In other cases, pure mathematicians have published research papers on seemingly useless mathematical ideas, which have then become vital in applications much later.

In *Everybody Counts*: *A report to the nation on the future of Mathematical Education* by the National Academy of Sciences (National Academy Press, 1989), there is an excellent section on the Nature of Mathematics. It includes:

> "Even the most esoteric and abstract parts of mathematics - number theory and logic, for example - are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago, the leading British mathematician G.H. Hardy could boast that number theory was the most pure and least useful part of mathematics. Today, Hardy's mathematics is studied as an essential prerequisite to many applications, including control of automated systems, data transmission from remote satellites, protection of financial records, and efficient algorithms for computation."

4 Should we only study the mathematics required to enter our chosen profession?

5 Why should we explore mathematics for its own sake, rather than to address the needs of science?

REVIEW SET 8A

1 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

2 Determine whether the given point satisfies the exponential function:

a $y = 4^x - 1$ $(2, 15)$ **b** $f(x) = 5 \times 3^{-x}$ $(-1, \frac{5}{3})$

3 Consider the exponential function $y = \left(\frac{1}{2}\right)^x + 2$.

a Copy and complete the table of values:

x	-3	-2	-1	0	1	2	3
y							

b Hence graph the function.

c Copy and complete:

i As $x \to \infty$, $y \to$ ...... **ii** As $x \to -\infty$, $y \to$ ......

d Find the horizontal asymptote.

4 On the same set of axes, draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

5 Find the y-intercept of:

a $f(x) = 4^x + 2$ **b** $f(x) = 2 \times 5^{-x} - 6$ **c** $f(x) = -\frac{1}{3} \times 2^x + 5$

6 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm1, \pm2$. **b** Discuss y as $x \to \pm\infty$.

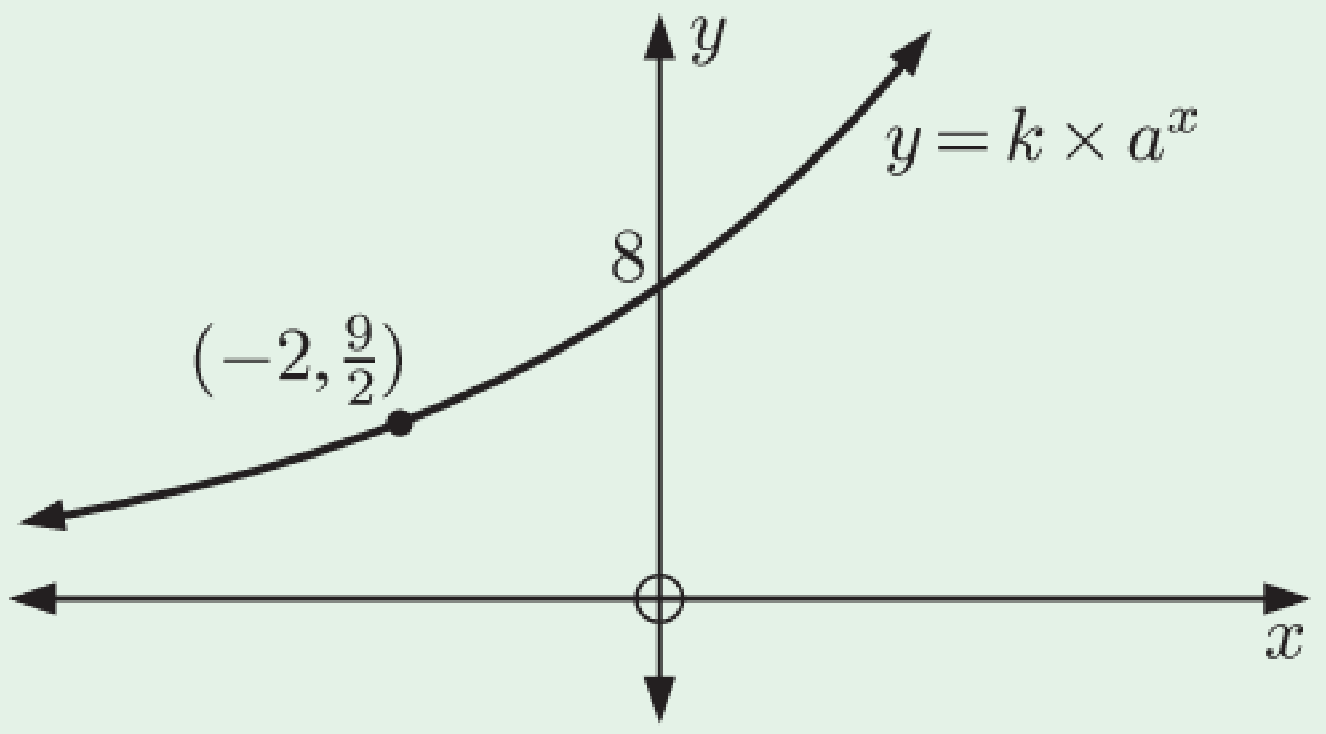
c Sketch the graph of $y = 3^x - 5$. **d** State the equation of any asymptote.

7 This graph shows the exponential function $y = k \times a^x$, where a and k are constants, $a > 0$.

a Find k and a.

b State the equation of the horizontal asymptote.

c Find the value of y when $x = 2$.



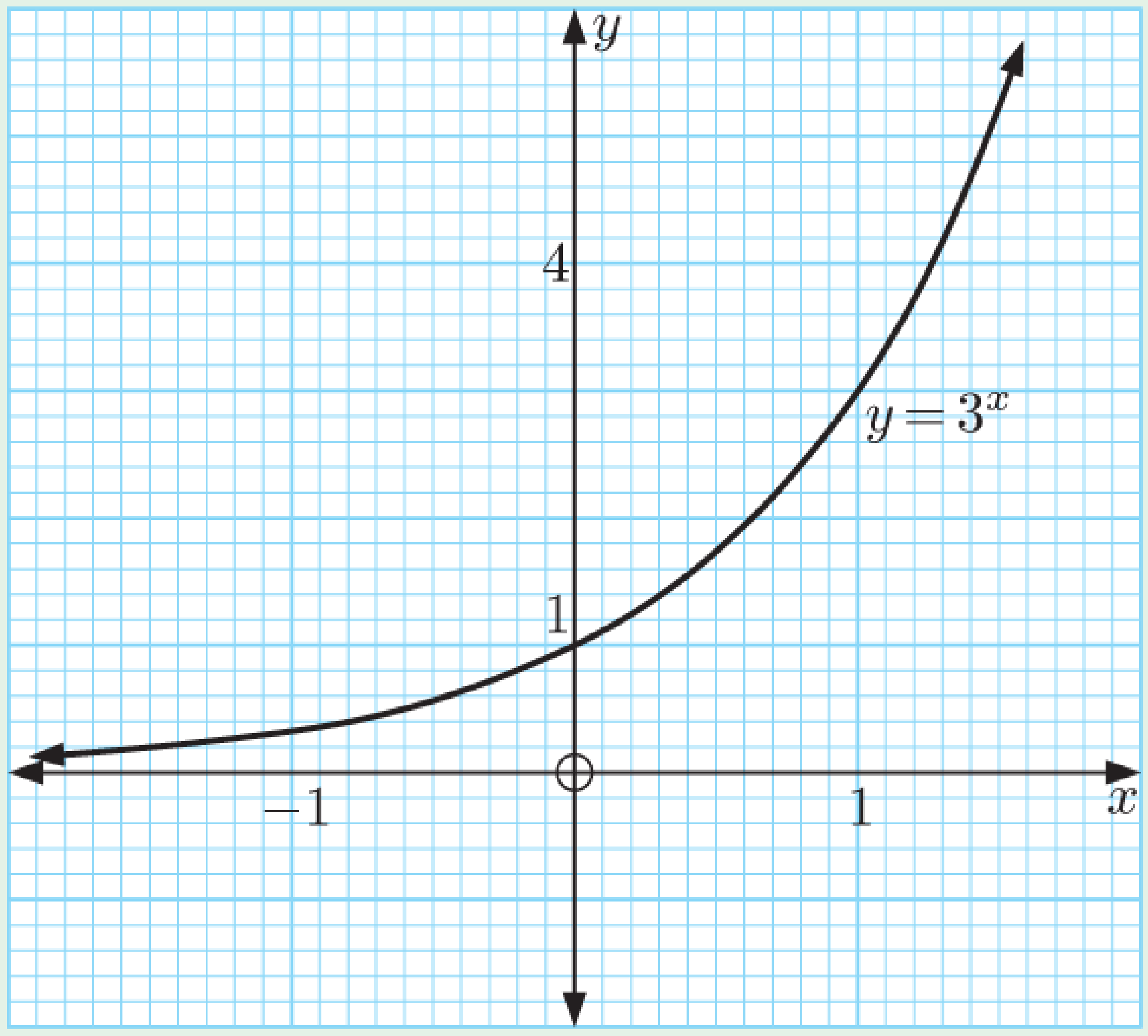
8 Consider the graph of $y = 3^x$ alongside.

a Use the graph to estimate the value of:

i $3^{0.7}$ **ii** $3^{-0.5}$

b Use the graph to estimate the solution to:

i $3^x = 5$ **ii** $3^x = \frac{1}{2}$



9 Use technology to solve:

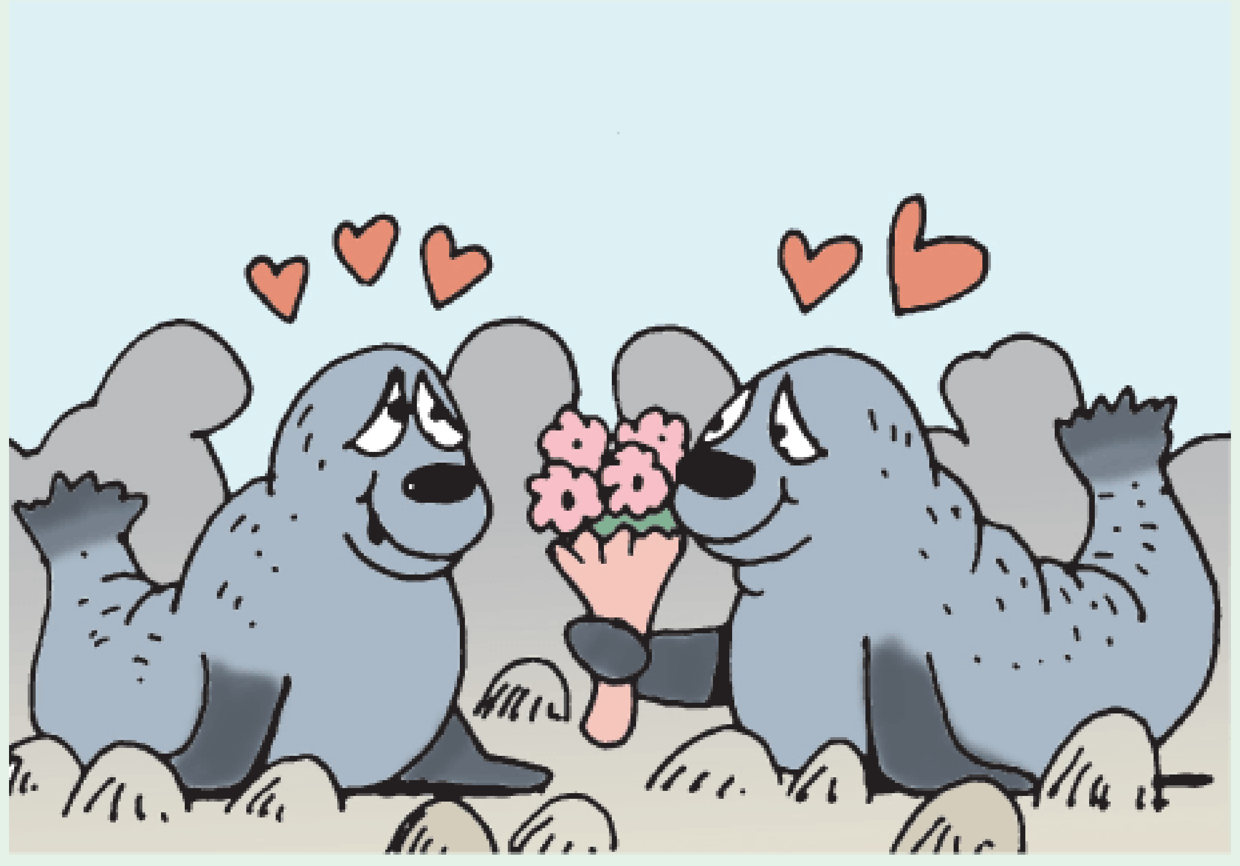
a $5^x = 1000$ **b** $15 \times (1.6)^x = 80$ **c** $400 \times (0.98)^x = 70$

10 A population of seals is given by $P(t) = 80 \times (1.15)^t$ where t is the time in years, $t \geqslant 0$.

a Find the initial population.

b Find the time required for the population to double in size.

c Find the percentage increase in population during the first 4 years.



11 The value of a computer t years after it is purchased is given by $V = k \times 0.6^t + c$ pounds, where k and c are constants.

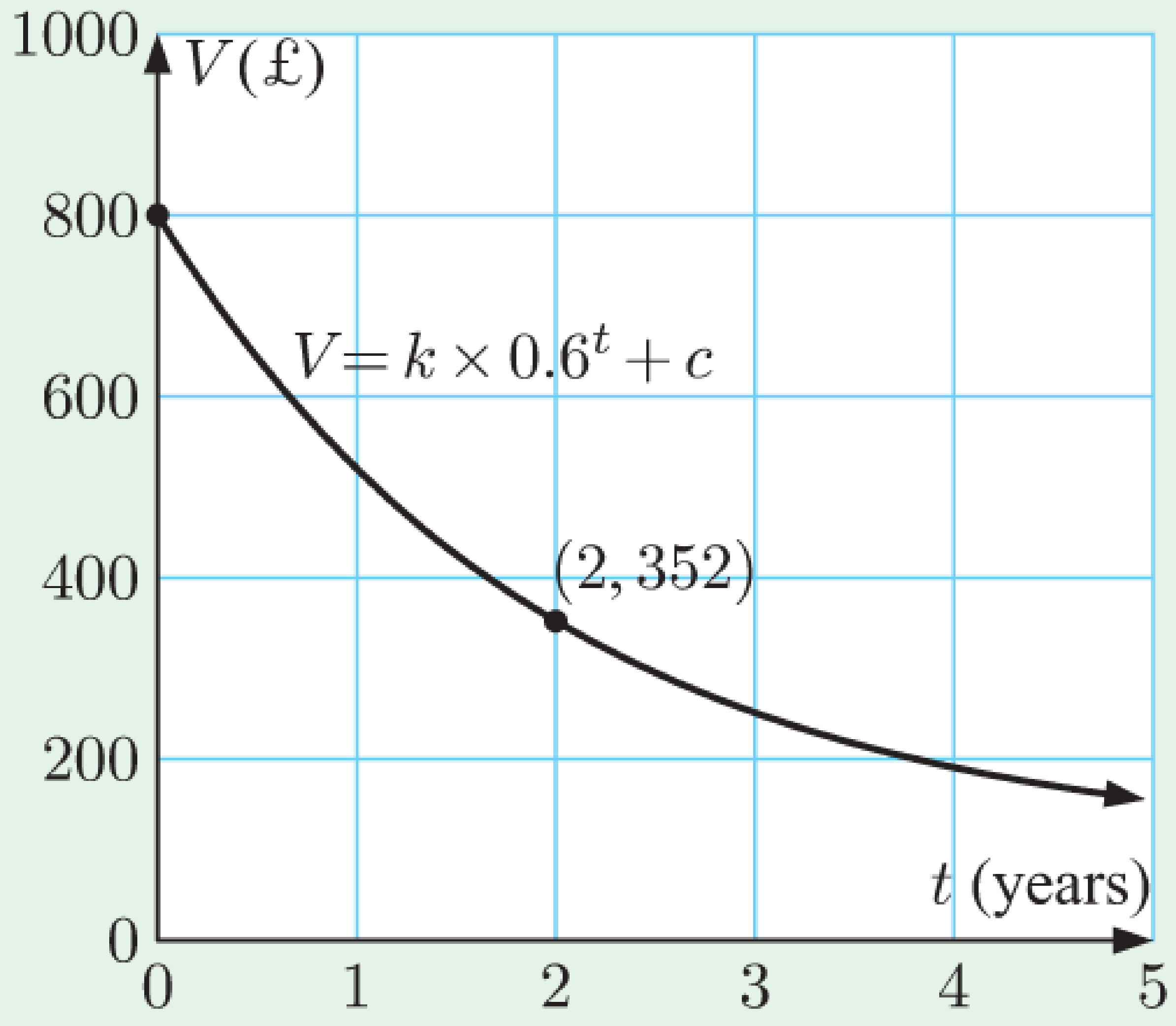
The model is graphed alongside.

a Use technology to find k and c.

b Warren thinks that the value of the computer decreases by 40% each year. Is he correct? Explain your answer.

c Find the value of the computer after 3 years.

d State the equation of the horizontal asymptote and explain what it means in this situation.



12 Find, to 3 significant figures, the value of:

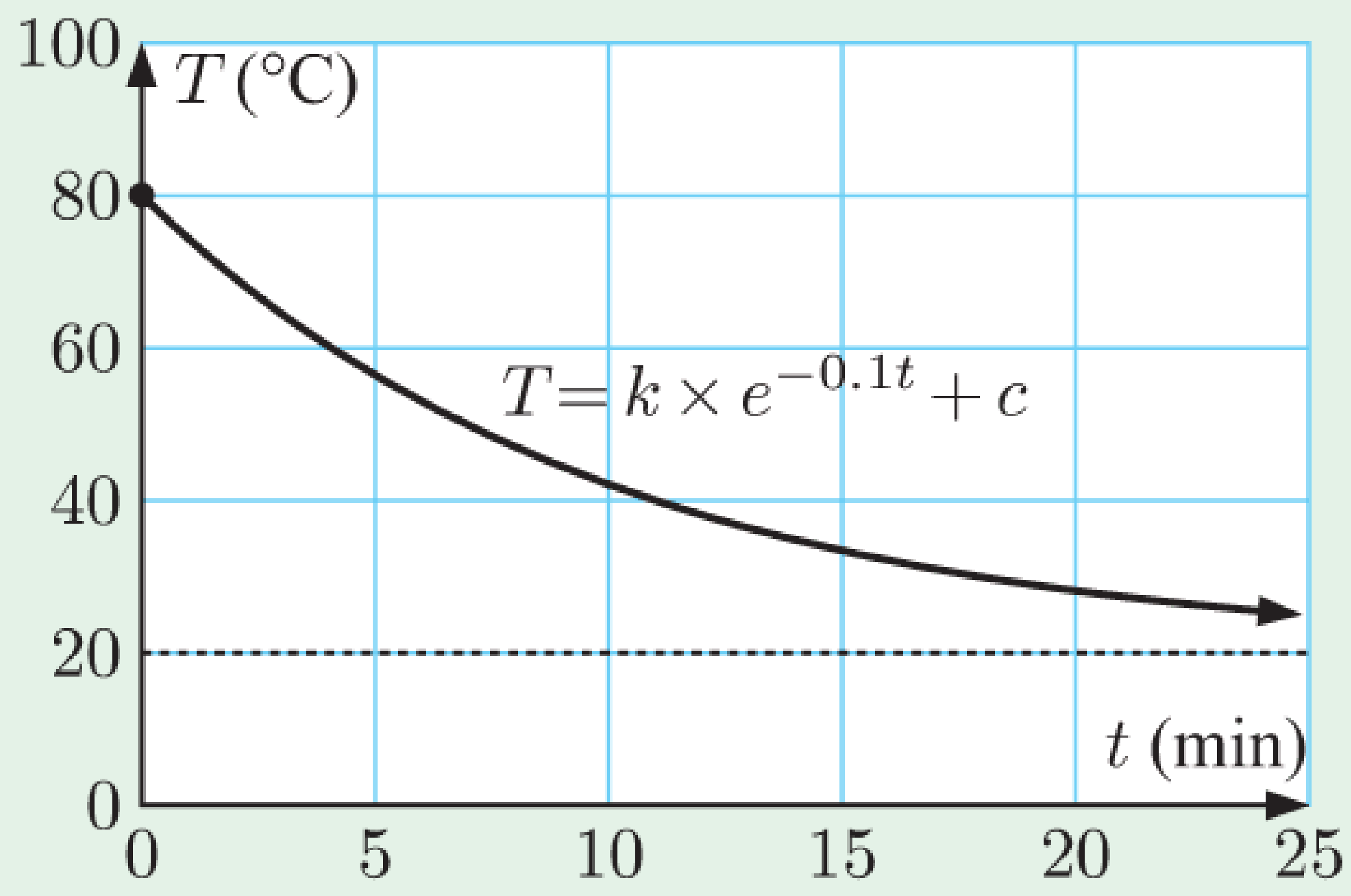
a e^4 **b** e^{-2} **c** $10e^{3.5}$ **d** $40e^{-2.53}$

13 Sketch, on the same set of axes, the graphs of:

a $y = e^x$ and $y = e^x - 2$ **b** $y = e^x$ and $y = 4e^x$

c $y = e^x$ and $y = -\frac{1}{2}e^x$ **d** $y = e^x$ and $y = e^{\frac{x}{3}}$

14 This graph shows the temperature of a mug of water t minutes after it has been poured from a kettle.



- **a** Find the exponential model connecting T and t.
- **b** Find the temperature of the water after 10 minutes.
- **c** How long will it take for the temperature of the water to fall to 30°C?

15 Use your calculator to evaluate, correct to 3 decimal places:

a $\log 27$ **b** $\log(0.58)$ **c** $\log 400$ **d** $\ln 40$

16 Write in the form 10^x, giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

17 Suppose $\ln k > 1$ but $\log k < 1$. Between which two values must k lie?

18 The point P has x-coordinate 2. It lies on the graph of the function $f(x) = 10^x$.

- **a** State the y-coordinate of P.
- **b** State the coordinates of the corresponding point on the inverse function $f^{-1}(x) = \log x$.

19 If a substance decays by $k\%$ per year, its half-life is given by $H = \dfrac{\ln 0.5}{\ln(1 - \frac{k}{100})}$ years.

- **a** Find the half-life of a substance which decays by:
  - **i** 5% per year **ii** 2.2% per year.
- **b** Substance A decays by 1.5% per year, whereas substance B decays by 0.7% per year. How much longer is substance B's half-life than substance A's half-life?
- **c** A substance has a half-life of 30 years. By what percentage does it decay each year?

The half-life of a substance is the time it takes to reduce to half of its original weight.

REVIEW SET 8B

1 For the function $g(x) = 3^{-x} - 2$, find: **a** $g(0)$ **b** $g(1)$ **c** $g(-1)$ **d** $g(3)$

2 For the function $f(x) = 3 \times 2^x - 12$, show that:

a the y-intercept is -9 **b** the x-intercept is 2.

3 Consider the exponential function $y = \frac{1}{2} \times 4^x$.

- **a** Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ for the function.
- **b** Hence graph the function.

4 State the equation of the horizontal asymptote of:

a $y = -\frac{1}{2} \times 5^x + 3$ **b** $y = 2^{-x} - 4$ **c** $y = 6 - 8 \times \left(\frac{1}{3}\right)^x$

5 Consider $f(x) = -2 \times 3^x - 4$.

a Find $f(-2)$ and $f(2)$.

b For the graph of $y = f(x)$, determine:

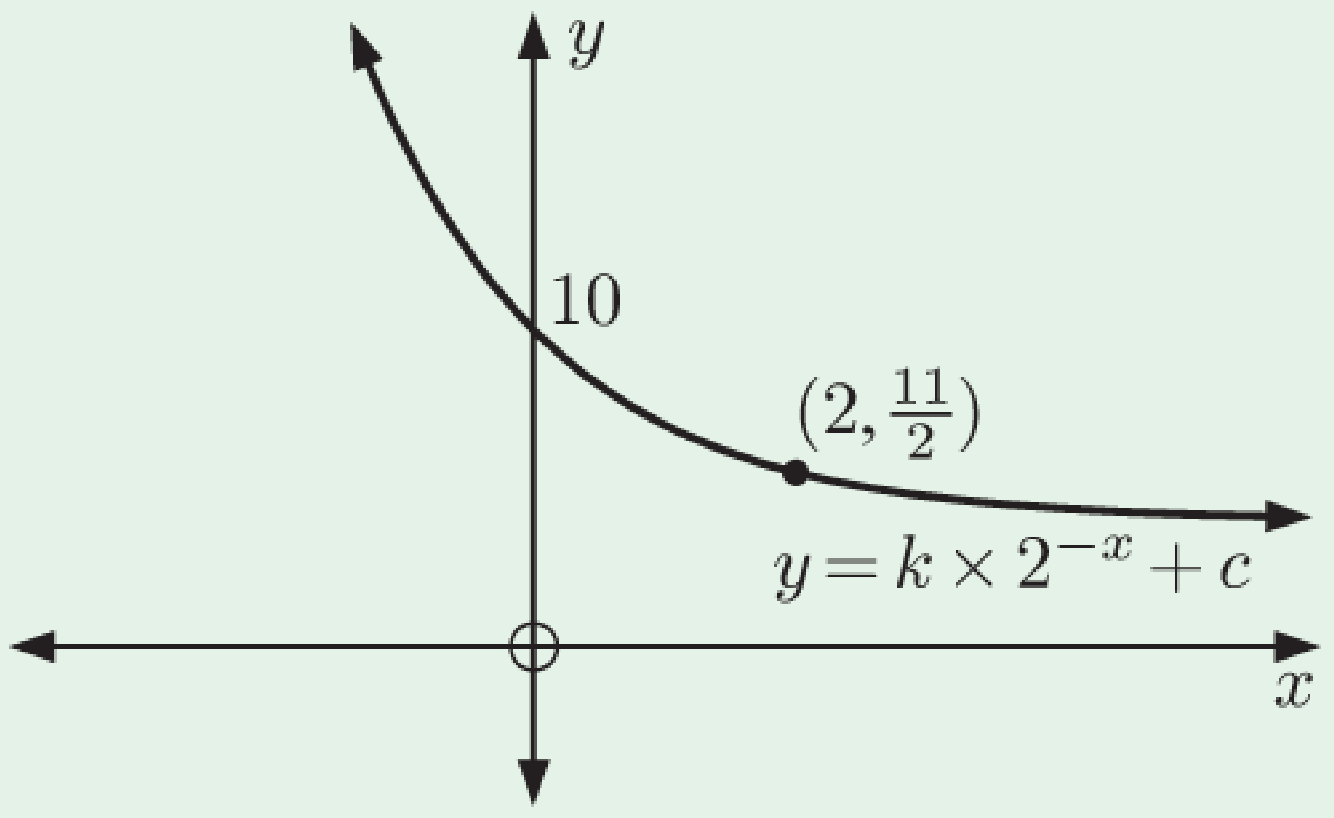
i the y-intercept **ii** the horizontal asymptote.

c Sketch $y = f(x)$, showing the details found in **a** and **b**.

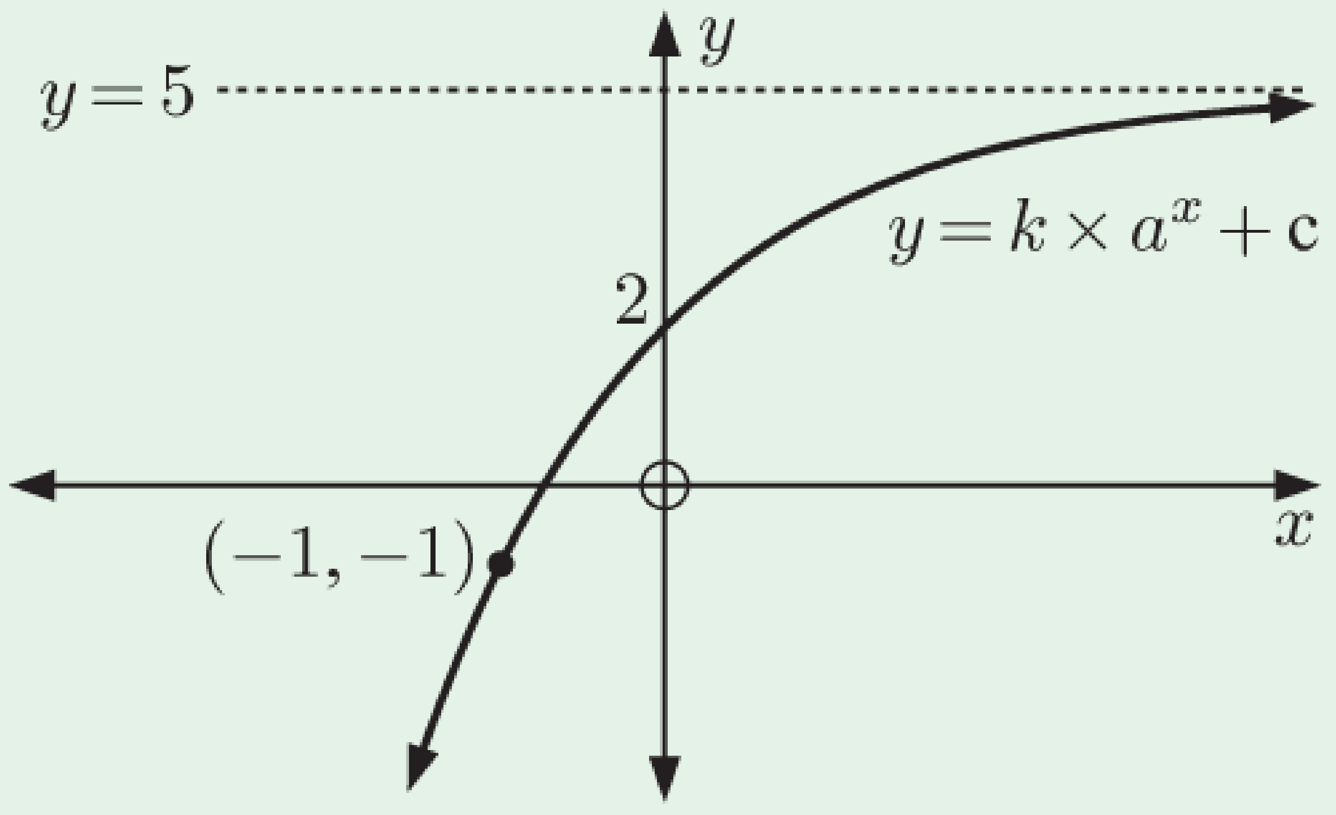
d State the domain and range of $y = f(x)$.

6 Find the exponential model in the graph:

a



b



7 An exponential model of the form $y = k \times 5^{-x} + c$ has x-intercept 1 and y-intercept 8.

a Find the exponential model. **b** Find the value of y when $x = -1$.

8 Solve using technology:

a $4^x = 30$ **b** $2 \times (1.6)^x = 7$ **c** $50 \times (0.65)^x = 15$

9 For what values of k does the equation $4 \times (0.3)^x - 2 = k$ have:

a 1 solution **b** no solutions?

10 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t for $t \geqslant 0$.

d Hence find the time taken for the weight to reduce to 100 grams.

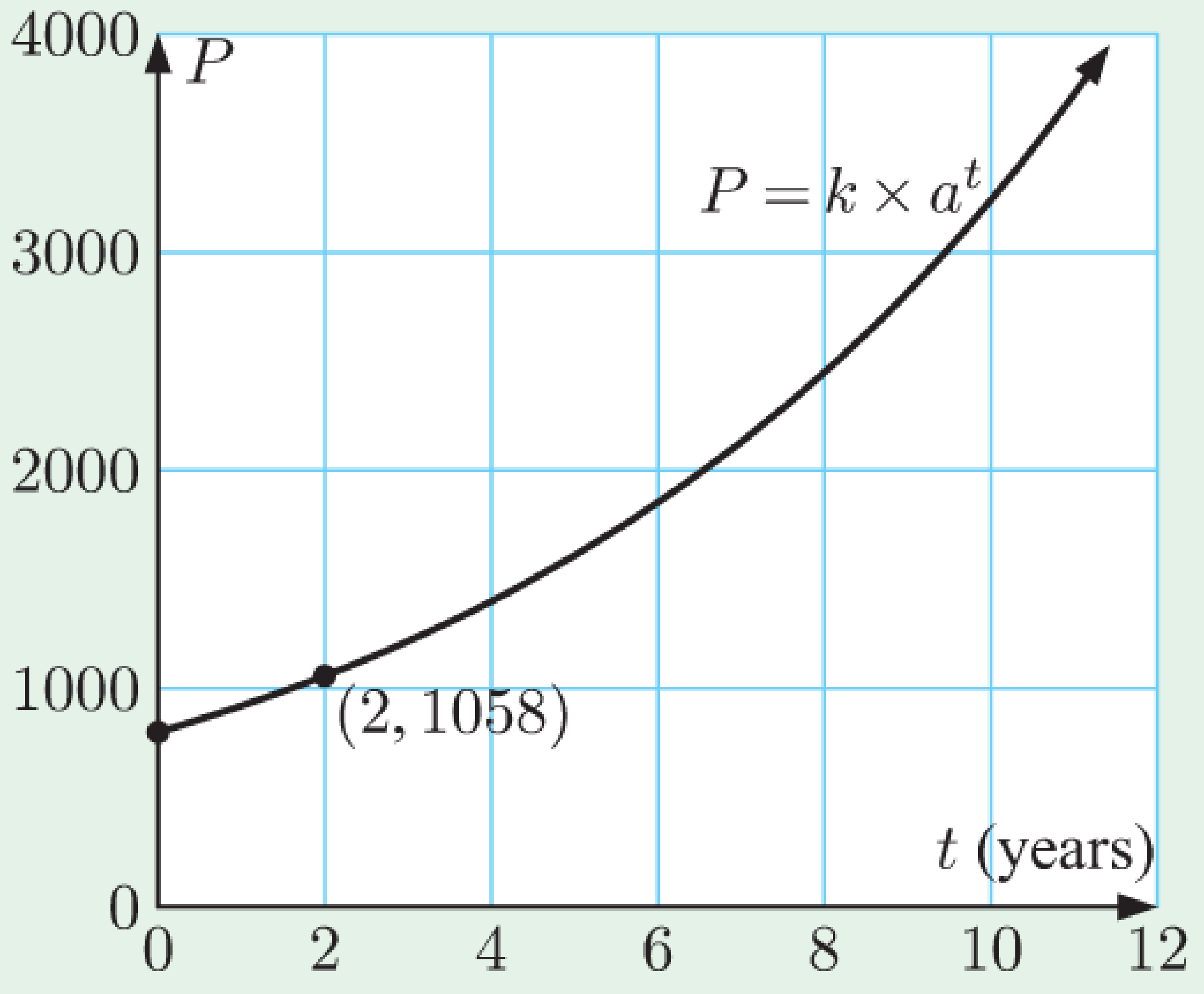
11 When a species of bird is introduced onto an island, its population increases by 15% each year. The population t years after introduction is given by $P = k \times a^t$, where k and a are constants, and $t \geqslant 0$.

a State the value of a.

b Find the value of k and interpret your answer.

c Find the population after 5 years.

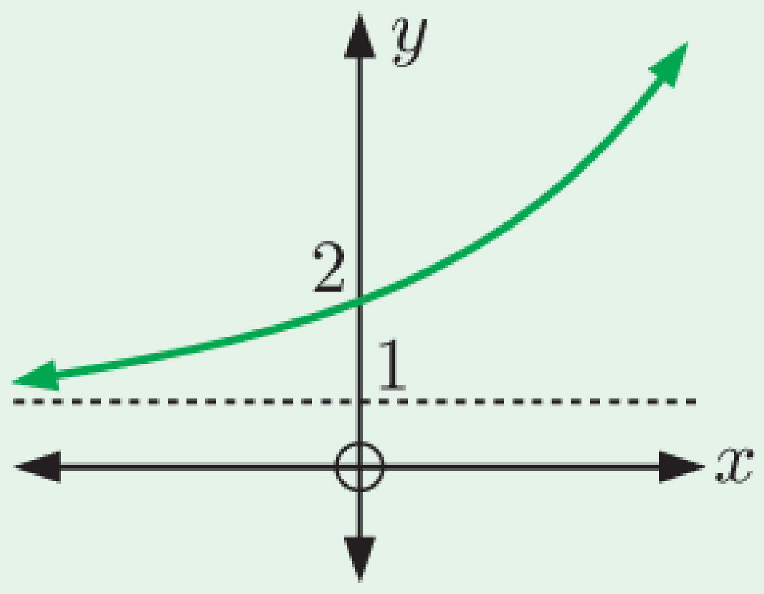
d Do you think this model is likely to be accurate for very large values of t? Explain your answer.



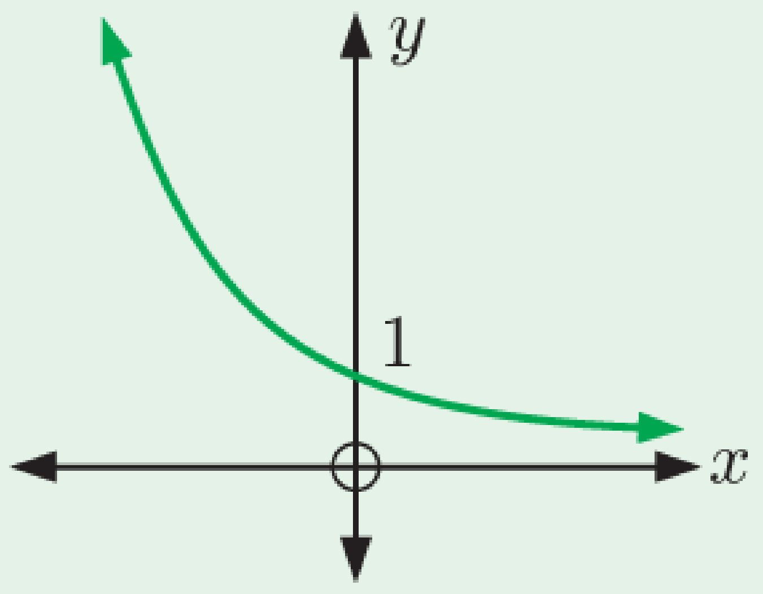
12 Match each equation to its corresponding graph:

a $y = -e^x$ **b** $y = 3 \times 2^x$ **c** $y = e^x + 1$ **d** $y = 3^{-x}$ **e** $y = -e^{-x}$

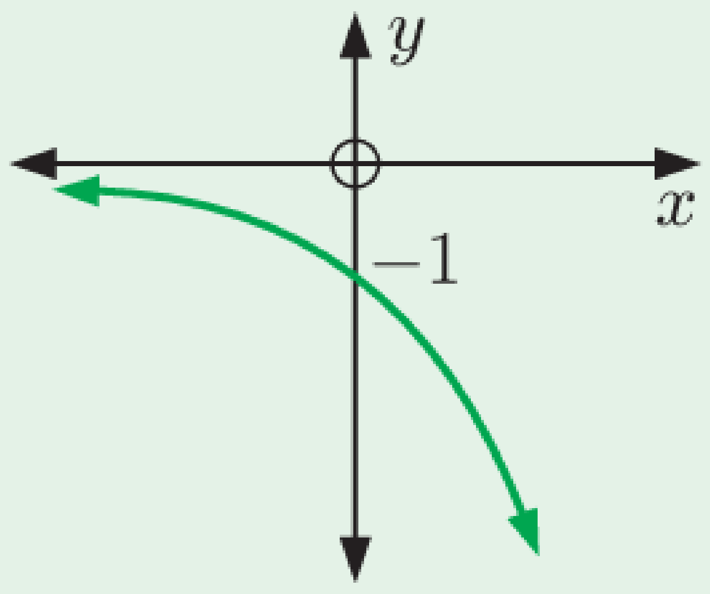
A



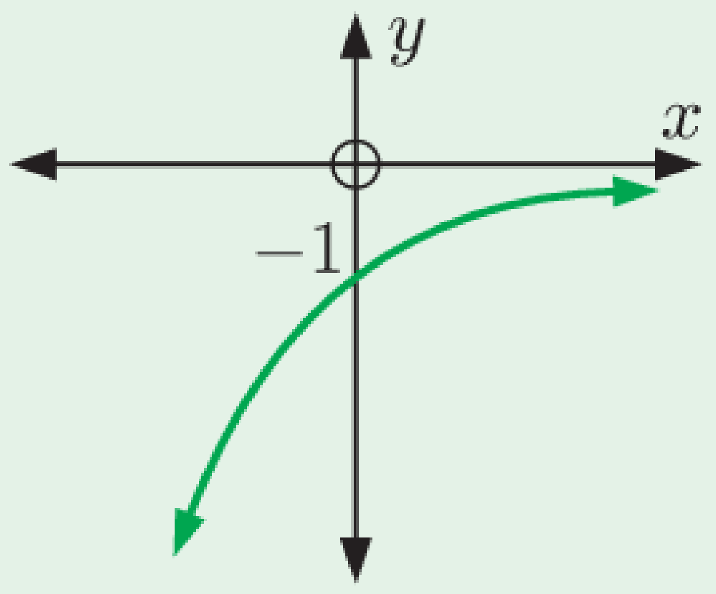
B



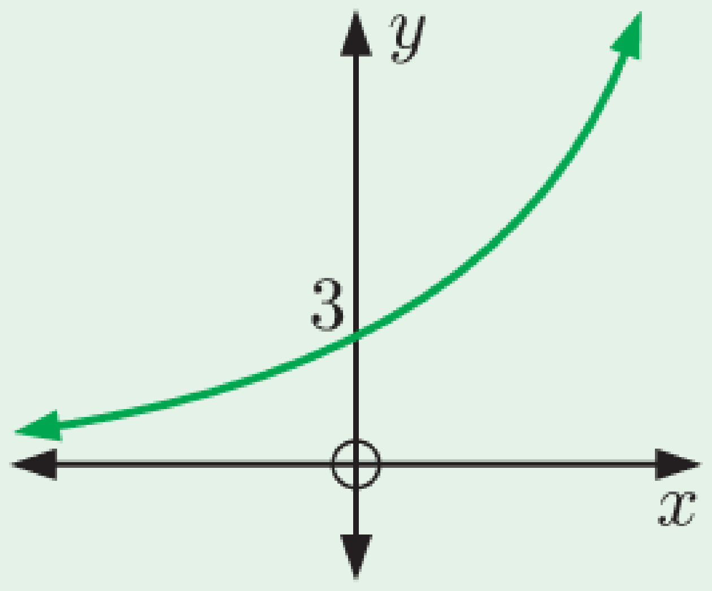
C



D



E

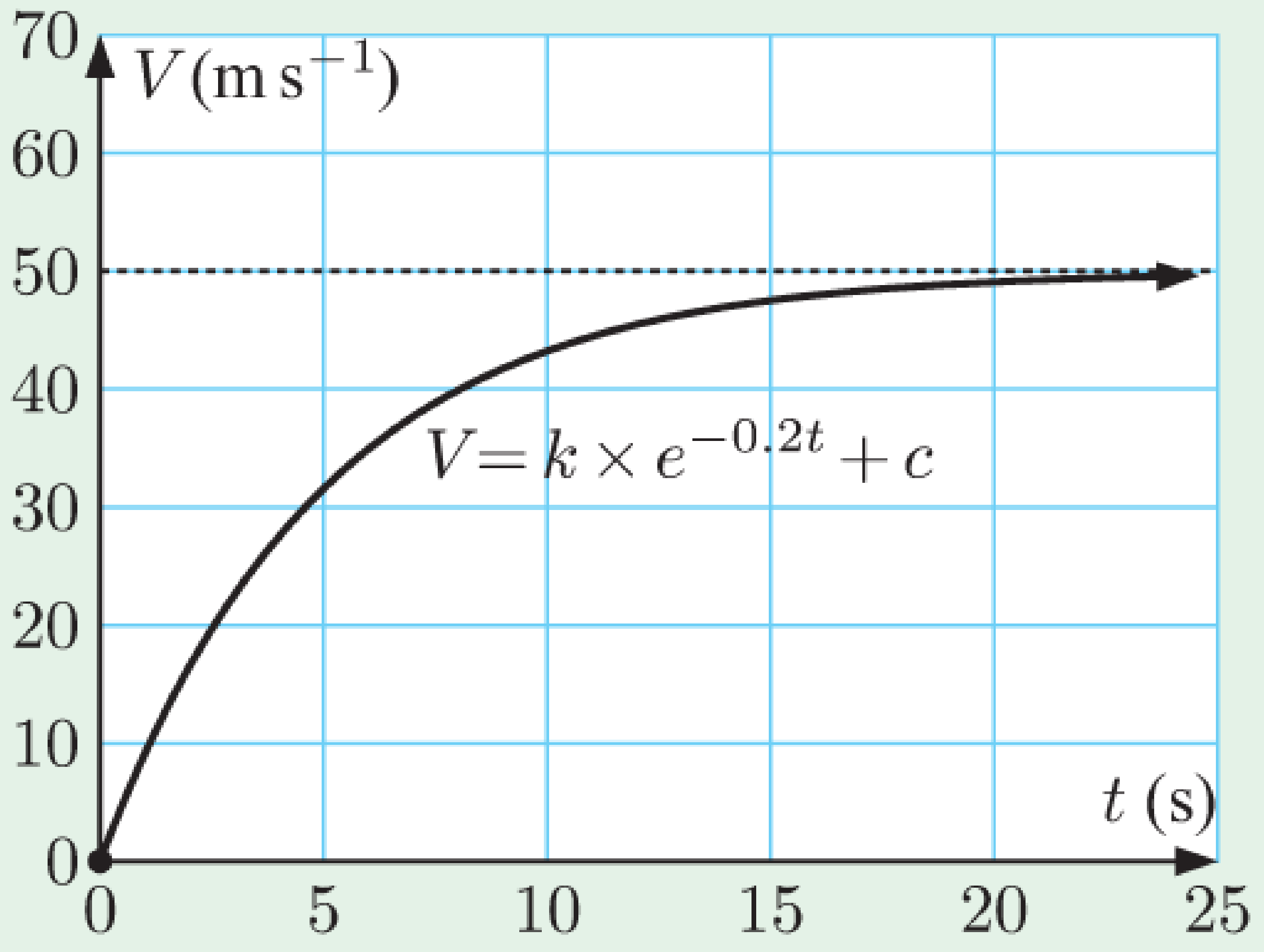


13 **a** Use technology to help sketch the graph of $f(x) = 3 - e^x$.

b State the domain and range of the function.

c Describe the behaviour of the function as $x \to \pm\infty$.

14 A skydiver jumps from an aeroplane. His speed of descent is given by $V = k \times e^{-0.2t} + c$ m s^{-1}, where k and c are constants and t is the time in seconds.



a Use the information in the graph to find k and c.

b Find the speed of the skydiver after 4 seconds.

c How long will it take for the skydiver's speed to reach 40 m s^{-1}?

15 Without using a calculator, find:

a $\log 10\,000$ **b** $\ln\left(\frac{1}{e^5}\right)$ **c** $\log\left(\frac{10^c}{1000}\right)$ **d** $e^{-\ln 4}$

16 Use your calculator to find, correct to 4 significant figures:

a $\log 125$ **b** $\log(0.03)$ **c** $\ln 19$ **d** $\ln\left(\frac{2}{3}\right)$

17 The apparent magnitude of a star with brightness b Watts per m^2 is given by $M = -2.5 \log\left(\frac{b}{2.84 \times 10^{-8}}\right)$.

a The Sun has brightness 1.4×10^3 Watts per m^2. Find the apparent magnitude of the Sun.

b The star Alpha Centauri has apparent magnitude -0.27. Find the brightness of Alpha Centauri.

18 Write in the form e^x, where x is correct to 4 decimal places:

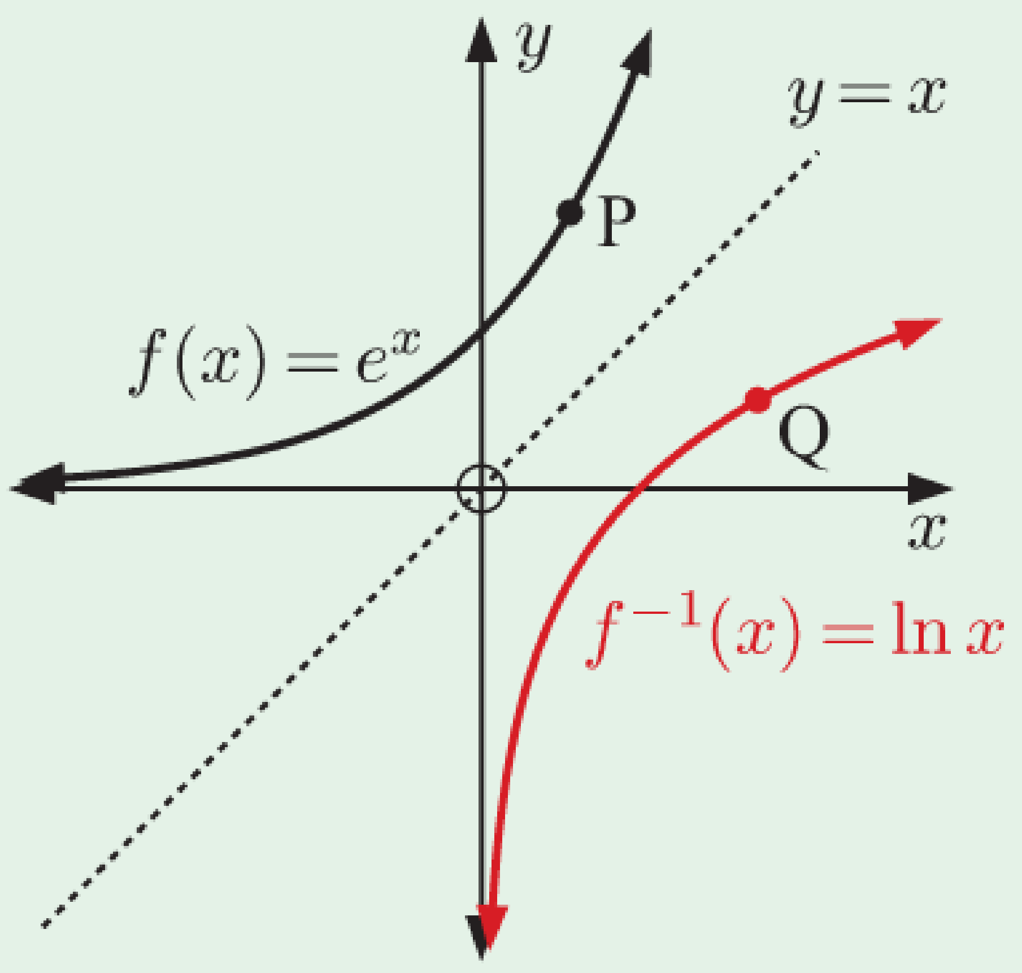
a 20 **b** 3000 **c** 0.075

19 The graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are shown alongside.

The point P has x-coordinate $\ln 2$.

a Find the y-coordinate of P.

b State the coordinates of Q, the reflection of P in the line $y = x$.



Chapter 9

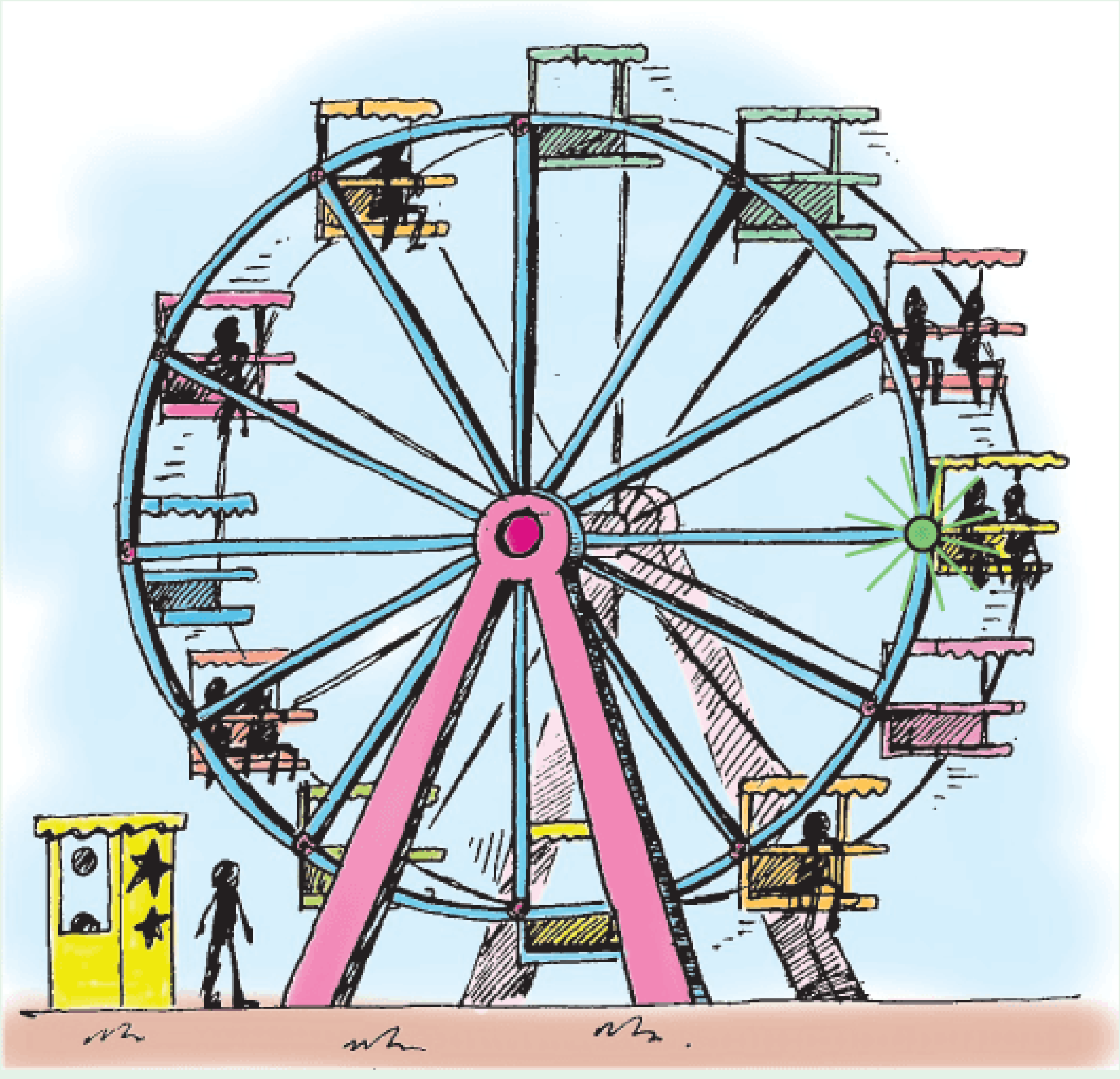
Trigonometric functions

Contents:

- **A** The unit circle
- **B** Periodic behaviour
- **C** The sine and cosine functions
- **D** General sine and cosine functions
- **E** Modelling periodic behaviour

OPENING PROBLEM

A Ferris wheel rotates anticlockwise at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From his viewing point next to the ticket booth, Andrew is watching a green light on the perimeter of the wheel. He notices that the green light moves in a circle. It takes 100 seconds for a full revolution.



Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

- in front of the wheel
- a side-on position
- above the wheel.

DEMO

You can then observe graphs of the green light's position as the wheel rotates at a constant rate.

Things to think about:

a Andrew estimates how high the light is above ground level at two second intervals. What will a graph of this data look like? Assume that the light is initially in the position shown.

b Andrew then estimates the horizontal position of the light at two second intervals. What will a graph of this data look like?

c What similarities and differences will there be between your two graphs?

d Can you write a function which will give the:

  i height of the light at any time t seconds

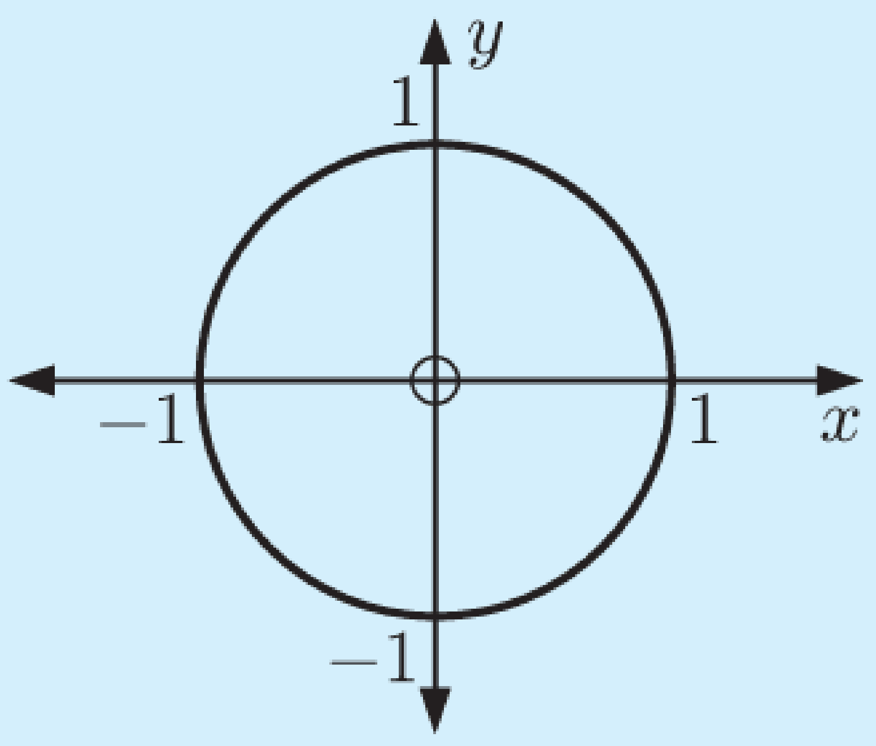
  ii horizontal displacement of the light at any time t seconds?

A THE UNIT CIRCLE

When we introduced non-right angled triangle trigonometry, we used the **unit circle** to give meaning to the trigonometric ratios for obtuse angles.

The **unit circle** is the circle with centre $(0, 0)$ and radius 1 unit.

The equation of the unit circle is $x^2 + y^2 = 1$.

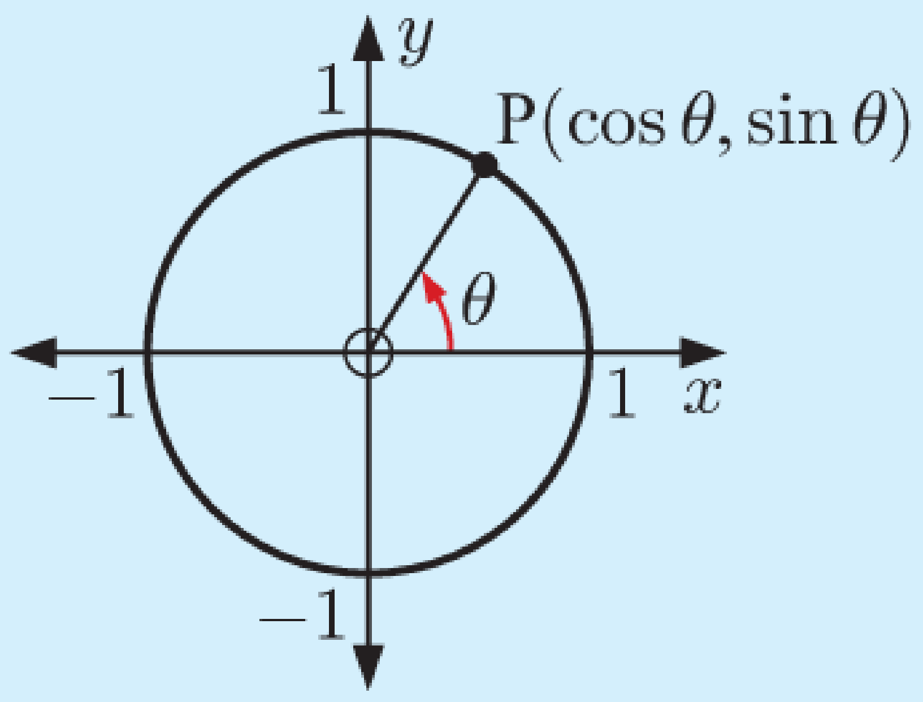


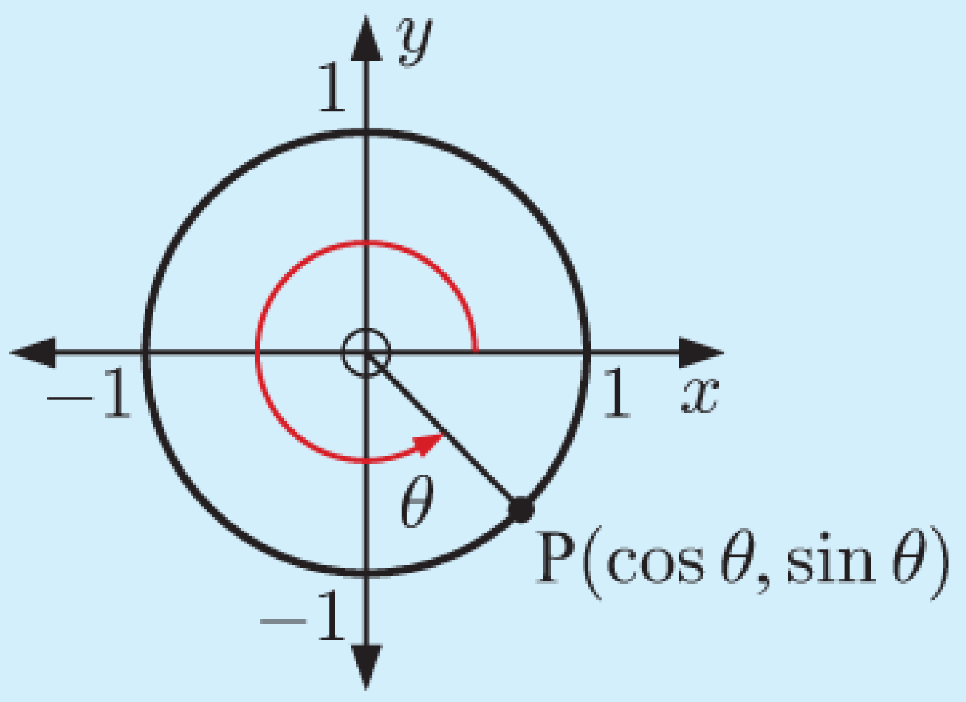
We now extend our definitions of sine and cosine to include *all* angles. This will allow us to study the sine and cosine *functions*.

DEFINITION OF SINE AND COSINE

If P is any point on the unit circle such that [OP] makes an angle θ measured anticlockwise from the positive x-axis:

- $\boldsymbol{\cos\theta}$ is the x-coordinate of P
- $\boldsymbol{\sin\theta}$ is the y-coordinate of P


y
1
P(cos θ, sin θ)
θ
−1
1 x
−1


y
1
−1
1 x
θ
P(cos θ, sin θ)
−1

For all points on the unit circle, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$. We therefore conclude:

For any angle θ:

- $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$
- $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

PERIODICITY OF THE TRIGONOMETRIC RATIOS

Since there are 360° in a full revolution, if we add any integer multiple of 360° to θ, then the position of P on the unit circle is unchanged.

For any $k \in \mathbb{Z}$, $\boldsymbol{\cos(\theta + 360k^\circ) = \cos\theta}$ and $\boldsymbol{\sin(\theta + 360k^\circ) = \sin\theta}$.

This **periodic** feature is an important property of the trigonometric ratios.

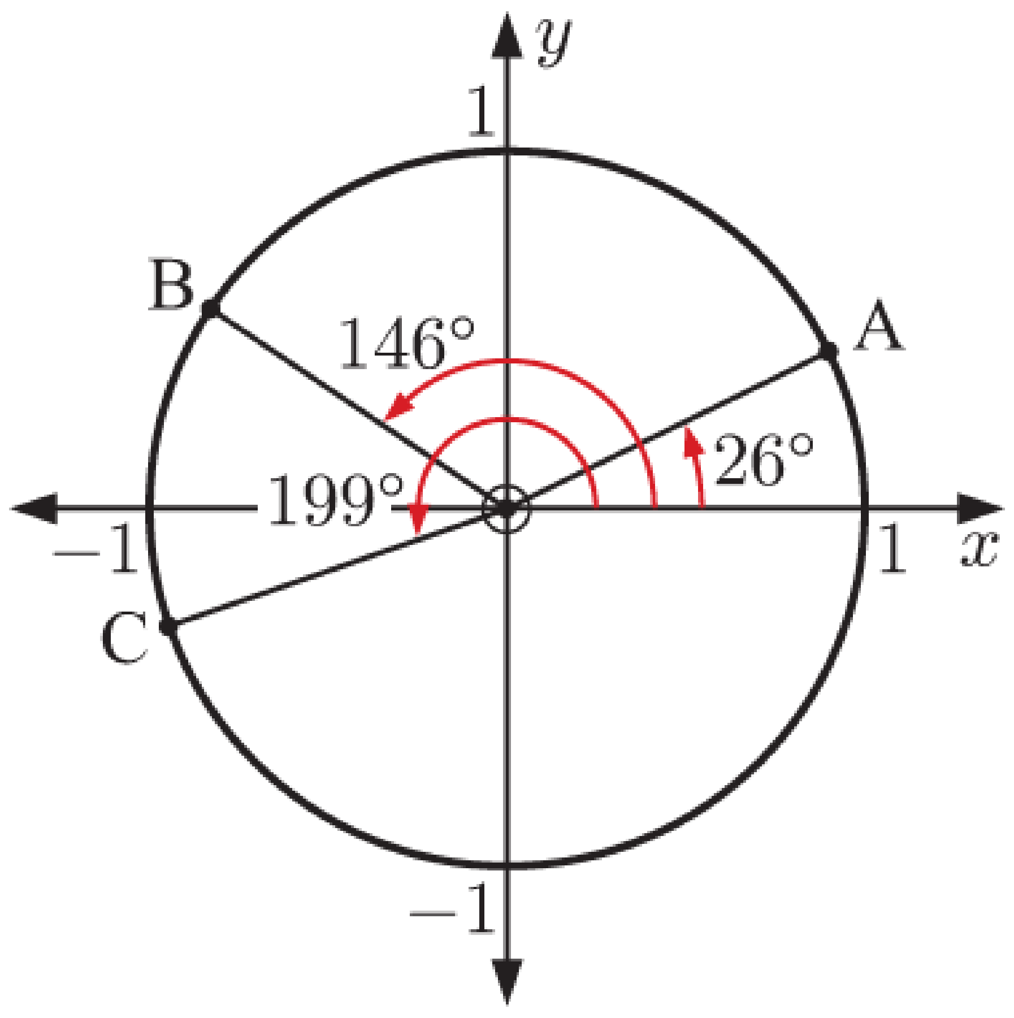
EXERCISE 9A

1 For each unit circle illustrated:

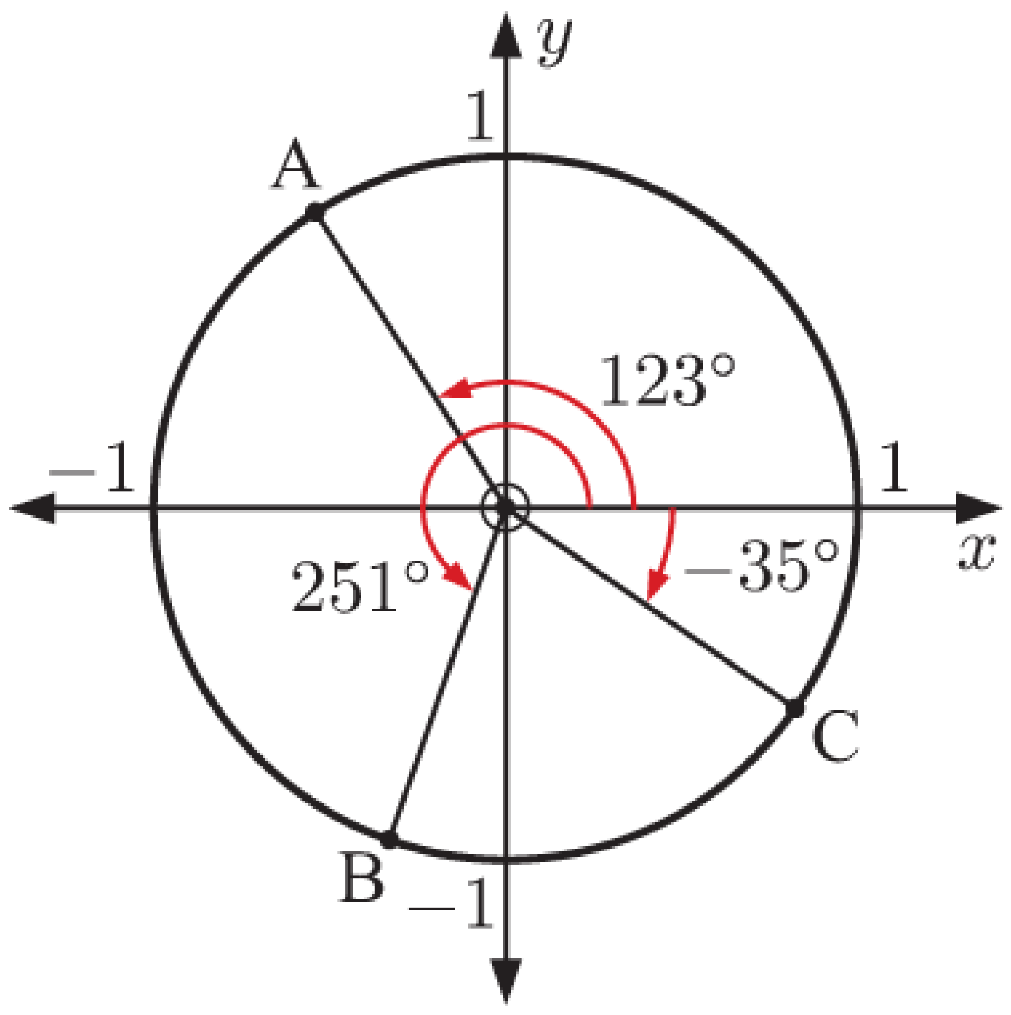
i State the exact coordinates of points A, B, and C in terms of sine and cosine.

ii Use your calculator to give the coordinates of A, B, and C correct to 3 decimal places.

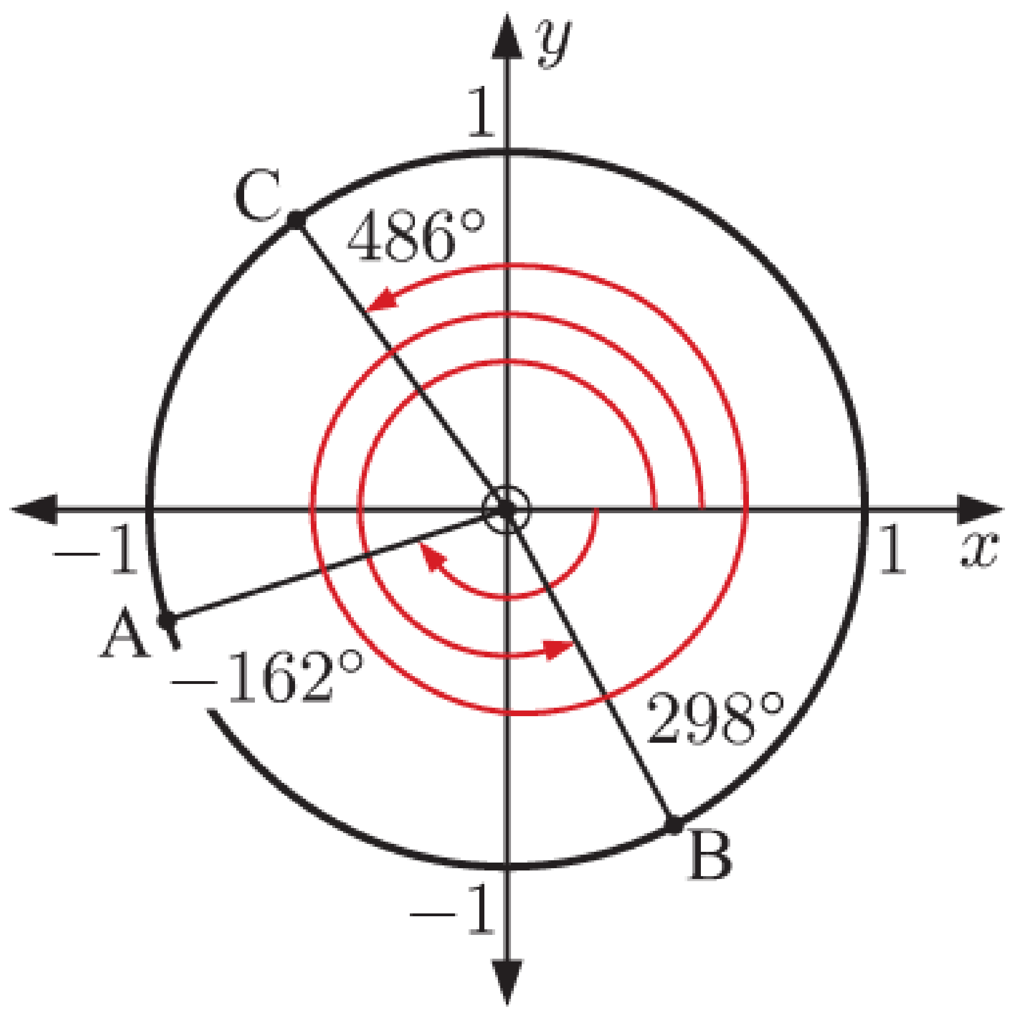
a

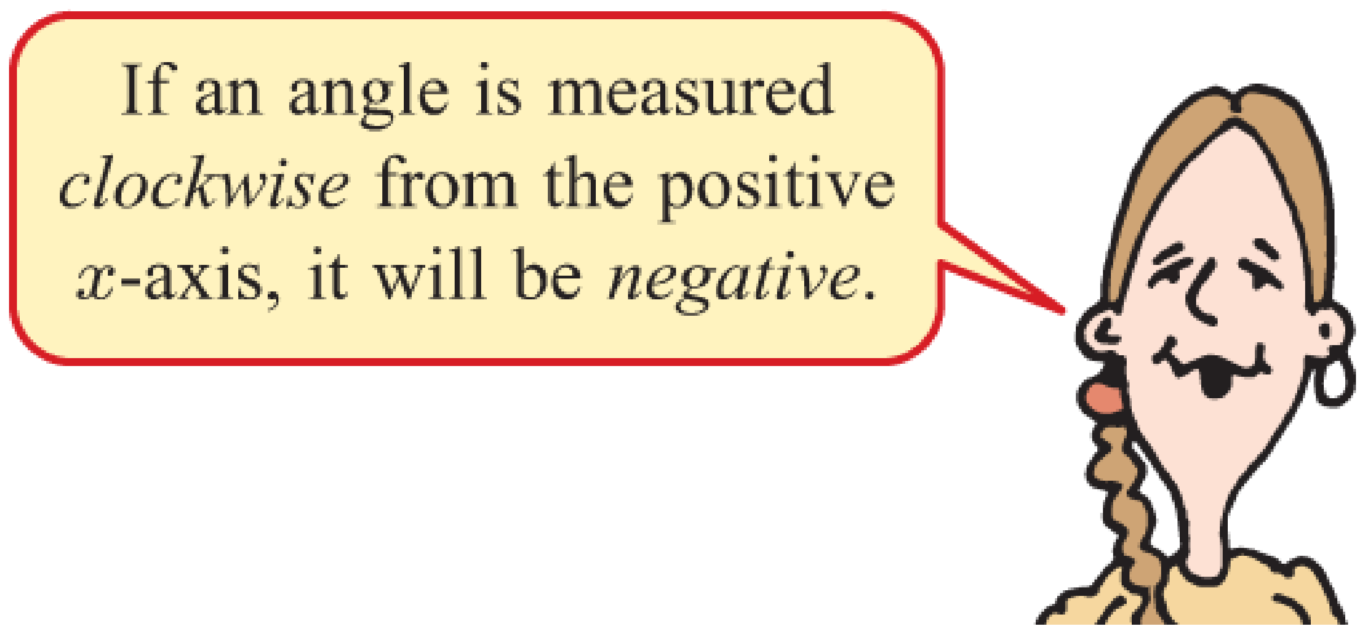

y
1
B
146°
A
26°
199°
−1
1 x
C
−1

b


y
1
A
123°
−1
1
x
251°
−35°
C
B
−1

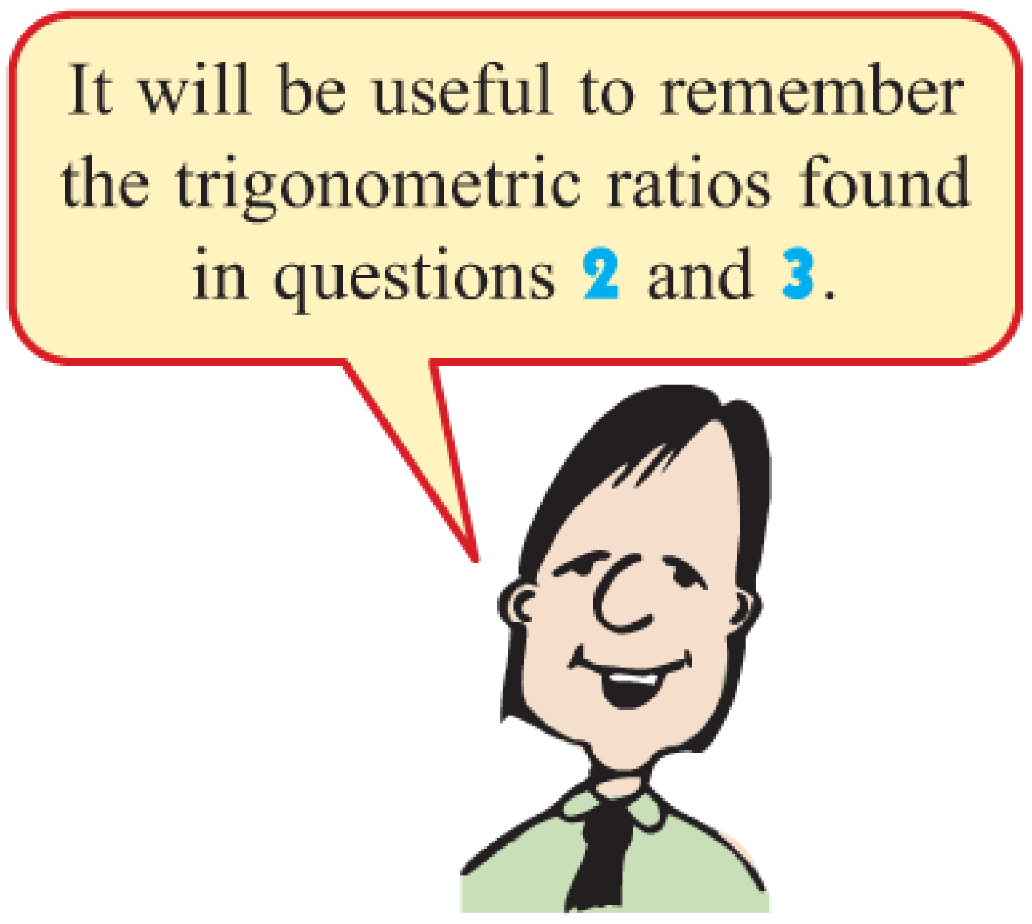
c


y
1
C
486°
−1
1 x
A
−162°
298°
B
−1


If an angle is measured *clockwise* from the positive x-axis, it will be *negative*.

2 With the aid of a unit circle, complete the following table:

θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$					
$\cos\theta$					



It will be useful to remember the trigonometric ratios found in questions **2** and **3**.

3 **a** Use your calculator to evaluate:

i $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. Use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
sine							
cosine							

4 **a** Copy and complete:

Quadrant	*Degree measure*	$\cos\theta$	$\sin\theta$
1	$0° < \theta < 90°$	positive	positive
2			
3			
4			

b In which quadrants are the following true?

i $\cos\theta$ is positive.
ii $\cos\theta$ is negative.
iii $\cos\theta$ and $\sin\theta$ are both negative.
iv $\cos\theta$ is negative and $\sin\theta$ is positive.

5 **a** State the range of possible values which $\sin\theta$ can take.

b Find the value(s) of θ on the interval $0° \leqslant \theta \leqslant 360°$ for which $\sin\theta$ is:

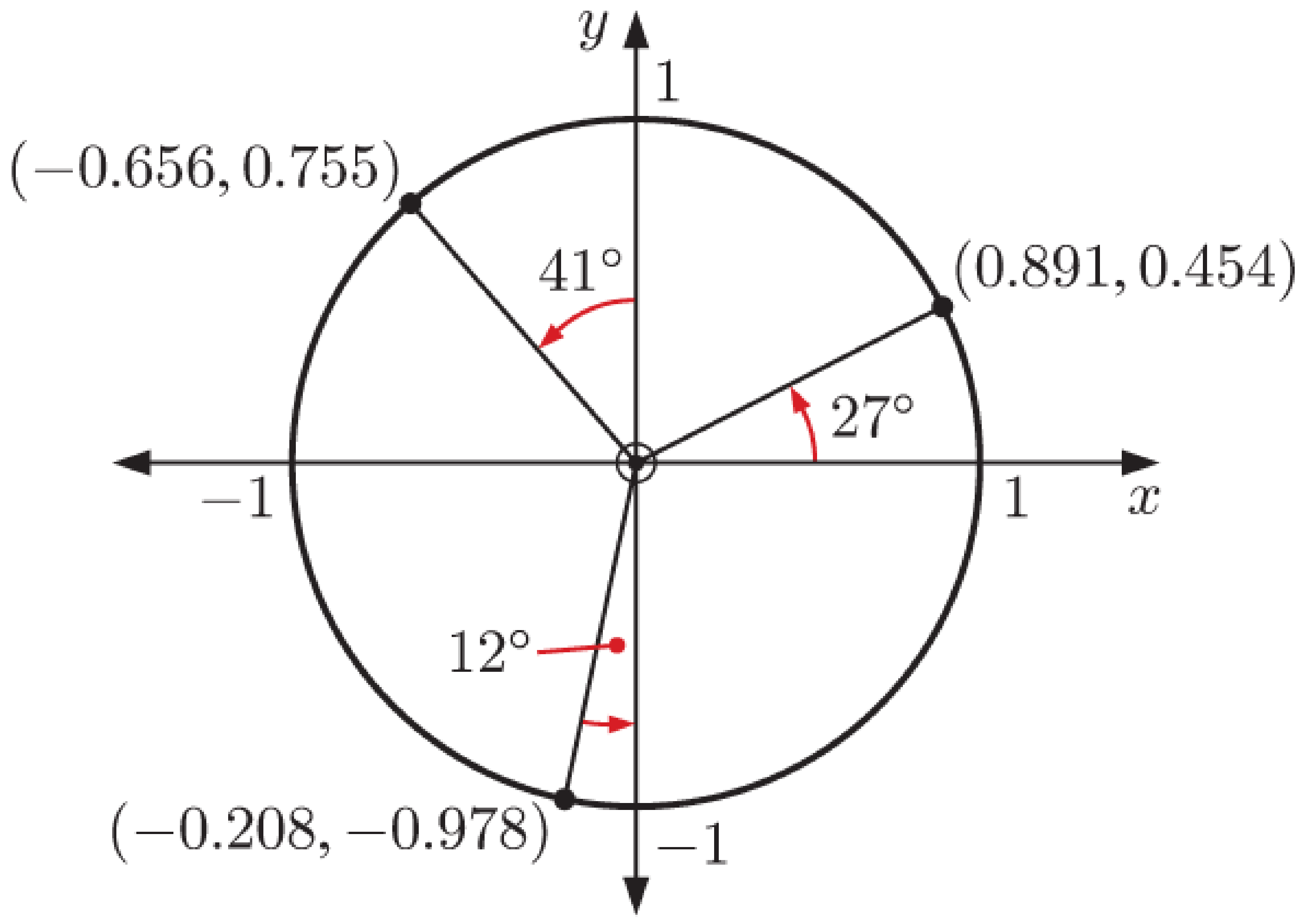
i maximised
ii minimised
iii zero
iv positive
v negative.

6 Explain why:

a $\cos 400° = \cos 40°$
b $\sin 130° = \sin(-230°)$
c $\cos 70° = \cos 790°$

7 The coordinates on the unit circle alongside have been rounded to 3 decimal places.
Use the unit circle to determine the value of:

a $\cos 27°$
b $\sin 27°$
c $\sin 131°$
d $\cos 258°$
e $\cos 387°$
f $\cos 491°$
g $\sin(-102°)$
h $\sin 747°$



B PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. For example, in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

These phenomena illustrate variable behaviour which is repeated over time. The repetition may be called **periodic**, **oscillatory**, or **cyclic** in different situations.

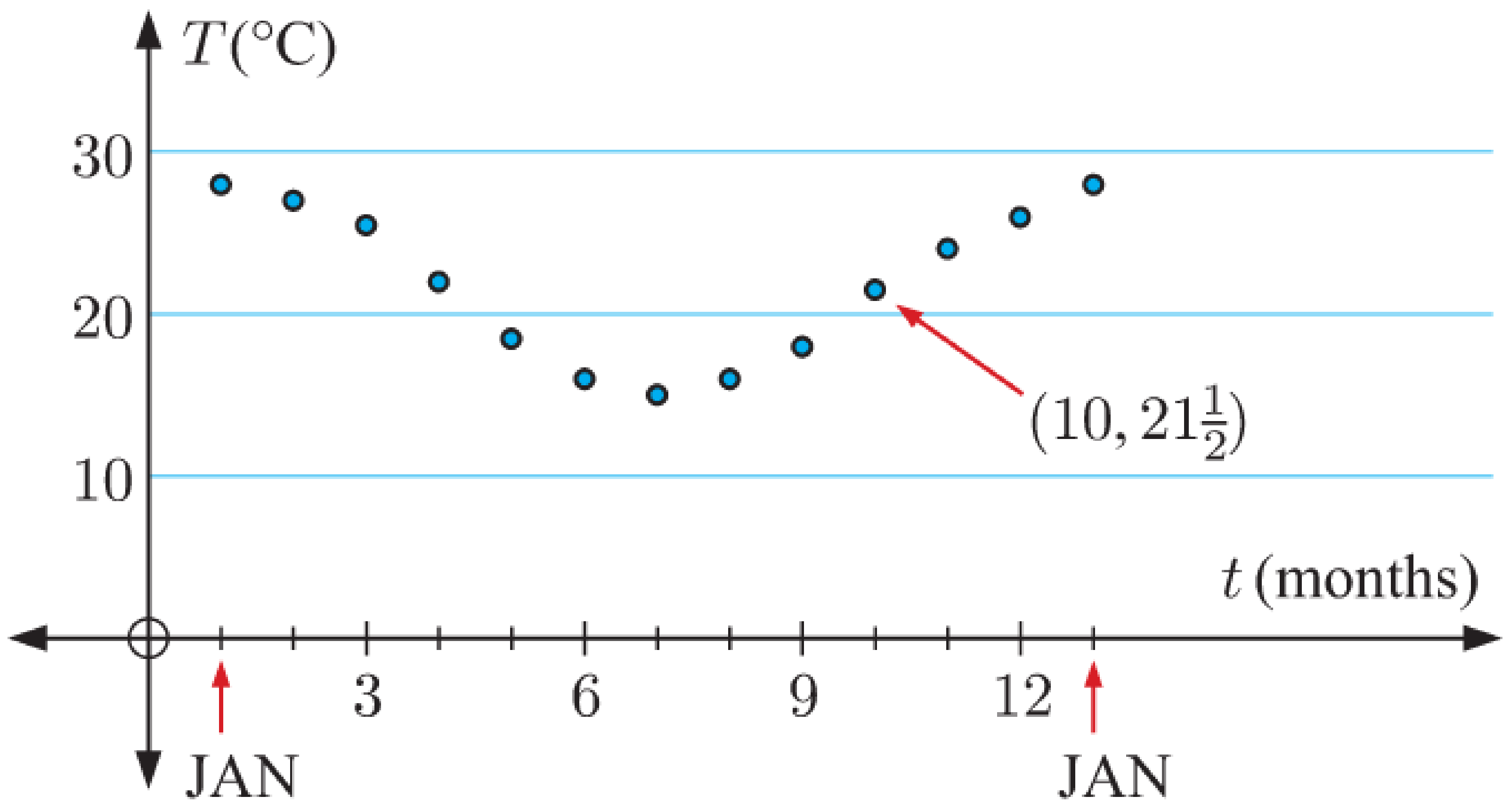
In this Chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

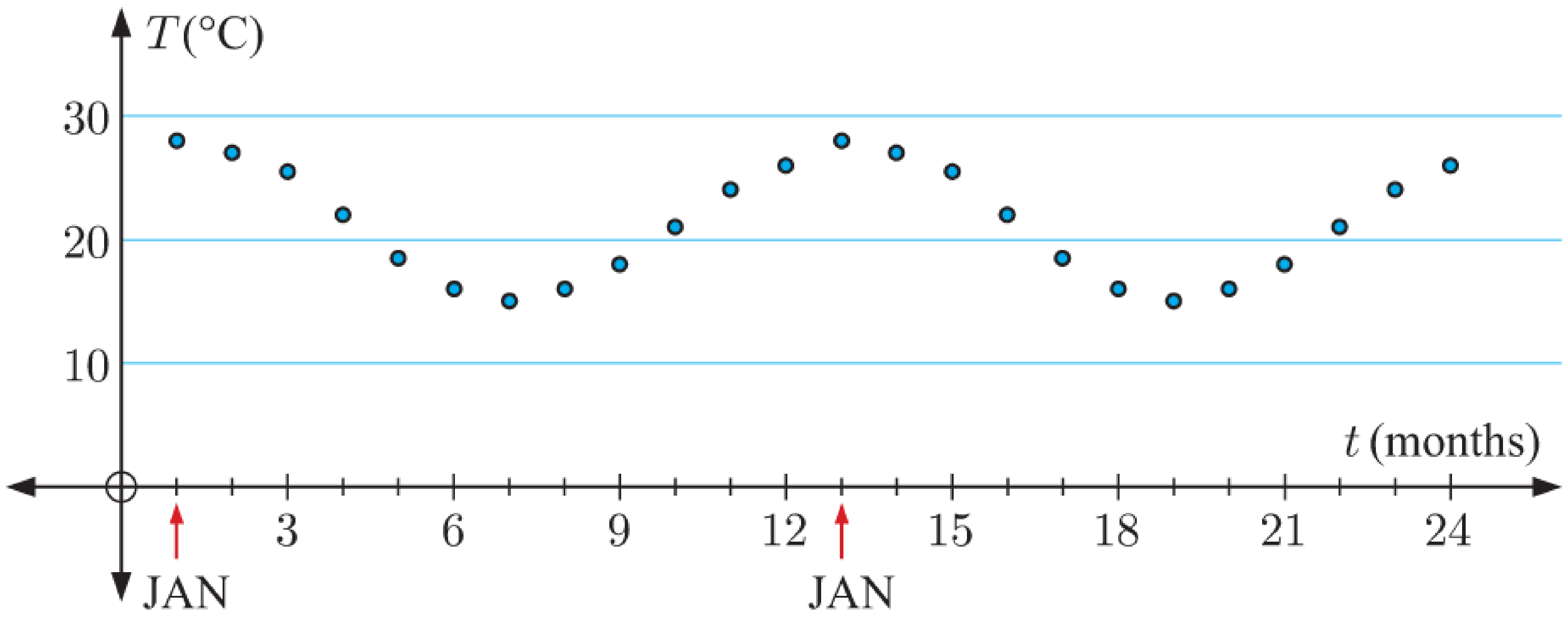
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (T °C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

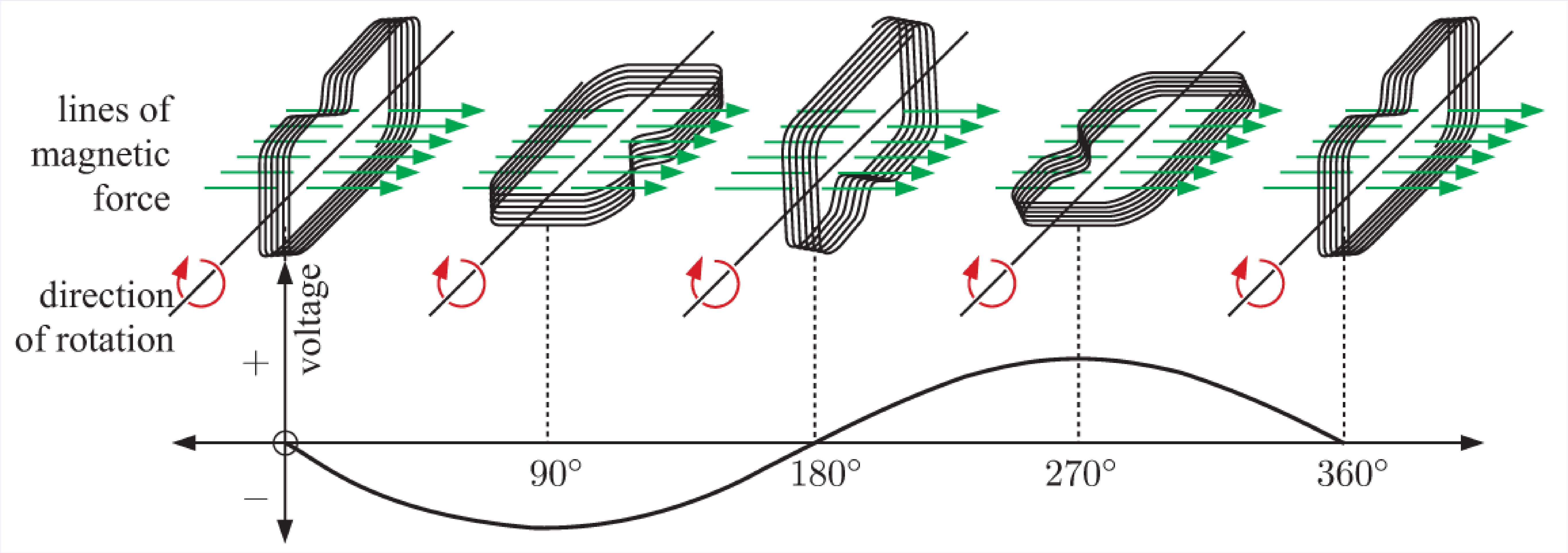
On the scatter diagram alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the Chapter we will be able to establish a **periodic function** which approximately fits this set of points.



HISTORICAL NOTE



In 1831, **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire through 360° in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values in a periodic function called a **sine wave**.

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length.

The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period p if $\boldsymbol{f(x+p)=f(x)}$ for all x, and p is the smallest positive value for this to be true.

A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

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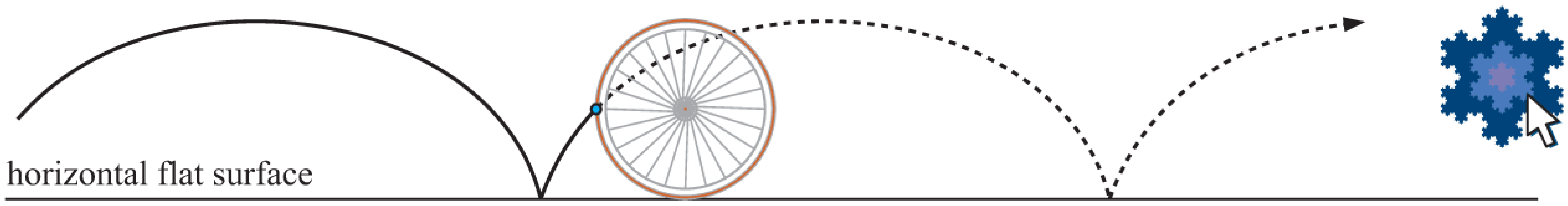


ACTIVITY PERIODIC FUNCTIONS

Use a **graphing package** to examine the function $f(x)=x-\lfloor x\rfloor$ where $\lfloor x\rfloor$ is "the largest integer less than or equal to x".

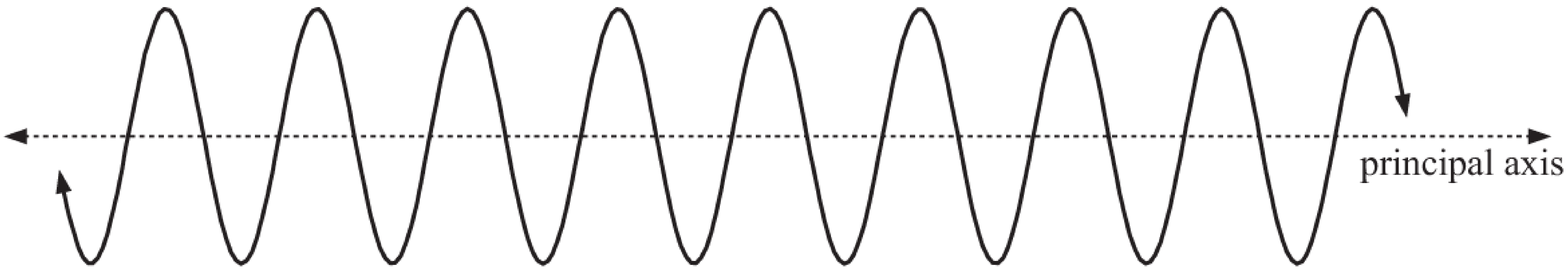
In the graphing package, you type $\lfloor x\rfloor$ as floor(x).

Is $f(x)$ periodic? What is its period?

GRAPHING PACKAGE

WAVES

In this course we are mainly concerned with periodic phenomena which show a wave pattern:



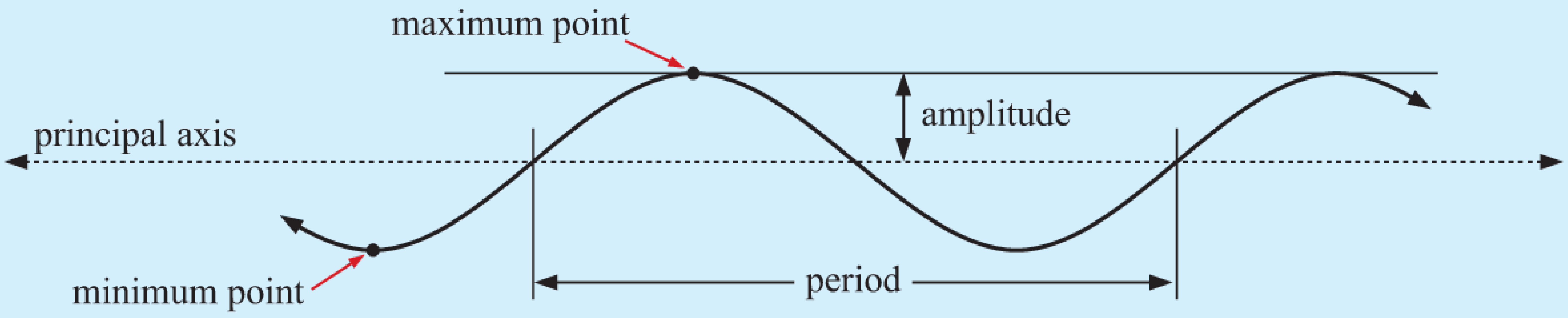
A **wave** oscillates about a horizontal line called the **principal axis** or **mean line**.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

If the maximum and minimum values of the wave are **max** and **min** respectively, then the principal axis has equation $y = \dfrac{\textbf{max} + \textbf{min}}{2}$.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

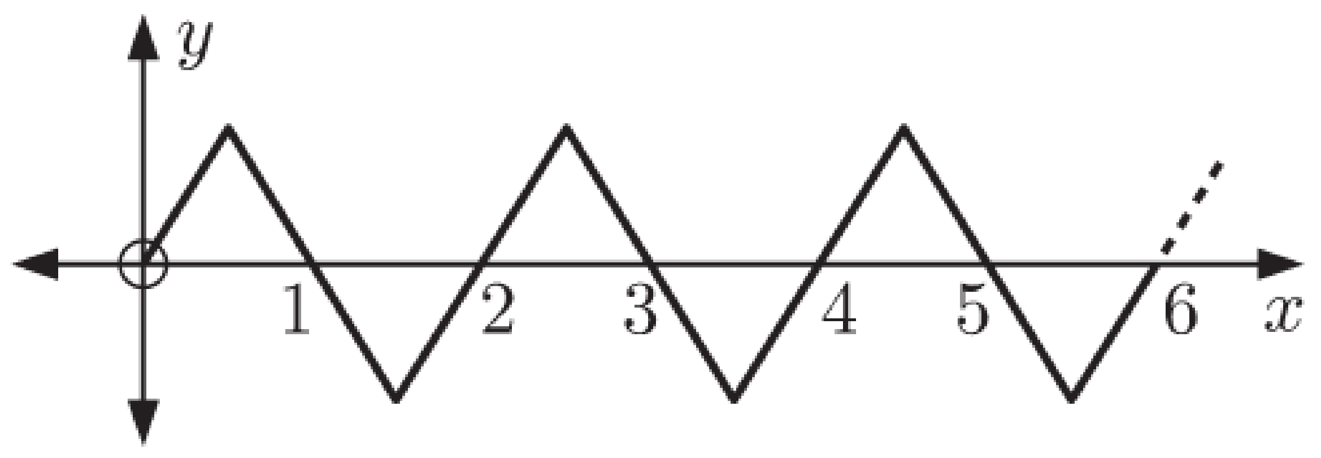
$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{2}$$



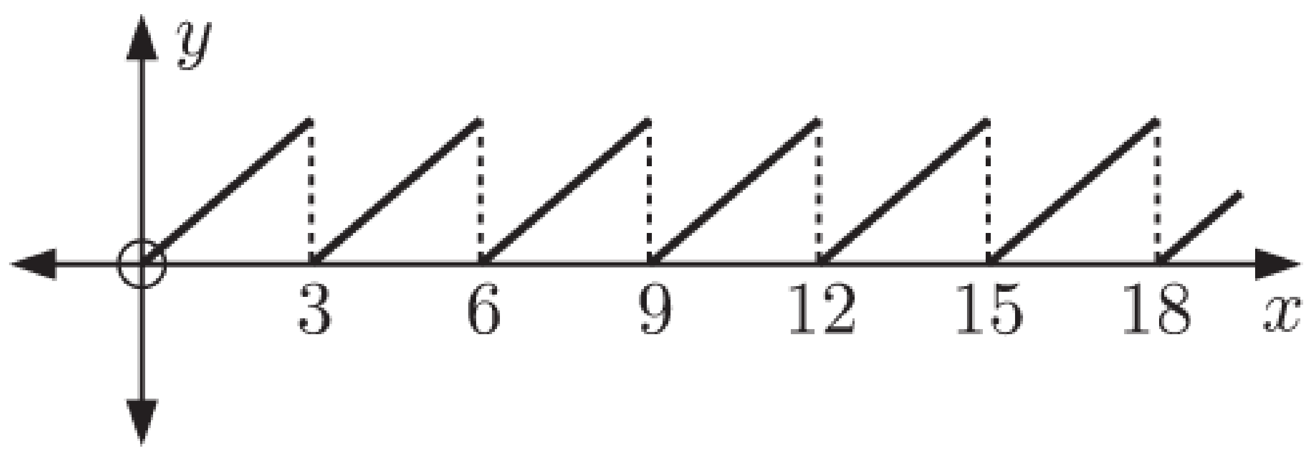
EXERCISE 9B

1 Which of these graphs show periodic behaviour?

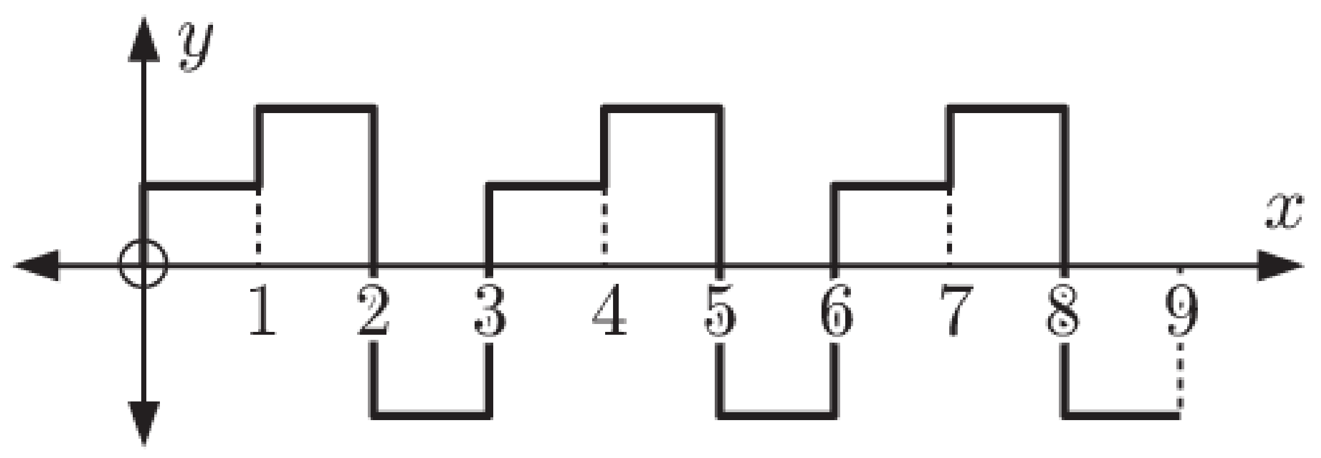
a



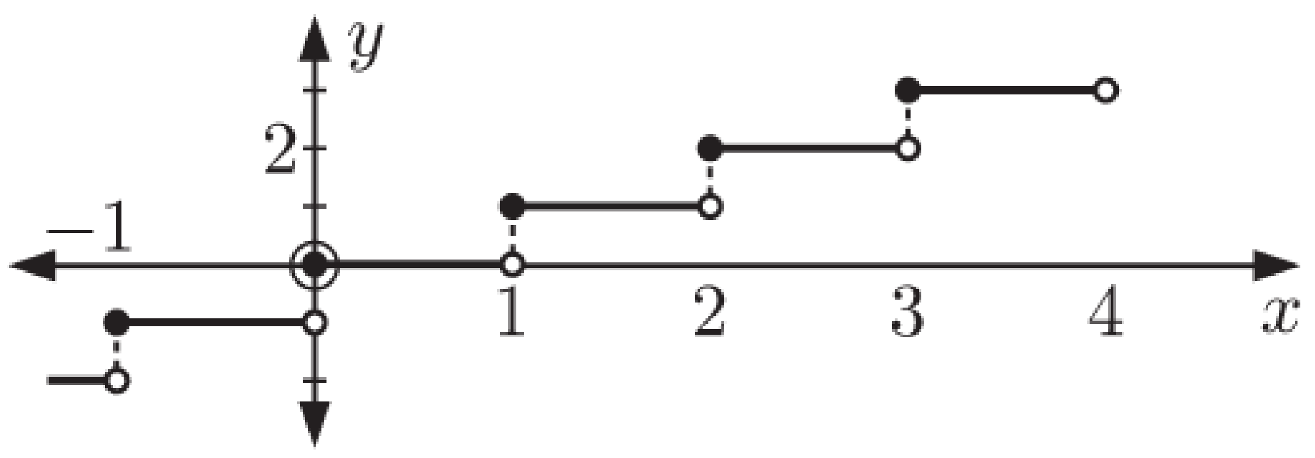
b



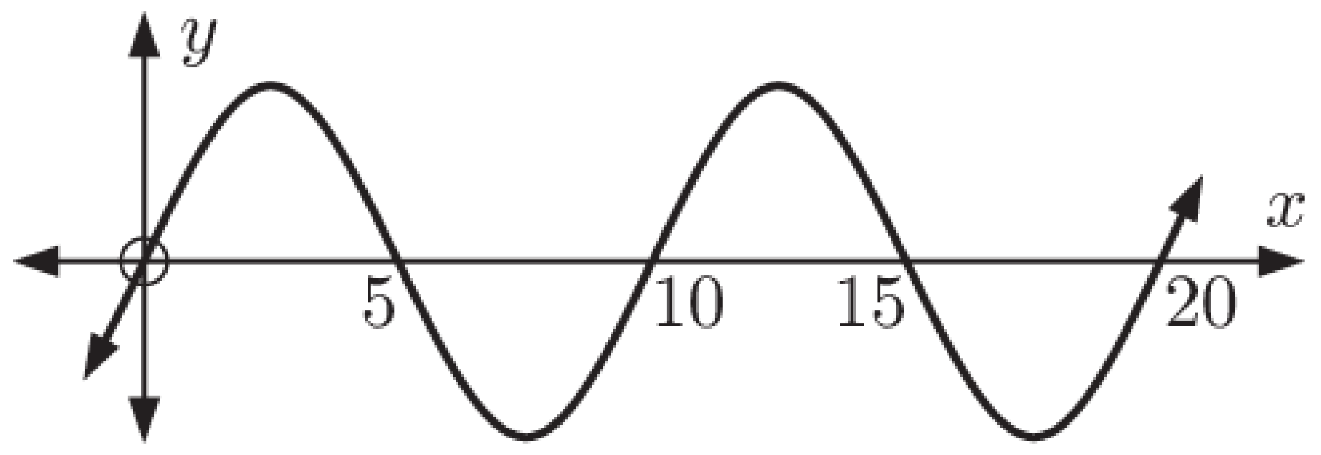
c



d



e



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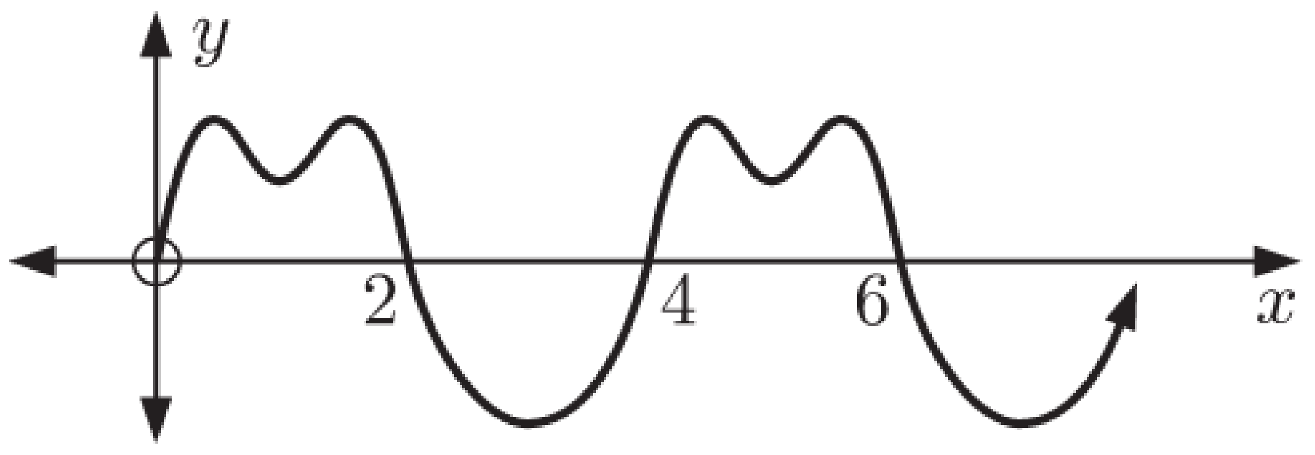
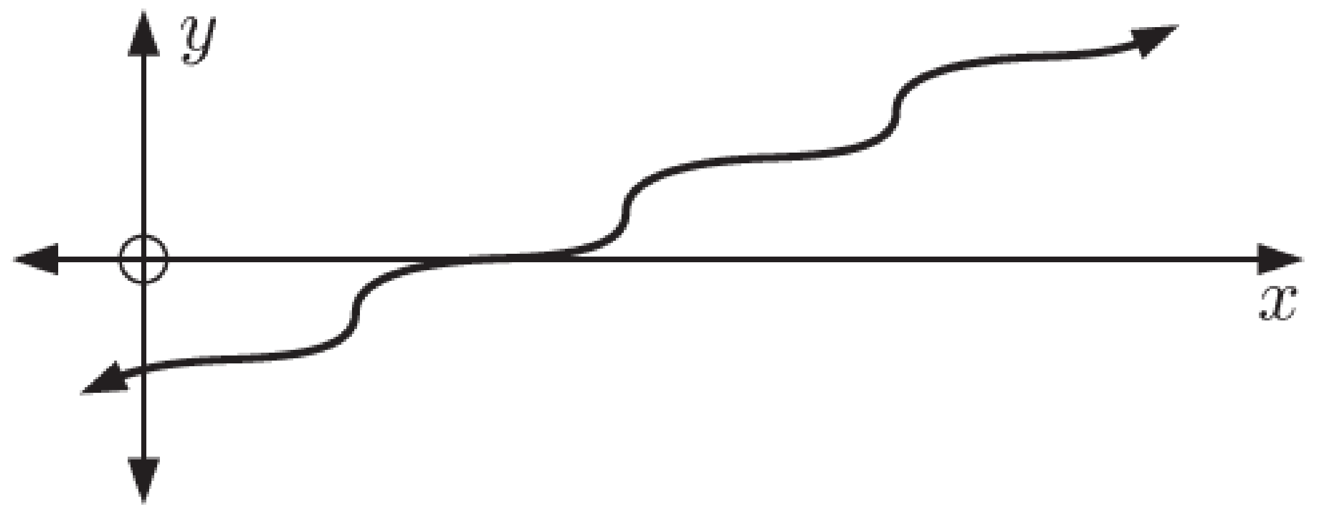
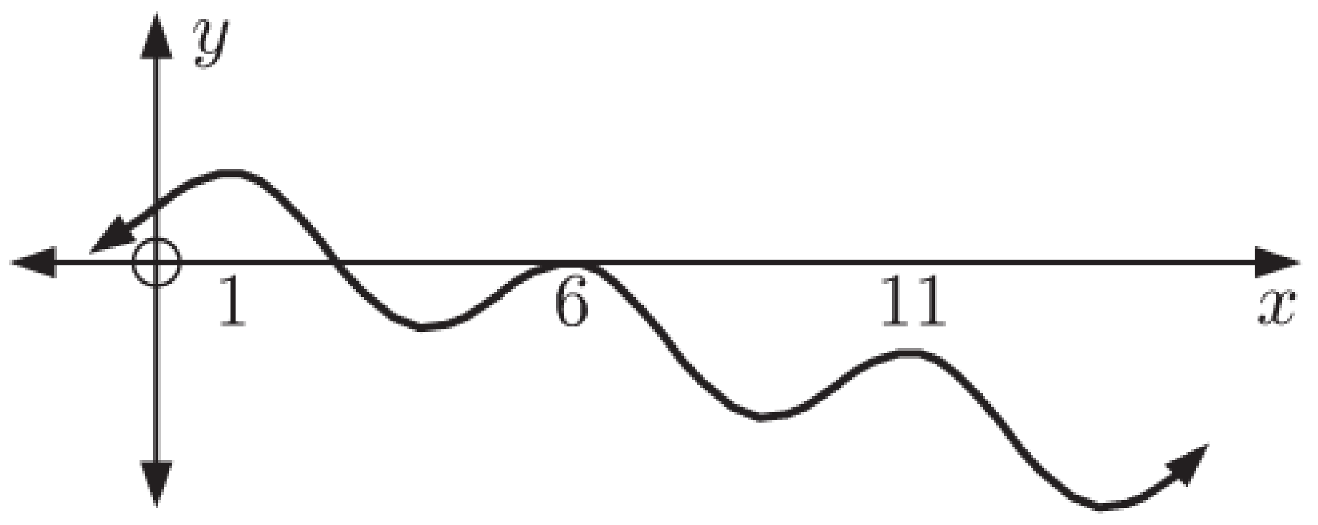


g



h



2 Draw a scatter diagram for each set of data below. Is there any evidence to suggest the data is periodic?

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

GRAPHICS CALCULATOR INSTRUCTIONS

3 Paul spun the wheel of his bicycle. The following tabled values show the height above the ground of a point on the wheel at various times.

Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against time.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

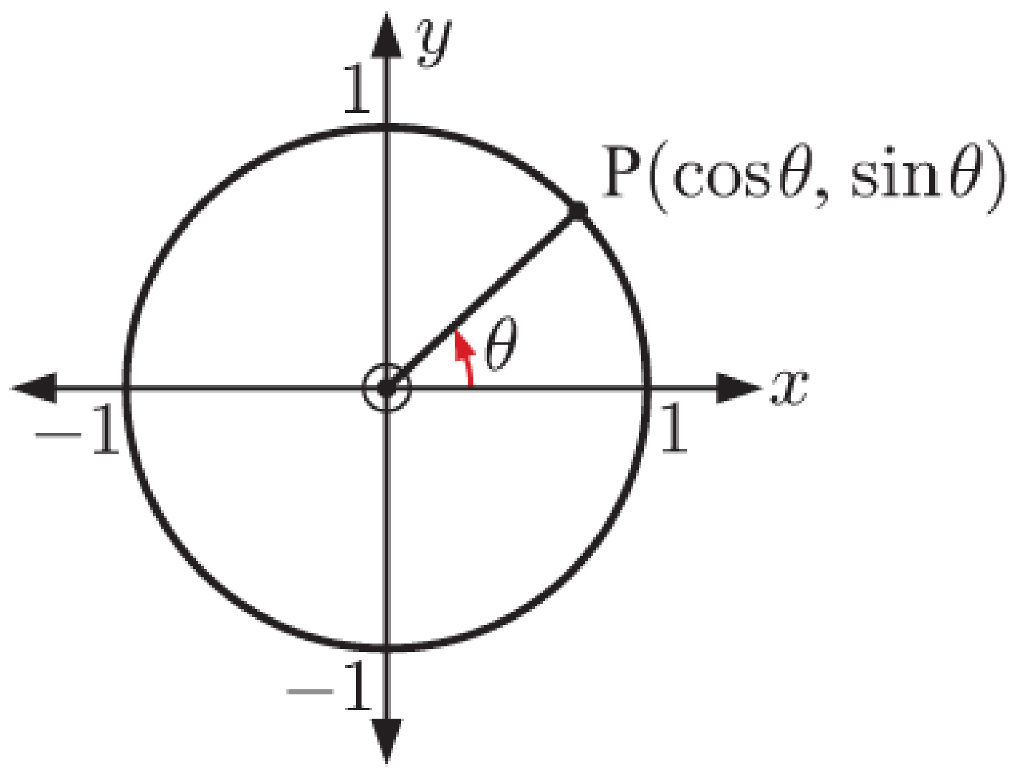
C THE SINE AND COSINE FUNCTIONS

A **trigonometric function** is a function which involves one of the trigonometric ratios.

Consider the point P$(\cos\theta, \sin\theta)$ on the unit circle.

As θ increases, the point P moves around the unit circle, and the values of $\cos\theta$ and $\sin\theta$ change.

We can draw the graphs of $y = \sin\theta$ and $y = \cos\theta$ by plotting the values of $\sin\theta$ and $\cos\theta$ against θ.

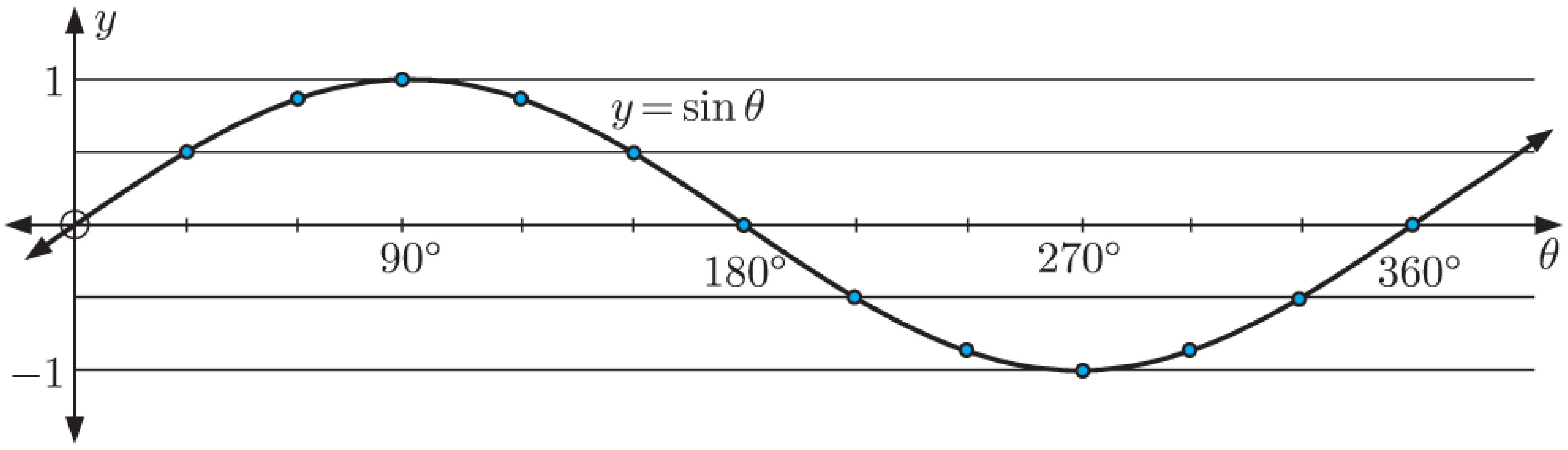


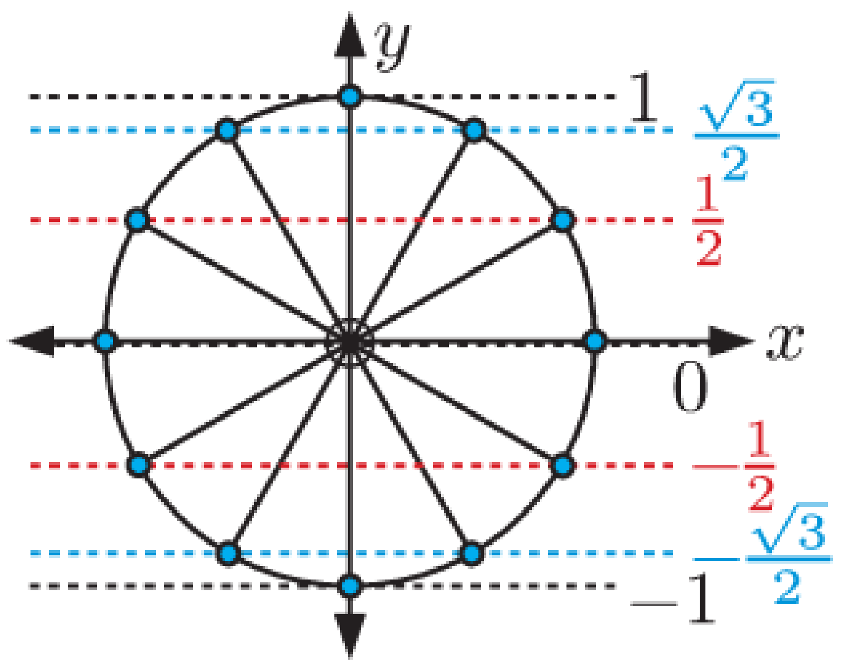
THE GRAPH OF $y = \sin\theta$

By considering the y-coordinates of the points on the unit circle at intervals of 30°, we can create a table of values for $\sin\theta$:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Plotting $\sin\theta$ against θ gives:



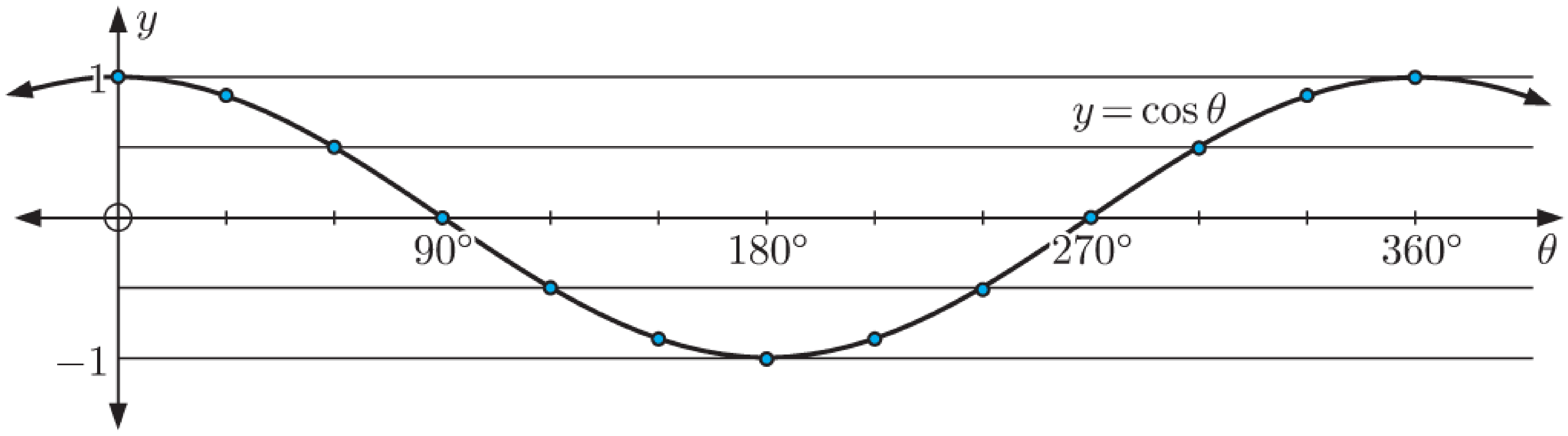


THE GRAPH OF $y = \cos\theta$

By considering the x-coordinates of the points on the unit circle at intervals of 30°, we can create a table of values for $\cos\theta$:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

Plotting $\cos\theta$ against θ gives:



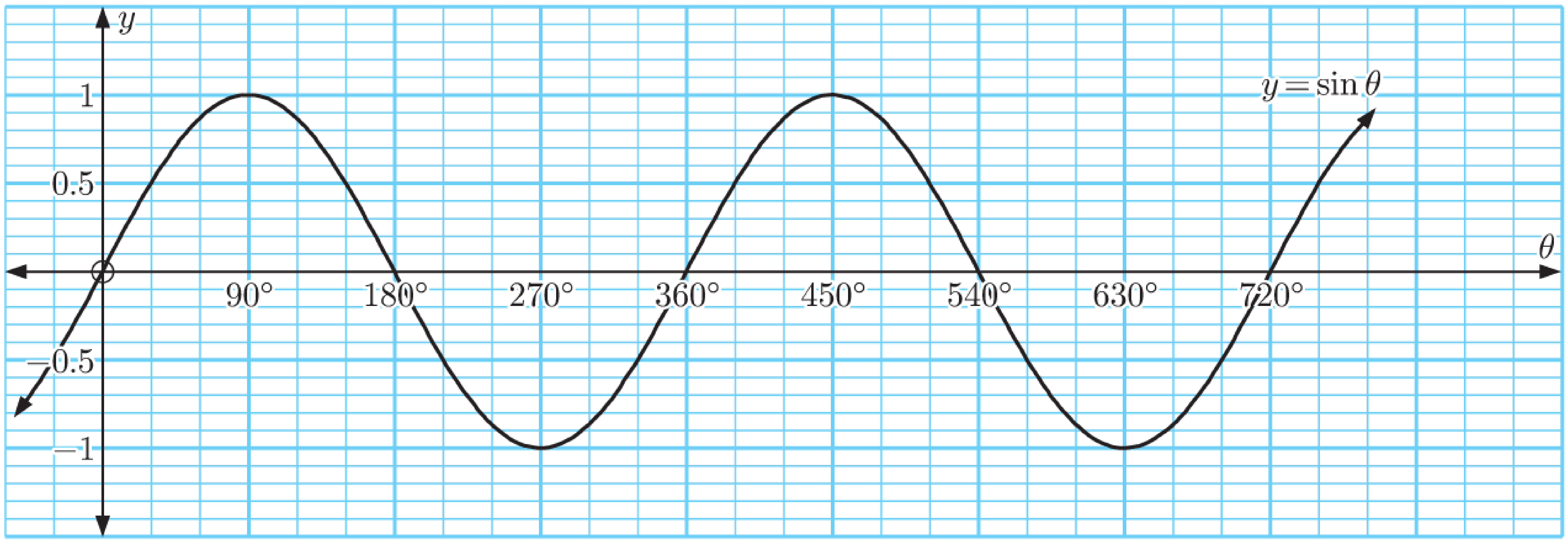
The graph of $y = \cos\theta$ shows the x-coordinate of P as P moves around the unit circle.

Once we reach 360°, the point P has completed a full revolution of the unit circle, so the patterns for both sine and cosine will repeat.

DEMO

EXERCISE 9C

1 Below is an accurate graph of $y = \sin\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0° \leqslant \theta \leqslant 720°$ for which:

i $\sin\theta = 0$ **ii** $\sin\theta = -1$ **iii** $\sin\theta = \frac{1}{2}$ **iv** $\sin\theta = \frac{\sqrt{3}}{2}$

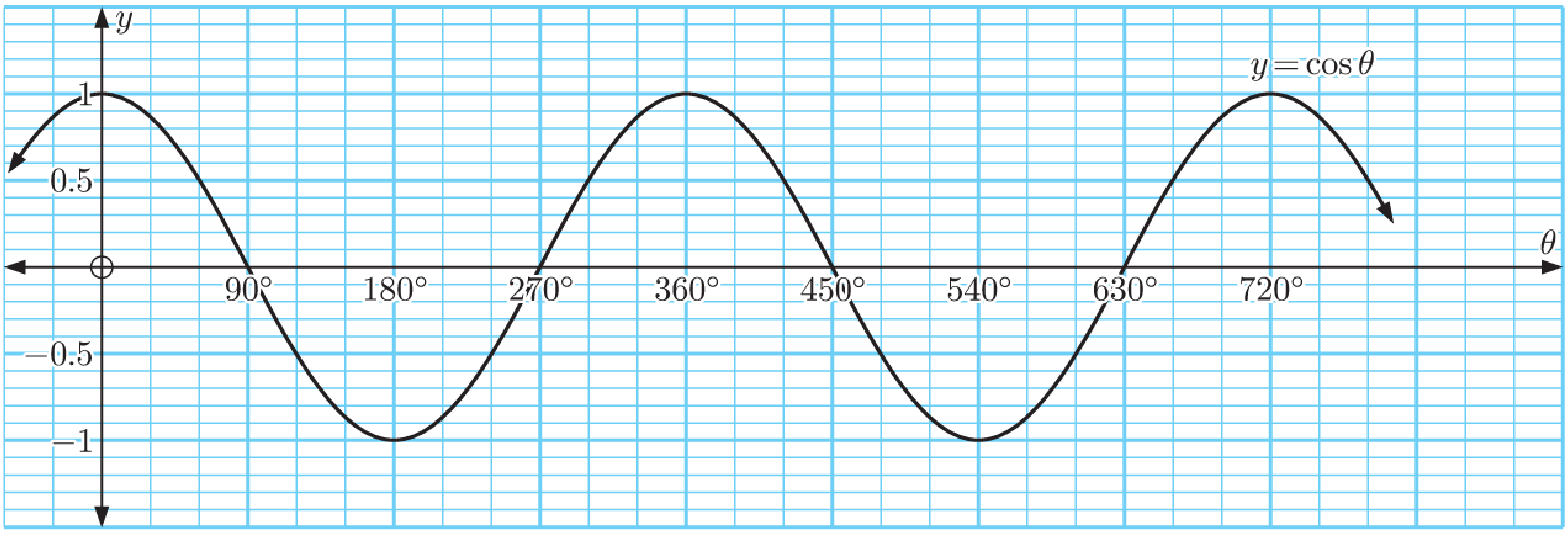
c Use the graph to estimate the values of θ on $0° \leqslant \theta \leqslant 720°$ for which $\sin\theta = 0.3$.

d Find the intervals on $0° \leqslant \theta \leqslant 720°$ where $\sin\theta$ is:

i positive **ii** negative.

e Find the range of the function.

2 Below is an accurate graph of $y = \cos\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0° \leqslant \theta \leqslant 720°$ for which:

  i $\cos\theta = 0$ **ii** $\cos\theta = 1$ **iii** $\cos\theta = -\frac{1}{2}$ **iv** $\cos\theta = -\frac{1}{\sqrt{2}}$

c Use the graph to estimate the values of θ on $0° \leqslant \theta \leqslant 720°$ for which $\cos\theta = 0.3$.

d Find the intervals on $0° \leqslant \theta \leqslant 720°$ where $\cos\theta$ is:

  i positive **ii** negative.

e Find the range of the function.

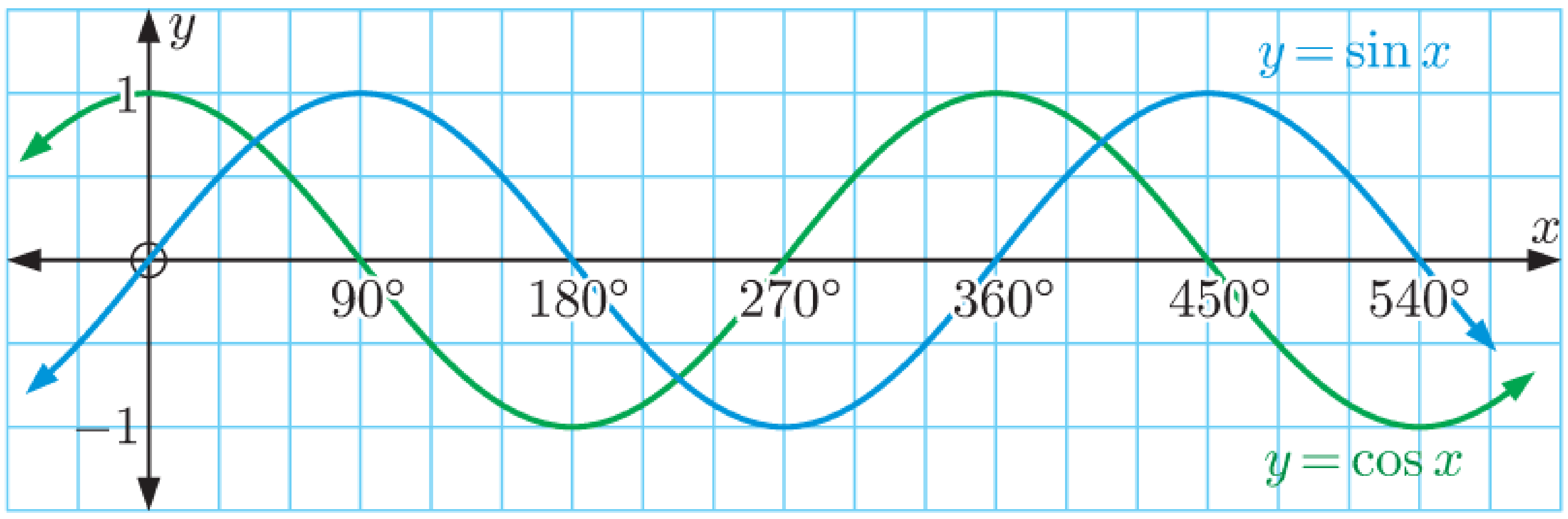
D GENERAL SINE AND COSINE FUNCTIONS

Now that we are familiar with the graphs of $y = \sin\theta$ and $y = \cos\theta$, we can use transformations to graph more complicated trigonometric functions.

Instead of using θ, we will now use x to represent the angle variable. This is just for convenience, so we are dealing with the familiar function form $y = f(x)$.

For the graphs of $y = \sin x$ and $y = \cos x$:

- The **domain** is $\{x \mid x \in \mathbb{R}\}$.
- The **period** is 360°.
- The **principal axis** is the line $y = 0$.
- The **range** is $\{y \mid -1 \leqslant y \leqslant 1\}$.
- The **amplitude** is 1.



DISCUSSION

How are the graphs of sine and cosine related?

In the following **Investigation**, we will use technology to draw graphs related to $y = \sin x$.

INVESTIGATION 1 FAMILIES OF TRIGONOMETRIC FUNCTIONS

GRAPHING PACKAGE

What to do:

1 **a** Use the graphing package to graph on the same set of axes:

- **i** $y = \sin x$
- **ii** $y = 2\sin x$
- **iii** $y = \frac{1}{2}\sin x$
- **iv** $y = -\sin x$
- **v** $y = -\frac{1}{3}\sin x$
- **vi** $y = -\frac{3}{2}\sin x$

b For graphs of the form $y = a\sin x$, comment on the significance of:

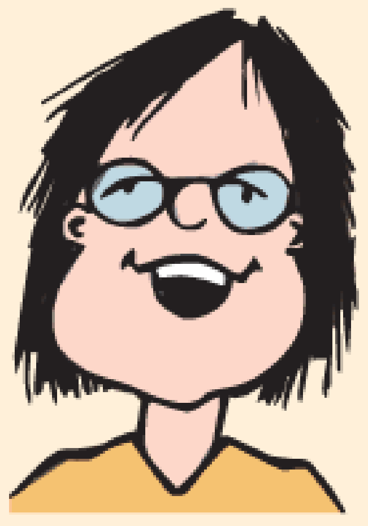
- **i** the sign of a
- **ii** the size of a, or $|\,a\,|$.

2 **a** Use the graphing package to graph on the same set of axes:

- **i** $y = \sin x$
- **ii** $y = \sin 2x$
- **iii** $y = \sin \frac{x}{2}$
- **iv** $y = \sin 3x$

b For graphs of the form $y = \sin bx$, $b > 0$, what is the period of the graph?

$|\,a\,|$ is the size of a, ignoring its sign. So, $|\,2\,| = 2$ and $\left|-\frac{1}{3}\right| = \frac{1}{3}$.



3 **a** Graph on the same set of axes:

- **i** $y = \sin x$
- **ii** $y = \sin x + 2$
- **iii** $y = \sin x - 2$

b What translation moves $y = \sin x$ to $y = \sin x + d$?

c What is the principal axis of $y = \sin x + d$?

4 What sequence of transformations maps $y = \sin x$ onto $y = a\sin(bx) + d$?

From the **Investigation** you should have observed the following properties about the **general sine function** $y = a\sin(bx) + d$:

$$\boldsymbol{y = a\sin(bx) + d}$$

affects **amplitude**

affects **period**

affects **vertical translation**

- The amplitude is $|\,a\,|$.
- The period is $\dfrac{360^\circ}{b}$ for $b > 0$.
- The principal axis is $y = d$.
- $y = a\sin(bx) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor a and a horizontal stretch with scale factor $\frac{1}{b}$, followed by a vertical translation of d units.

DEMO

Click on the icon to obtain a demonstration for the general sine function.

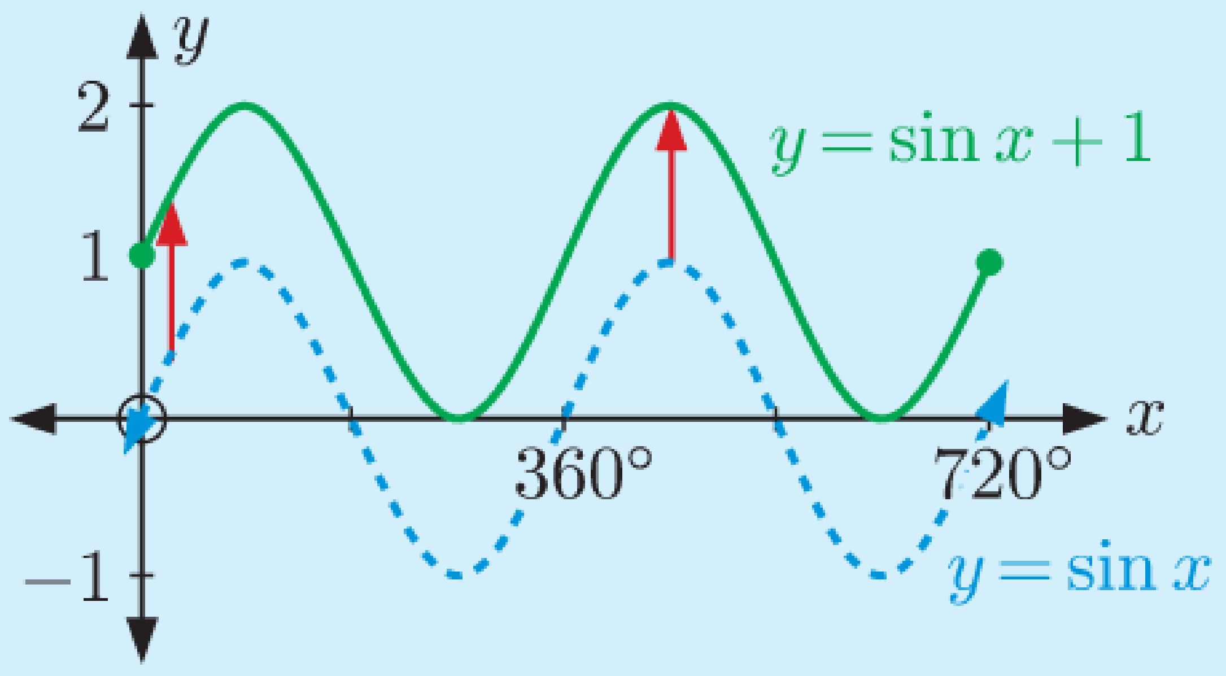
The properties of the **general cosine function** $y = a\cos(bx) + d$ are the same as those of the general sine function.

Example 1 Self Tutor

Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

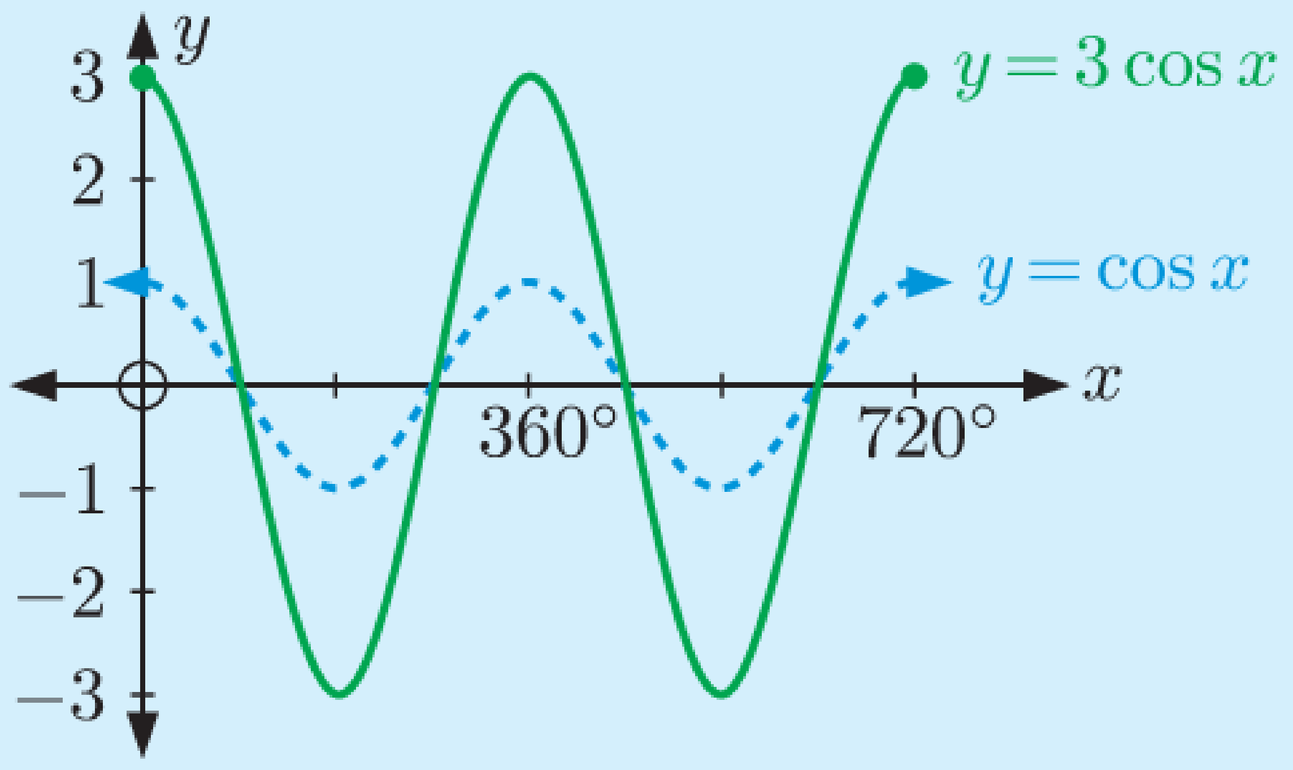
a $y = \sin x + 1$ **b** $y = 3\cos x$ **c** $y = \sin 2x$

a We translate $y = \sin x$ 1 unit upwards.



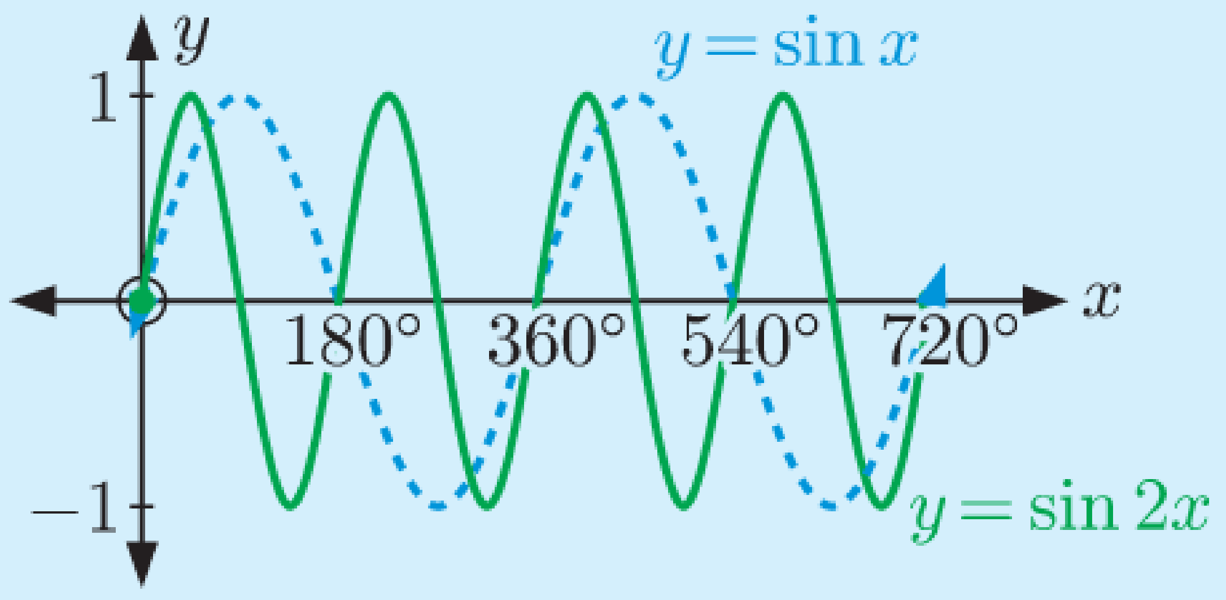
b We stretch $y = \cos x$ vertically with scale factor 3.

$\therefore$ $y = 3\cos x$ has amplitude 3.



c We stretch $y = \sin x$ horizontally with scale factor $\frac{1}{2}$.

$\therefore$ $y = \sin 2x$ has period $\dfrac{360°}{2} = 180°$.



EXERCISE 9D

1 Find the amplitude of:

a $y = 4\sin x$ **b** $y = -2\cos x + 1$ **c** $y = -\frac{1}{3}\sin 2x$

2 Find the period of:

a $y = \cos 3x$ **b** $y = 5\sin 4x + 2$ **c** $y = -\cos \frac{x}{2}$

3 Find the principal axis of:

a $y = \sin x - 3$ **b** $y = -2\cos x + 5$ **c** $y = \frac{1}{4}\sin 15x$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 90° **b** 24° **c** 1440°

5 State the maximum and minimum value of:

a $y = 4\cos 2x$ **b** $y = 3\cos x + 5$ **c** $y = -2\cos x - 4$

6 State the transformation which maps $y = \sin x$ onto:

a $y = \sin x - 1$ **b** $y = 2\sin x$ **c** $y = \sin 4x$

7 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = \sin x + 3$ **b** $y = 3\sin x$ **c** $y = \sin x - 2$

d $y = -2\sin x$ **e** $y = \sin 3x$ **f** $y = \sin \frac{x}{2}$

8 State the transformation which maps $y = \cos x$ onto:

a $y = \frac{1}{2}\cos x$ **b** $y = -\cos x$ **c** $y = \cos x + 3$

9 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = \cos x - 1$ **b** $y = 2\cos x$ **c** $y = \cos x + 3$

d $y = -\frac{1}{3}\cos x$ **e** $y = \cos 2x$ **f** $y = \cos\frac{3x}{2}$

10 **a** Sketch the curve $y = 4\sin x$ for $0° \leqslant x \leqslant 360°$.

b Find the value of y when:

i $x = 150°$ **ii** $x = 315°$

Mark these points on your graph in **a**.

11 For what values of d does the graph of $y = 3\cos x + d$ lie:

a entirely above the x-axis **b** entirely below the x-axis

c partially above and partially below the x-axis?

12 For the function $y = 4\sin 3x + 2$, state the:

a amplitude **b** period **c** range.

Example 2 **Self Tutor**

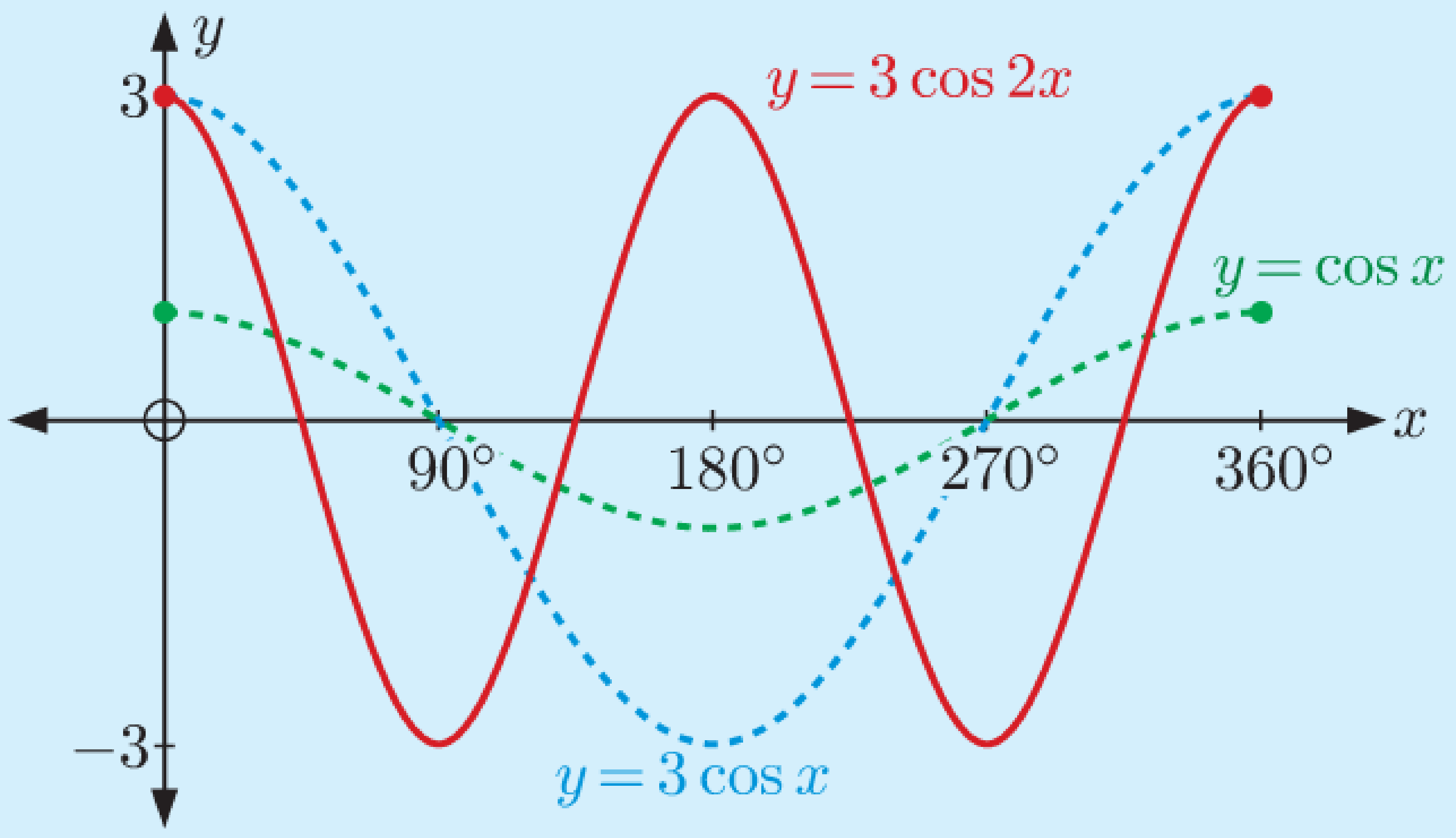
Sketch the graph of $y = 3\cos 2x$ for $0° \leqslant x \leqslant 360°$.

$a = 3$, so the amplitude is $|3| = 3$.

$b = 2$, so the period is

$$\frac{360°}{b} = \frac{360°}{2} = 180°.$$

We stretch $y = \cos x$ vertically with scale factor 3 to give $y = 3\cos x$, then stretch $y = 3\cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = 3\cos 2x$.



13 State the transformations which map $y = \sin x$ onto:

a $y = 2\sin 3x$ **b** $y = 3\sin x - 5$ **c** $y = -2\sin x$

14 Sketch the graph of the following for $0° \leqslant x \leqslant 360°$:

a $y = 2\cos x + 1$ **b** $y = \sin 2x + 3$ **c** $y = -2\cos x$

d $y = \frac{1}{2}\cos 3x$ **e** $y = 3\sin 4x + 7$ **f** $y = -\cos\frac{1}{2}x + 1$

15 **a** Sketch the graph of $y = 6\sin x + 10$ for $0° \leqslant x \leqslant 720°$.

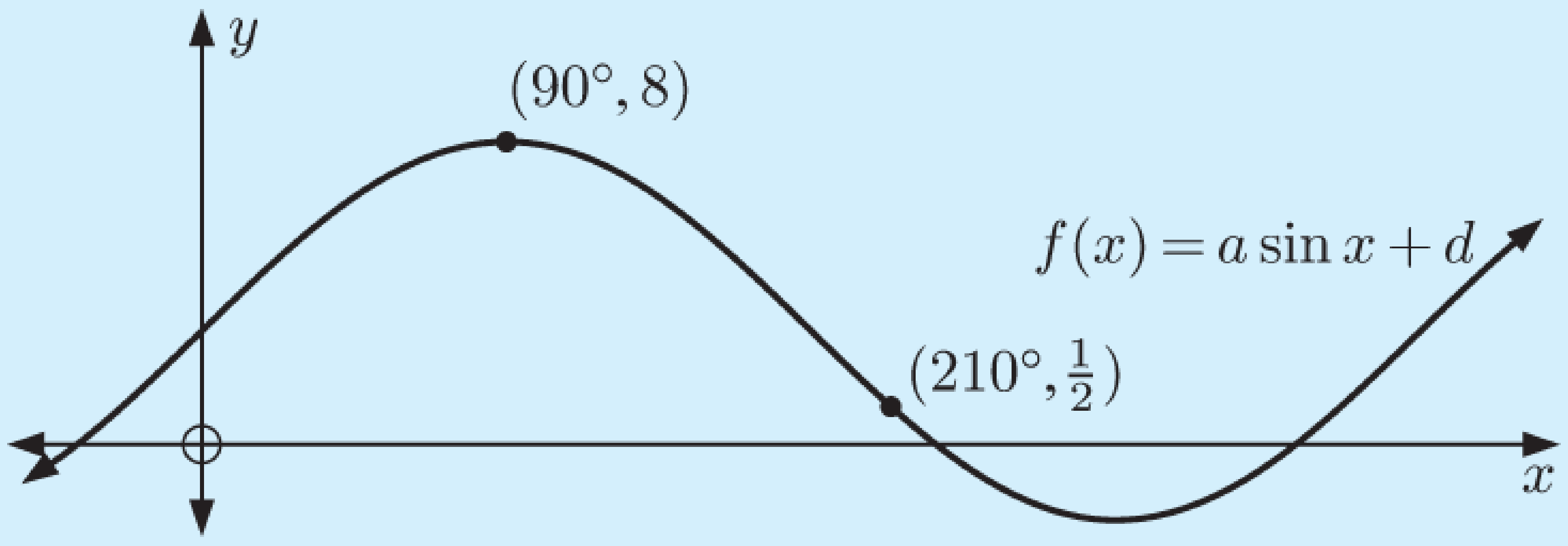
b Find the value of y when $x = 30°$.

c Find the maximum value of y, and the values of x where the maximum occurs.

d Find the minimum value of y, and the values of x where the minimum occurs.

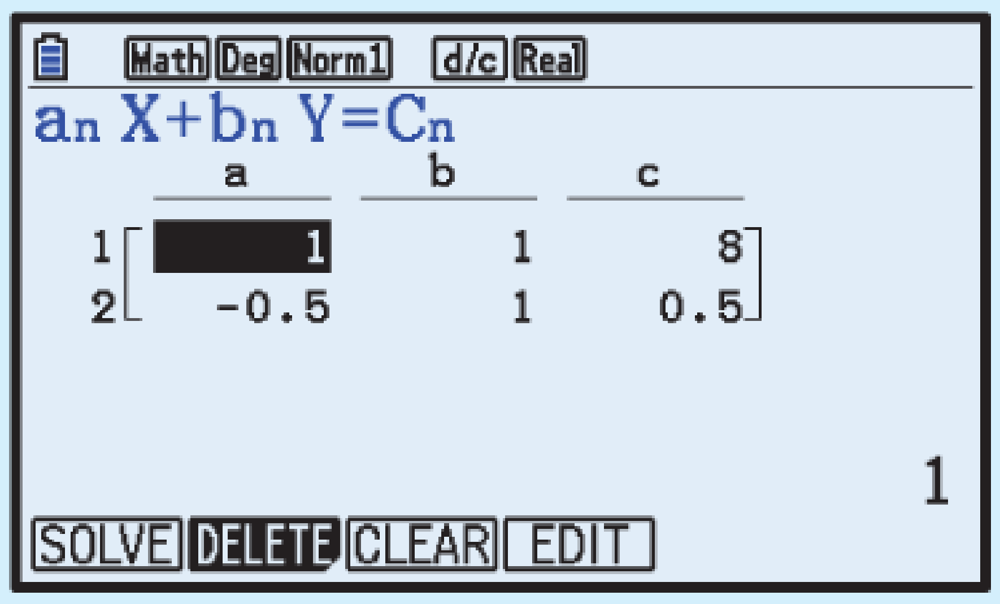
Example 3 **Self Tutor**

Find the unknowns in this function:

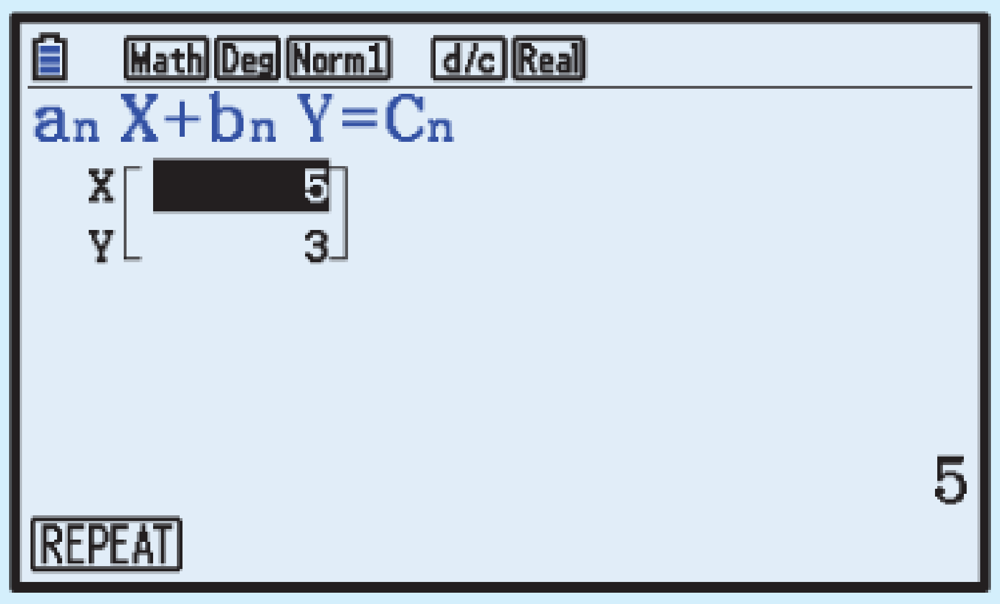


$$f(90^\circ) = 8$$
$$\therefore \quad a\sin 90^\circ + d = 8$$
$$\therefore \quad a + d = 8 \quad .... (1)$$

$$f(210^\circ) = \tfrac{1}{2}$$
$$\therefore \quad a\sin 210^\circ + d = \tfrac{1}{2}$$
$$\therefore \quad -\tfrac{1}{2}a + d = \tfrac{1}{2} \quad .... (2)$$


Math Deg Norm1 d/c Real
SOLVE DELETE CLEAR EDIT

```
anX+bnY=Cn
      a        b        c
1[    1        1        8]
2[ -0.5        1      0.5]
                            1
```

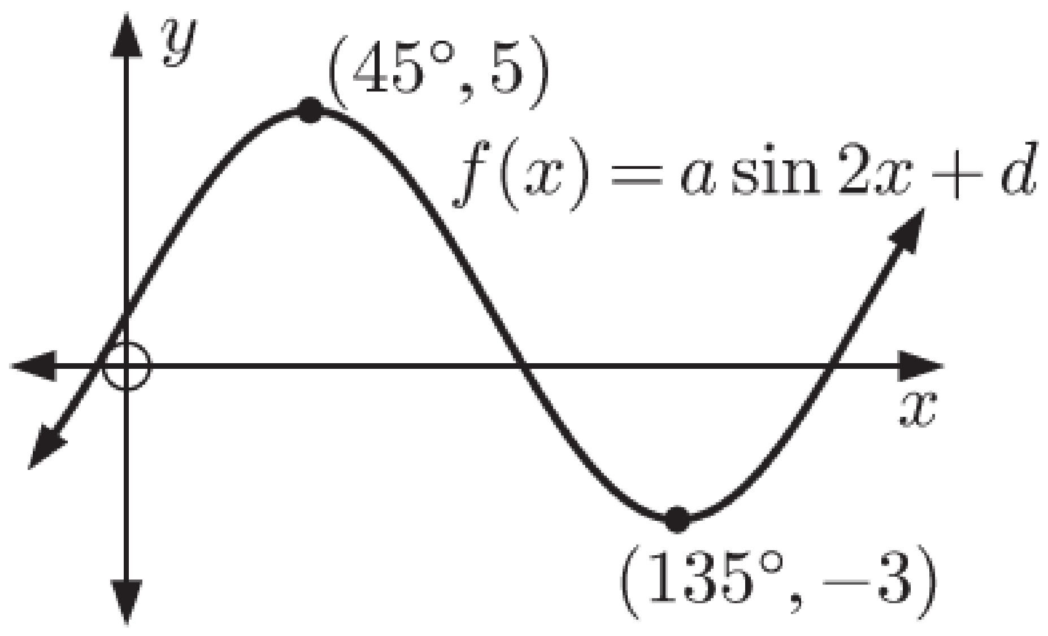

Math Deg Norm1 d/c Real
REPEAT

```
anX+bnY=Cn
X[   5]
Y[   3]
                            5
```

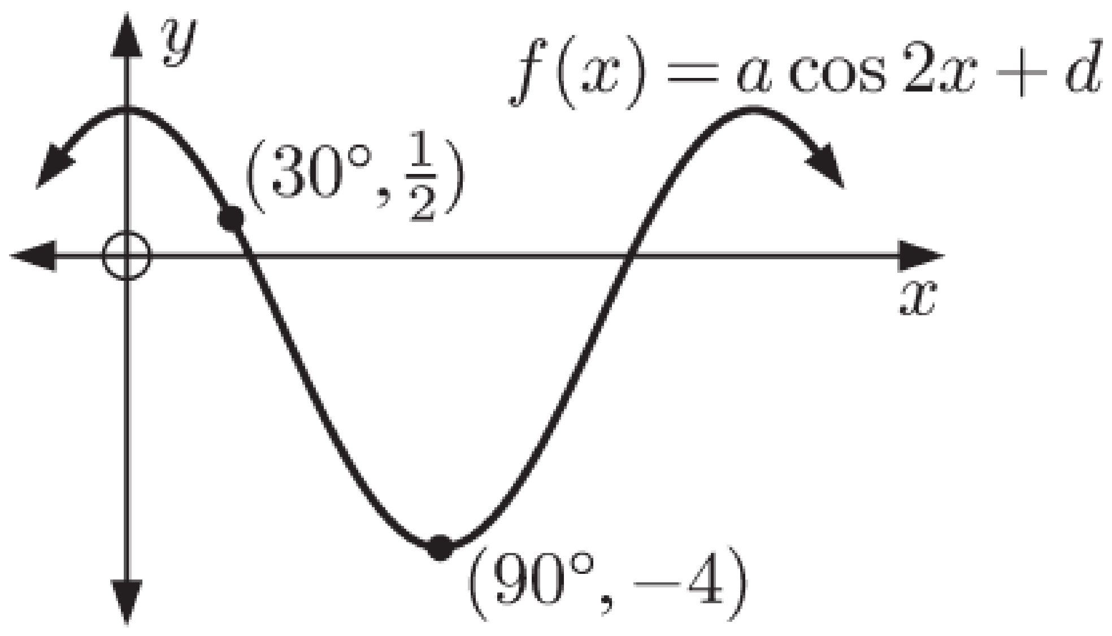
Solving (1) and (2) simultaneously using technology gives $a = 5$ and $d = 3$.

16 Find the unknowns in each function:

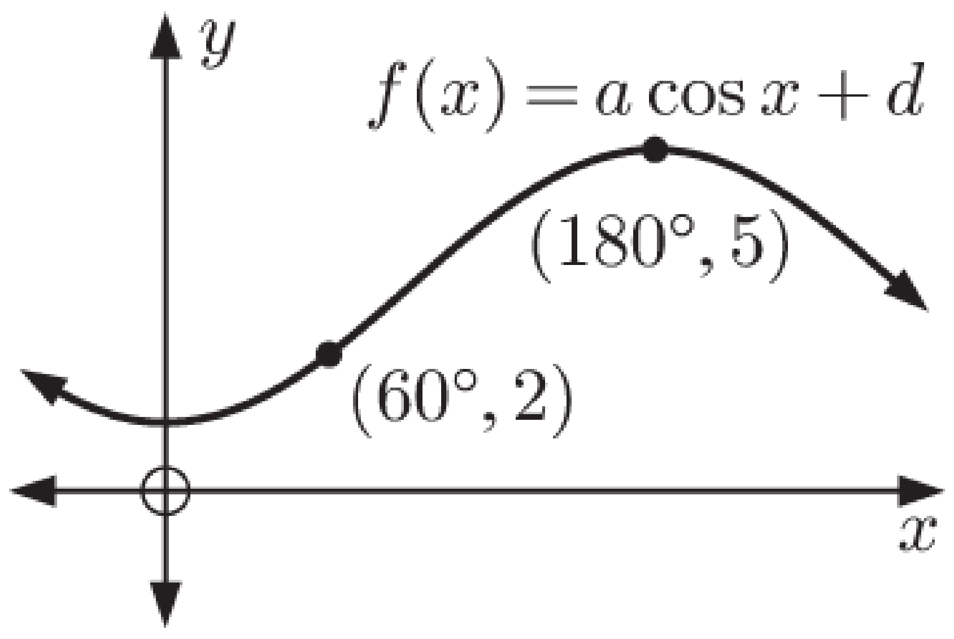
a



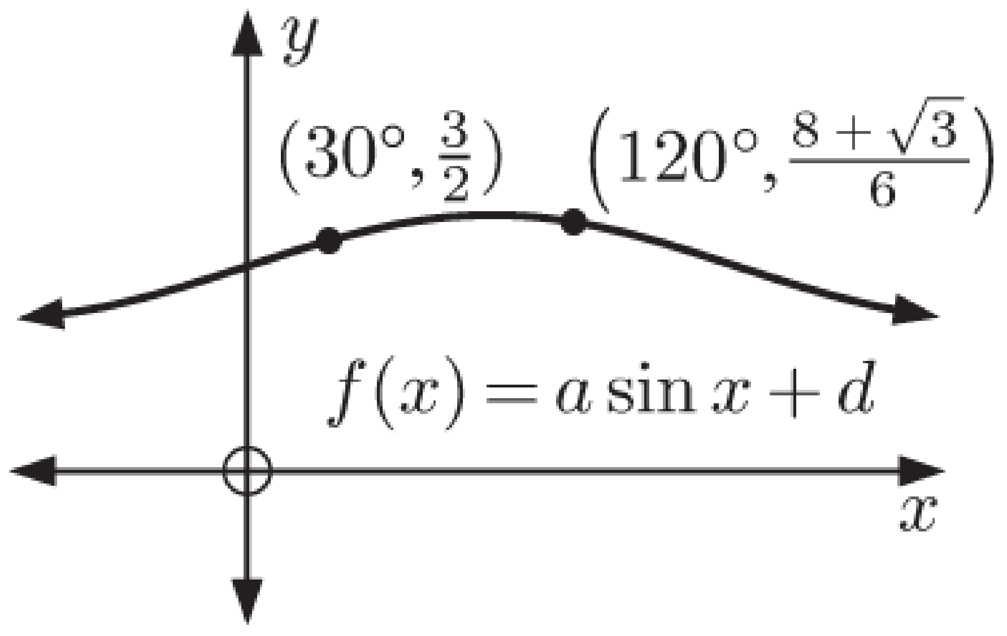
b



c

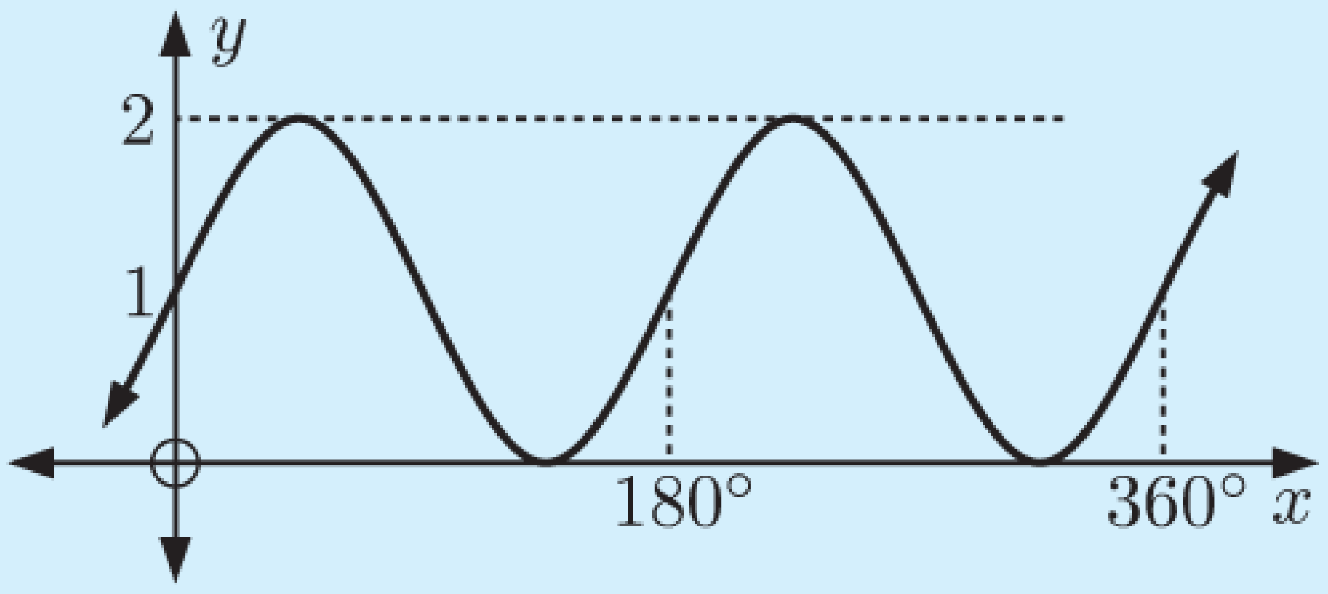


d



Example 4 **Self Tutor**

Find the equation of this sine function:



The amplitude is 1, so $a = 1$.

The period is 180°, so $\dfrac{360^\circ}{b} = 180^\circ$

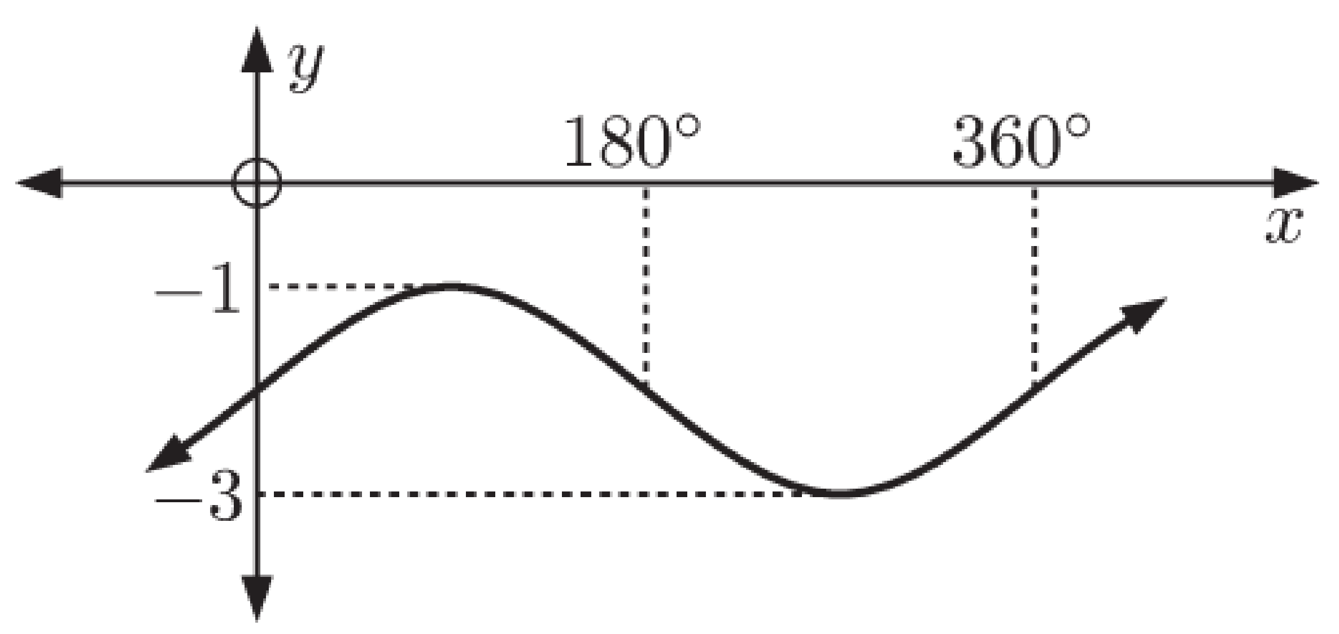
$$\therefore \quad b = 2.$$

The principal axis is $y = 1$, so $d = 1$.

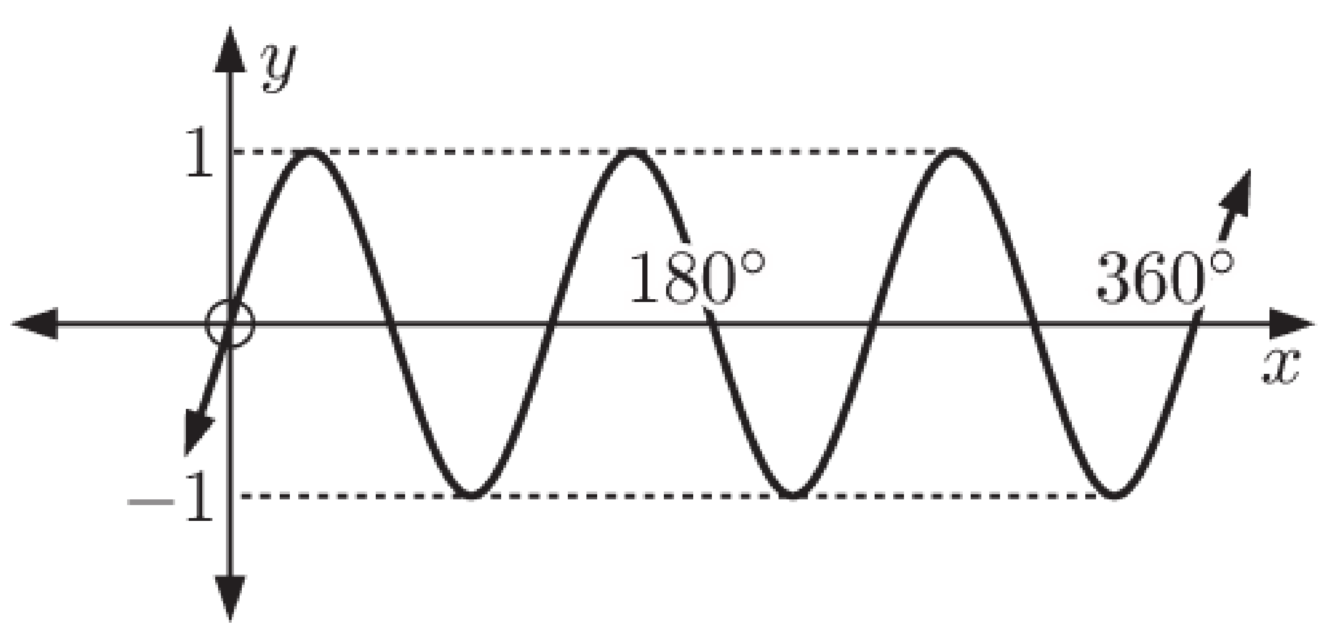
The equation of the function is $y = \sin 2x + 1$.

17 Find the equation of each sine function:

a



b



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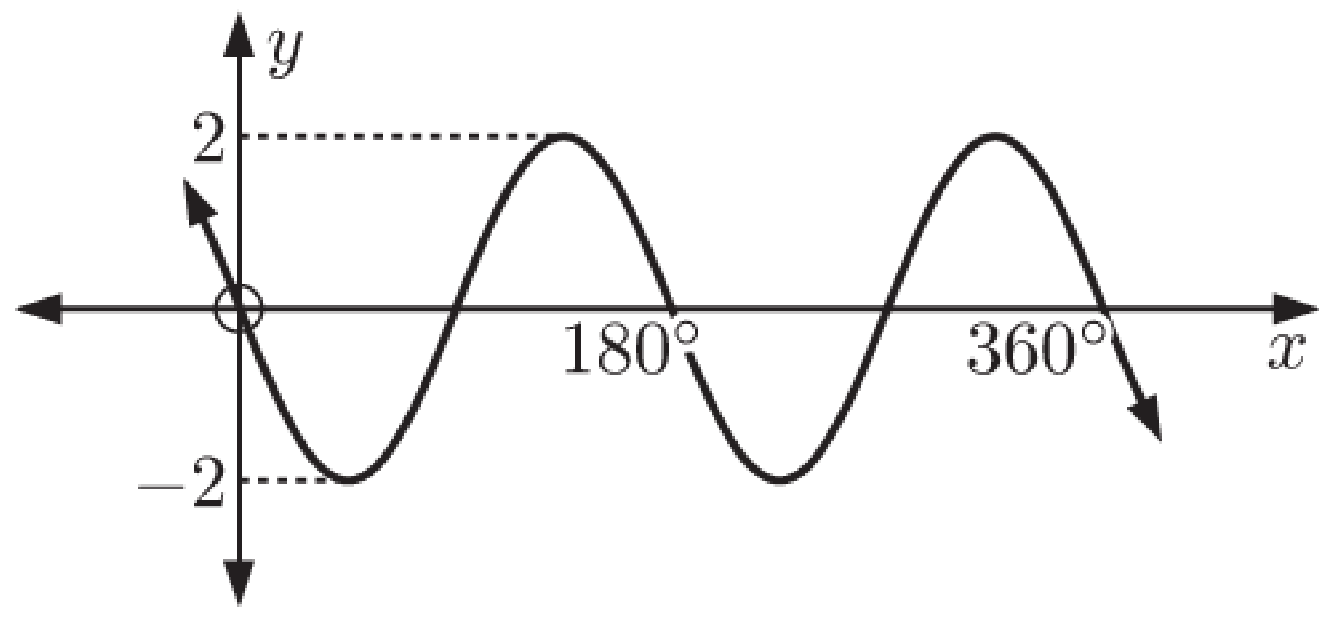
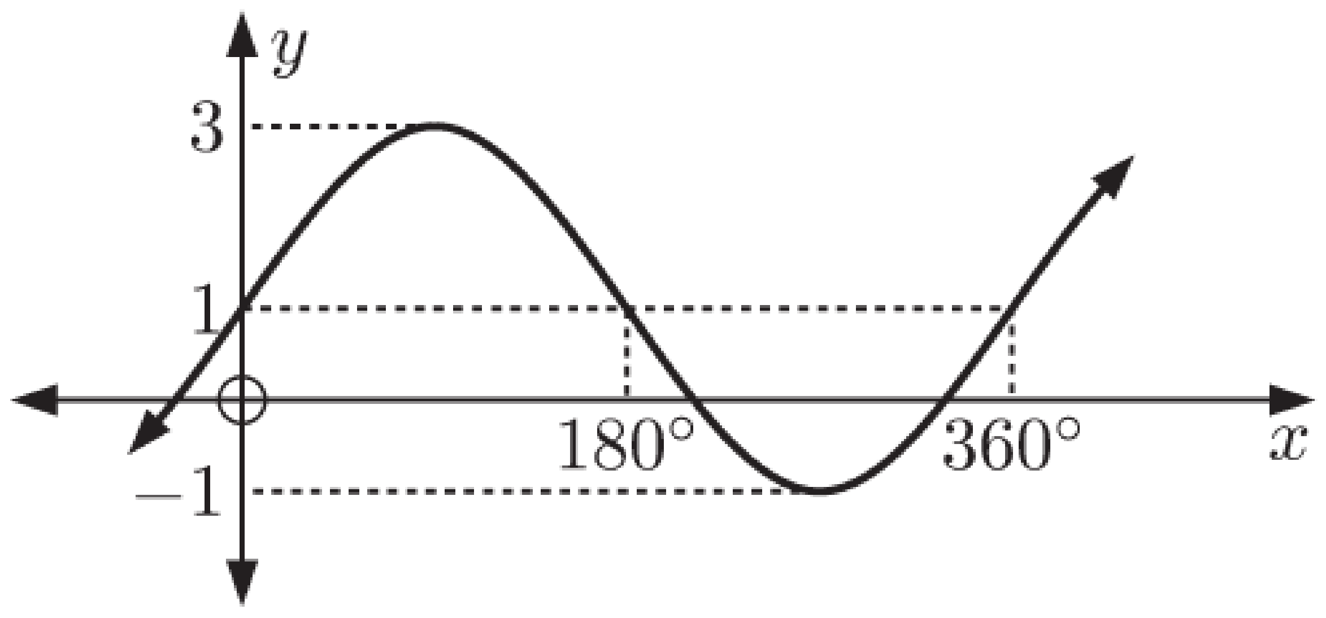


d



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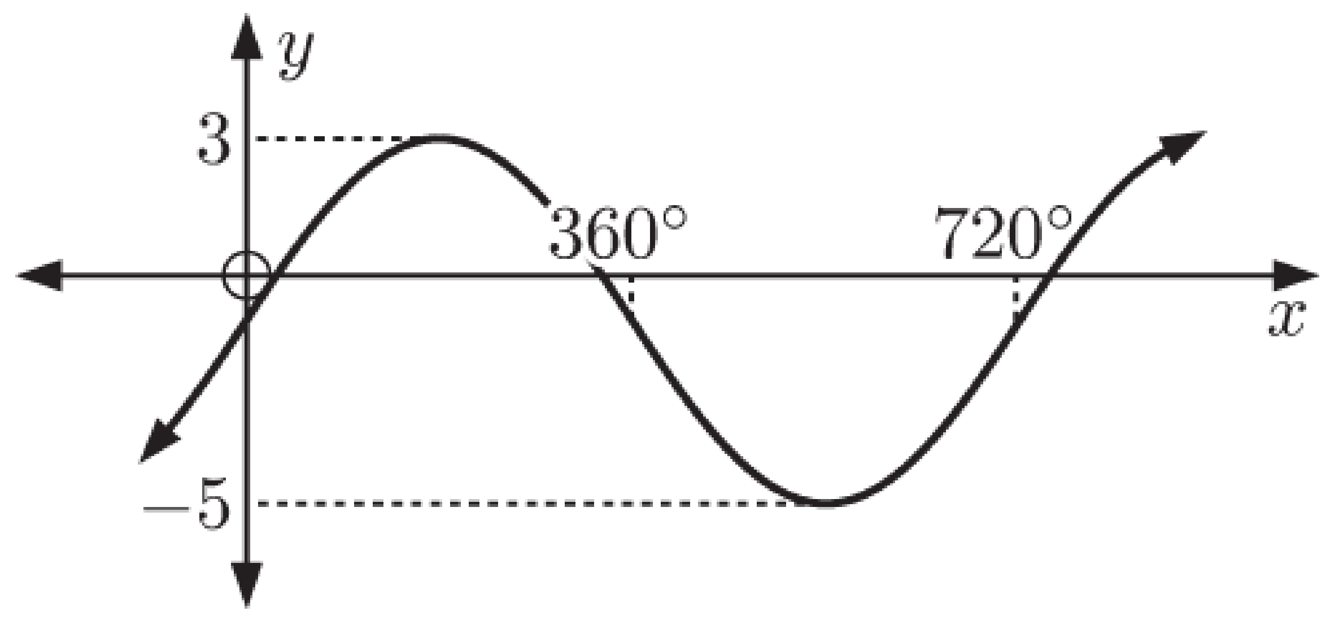

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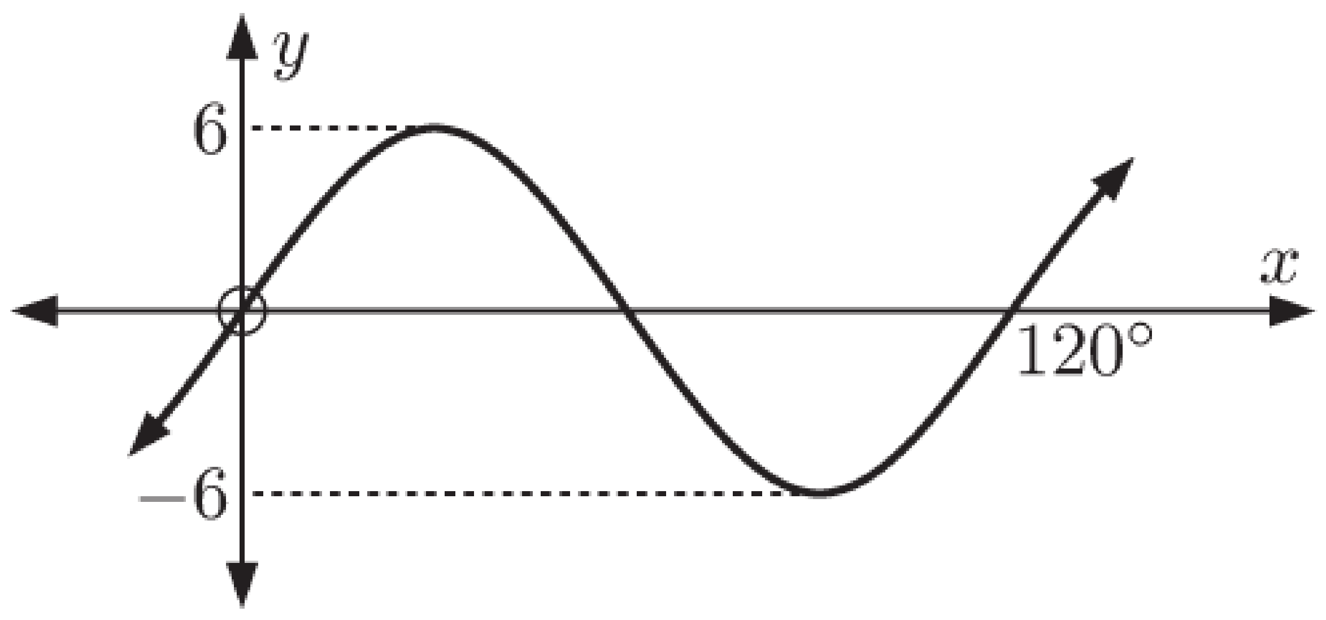
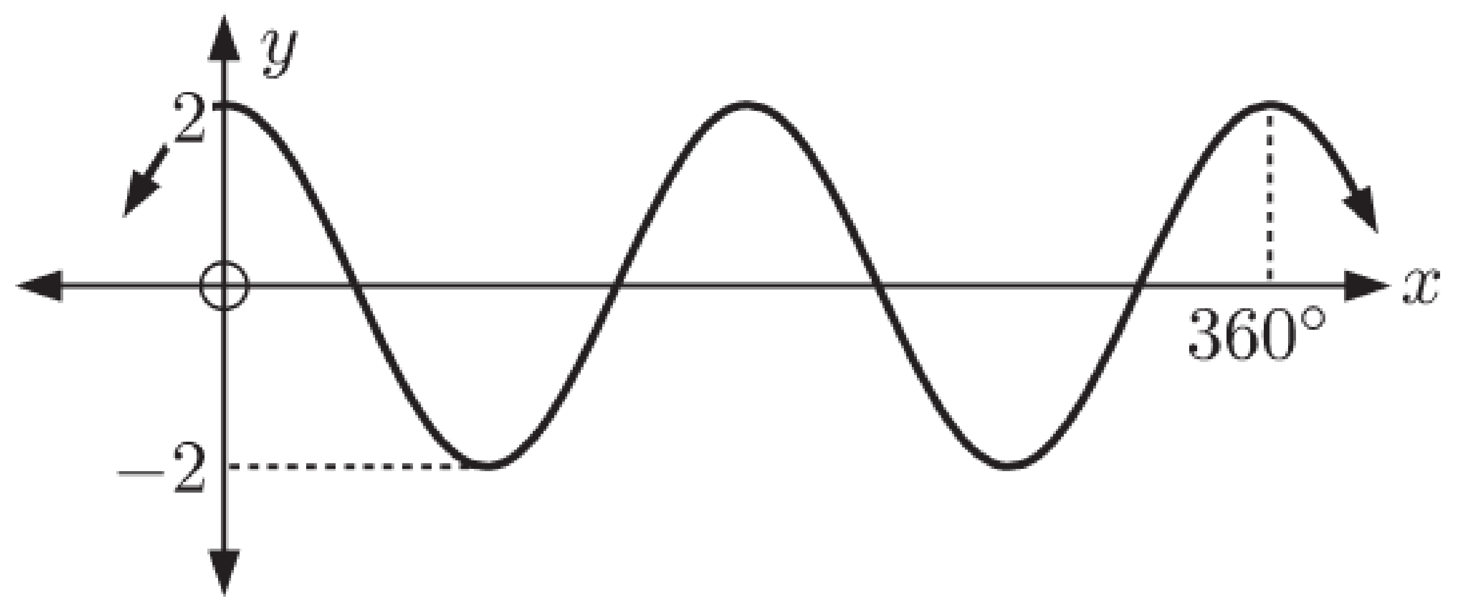
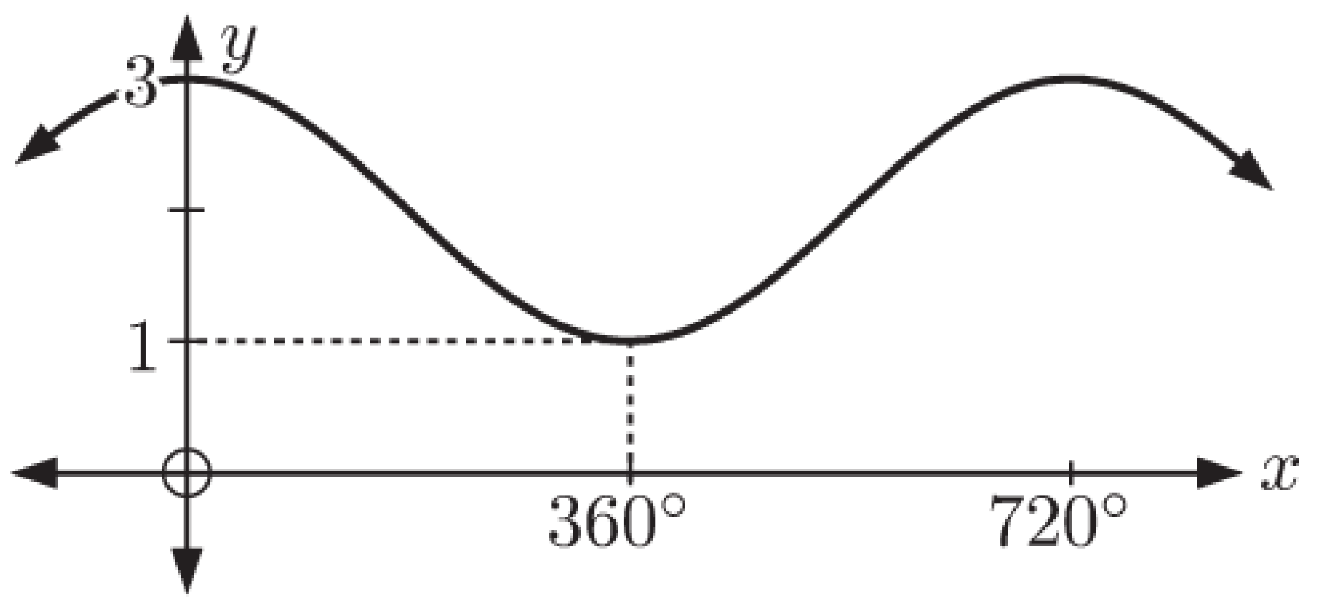


18 Find the cosine function shown in the graph:

a



b



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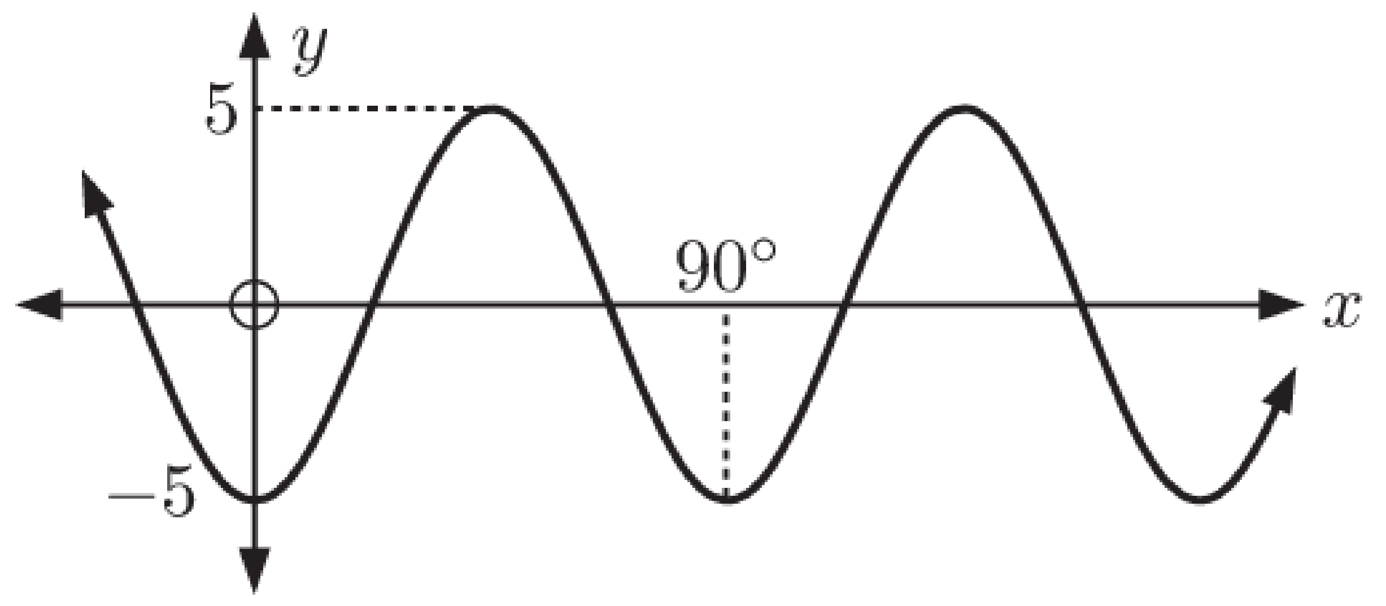

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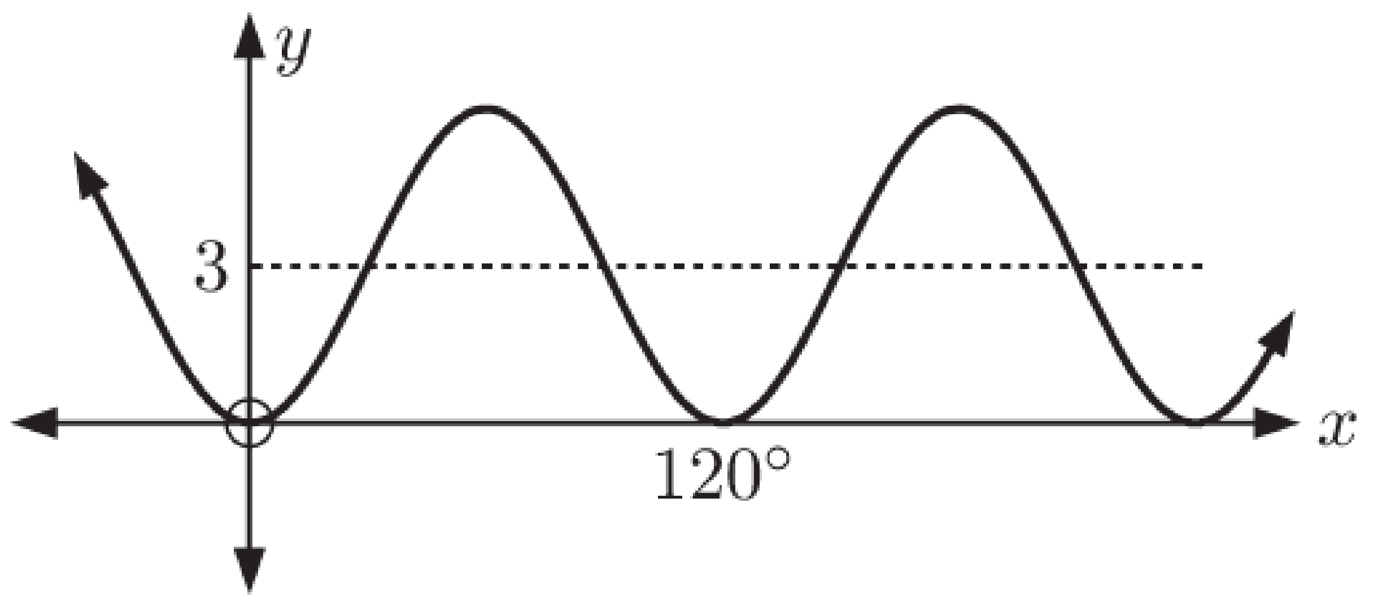


E MODELLING PERIODIC BEHAVIOUR

The sine and cosine functions are both referred to as **sinusoidal** functions. They can be used to model many periodic phenomena in the real world.

In some cases such as the movement of hands on a clock, the models we find will be almost exact. In other cases, such as the maximum daily temperature of a city of a year, the model will be less accurate.

Example 5 **Self Tutor**

The average daytime temperature for a city is given by the function $D(t) = 5\cos(30t)^\circ + 20$ °C, where t is the time in months after January.

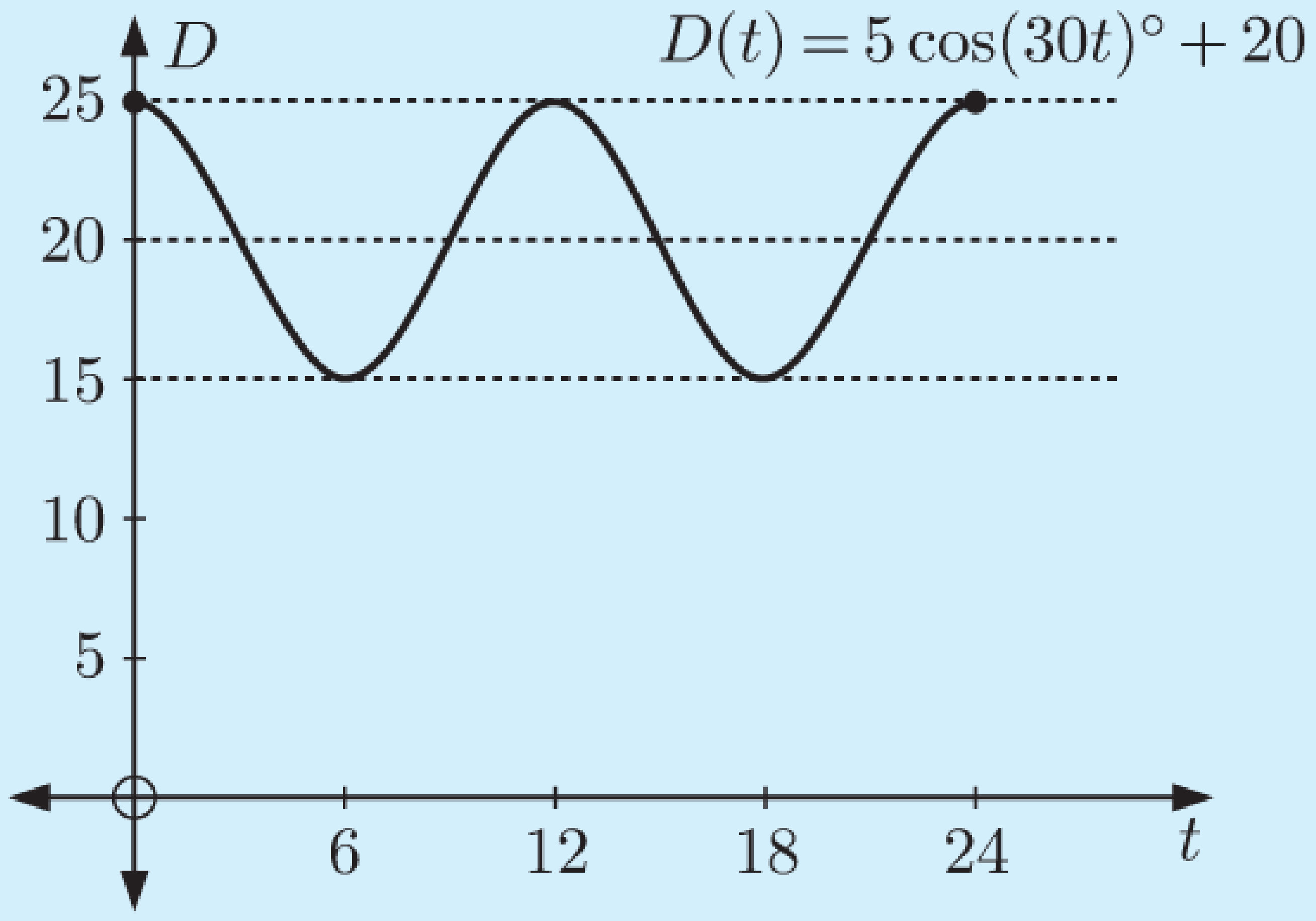
a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the average daytime temperature during May.

c Find the minimum average daytime temperature, and the month in which it occurs.

a For $D(t) = 5\cos(30t)^\circ + 20$:

- the amplitude is 5
- the period is $\frac{360}{30} = 12$ months
- the principal axis is $D = 20$.



b May is 4 months after January.

When $t = 4$, $D = 5 \times \cos 120^\circ + 20$
$= 5 \times (-\frac{1}{2}) + 20$
$= 17.5$

So, the average daytime temperature during May is 17.5°C.

c The minimum average daytime temperature is $20 - 5 = 15$°C, which occurs when $t = 6$ or 18.

So, the minimum average daytime temperature occurs during July.

EXERCISE 9E

1 The temperature inside Vanessa's house t hours after midday is given by the function $T(t) = 6\sin(15t)^\circ + 26$ °C.

a Sketch the graph of T against t for $0 \leqslant t \leqslant 24$.

b Find the temperature inside Vanessa's house at:

i midnight **ii** 2 pm.

c Find the maximum temperature inside Vanessa's house, and the time at which it occurs.

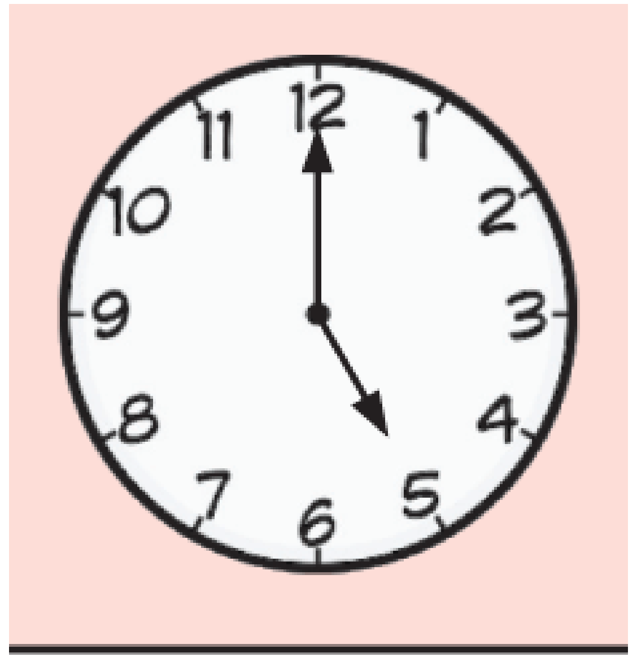
2 The depth of water in a harbour t hours after midnight is $D(t) = 4\cos(30t)^\circ + 6$ metres.

a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the highest and lowest depths of the water, and the times at which they occur.

c A boat requires a water depth of 5 metres to sail in. Will the boat be able to enter the harbour at 8 pm?

3 The tip of a clock's minute hand is $H(t) = 15\cos(6t)^\circ + 150$ cm above ground level, where t is the time in minutes after 5 pm.



a Sketch the graph of H against t for $0 \leqslant t \leqslant 180$.

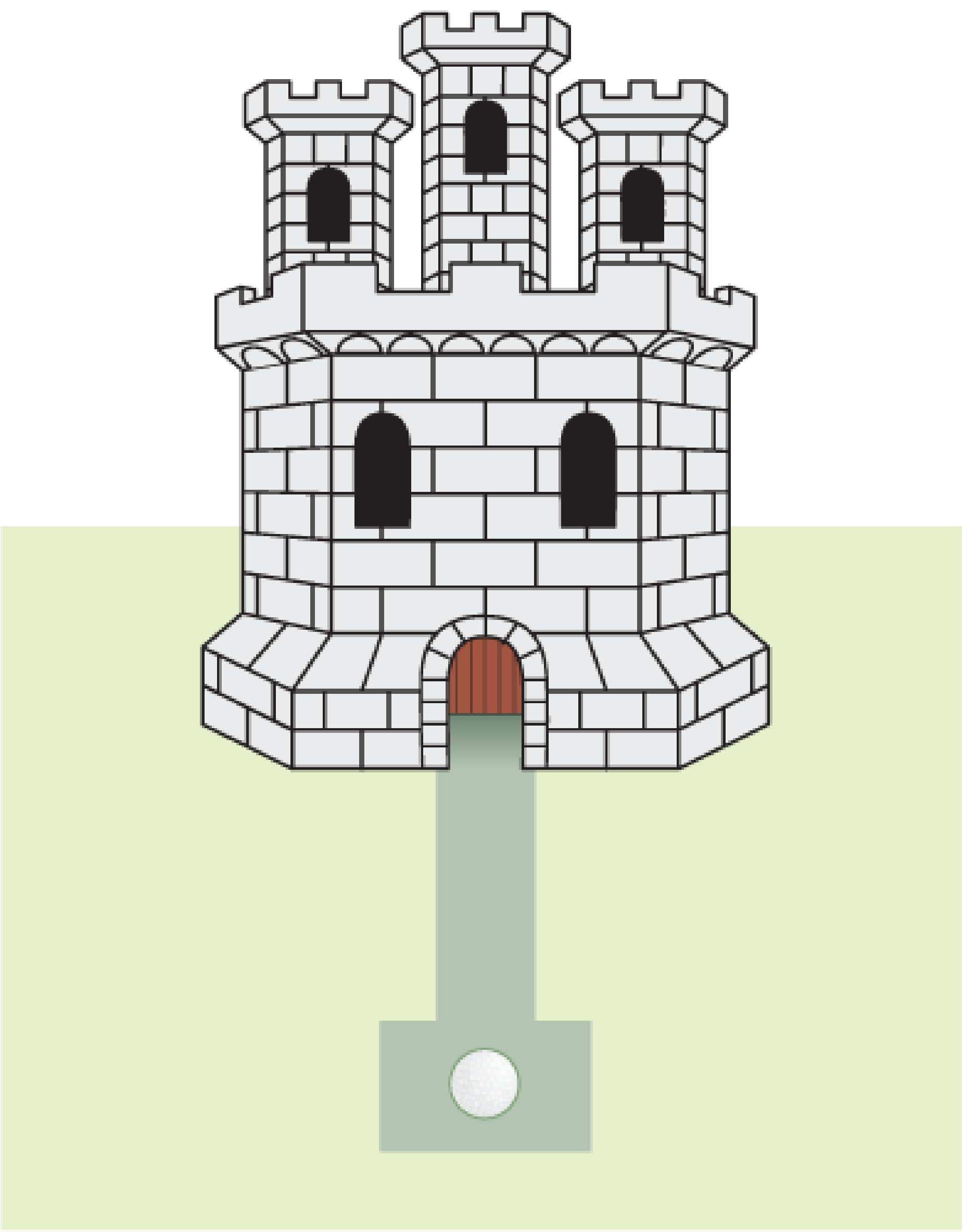
b Find the length of the minute hand.

c Find, rounded to 1 decimal place, the height of the minute hand's tip at:

i 5:08 pm **ii** 5:37 pm **iii** 5:51 pm **iv** 6:23 pm

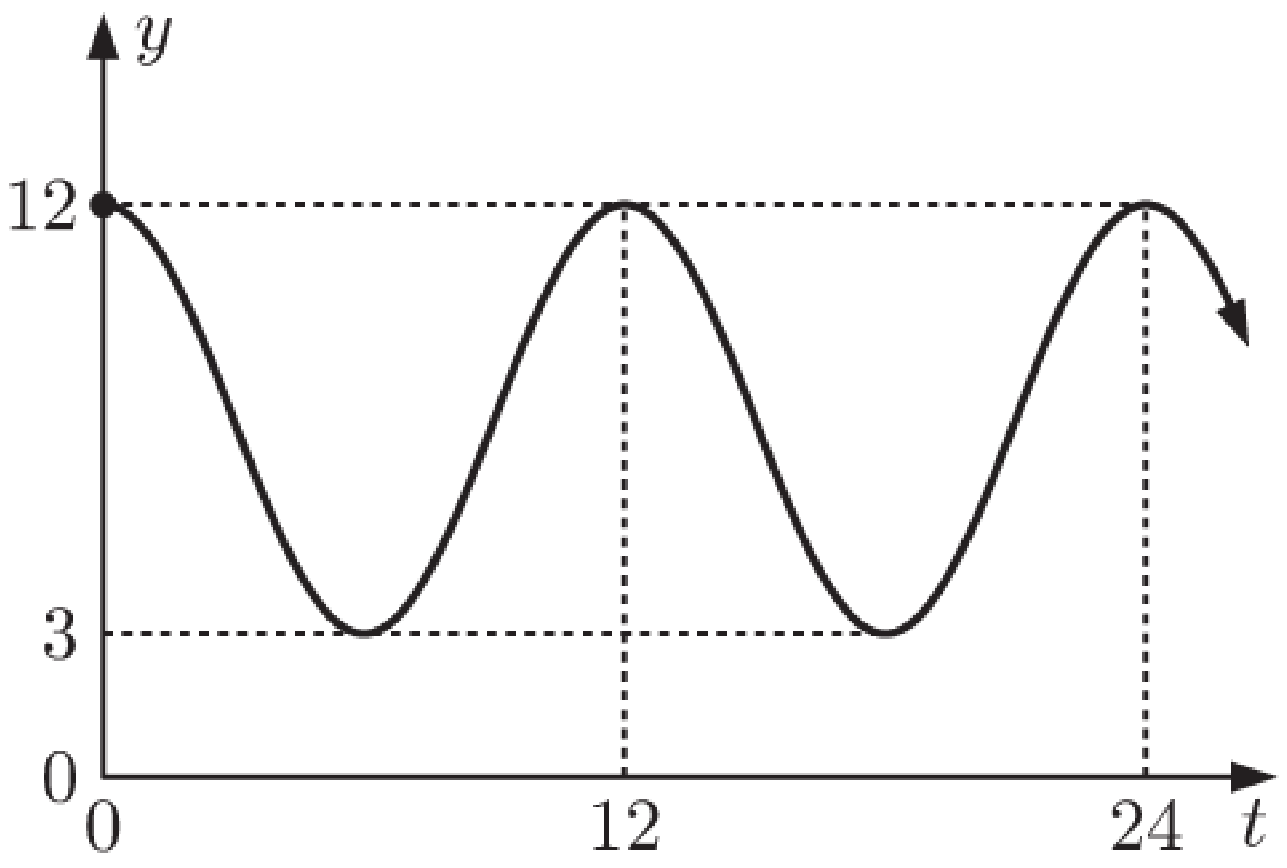
4 On a mini-golf hole, golfers must putt the ball through a castle's entrance. The entrance is protected by a gate which moves up and down.

The height of the gate above the ground t seconds after it touches the ground is $H(t) = -4\cos(45t)^\circ + 4$ cm.



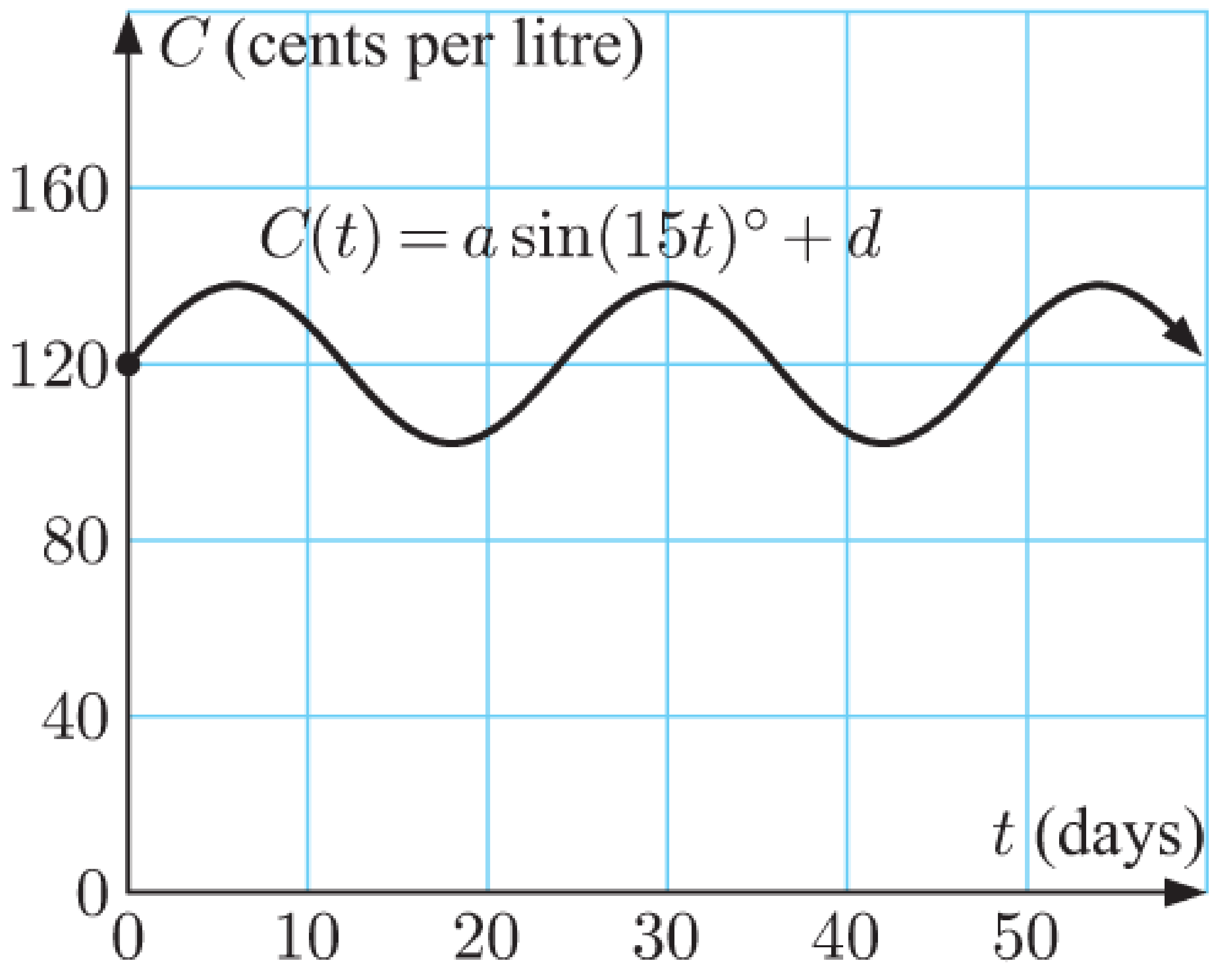
- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 16$.
- **b** Find the height of the gate above the ground 2 seconds after the gate touches the ground.
- **c** Eric is using a golf ball with radius 2.14 cm. He putts the ball 1 second after the gate touches the ground, and the ball takes 5.3 seconds to reach the castle's entrance. Will the ball pass through the entrance?

5 The average UV index in Santiago, Chile, is given by the function $y = a\cos(bt)^\circ + d$, where t is the time in months after January. Its graph is shown alongside.



- **a** Explain why $b = 30$.
- **b** Find the values of a and d.
- **c** Find the average UV index for Santiago during May. Give your answer to the nearest whole number.

6 The price of petrol at a particular service station is modelled by $C(t) = a\sin(15t)^\circ + d$ cents per litre, where t is the time in days. The model is graphed alongside.



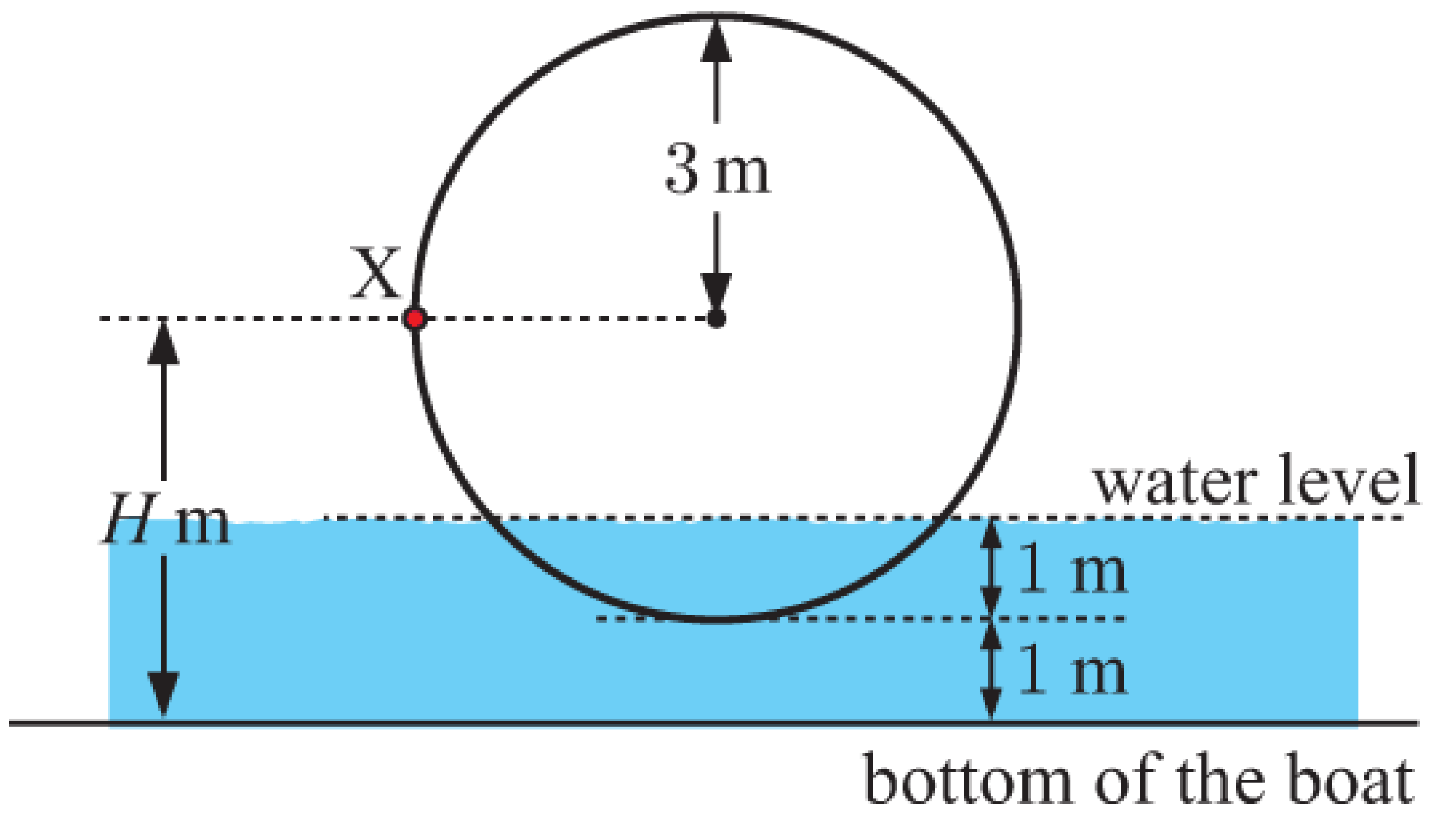
- **a** State the value of d.
- **b** How often does the cycle of petrol prices repeat itself?
- **c** Given that the petrol price on day 10 was 129 cents per litre, find the value of a.
- **d** Find the minimum price of petrol at this service station.
- **e** Find the petrol price on day 17.

7 The amount of feed available for grazing animals on a farm fluctuates in a yearly cycle. The amount available on day t is $F(t) = a\sin(bt)^\circ + d$ tonnes. On day 40 there are 10.63 tonnes available, and on day 132 there are 11.32 tonnes available.

- **a** Find the value of b, assuming there are 365 days in a year.
- **b** Use technology to find a and d.
- **c** Sketch the graph of $F(t)$.
- **d** Find the amount of feed available on day 252 of the year.
- **e** Find the minimum and maximum amounts of available feed during the year.

8 Answer the **Opening Problem** on page **218**.

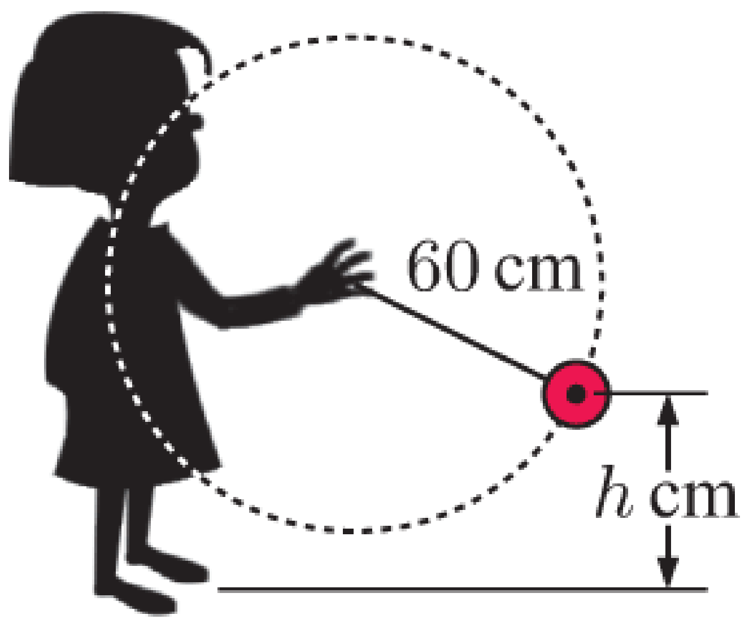
9 A paint spot X lies on the outer rim of the wheel of a paddle-steamer.
The wheel has radius 3 m.
It rotates clockwise at a constant rate, and X is seen entering the water every 4 seconds.
H is the distance of X above the bottom of the boat. At time $t = 0$, X is in the position shown.



a Find a sine model for H in the form $H(t) = a\sin(bt)^\circ + d$.

b Is X above or below the water after 6.5 seconds?

10 Fiona is spinning a yo-yo on a string. The string is 60 cm long, and the yo-yo completes two revolutions each second. Initially the yo-yo is at its highest point, 130 cm above ground level.



a Find a cosine model for the height of the yo-yo above ground level after t seconds.

b Hence find the height of the yo-yo after 3.9 seconds.

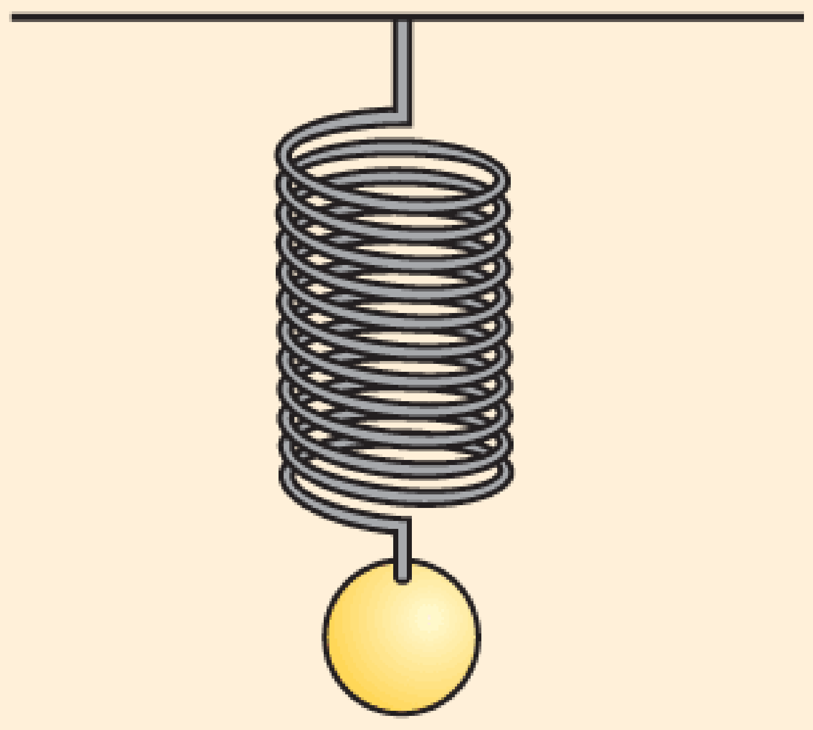
INVESTIGATION 2 MODELLING WITH TRIGONOMETRIC FUNCTIONS

Periodic phenomena occur all around us. If we are given a set of periodic data, we can use a trigonometric function to *model* this behaviour. The model can then be used to predict future behaviour.

Task 1: The undamped spring

An object is suspended from a spring. If the object is pulled below its resting position and then released, it will oscillate up and down.

The data below shows the height of the object relative to its rest position, at different times.



Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5

We will attempt to model the data with a trigonometric function of the form $H = a\cos(bt)^\circ + d$.

What to do:

1 **a** Draw a scatter diagram of the data.
b What features of the data suggest that a trigonometric model might be appropriate?

2 **a** State the *principal axis* of the oscillation.
b Hence determine the value of d.

3 **a** State the *amplitude* of the oscillation.
  b Use **a** and the initial conditions to determine the value of a.

4 **a** State the *period* of the oscillation.
  b Hence determine the value of b.

5 Write down the trigonometric function which models the height of the object over time.

6 Use your model to predict the height of the object after 4.25 seconds.

7 What do you think is unrealistic about this model? What would happen differently in reality? Watch the video to find out.

VIDEO

Task 2: The pendulum

This task is best performed in small groups.

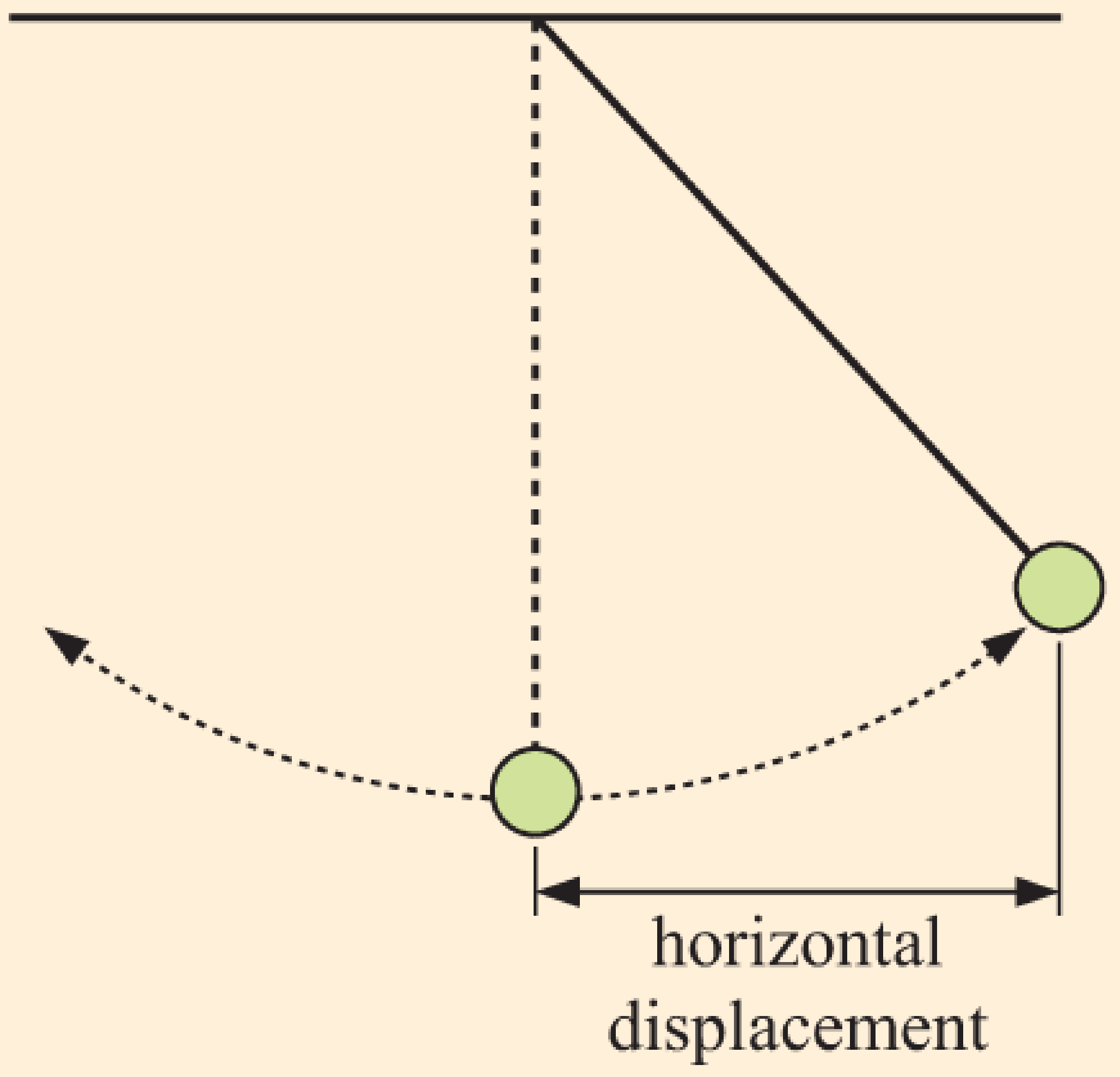
You will need: string, sticky tape, a ruler, a stopwatch, and a tennis ball.

What to do:

1 Cut a piece of string of length 30 cm. Attach one end of the string to the tennis ball, and the other end to your desk.

2 Hold the ball to one side, then release it, causing the ball to swing back and forth like a pendulum.

3 Using your stopwatch and ruler, measure the maximum and minimum horizontal displacement reached by the ball, and the times at which they occurred. You may need to repeat the experiment several times, but make sure the ball is released from the same position each time.



4 Use your data to find a trigonometric function which models the horizontal displacement of the ball over time.

5 What part of the function affects the *period* of the pendulum?

6 Repeat the experiment with strings of different length. Explore the relationship between the length of the string and the period of the pendulum.

RESEARCH

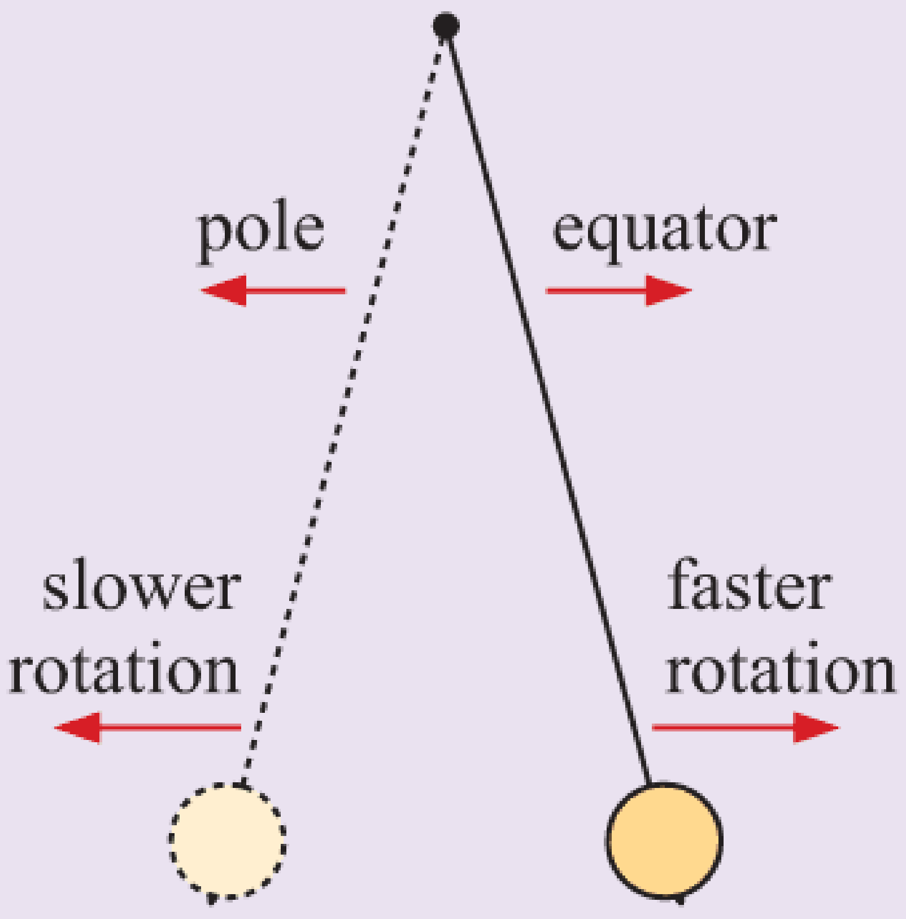
1 How accurately will a trigonometric function model the phases of the moon?

2 Are there any periodic phenomena which can be modelled by the *sum* of trigonometric functions?

HISTORICAL NOTE FOUCAULT'S PENDULUM

In 1851, French physicist **Léon Foucault** used a simple experiment at the Panthéon in Paris to show that the Earth rotates about an axis. The experiment used a brass-coated pendulum bob made from lead, and a 67 m long wire.

The pendulum was strung from the dome in the Panthéon and allowed to swing freely. Rather than swinging back and forth in a fixed plane, the pendulum rotated very slowly over time.



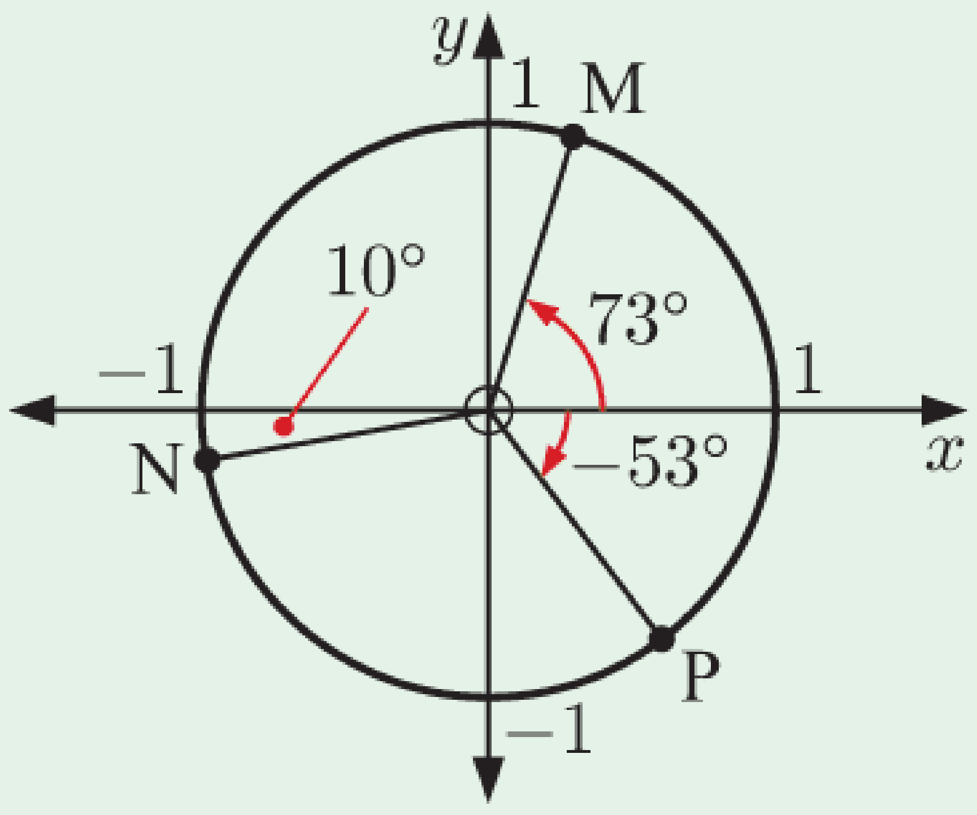
The Earth's rotation means that objects closer to the equator (and therefore further from the axis of rotation) rotate faster than objects closer to the poles. The bob will therefore have a higher rotational velocity when it swings closer to the equator than away from the equator, and this shift in velocity causes the pendulum to rotate.

The time a pendulum takes to perform a complete rotation is given by $T = \dfrac{24}{\sin\phi}$ hours, where ϕ is the angle of latitude of the pendulum's location.

For example, the angle of latitude at Paris is $\phi \approx 48.86°$, so a pendulum in Paris performs a complete rotation every $\dfrac{24}{\sin 48.86°} \approx 31.9$ hours. This closely matches Foucault's experiment.

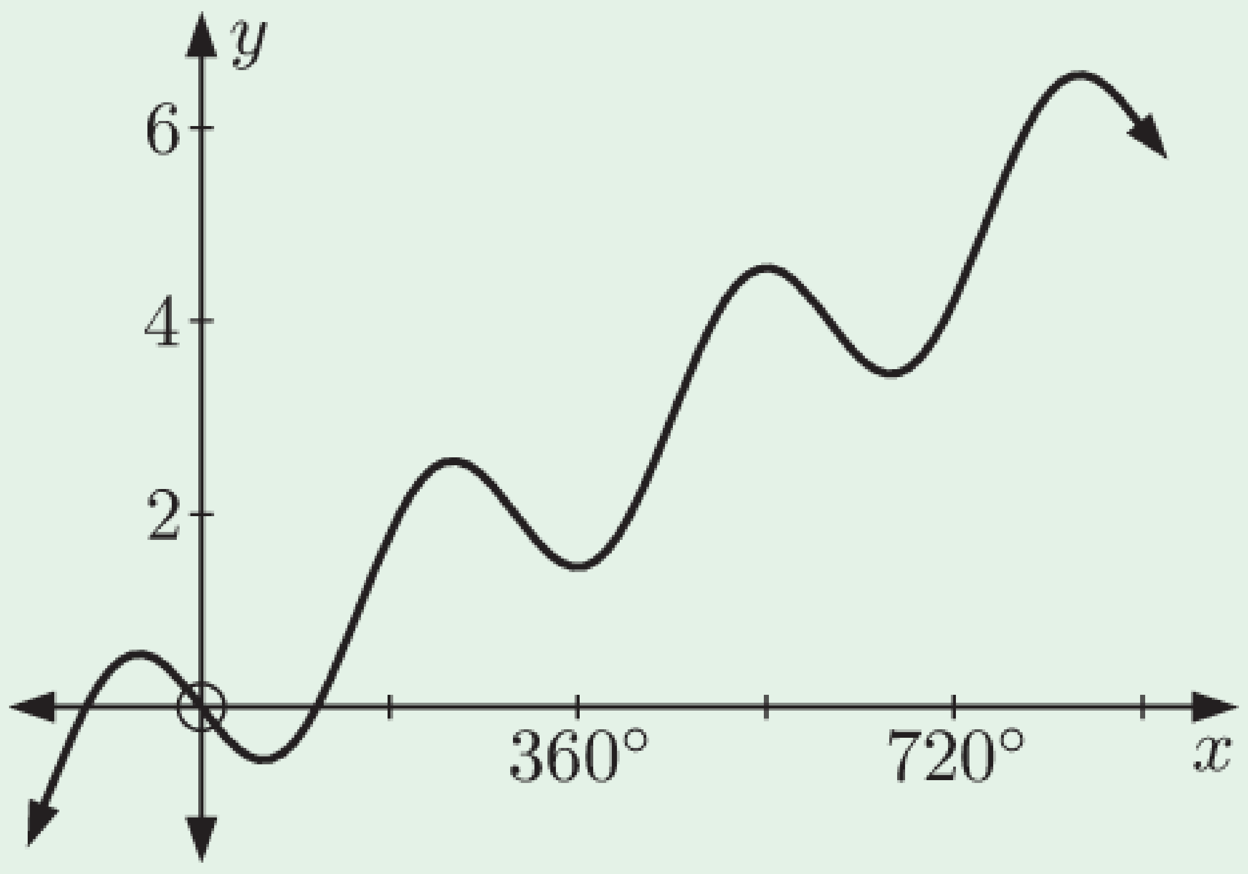
REVIEW SET 9A

1 Find the coordinates of the points M, N, and P on the unit circle.

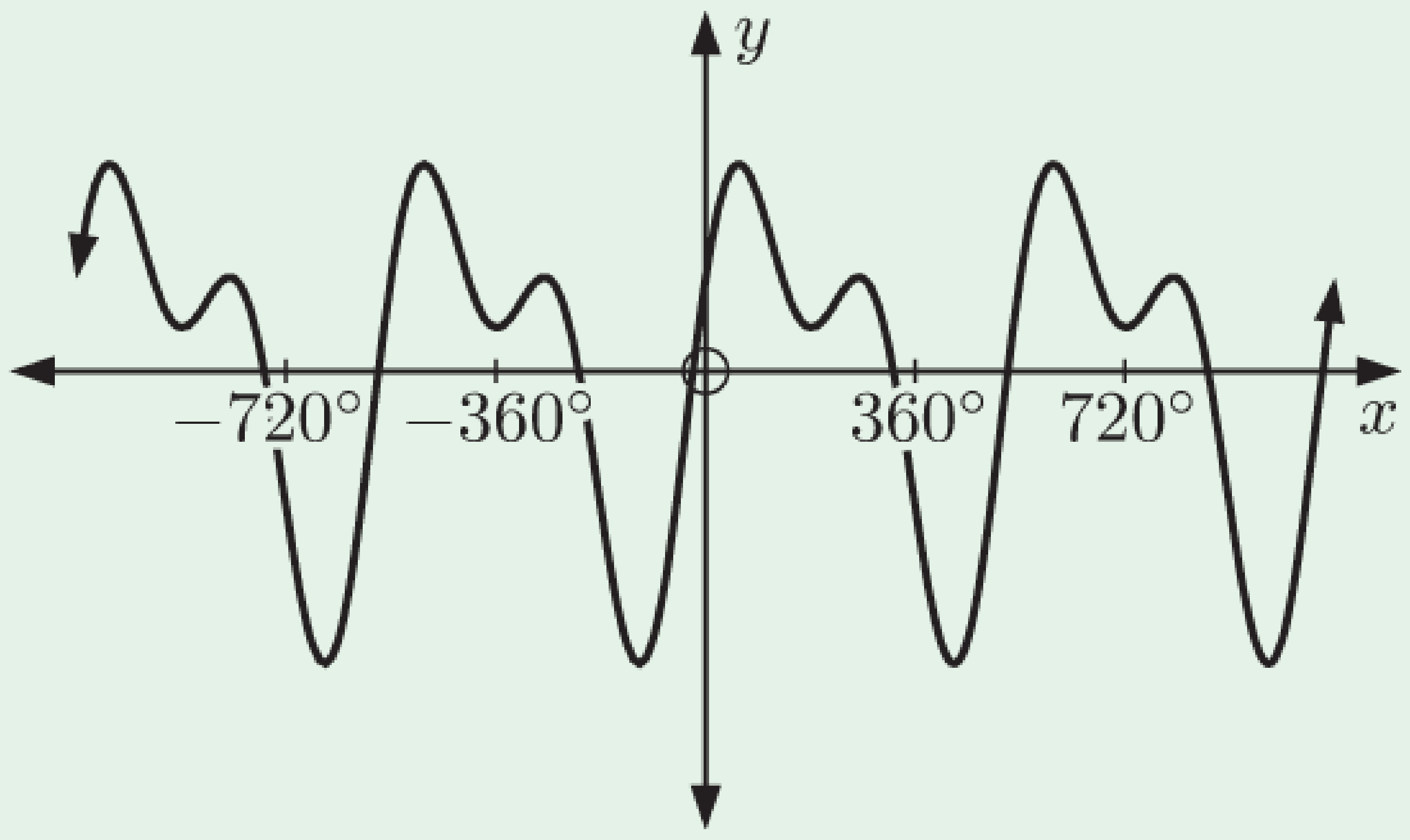


2 Which of the following graphs display periodic behaviour?

a



b



3 Find the amplitude of:

a $y = 5\cos 2x + 3$

b $y = -\frac{1}{4}\sin x - 2$

4 State the period of:

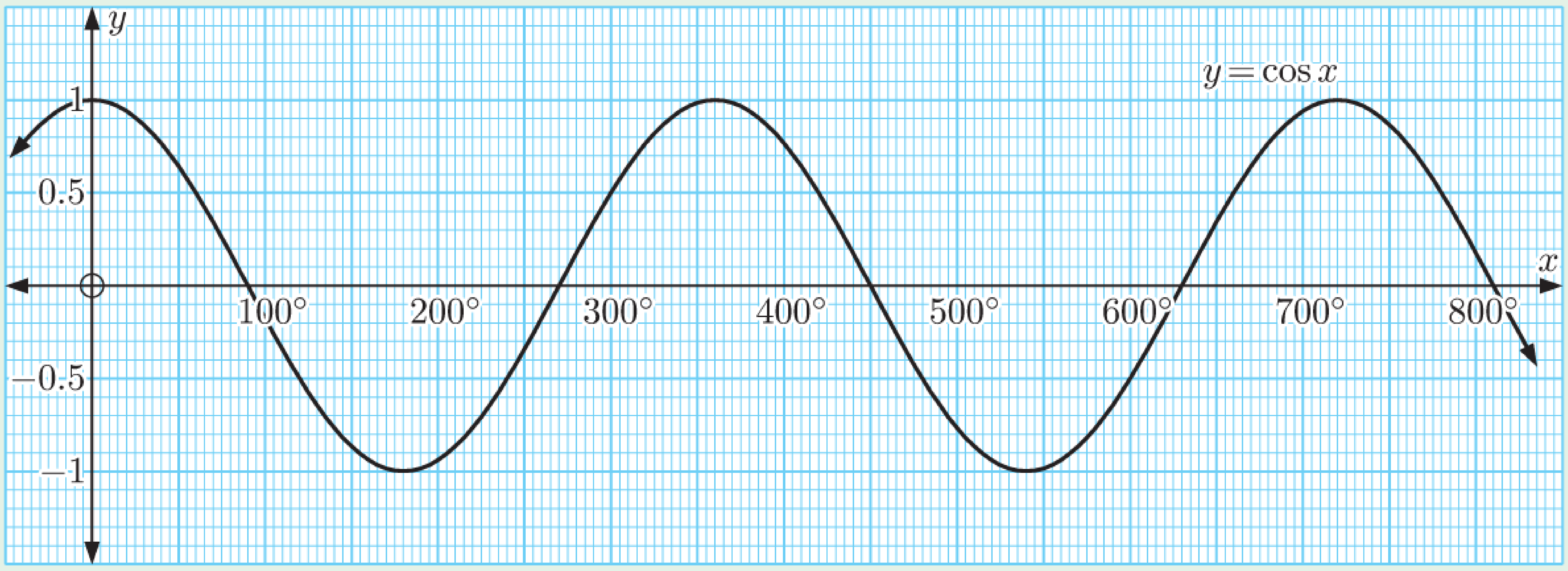
a $y = 4\sin\frac{x}{5}$

b $y = -2\cos 4x$

c $y = 4\cos\frac{x}{2} + 4$

d $y = \frac{1}{2}\sin 3x$

5



Use the graph of $y = \cos x$ to find the values of x on $0° \leqslant x \leqslant 800°$ for which:

a $\cos x = -0.4$

b $\cos x = 0.9$

6 Copy and complete:

Function	*Period*	*Amplitude*	*Range*
$y = -3\sin\frac{x}{4} + 1$			
$y = 2\cos 5x - 7$			

7 **a** Draw the graph of $y = \cos 3x$ for $0° \leqslant x \leqslant 720°$.

b Find the value of y when $x = 135°$. Mark this point on your graph.

8 State the transformations which map $y = \sin x$ onto $y = 3\sin 2x$.

9 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = \frac{1}{4}\sin x$

b $y = 2\cos x - 3$

c $y = \sin\frac{x}{3}$

d $y = \sin\frac{2x}{3} + 1$

e $y = -\cos 2x$

f $y = \frac{1}{2}\sin x - 2$

10 Consider the function $y = 5\sin 2x + 4$.

a Find the amplitude of the function.

b Find the principal axis of the function.

c Find the period of the function.

d Hence sketch the function for $0° \leqslant x \leqslant 360°$.

e Find the value of y when $x = 45°$.

11 **a** Sketch the graph of $y = -2\sin x + 7$ for $0° \leqslant x \leqslant 720°$.

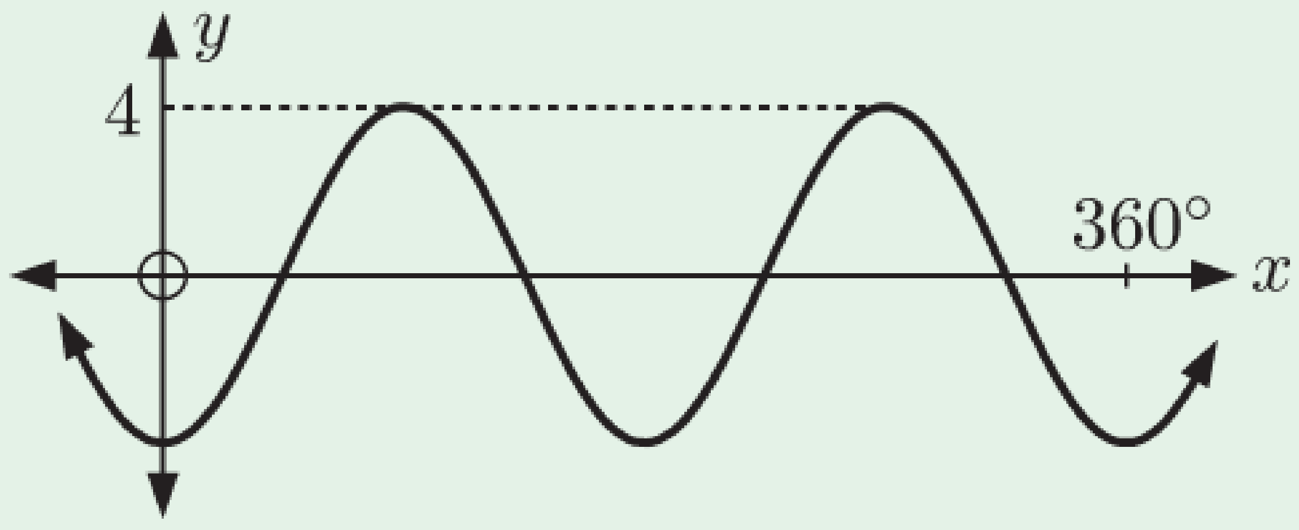
b Find the value of y when $x = 150°$.

c Find the maximum value of y, and the values of x for which the maximum occurs.

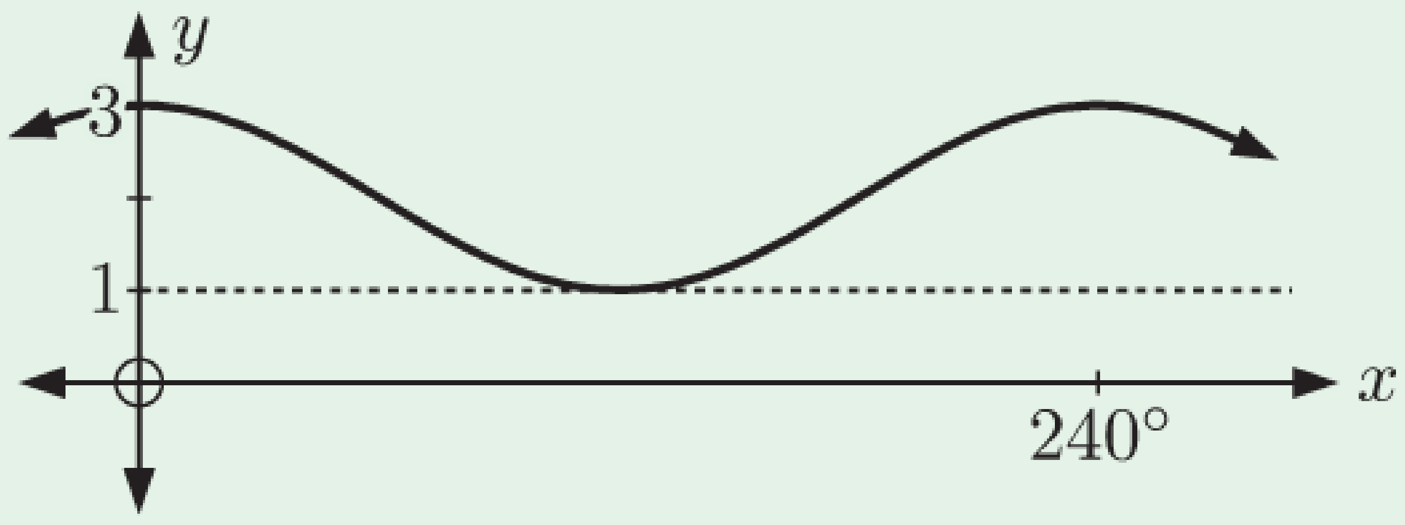
d Find the minimum value of y, and the values of x for which the minimum occurs.

12 Find the cosine function represented in each of the following graphs:

a

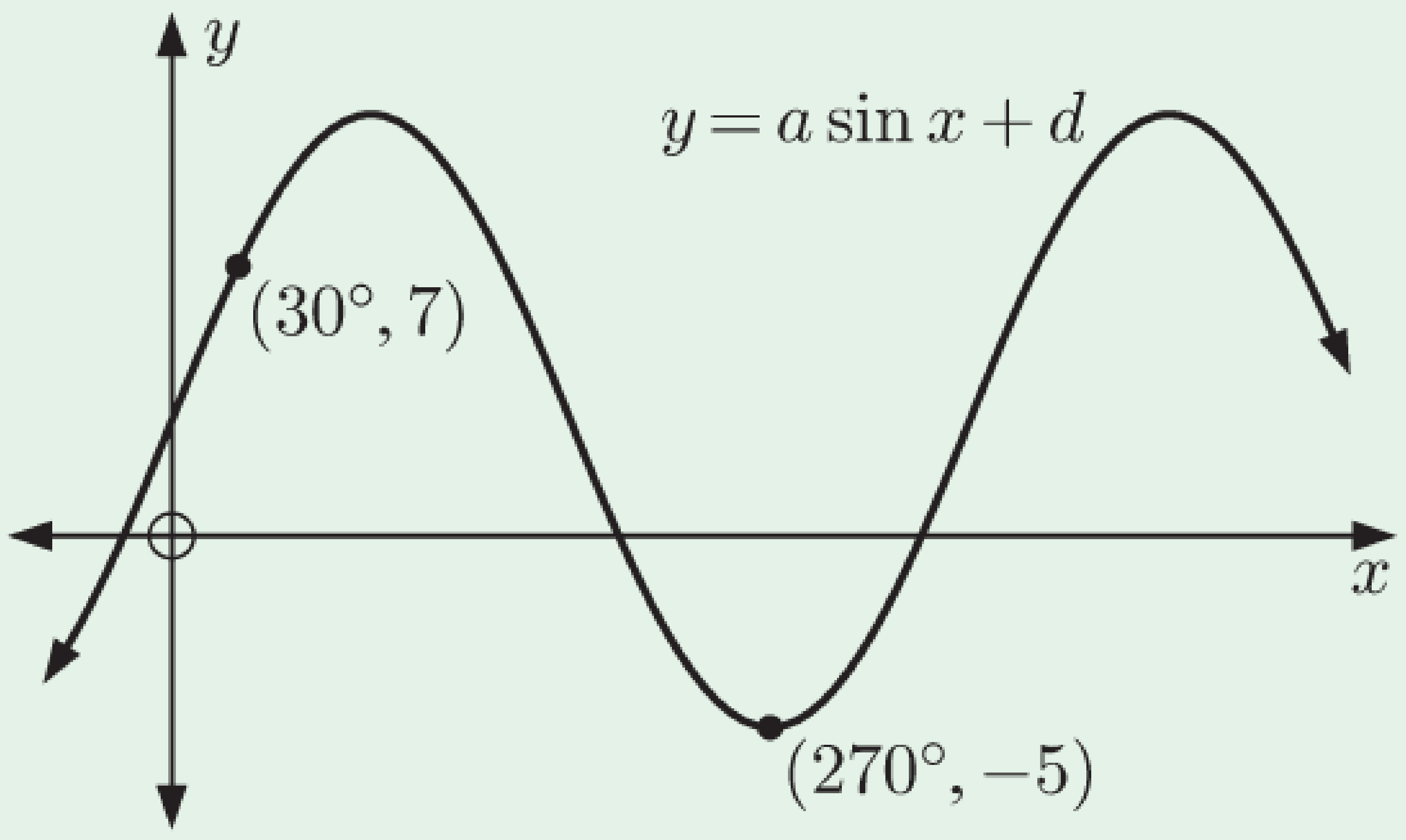


b



13 **a** Find the unknowns in this function.

b Hence determine the range of the function.



14 The fraction of the moon which is illuminated each night is given by the function $M(t) = \frac{1}{2}\cos(12t)^\circ + \frac{1}{2}$, where t is the time in days after January 1st.

a Sketch the graph of M against t for $0 \leqslant t \leqslant 60$.

b Find the fraction of the moon which is illuminated on the night of:

i January 6th
ii January 21st
iii January 27th
iv February 19th.

c How often does a full moon occur?

d On what dates during January and February is the moon not illuminated at all?

15 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours.

The height of the tide is given by $H(t) = a\cos(bt)^\circ + d$ metres, where t is the number of hours after the first high tide.

a Determine the values of a and b.

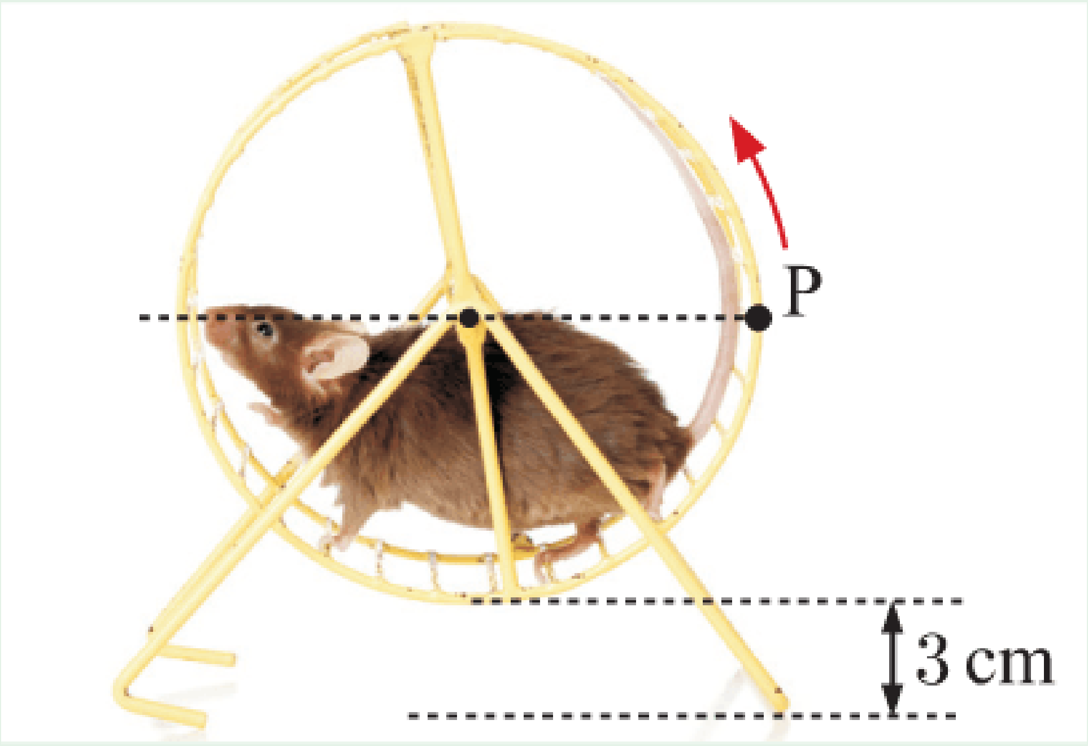
b Given that the high tide is 16.2 m, find the value of d.

c Sketch the graph of $H(t)$.

d Find the height of the tide 8 hours after the first high tide.

16 A hamster wheel has diameter 16 cm. When the hamster runs on the wheel, it rotates through one revolution each second.

Suppose a point P is marked on the rim of the wheel.



a Find a sine model $H = a\sin(bt)^\circ + d$ for the height of P in centimetres above ground level after t seconds. Assume that P is initially in the position shown on the diagram.

b Use your model to predict the height of P above ground level after 4.7 seconds.

REVIEW SET 9B

1 Explain why:

a $\cos 500° = \cos 140°$ **b** $\sin(-100°) = \sin 620°$

2 Consider the graph alongside.

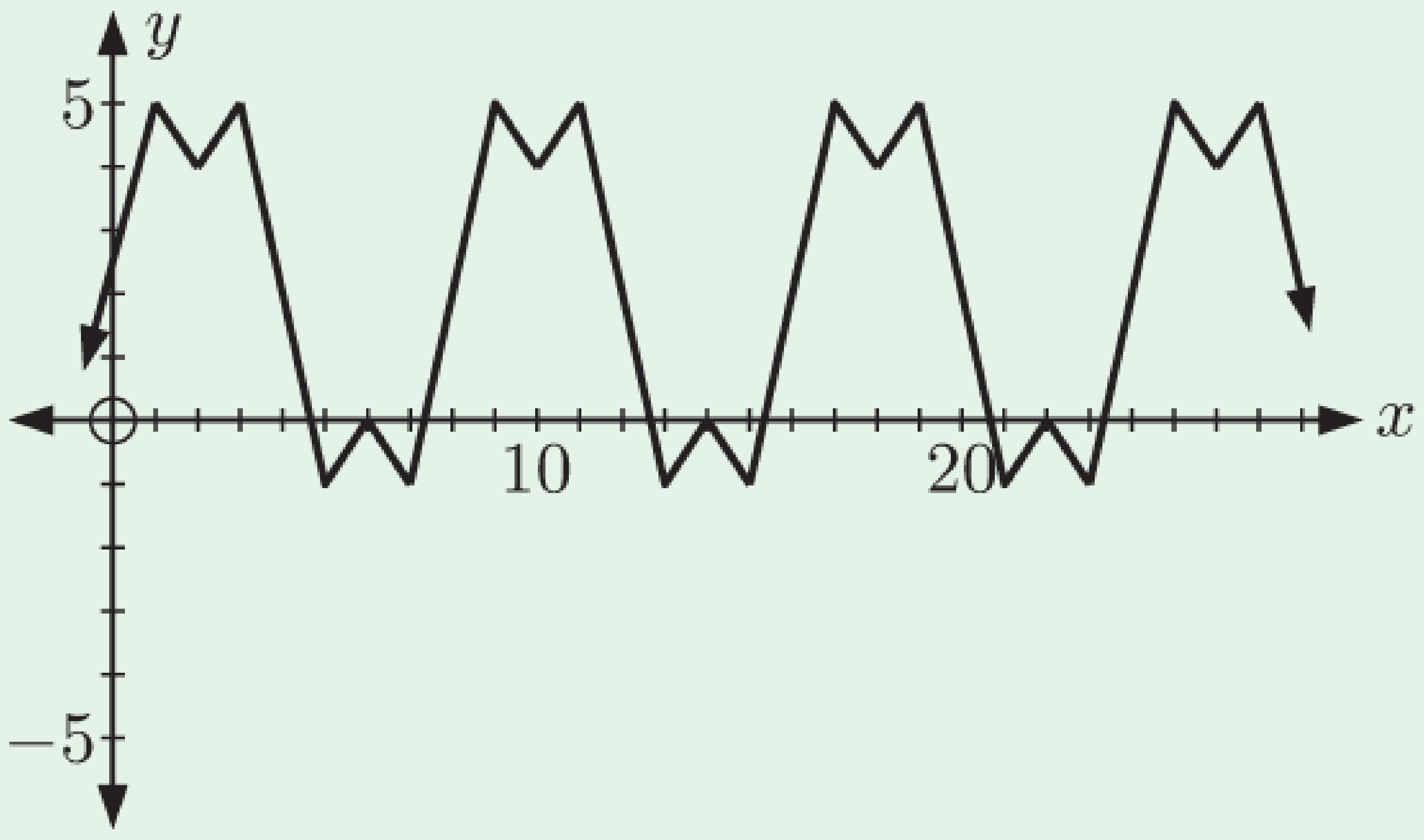
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value.



3 Find the period of:

a $y = 2\sin 3x$ **b** $y = -4\cos\frac{x}{2} - 1$

4 Find the principal axis of:

a $y = -\frac{1}{3}\sin x + 5$ **b** $y = 2\cos\frac{x}{3} - 4$

5 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a $1080°$ **b** $15°$ **c** $9°$

6 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = \frac{1}{3}\cos x + 1$

7 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = \cos x - 3$ **b** $y = -2\sin x$

c $y = 4\sin x - 1$ **d** $y = \frac{1}{2}\cos 2x$

8 State the transformations which map:

a $y = \cos x$ onto $y = \frac{1}{3}\cos x + 1$ **b** $y = \sin x$ onto $y = -\sin\frac{3}{2}x$.

9 Sketch the graph of the following for $0° \leqslant x \leqslant 360°$:

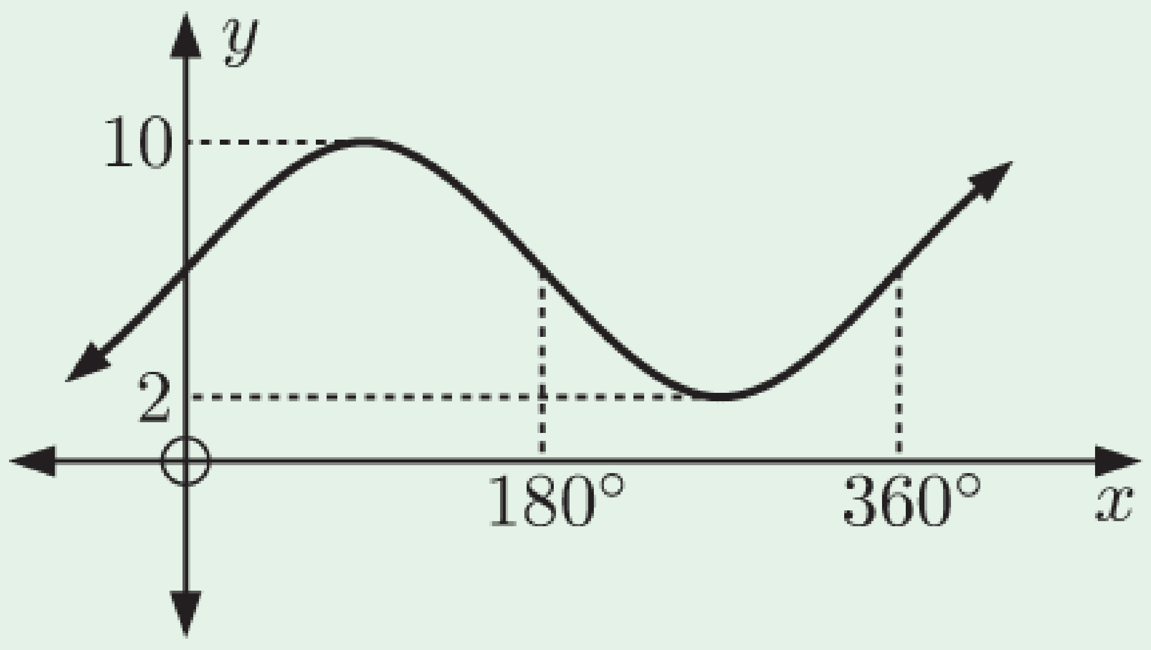
a $y = \frac{3}{2}\sin 3x$ **b** $y = 2\cos 3x - 1$

c $y = \sin 4x$ **d** $y = -\frac{1}{2}\cos 2x$

10 **a** Sketch the graph of $y = 3\sin 2x + 5$ for $0° \leqslant x \leqslant 360°$.

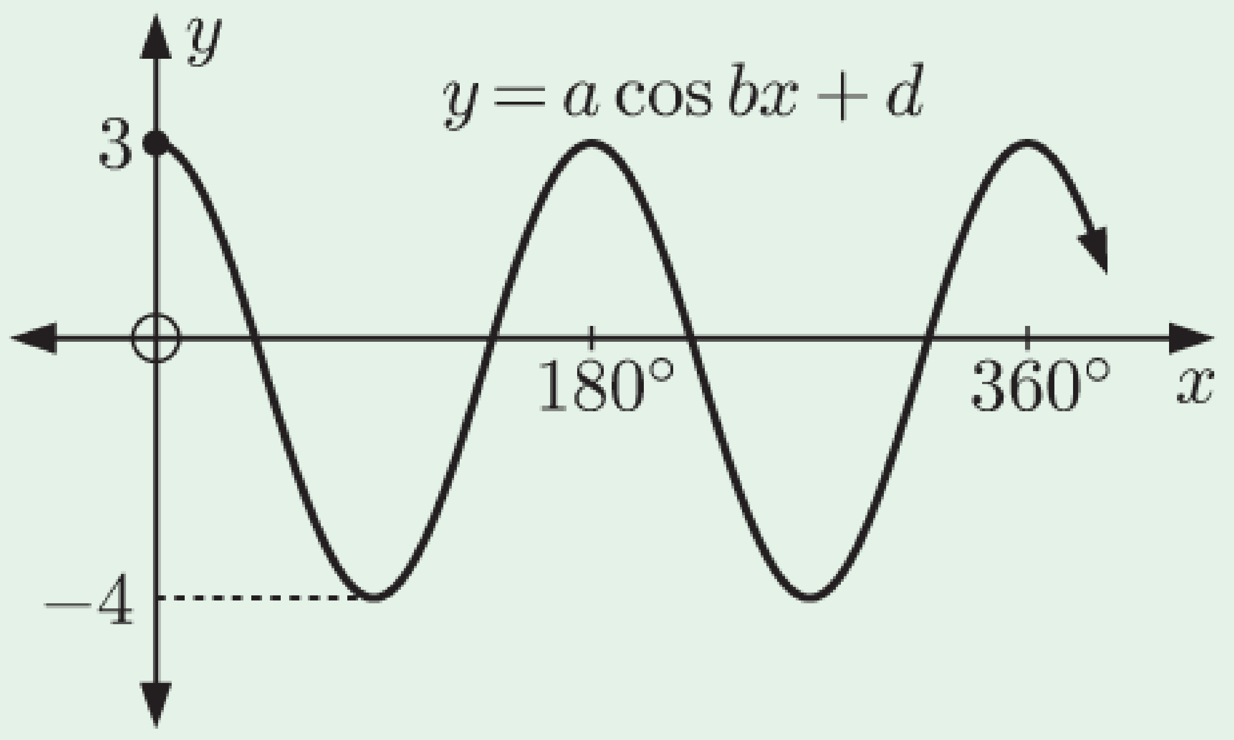
b Find the value of y when $x = 30°$.

11 Find the sine function shown in this graph.

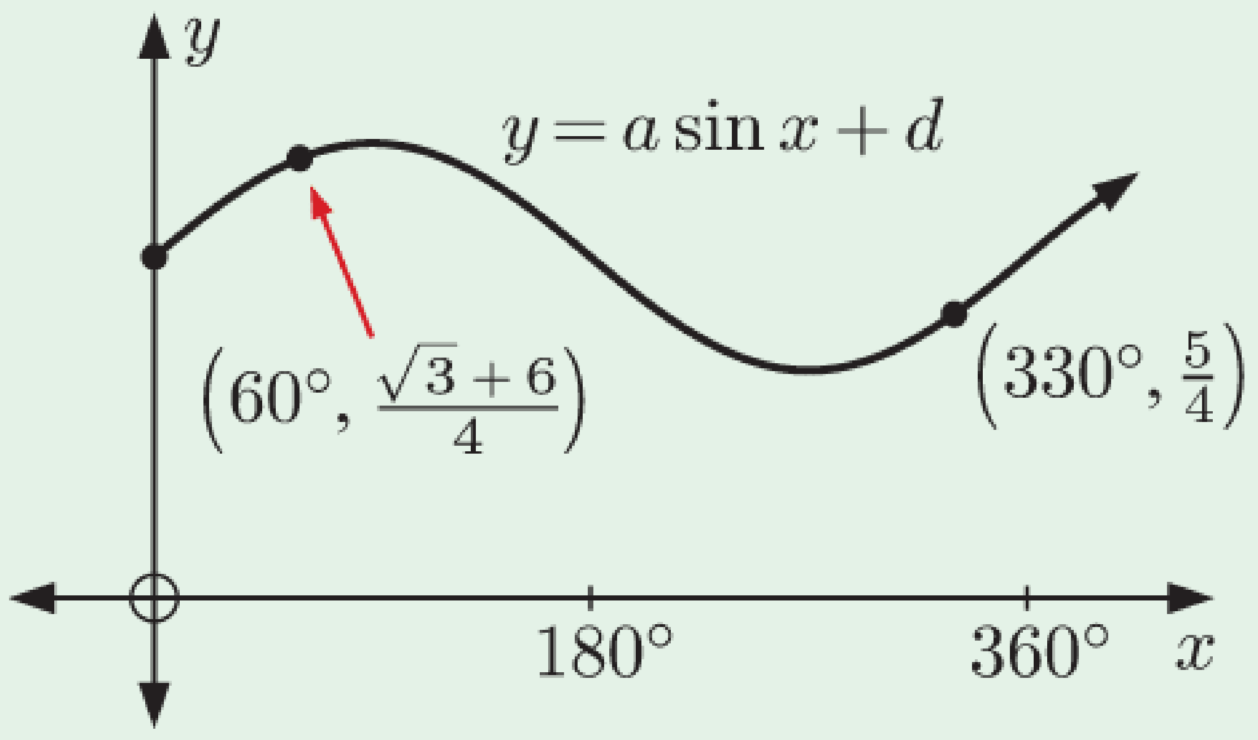


12 Find the unknowns in each function:

a

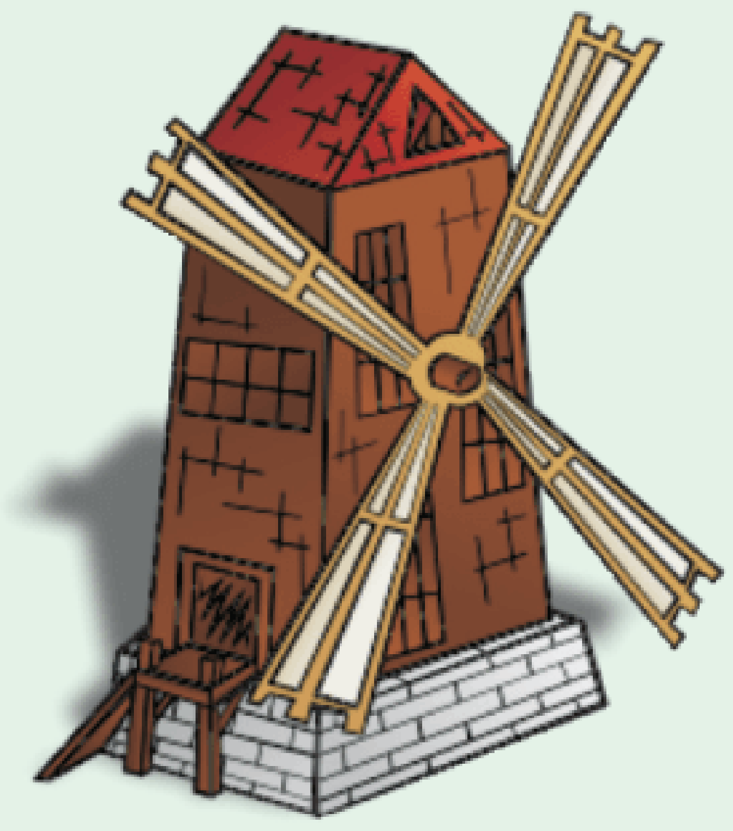


b

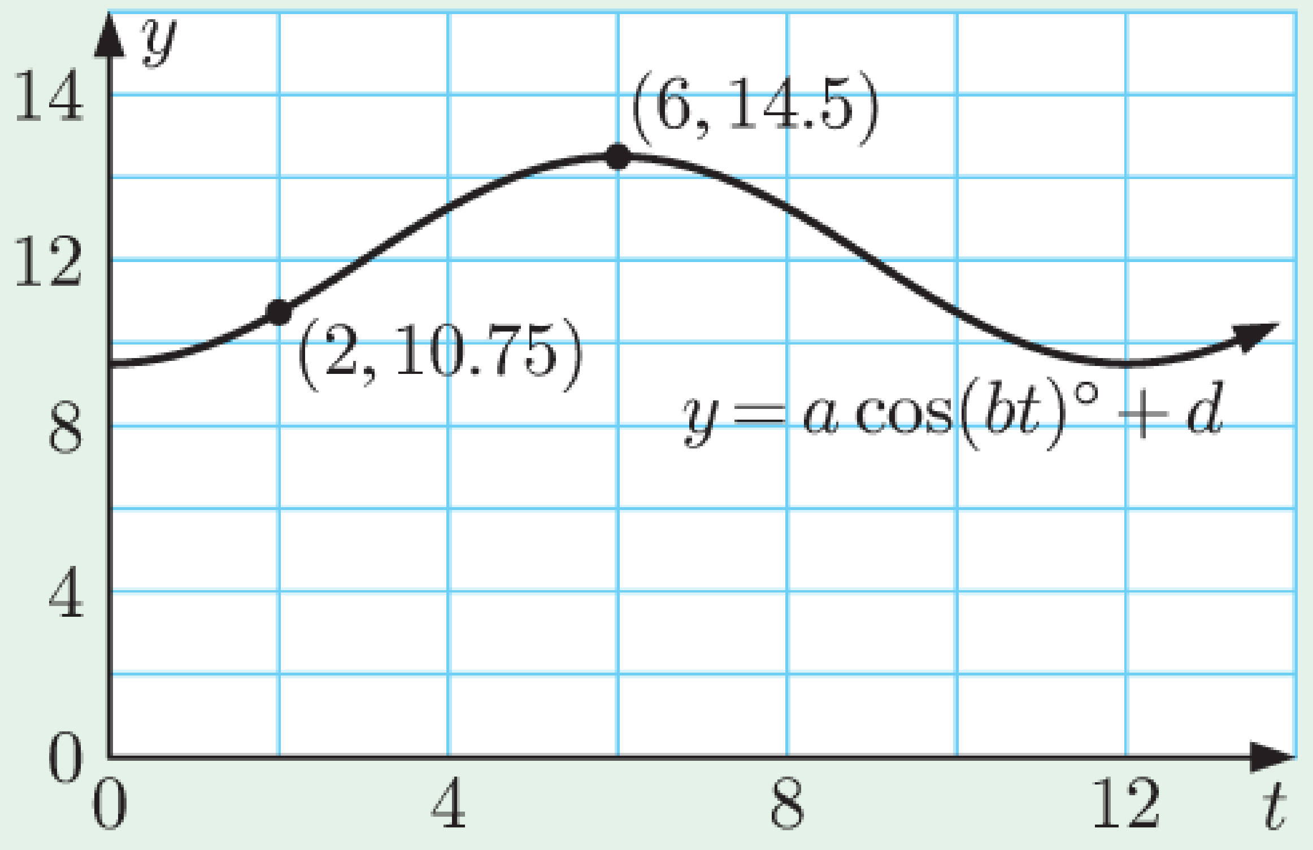


13 As the tip of a windmill's blade rotates, its height above ground is given by $H(t) = 10\sin(30t)° + 20$ metres, where t is the time in seconds.

- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 36$.
- **b** Find the height of the blade's tip after 9 seconds.
- **c** Find the minimum height of the blade's tip.
- **d** How long does the blade take to complete a full revolution?

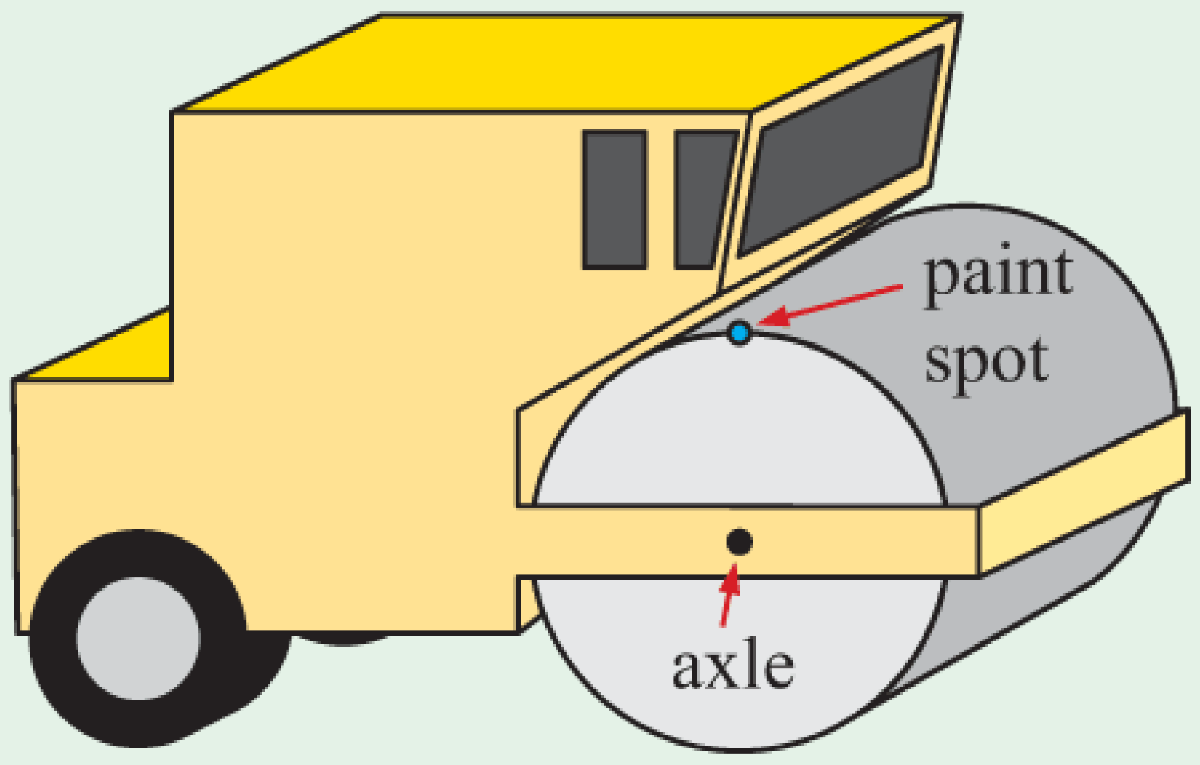


14 This graph shows the average number of daylight hours per day in Los Angeles. The model has the form $y = a\cos(bt)° + d$ hours, where t is the time in months after January.



- **a** Find the value of b.
- **b** Use the remaining information to find a and d.
- **c** Predict the average number of daylight hours in Los Angeles during September.
- **d** If we modelled the average number of daylight hours in a different city, which of the parameters a, b, and d would you expect to change? Explain your answer.

15 A steamroller has a spot of paint on its roller. As the steamroller moves, the spot rotates around the axle. The roller has radius 1 metre and completes one full revolution every 2 seconds.



- **a** What does the graph of the spot's height over time look like? Assume that the spot is initially in the position shown.
- **b** Construct a cosine model to describe the height of the paint spot over time.
- **c** Use your model to predict the height of the spot after 3.6 seconds.

Chapter 10

Differentiation

Contents:

A Rates of change
B Instantaneous rates of change
C Limits
D The gradient of a tangent
E The derivative function
F Differentiation
G Rules for differentiation

OPENING PROBLEM

Suppose we add water to a container at a constant rate. The depth of the water increases over time.

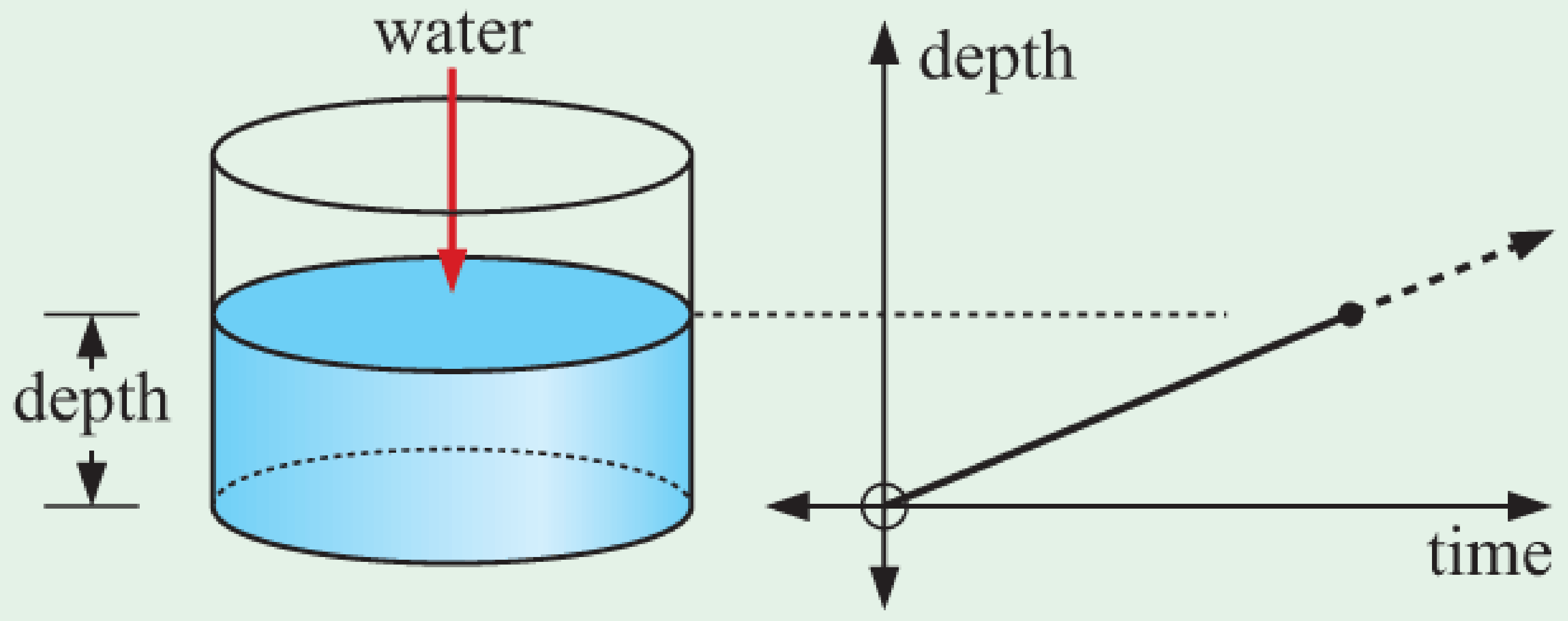
Things to think about:

a Think about the graph of the *volume of water added* against *time*.

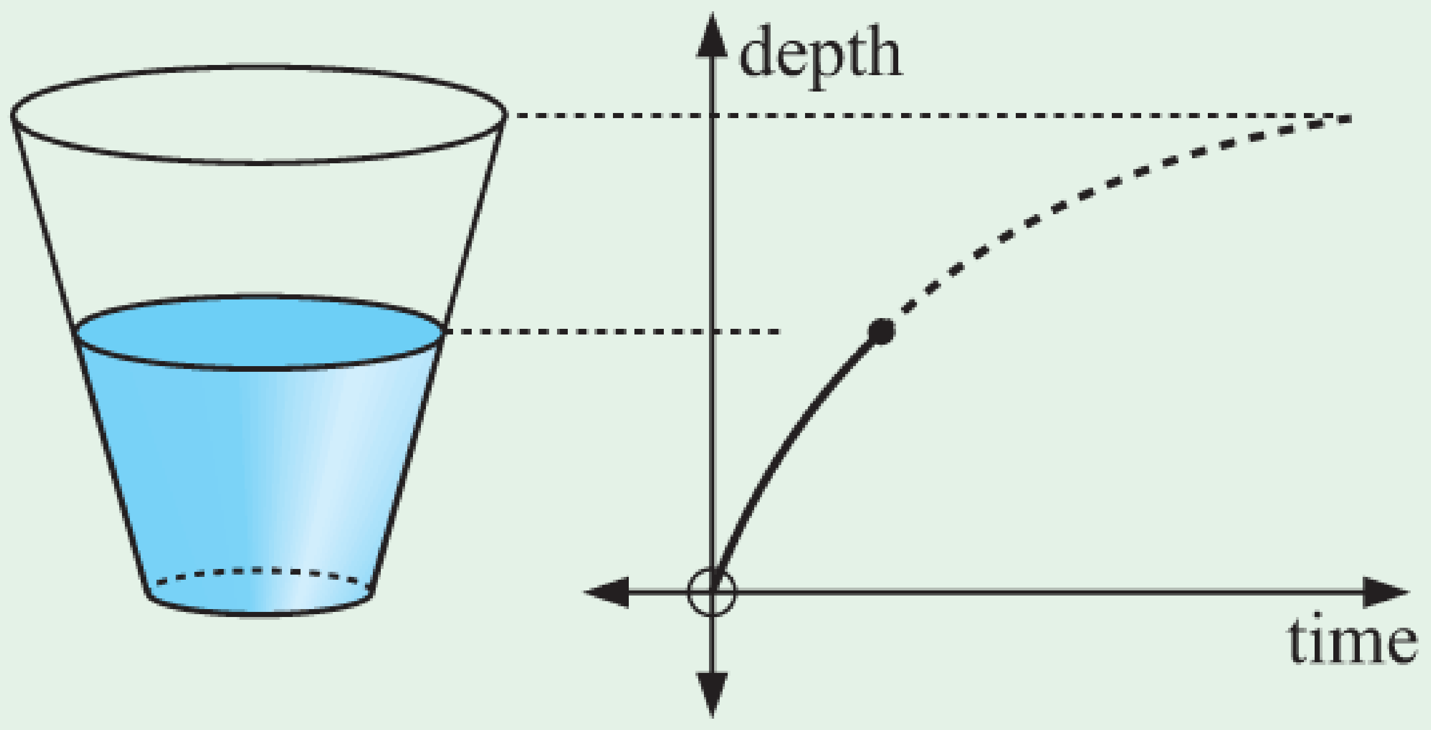
- **i** Can you explain why this graph is a straight line passing through the origin?
- **ii** What does the *gradient* of the straight line tell us?

b Think about the graph of the *depth of water* against *time*.

- **i** Can you explain why this graph is a straight line for a cylindrical container?

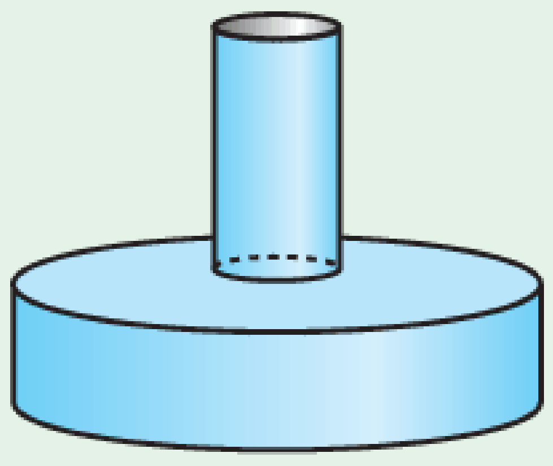


- **ii** Can you explain why this graph is *not* a straight line for the container shown?

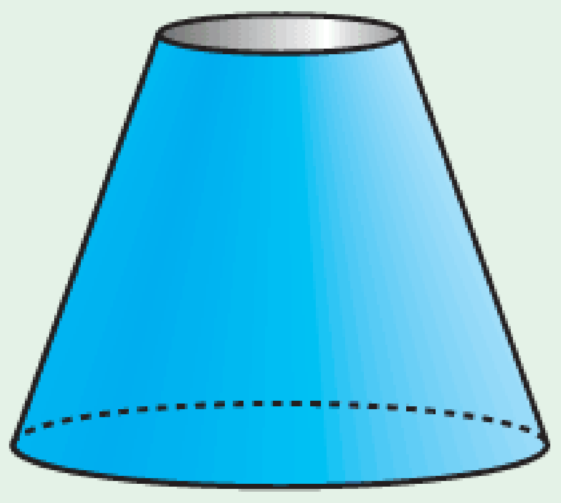


c By examining the shape of each container, can you predict the depth-time graph when water is added at a constant rate?

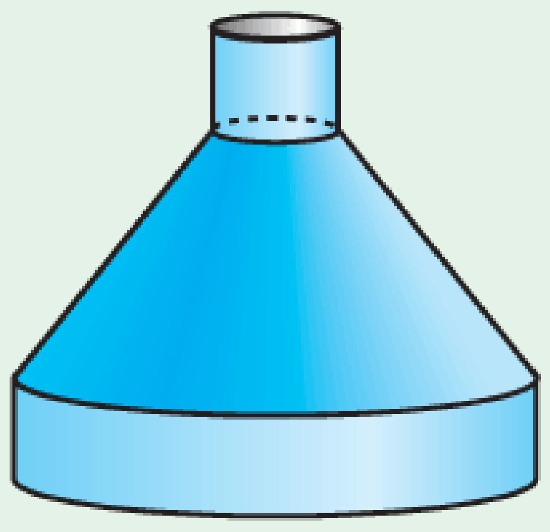
i



ii



iii



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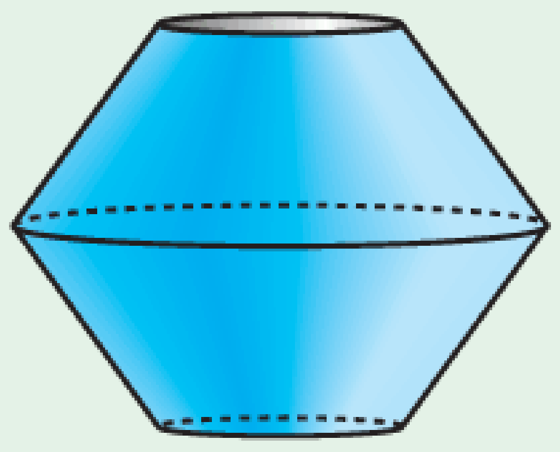
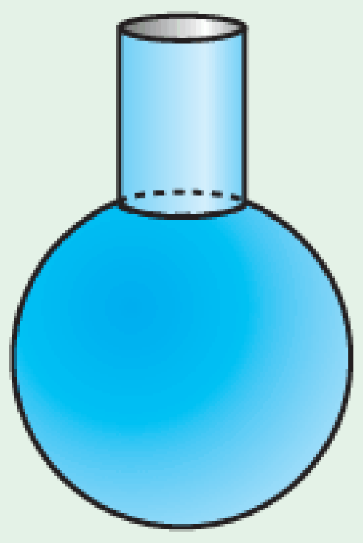


v

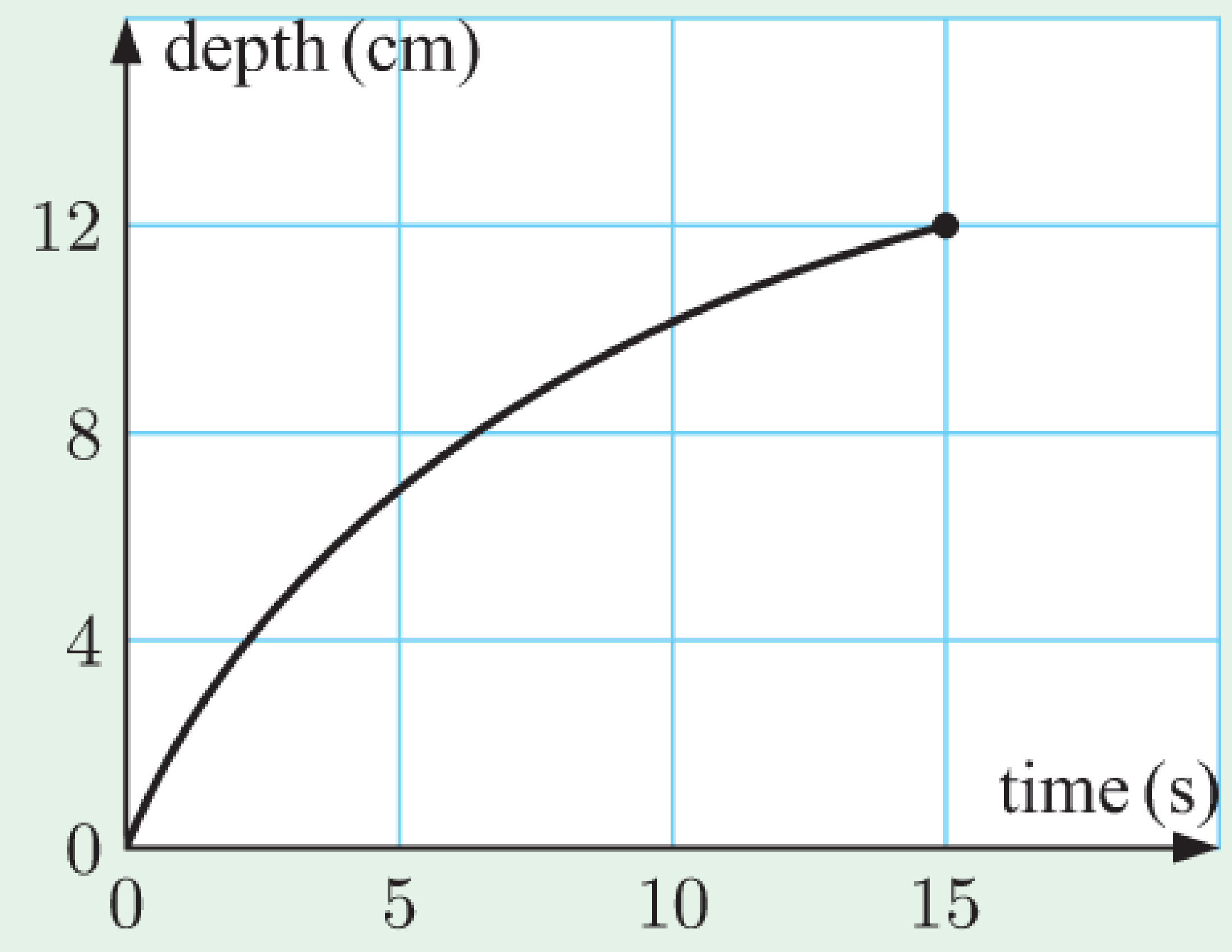


Use the water filling demonstration to check your answers.

DEMO

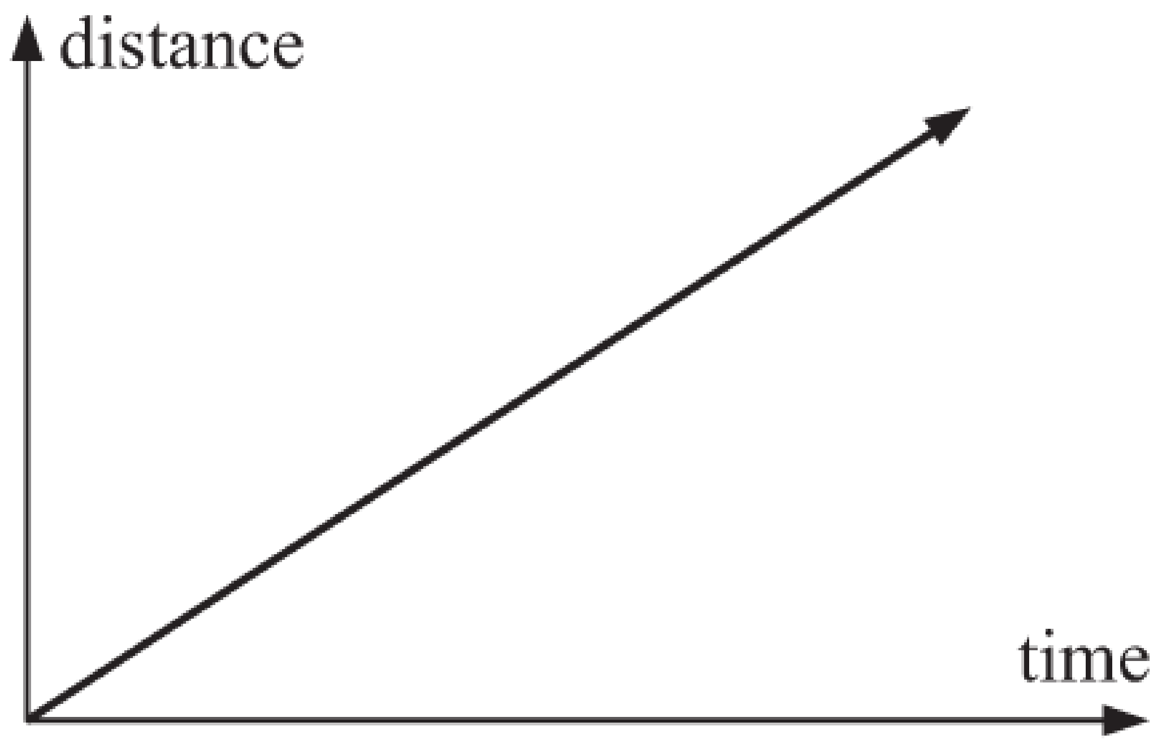
d Consider the depth-time graph alongside.

- **i** How can we measure the *average rate* at which the depth increases from $t = 5$ to $t = 10$ seconds?
- **ii** How can we measure the *instantaneous rate* at which the depth is increasing at the instant when $t = 8$ seconds?



If the relationship between two variables is a straight line, the gradient of the line tells us the rate at which one variable changes with respect to the other.

For example, in a travel graph showing distance against time, the gradient of a straight line segment tells us the *speed* of the object.



In the real world, rates such as speed are constantly changing. Their travel graphs are curves rather than straight lines. In order to calculate the instantaneous speed of an object, we need a branch of mathematics called **differential calculus**.

Differential calculus deals with **rates of change**. It has widespread applications in science, engineering, and finance.

A RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 runs per innings.
- Michael Jordan's average basketball scoring rate was 30.1 points per game.
- Rangi's average typing rate is 63 words per minute with an error rate of 2.3 errors per page.

Example 1 **Self Tutor**

Josef typed 213 words in 3 minutes and made 6 errors, whereas Marie typed 260 words in 4 minutes and made 7 errors. Compare their performances using rates.

$$\text{Josef's typing rate} = \frac{213 \text{ words}}{3 \text{ minutes}} = 71 \text{ words per minute.}$$

$$\text{Josef's error rate} = \frac{6 \text{ errors}}{213 \text{ words}} \approx 0.0282 \text{ errors per word.}$$

$$\text{Marie's typing rate} = \frac{260 \text{ words}}{4 \text{ minutes}} = 65 \text{ words per minute.}$$

$$\text{Marie's error rate} = \frac{7 \text{ errors}}{260 \text{ words}} \approx 0.0269 \text{ errors per word.}$$

$\therefore$ Josef typed at a faster rate, but Marie typed with greater accuracy.

EXERCISE 10A.1

1 Kirsten's resting pulse rate was measured at 62 beats per minute.

- **a** Explain exactly what this rate means.
- **b** How many heart beats would Kirsten expect to have each hour while she sleeps?

2 When Jana typed a 14 page document, she made eight errors. If an average page of typing has 380 words, find Jana's error rate in:

a errors per word **b** errors per 100 words.

3 Niko worked 12 hours for £148.20, whereas Marita worked 13 hours for £157.95. Who worked for the better hourly rate of pay?

4 New tyres have a tread depth of 8 mm. After driving for 32 178 km, the tread depth on Joanne's tyres was reduced to 2.3 mm. What was the wearing rate of the tyres in:

a mm per km travelled **b** mm per 10 000 km travelled?

5 We left Kuala Lumpur at 11:43 am and travelled to Penang, a distance of 350 km. We arrived there at 3:39 pm. Find our average speed in:

a $\text{km}\,\text{h}^{-1}$ **b** $\text{m}\,\text{s}^{-1}$.

CONSTANT RATES OF CHANGE

Suppose water from a hose is used to fill a swimming pool. The volume of water in the pool is recorded at 1 minute intervals in the table alongside.

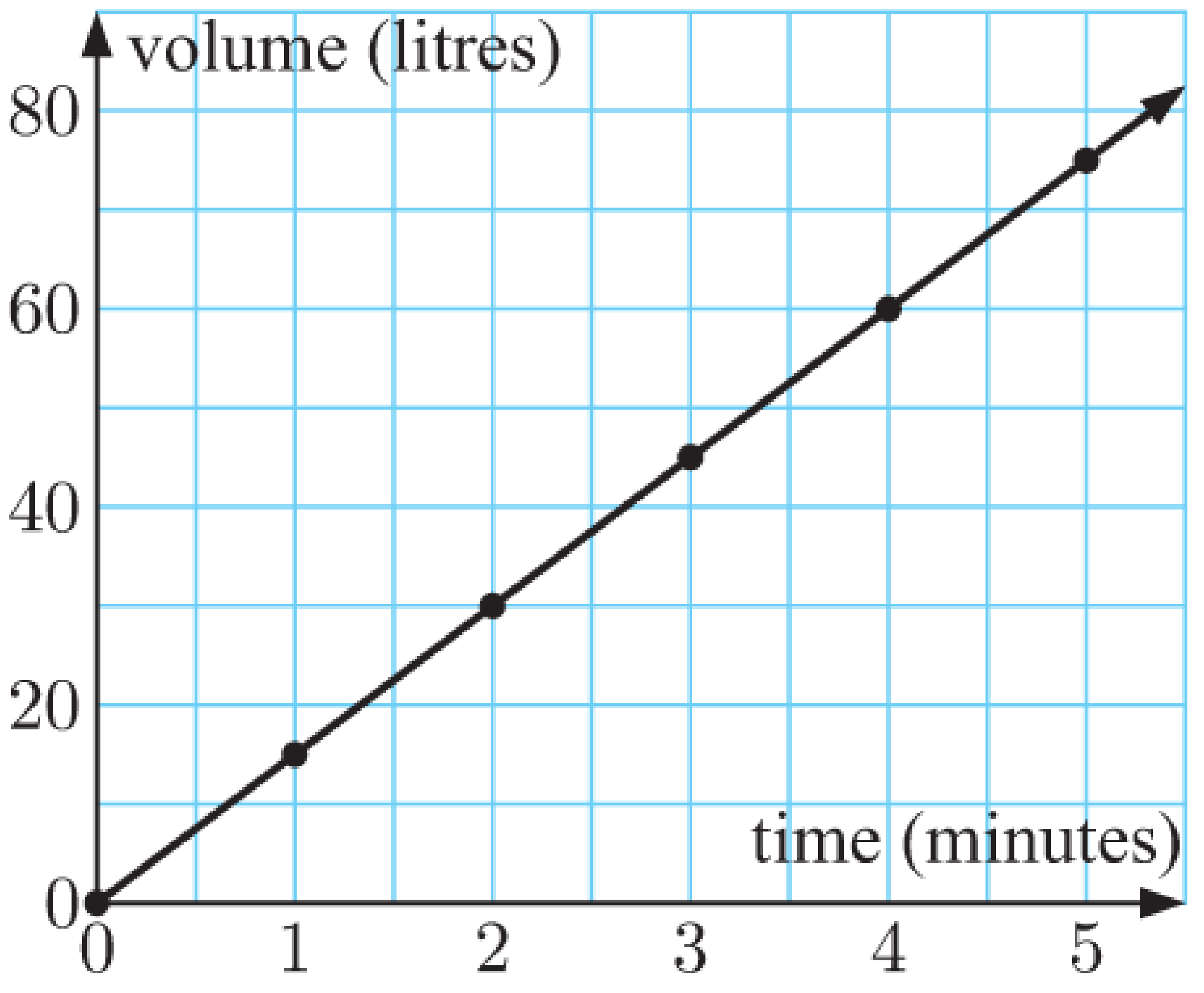
Time (minutes)	0	1	2	3	4	5
Volume (litres)	0	15	30	45	60	75

+15 +15 +15 +15 +15

Notice that the volume of water increases by the same amount each time interval. The rate of change in volume is constant, so the graph of *volume* against *time* is a straight line.

The **gradient** of the line gives the rate of change:

$$\text{rate of flow} = \frac{15-0}{1-0}$$
$$= 15 \text{ litres per minute}$$



EXERCISE 10A.2

1 The table alongside shows the distance travelled by a jogger at 30-second intervals.

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

a Is the jogger travelling at a constant speed? Explain your answer.

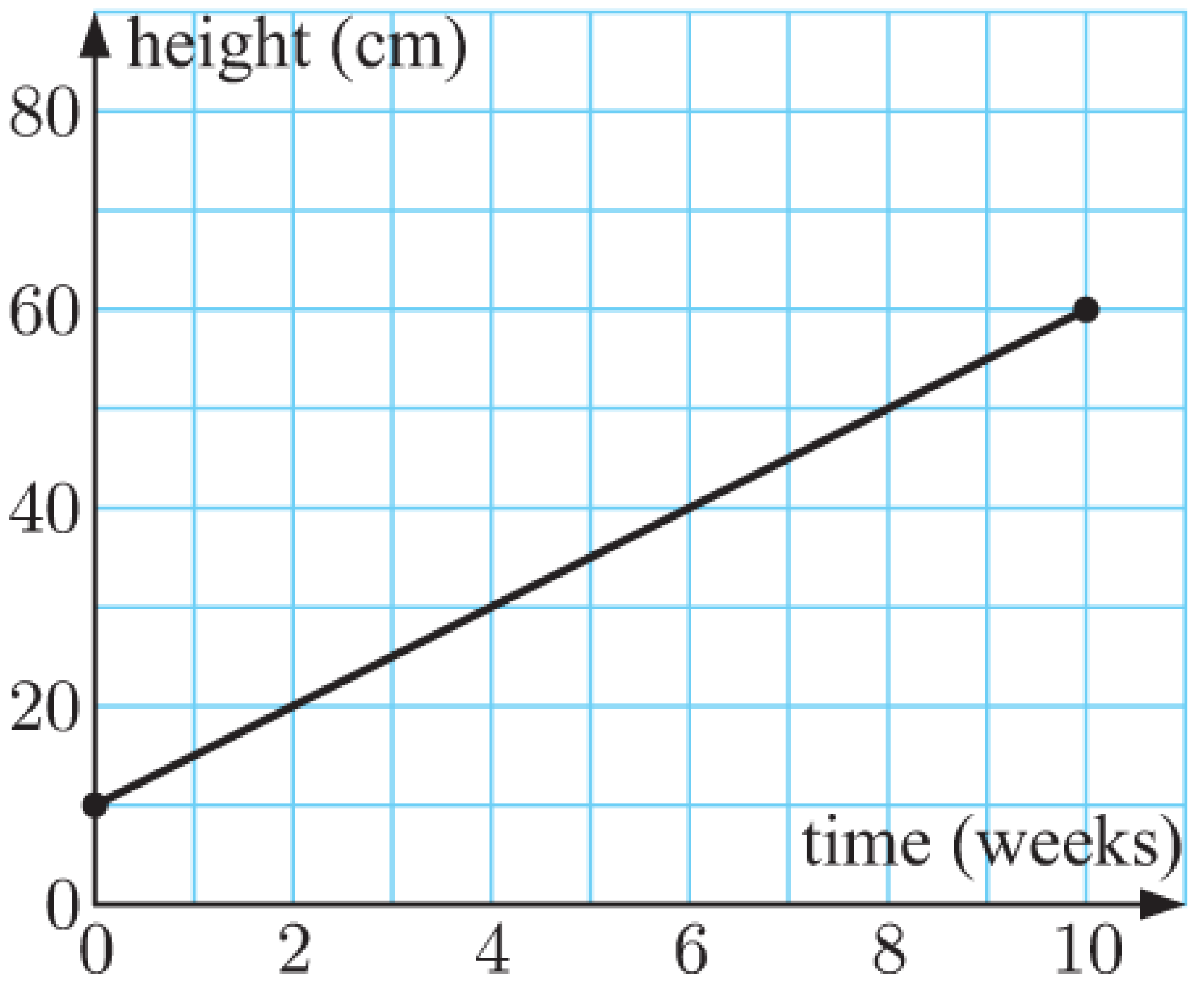
b Draw the graph of distance against time.

c Find the speed of the jogger in metres per second.

2 This graph shows the height of a seedling during its first 10 weeks.

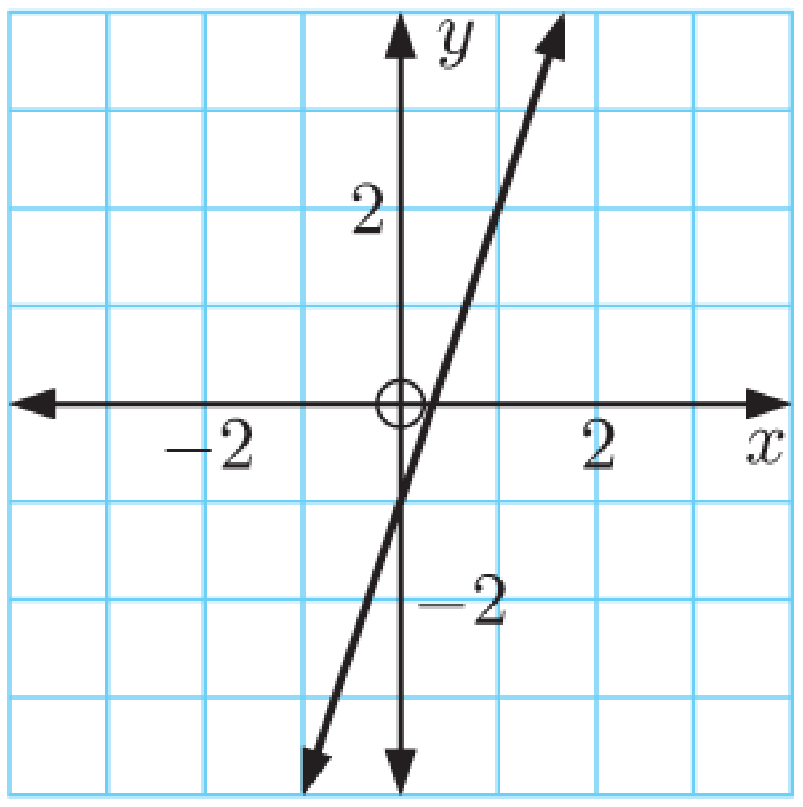
a Is the height changing at a constant rate? Explain your answer.

b Find the rate of change in height.

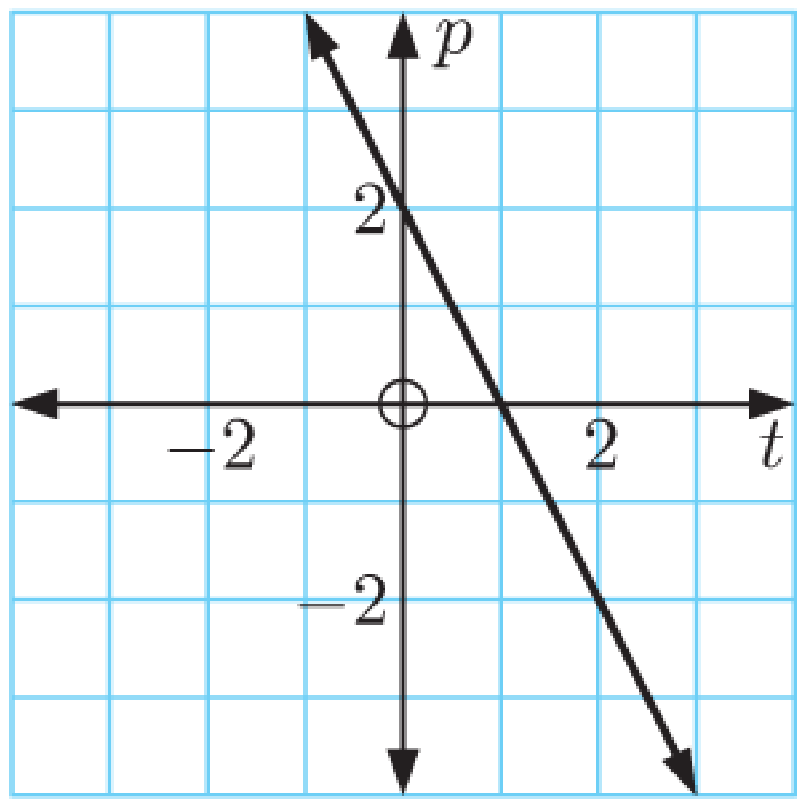


3 Find the rate of change for each function. Do not include units in your answer.

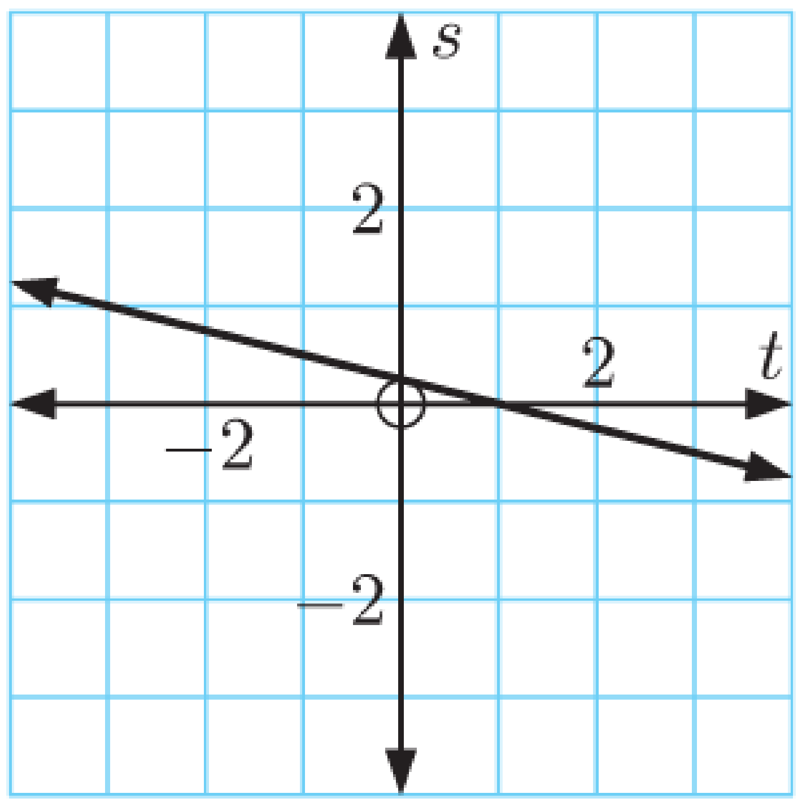
a



b

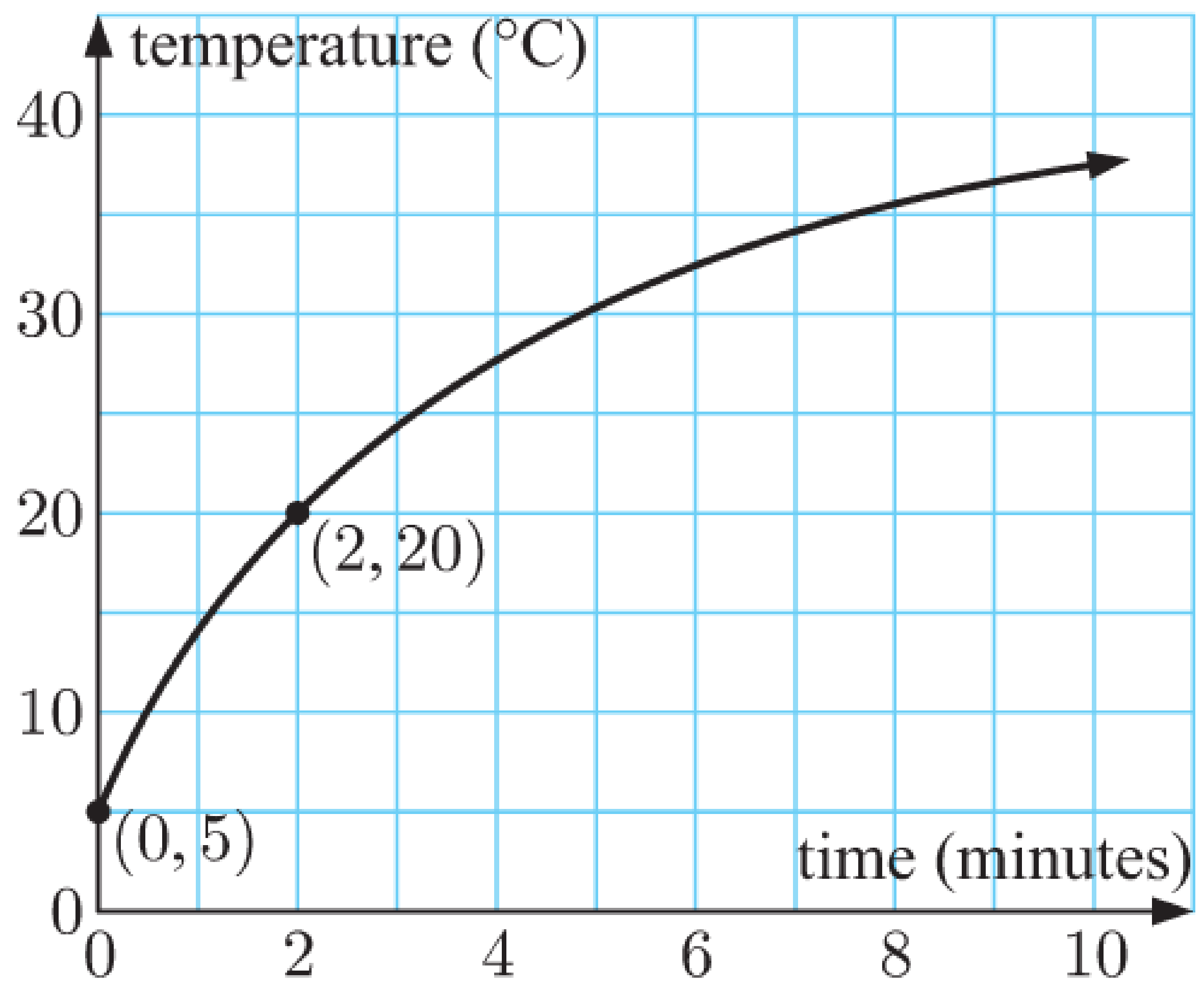


c



AVERAGE RATES OF CHANGE

This graph shows the temperature of a glass of water which is left in the sun. The graph is not a straight line, which means the rate of change in temperature is not constant. The temperature increases quickly at first, and then more slowly as time goes by.



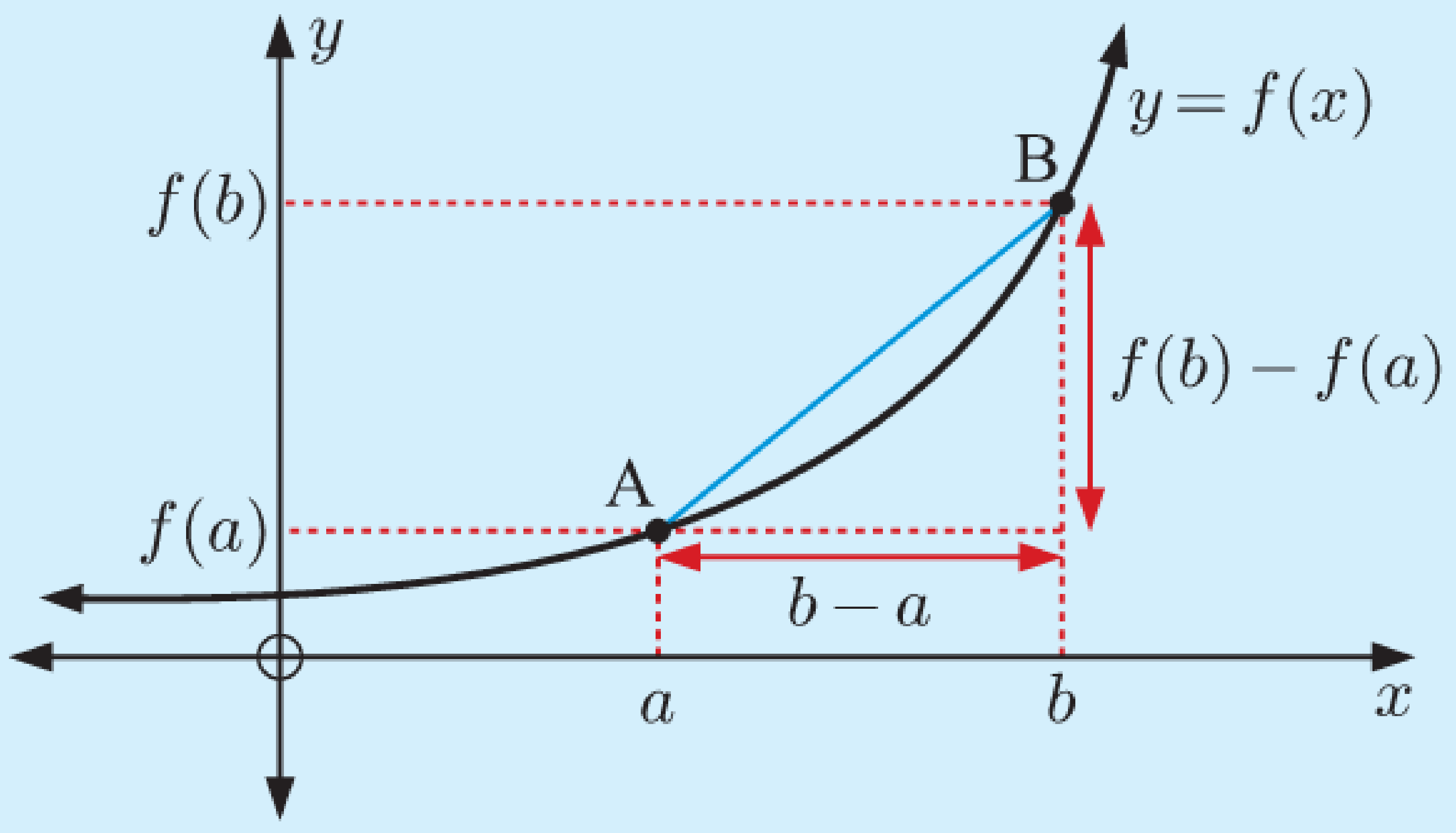
In such cases, we can find an **average rate of change** over a particular time interval.

For example, from time $t = 0$ to $t = 2$ minutes, the temperature increases from 5°C to 20°C. So, the average rate of change is $\dfrac{20-5}{2-0} = 7.5$°C per minute.

In the context of functions, we say that:

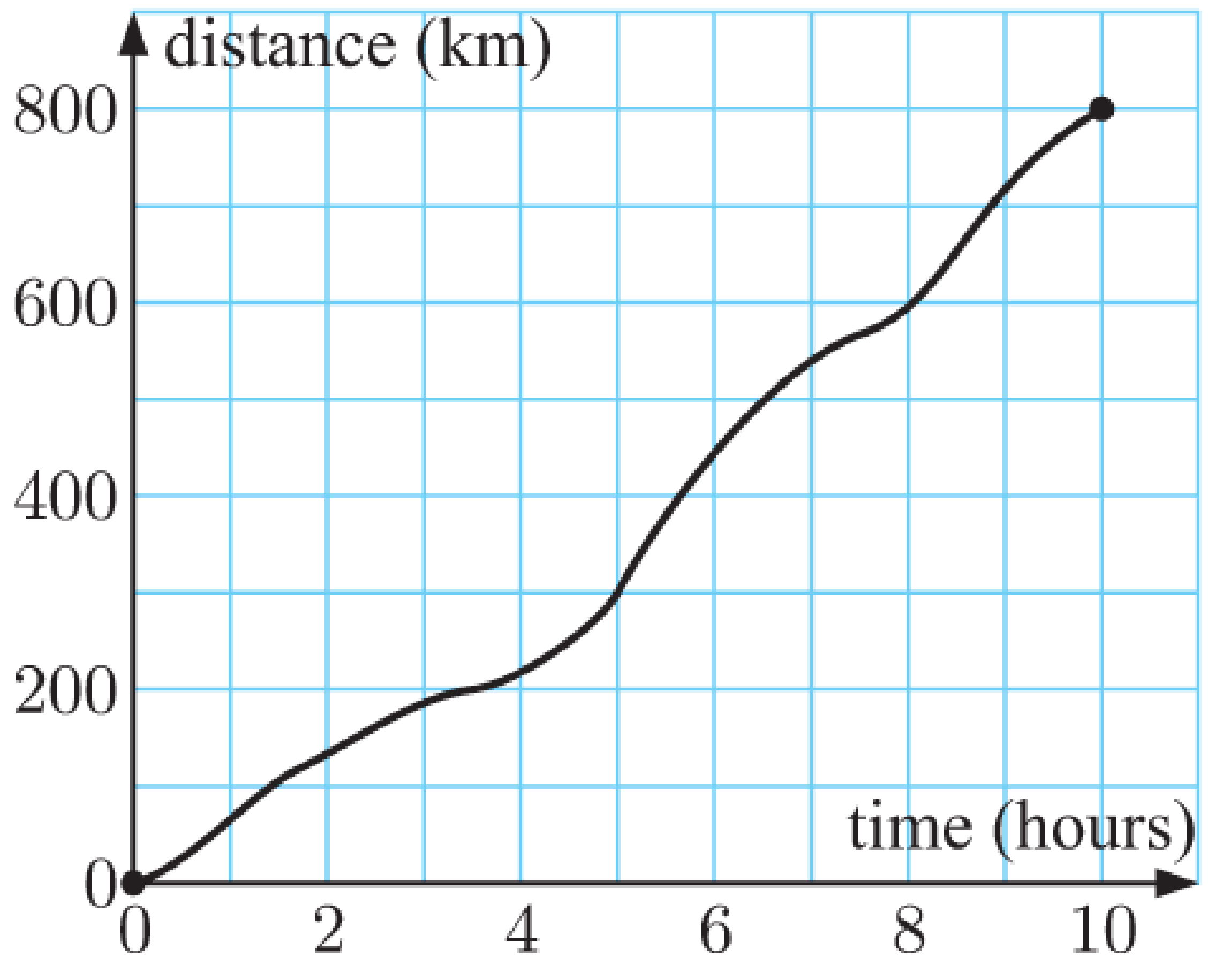
The **average rate of change** in $f(x)$ from $x = a$ to $x = b$ is $\dfrac{f(b)-f(a)}{b-a}$.

This is the **gradient of the chord [AB]**.



EXERCISE 10A.3

1 Aileen is driving from Amsterdam to Zurich. This graph shows the distance travelled against time.

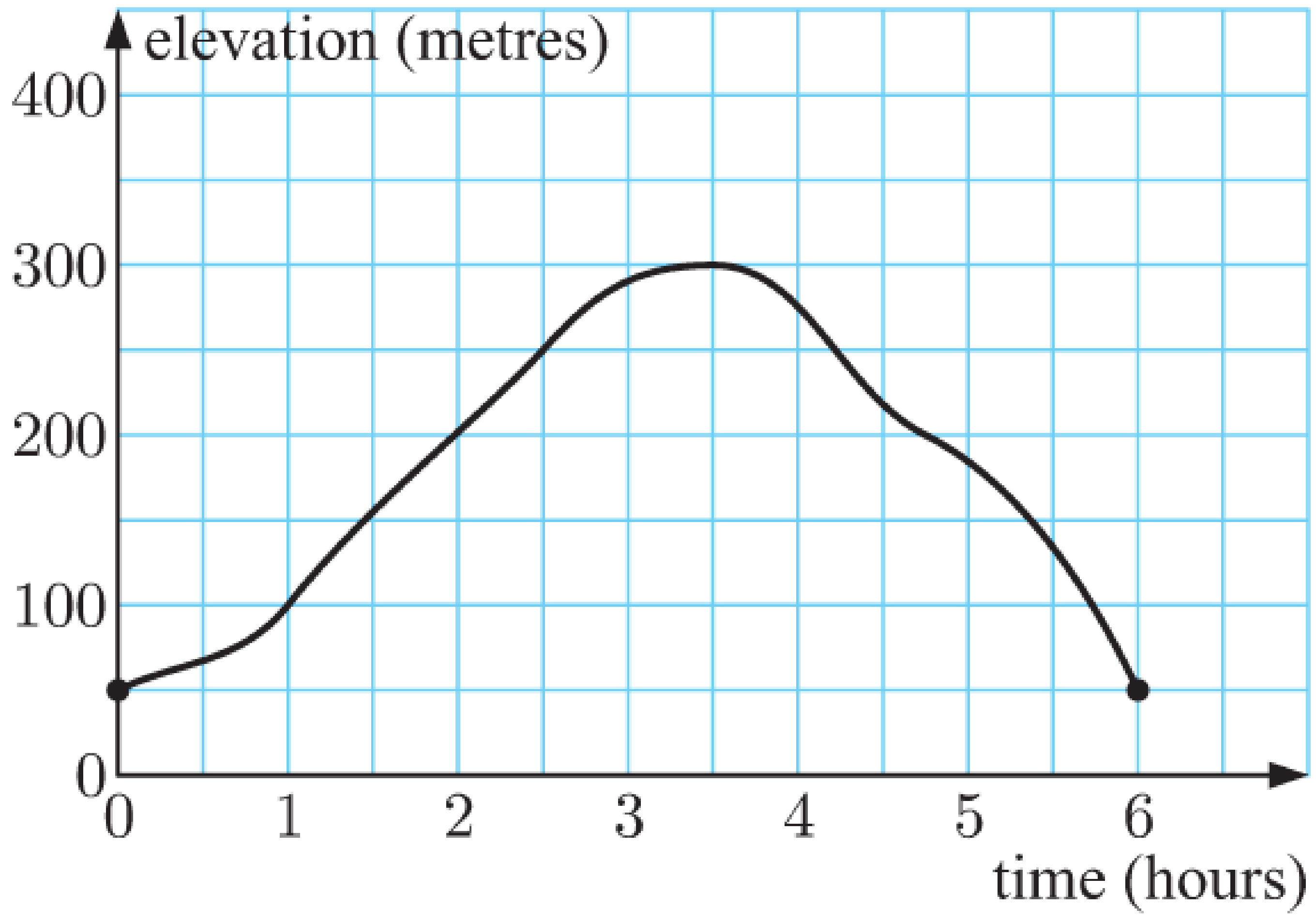


- **a** Did Aileen travel at constant speed? Explain your answer.
- **b** Find Aileen's average speed for:
  - **i** the first 5 hours
  - **ii** the final 5 hours.

2 Chris went hiking in the mountains. His elevation above sea level is shown on this graph.
Find Chris' average rate of change in elevation from:

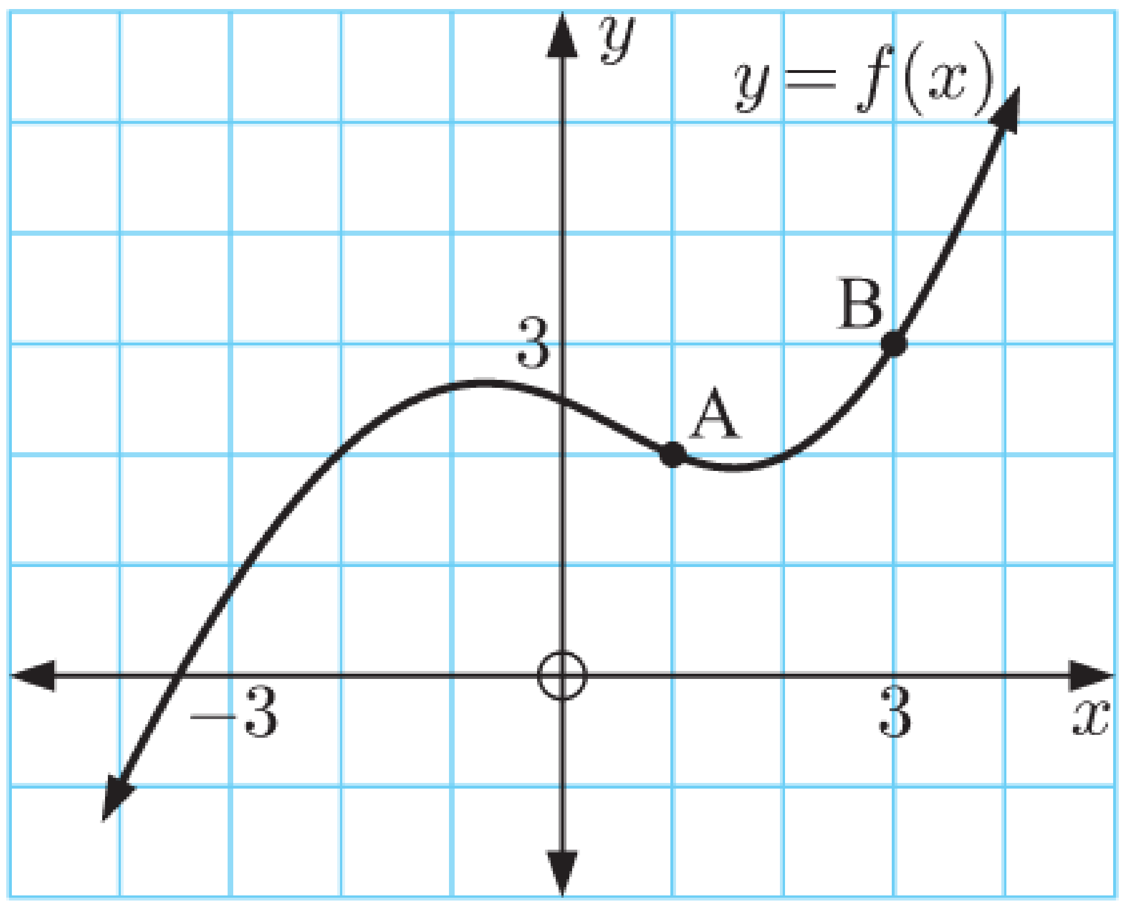
a $t = 1$ hour to $t = 2.5$ hours

b $t = 3.5$ hours to $t = 6$ hours.

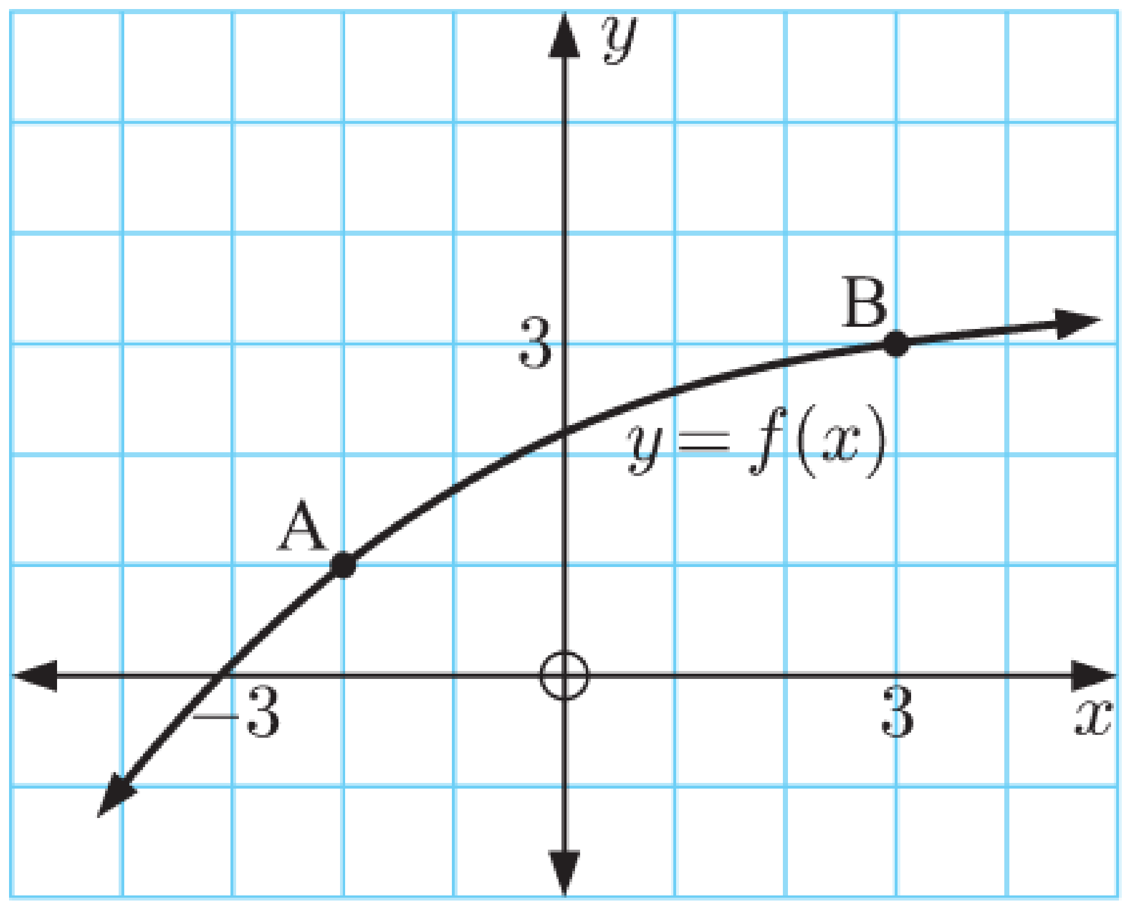


3 For each function, find the average rate of change in $f(x)$ from A to B:

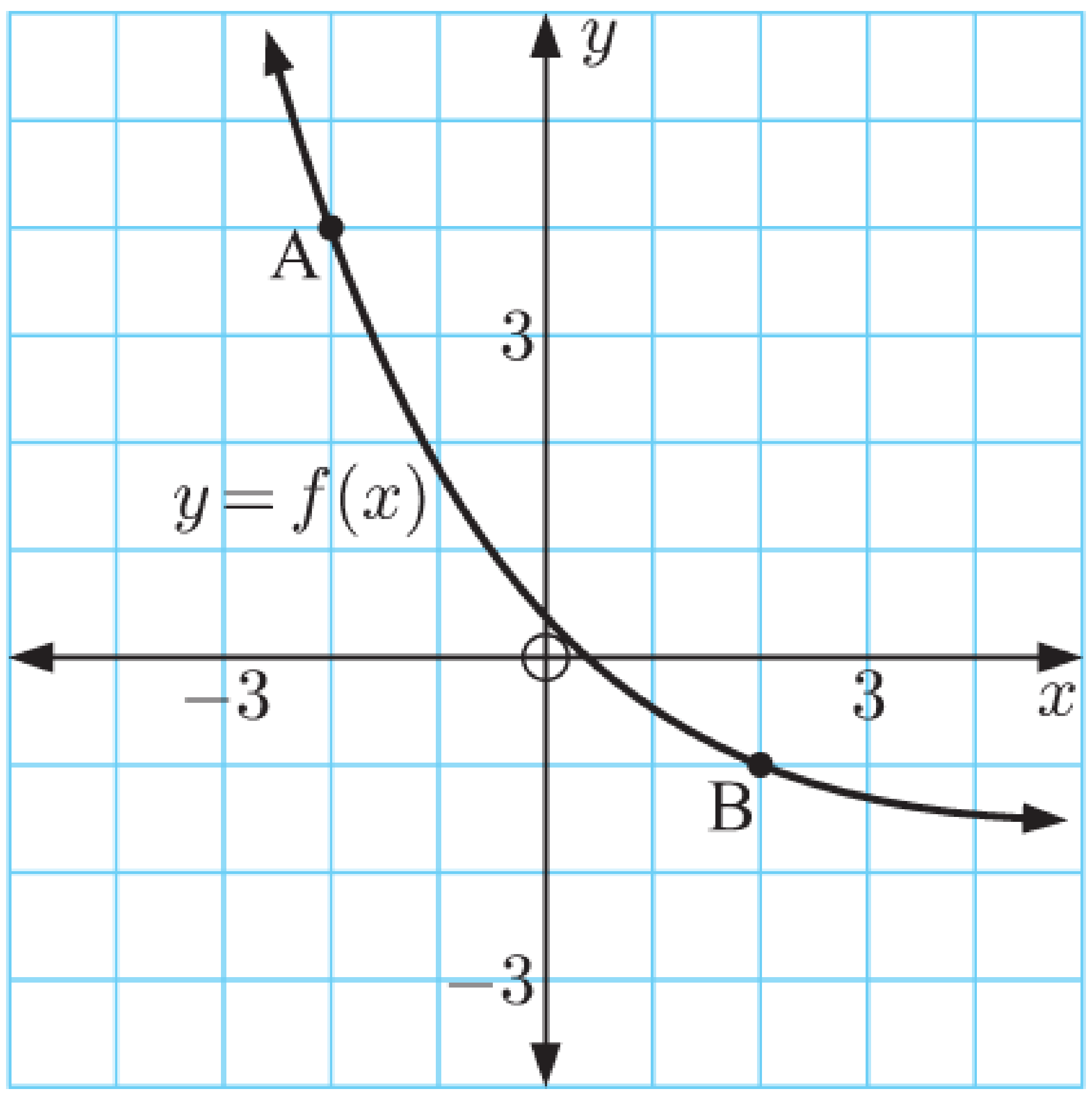
a



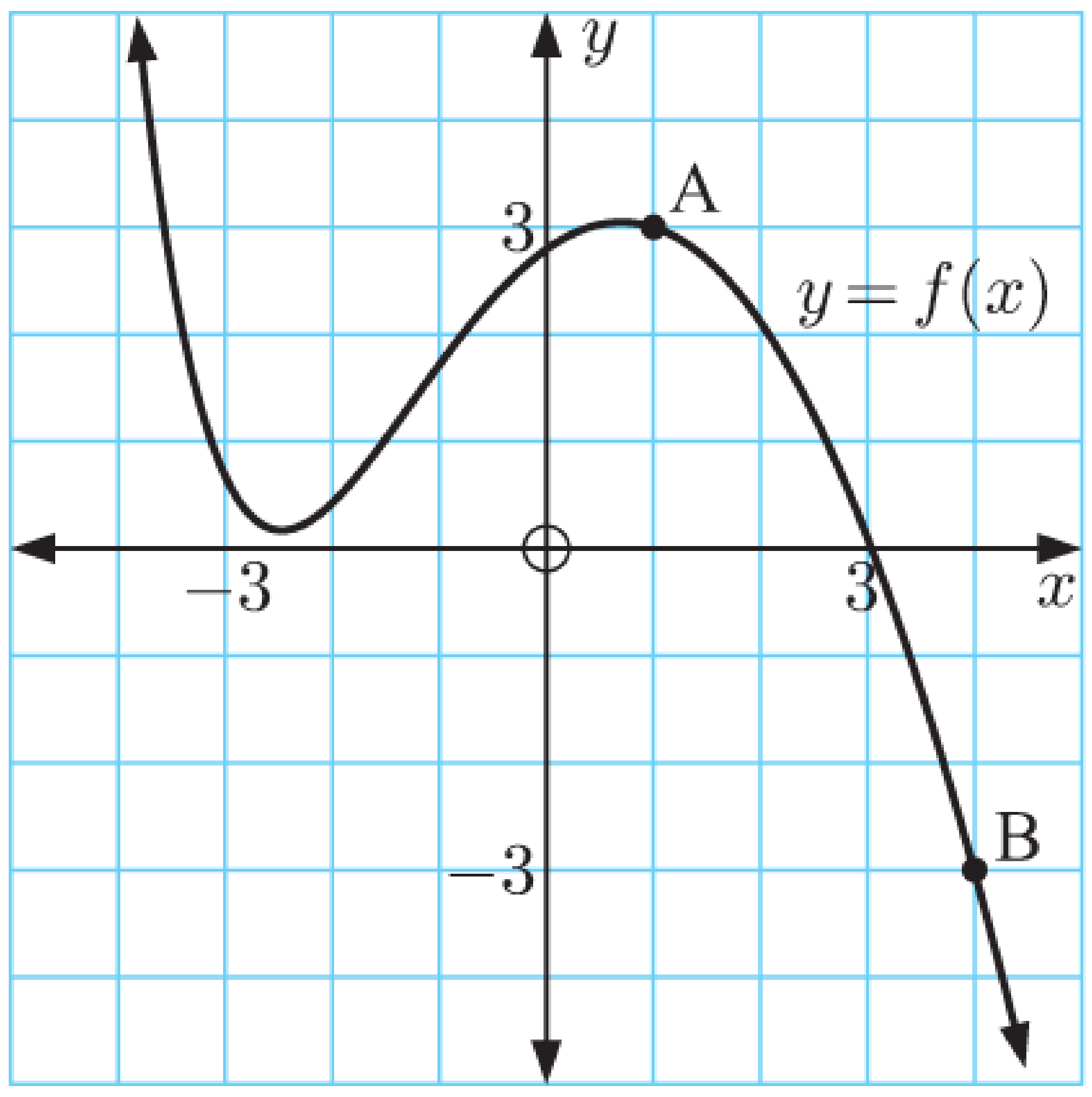
b



c



d

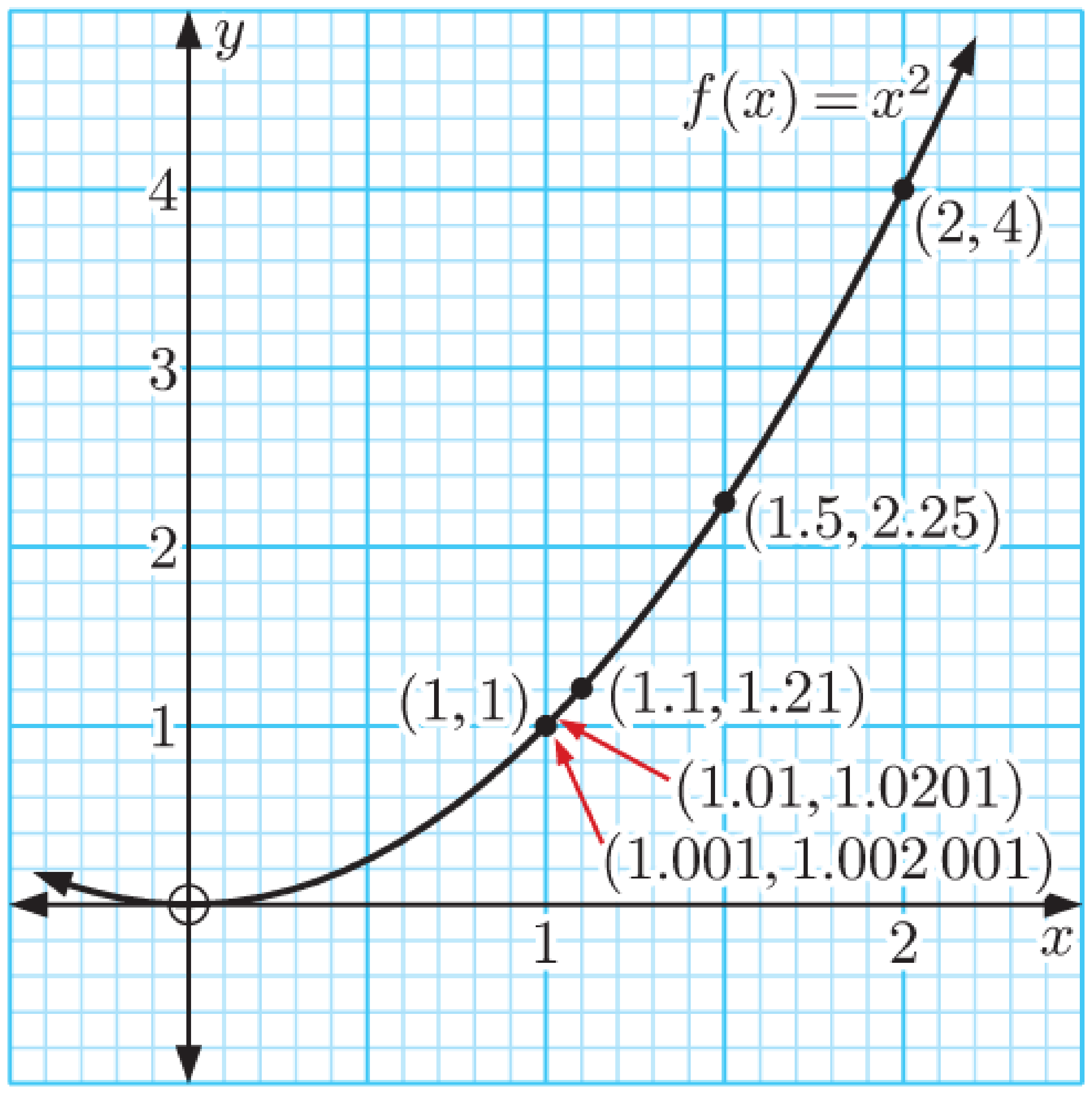


4 Consider the graph of $f(x) = x^2$.

a Find the average rate of change in $f(x)$ from:

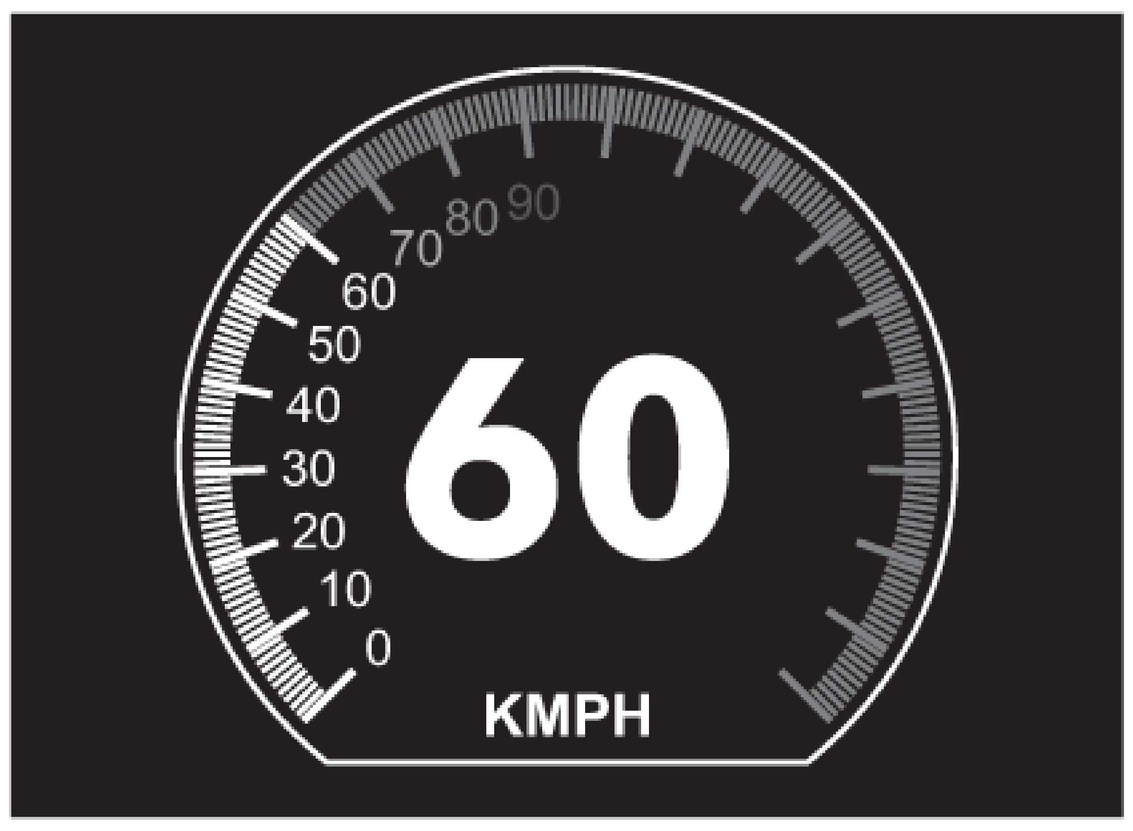
- **i** $x = 1$ to $x = 2$
- **ii** $x = 1$ to $x = 1.5$
- **iii** $x = 1$ to $x = 1.1$
- **iv** $x = 1$ to $x = 1.01$
- **v** $x = 1$ to $x = 1.001$

b Comment on your answers in **a**.



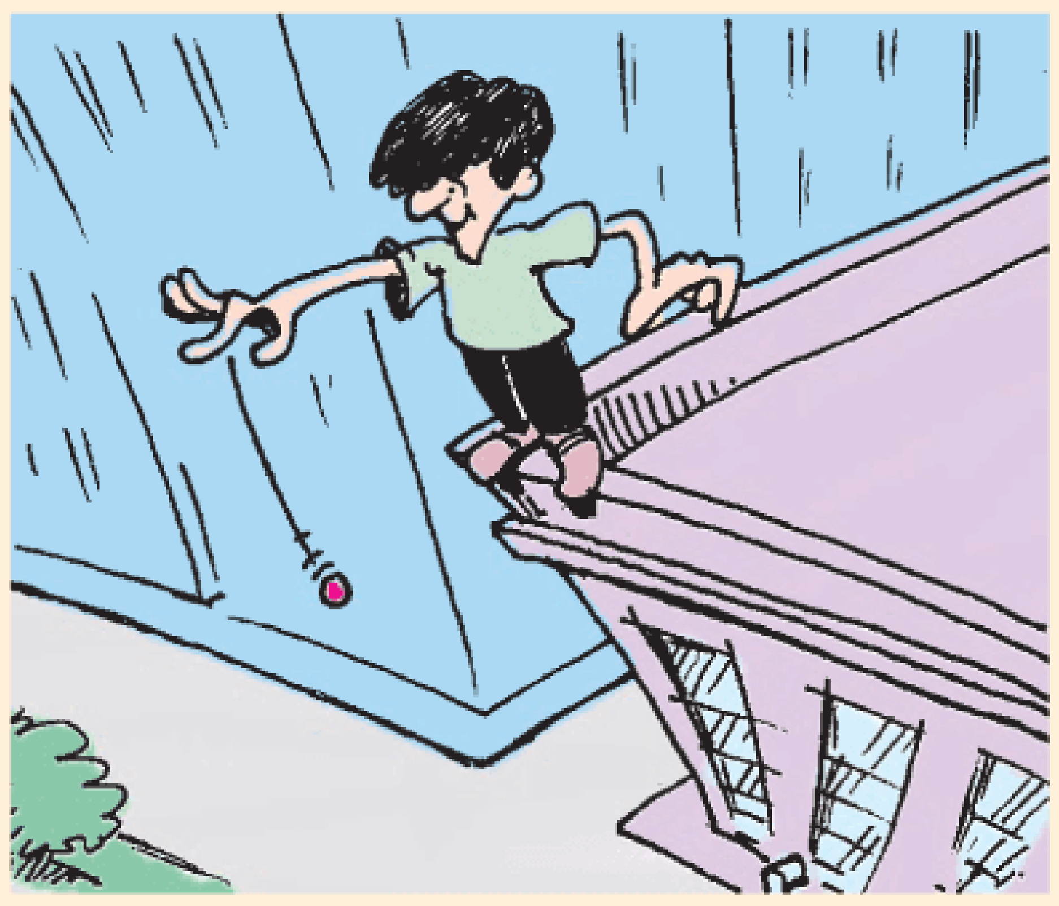
B INSTANTANEOUS RATES OF CHANGE

Suppose the speedometer in a car indicates that you are travelling at 60 km per hour. This is not an average speed, but an *instantaneous speed*. It is the speed at which you are travelling at that particular instant.

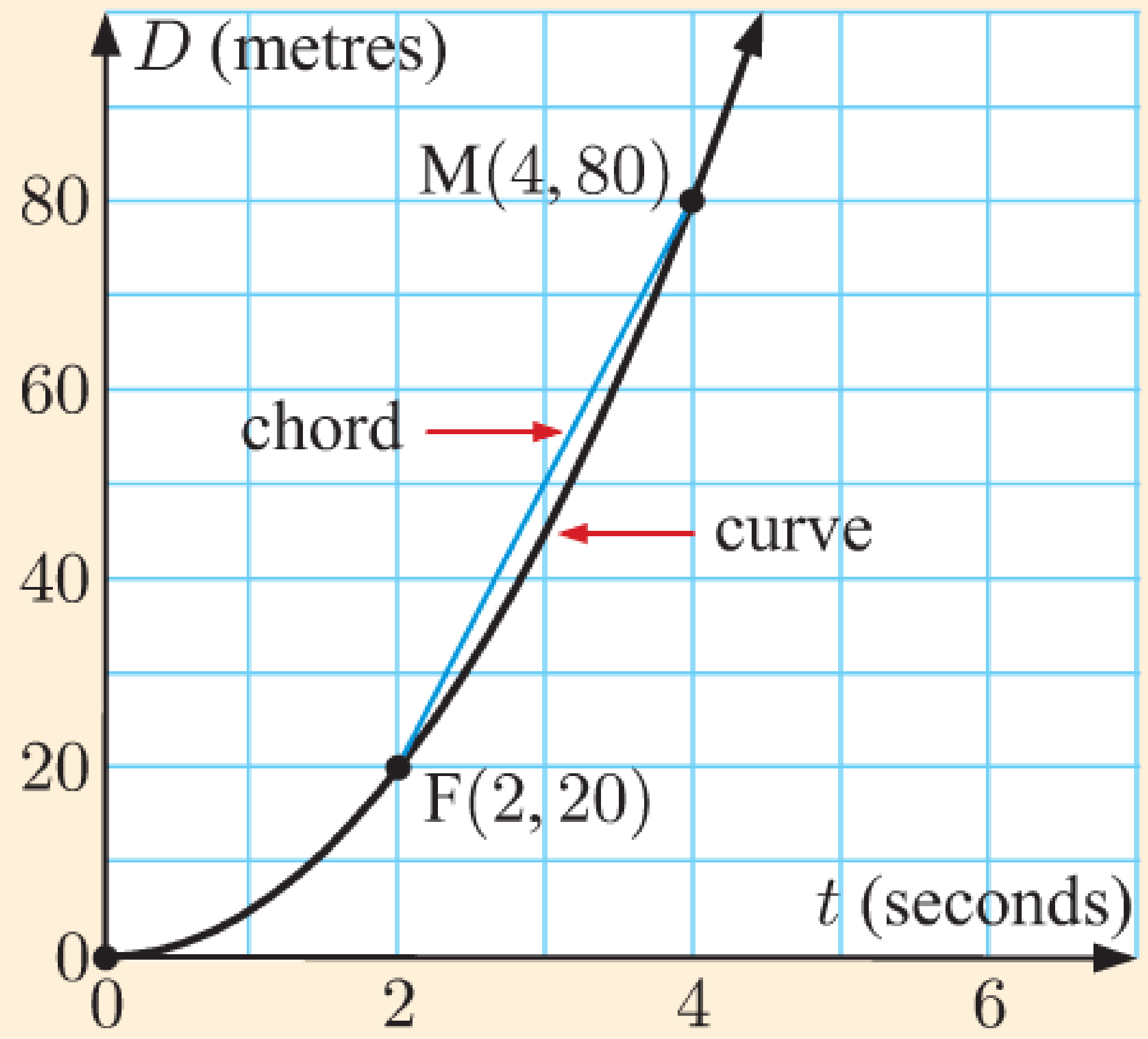


INVESTIGATION 1 INSTANTANEOUS SPEED

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time is obtained.



Consider a fixed point F on the curve when $t = 2$ seconds. We now choose another point M on the curve, and draw the line segment or **chord** [FM] between the two points. To start with, let M be the point when $t = 4$ seconds.



The *average* speed of the ball bearing over the time interval $2 \leqslant t \leqslant 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20)\text{ m}}{(4 - 2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

In this Investigation we will try to measure the *instantaneous* speed of the ball bearing when $t = 2$ seconds.

What to do:

1 Click on the icon to start the demonstration.
The gradient of the chord [FM] is shown. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see that the average speed is 30 m s^{-1}.

DEMO

2 Click on M and drag it slowly towards F. Copy and complete the table alongside, where M corresponds to the given value of t.

t	gradient of [FM]
4	30
3	
2.5	
2.1	
2.01	

3 Observe what happens as M reaches F. Explain why this is so.

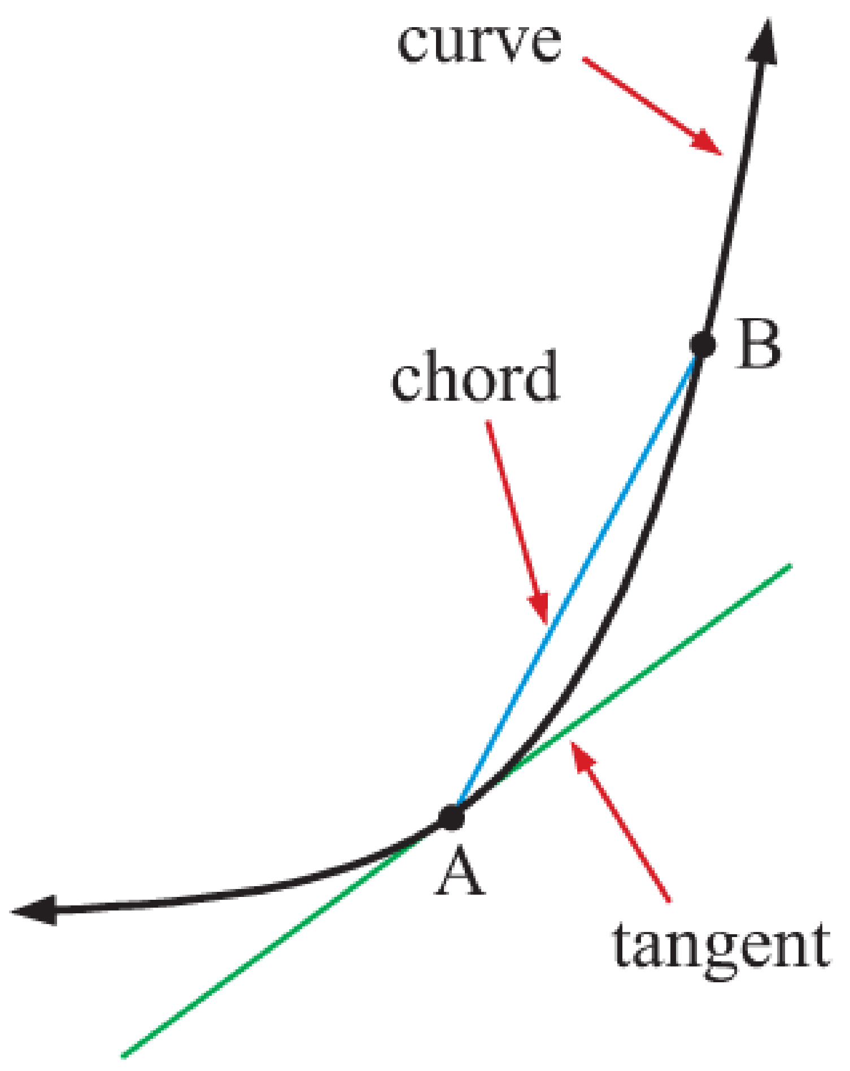
4 What do you suspect is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

5 Move M to the origin, and then slide it towards F from the left. Copy and complete the table, where M again corresponds to the given value of t.

t	gradient of [FM]
0	
1.5	
1.9	
1.99	

6 Do your results agree with those in **4**?

If A and B are two points on a function, the gradient of the chord [AB] is the average rate of change between these points.



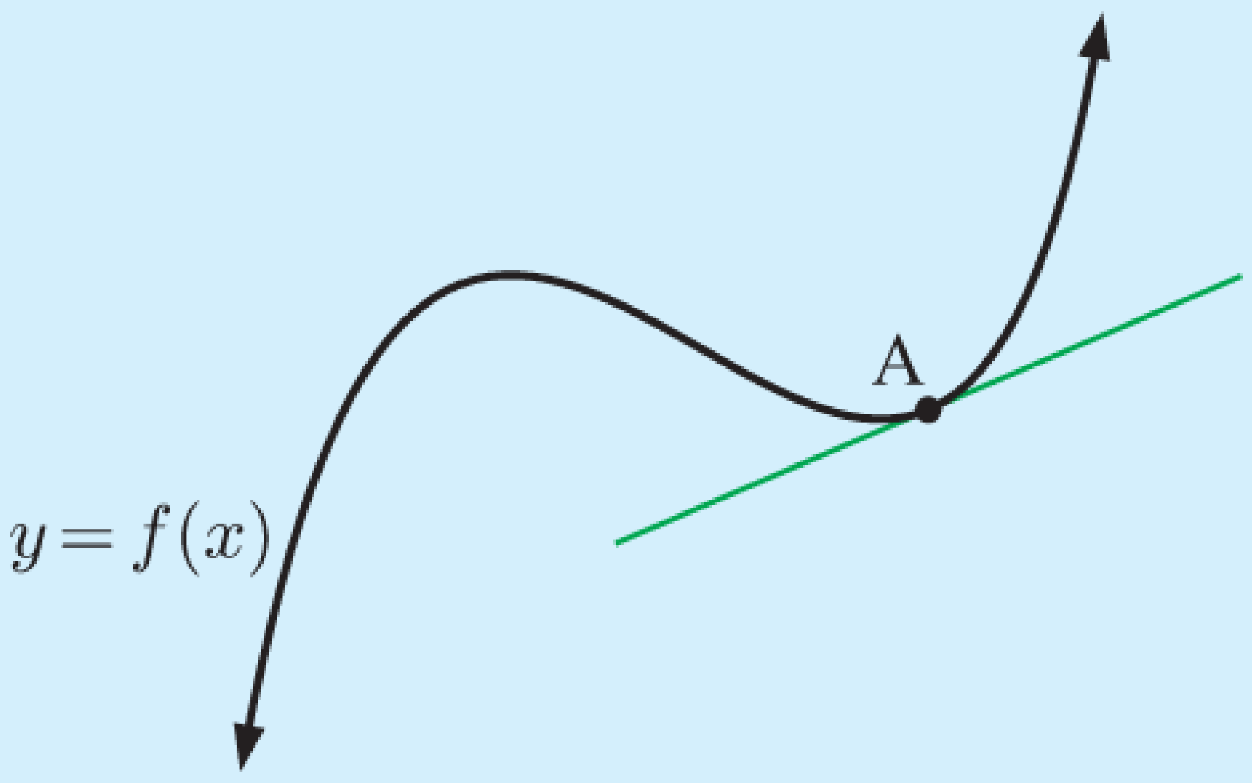
If we let B get closer and closer to A:

- the chord [AB] approaches the **tangent** which *touches* the curve at A
- the average rate of change from A to B approaches the *instantaneous* rate of change at A.

In particular, as B approaches A, the gradient of [AB] approaches the gradient of the tangent at A.

DEMO

The **instantaneous rate of change** in $f(x)$ at any point A on the curve is the **gradient of the tangent** at A.

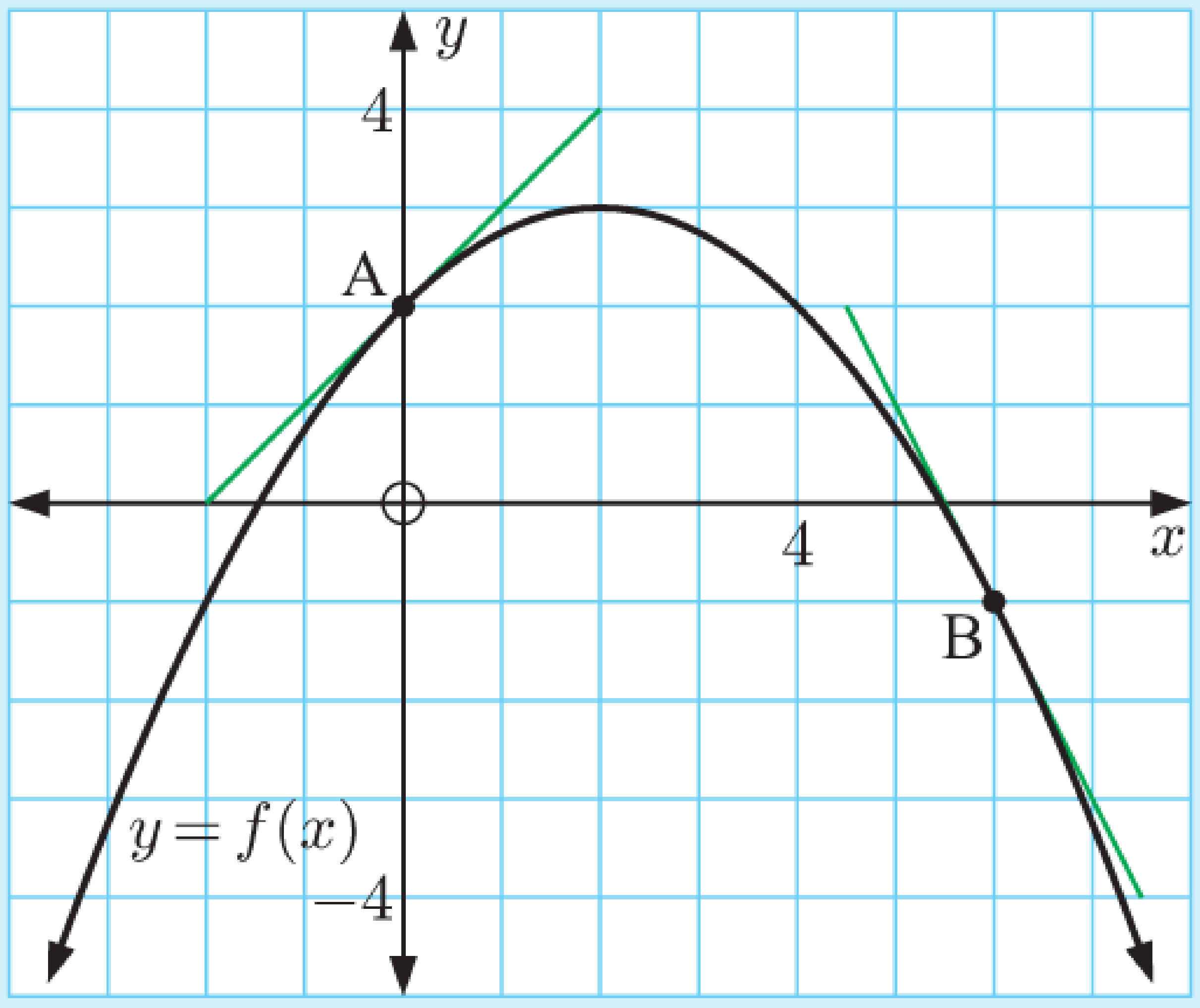


Example 2

Self Tutor

Use the tangents drawn to find the instantaneous rate of change in $y = f(x)$ at:

a A **b** B



a The tangent at A has gradient 1.
$\therefore$ the instantaneous rate of change at A is 1.

b The tangent at B has gradient -2.
$\therefore$ the instantaneous rate of change at B is -2.

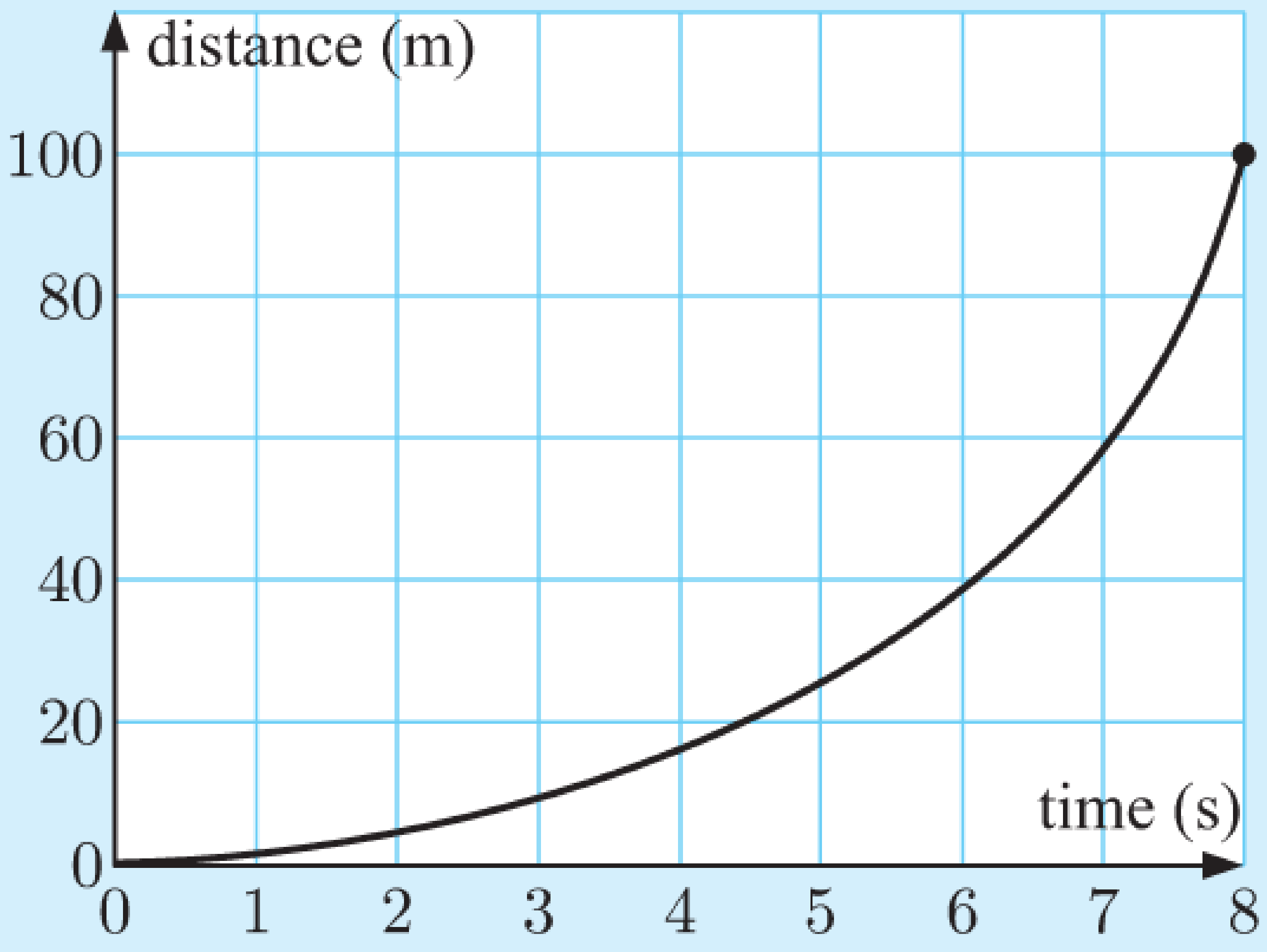
Example 3 **Self Tutor**

The graph alongside shows how a cyclist accelerates away from an intersection.

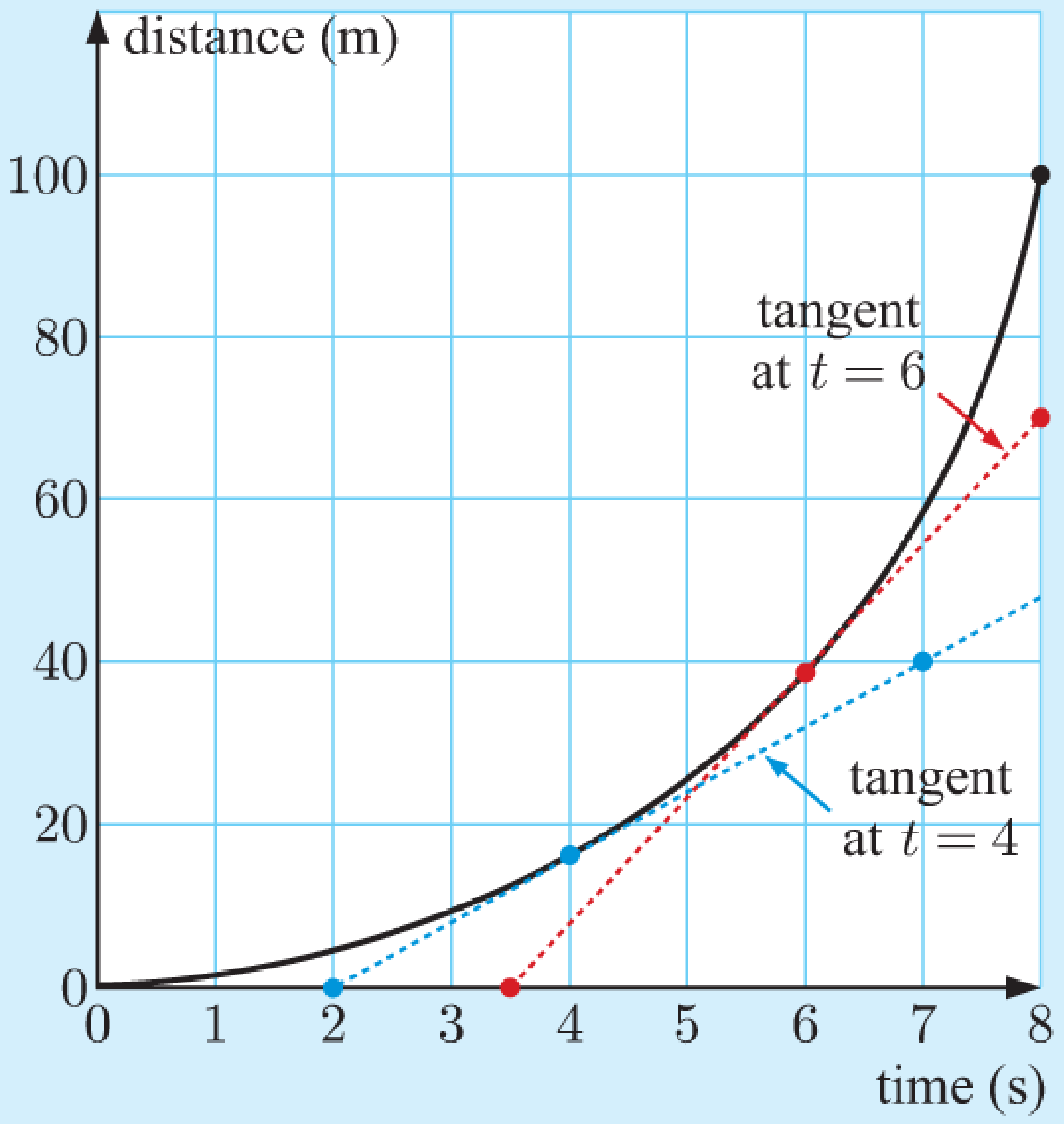
a Estimate the instantaneous speed of the cyclist after:

i 4 seconds **ii** 6 seconds.

b Describe what happens to the cyclist's speed over time.



a



i The tangent at $t = 4$ passes through $(2, 0)$ and $(7, 40)$.

$\therefore$ the instantaneous speed at $t = 4$

$= \dfrac{(40-0)\text{ m}}{(7-2)\text{ s}}$

$= \frac{40}{5}\ \text{m s}^{-1}$

$= 8\ \text{m s}^{-1}$

ii The tangent at $t = 6$ passes through $(3.5, 0)$ and $(8, 70)$.

$\therefore$ the instantaneous speed at $t = 6$

$= \dfrac{(70-0)\text{ m}}{(8-3.5)\text{ s}}$

$= \frac{70}{4.5}\ \text{m s}^{-1}$

$\approx 15.6\ \text{m s}^{-1}$

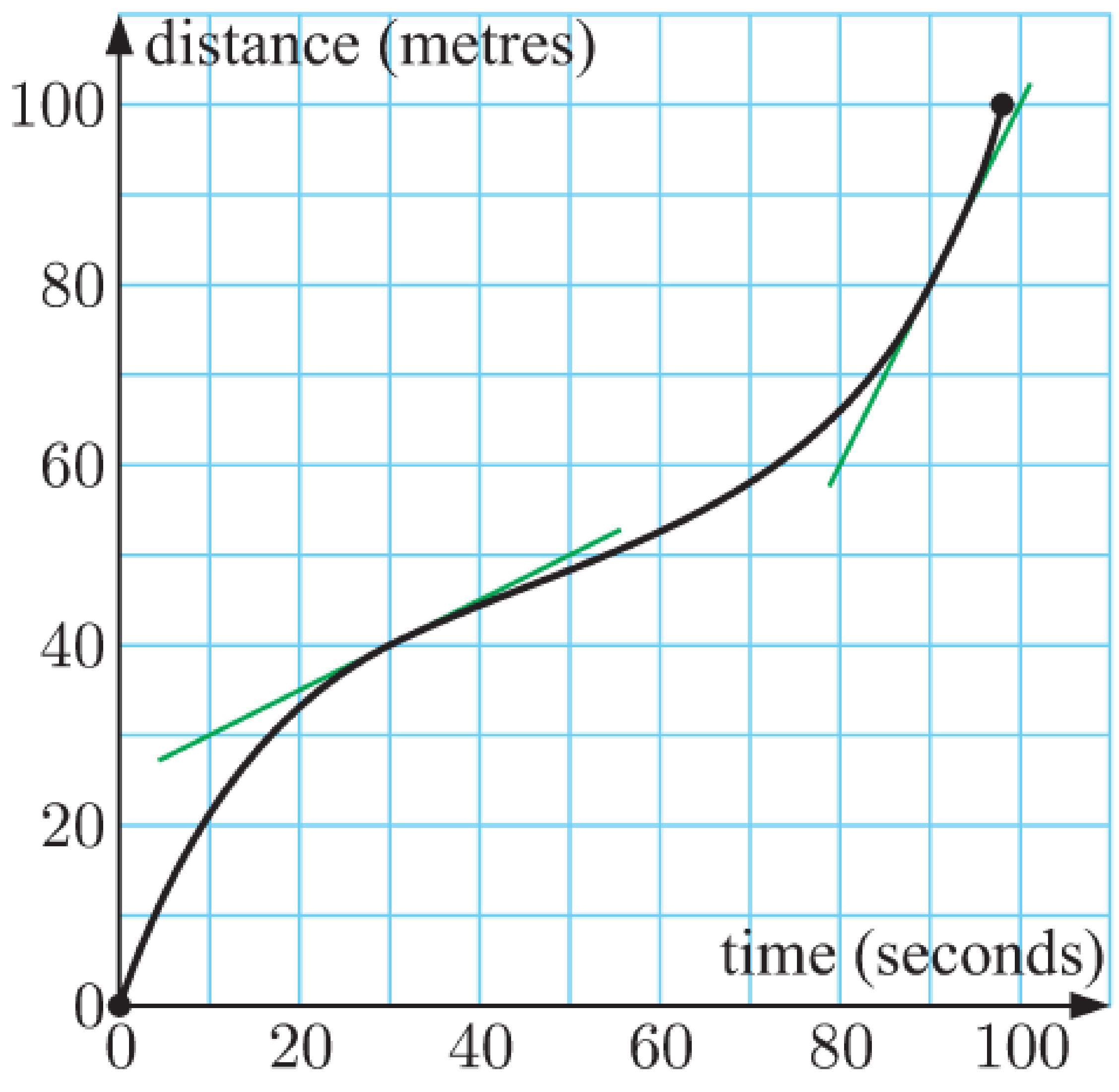
b As time increases, the tangent to the curve gets steeper and steeper. The gradient of the tangent is increasing, so the speed of the cyclist is increasing.

EXERCISE 10B

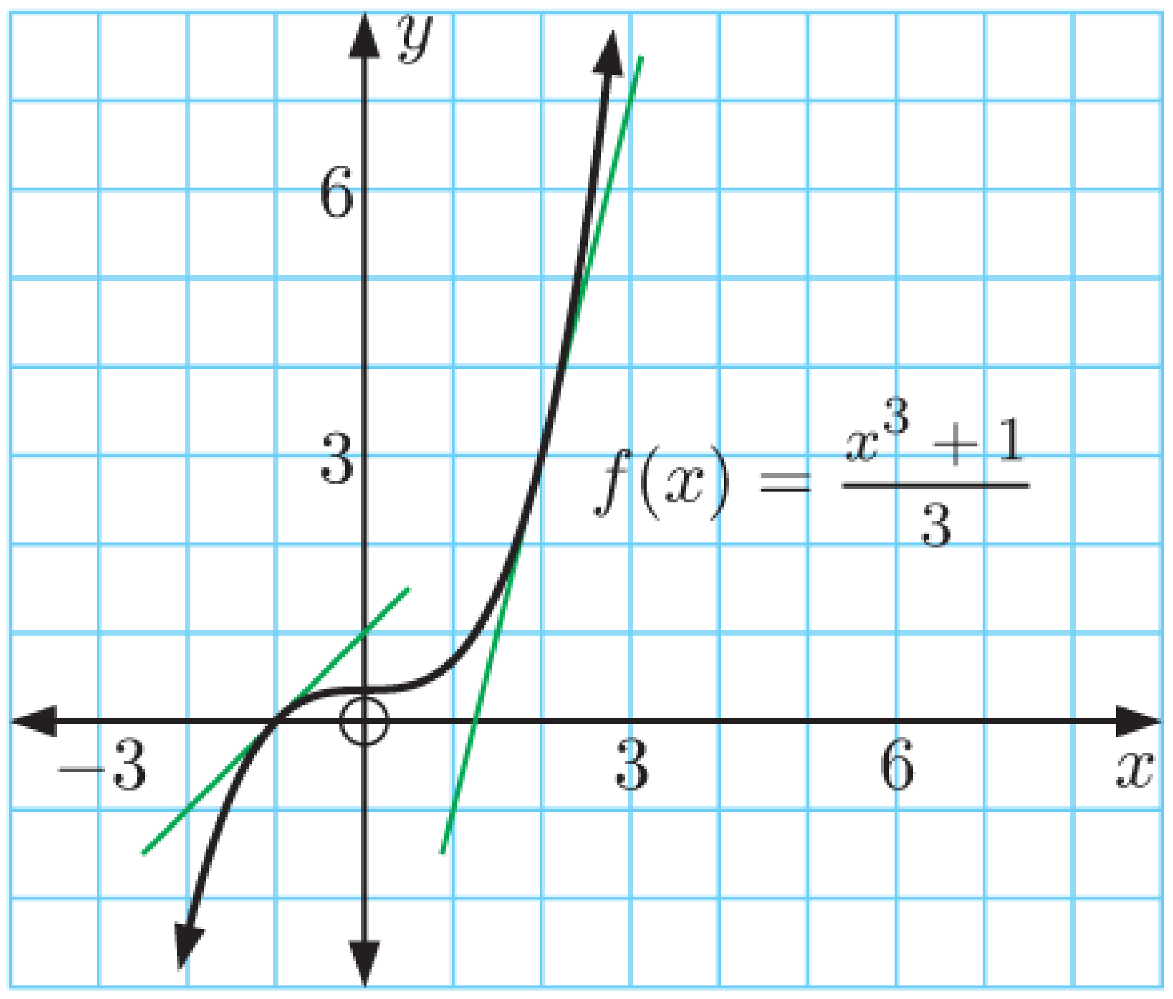
1 This graph shows the distance travelled by a swimmer in a pool.

Use the tangents drawn to find the swimmer's instantaneous speed after:

a 30 seconds **b** 90 seconds.



2



The graph of $f(x) = \frac{x^3+1}{3}$ is shown alongside.

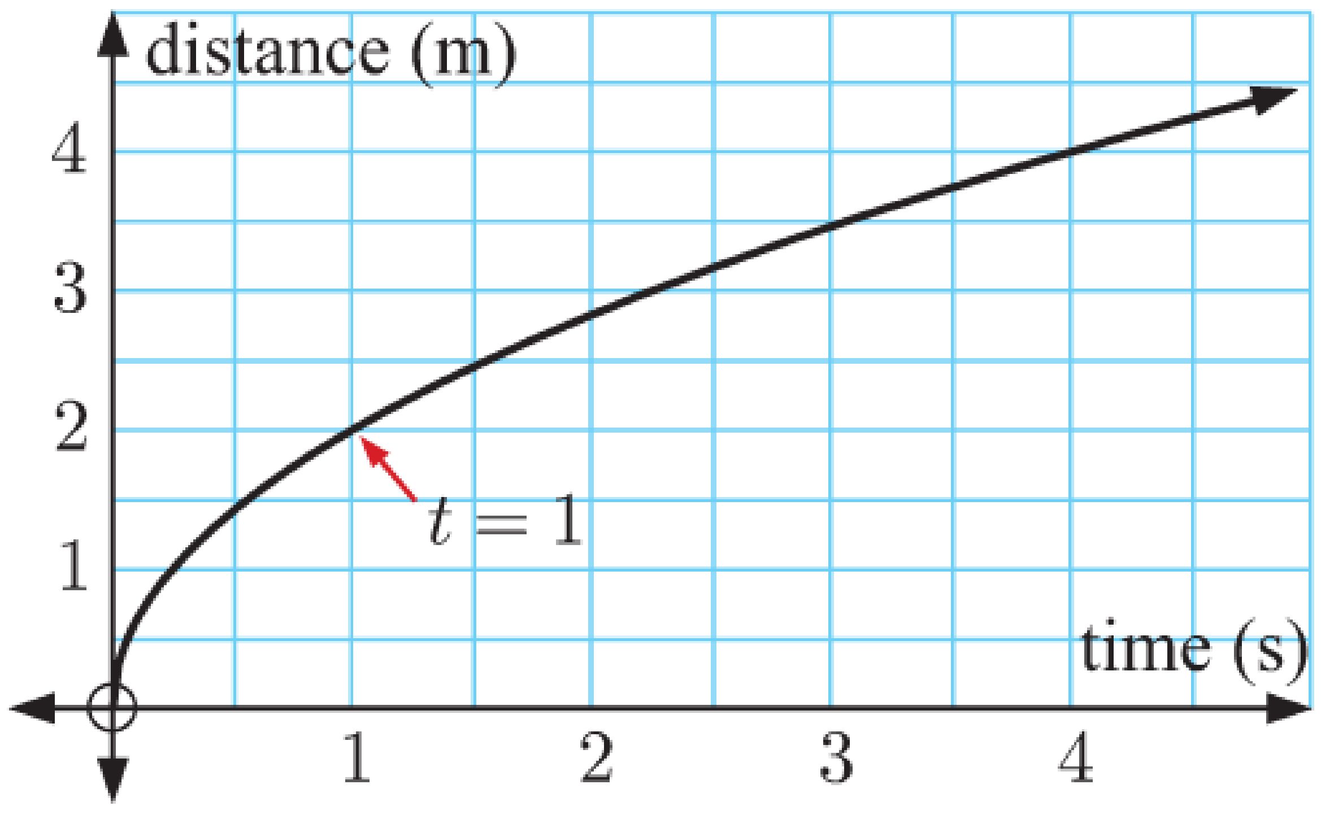
Use the tangents drawn to find the instantaneous rate of change in $f(x)$ at:

a $x = -1$ **b** $x = 2$.

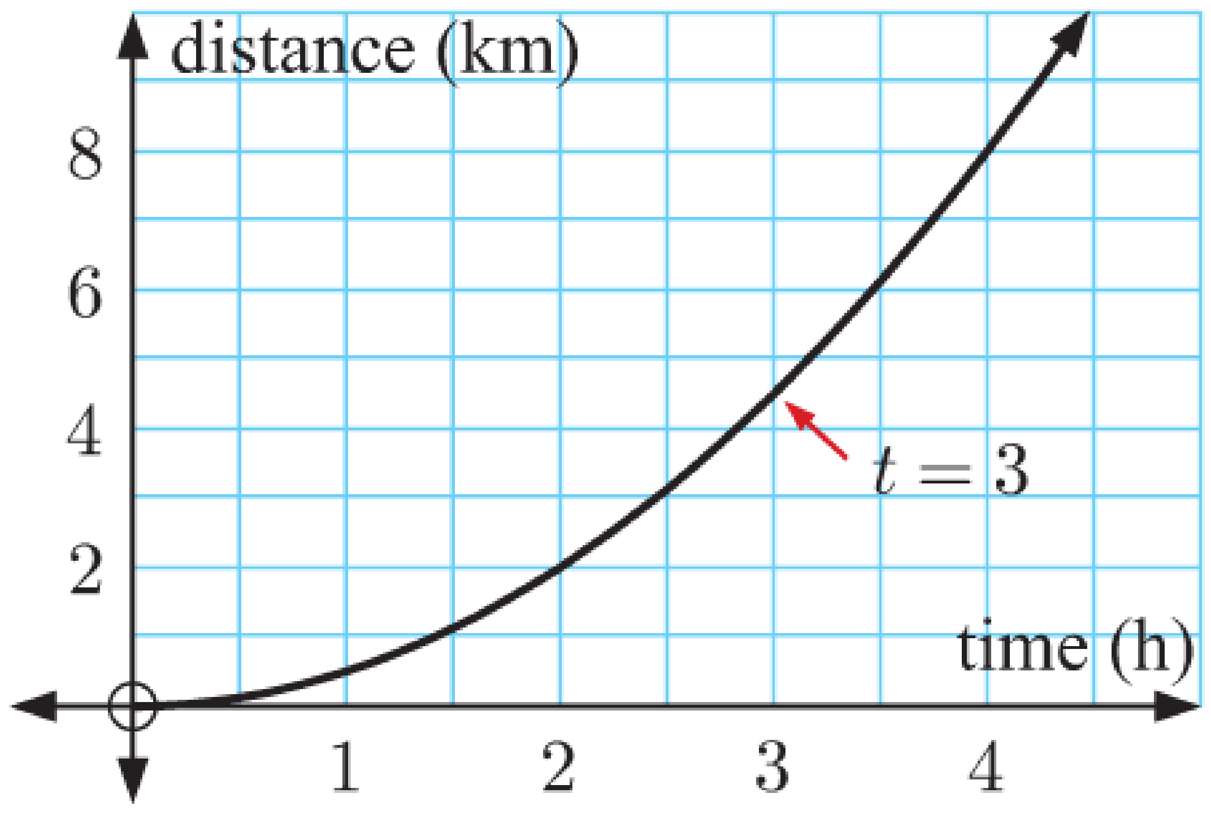
PRINTABLE GRAPHS

3 For each graph, estimate the rate of change at the point shown by the arrow. Make sure your answer has the correct units.

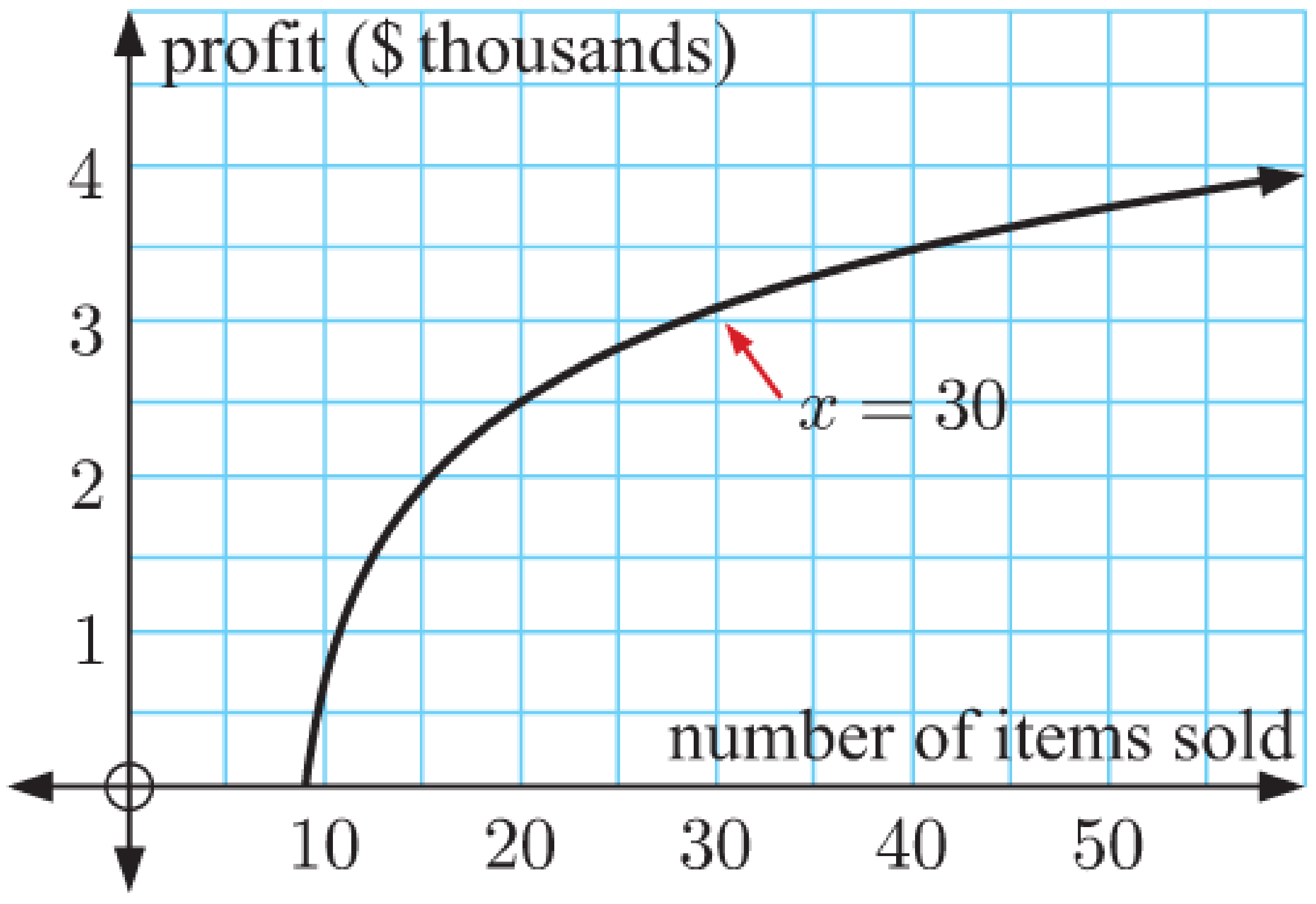
a



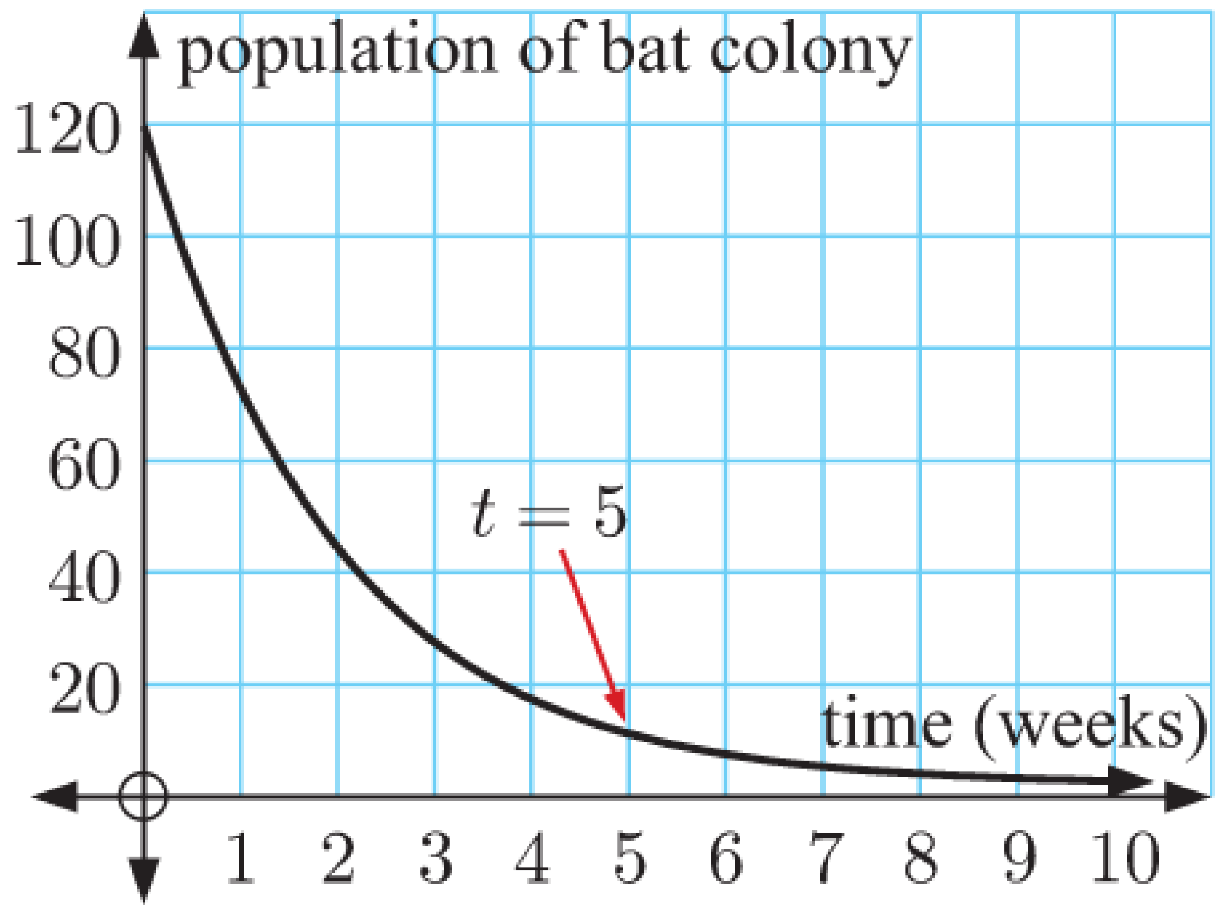
b



c

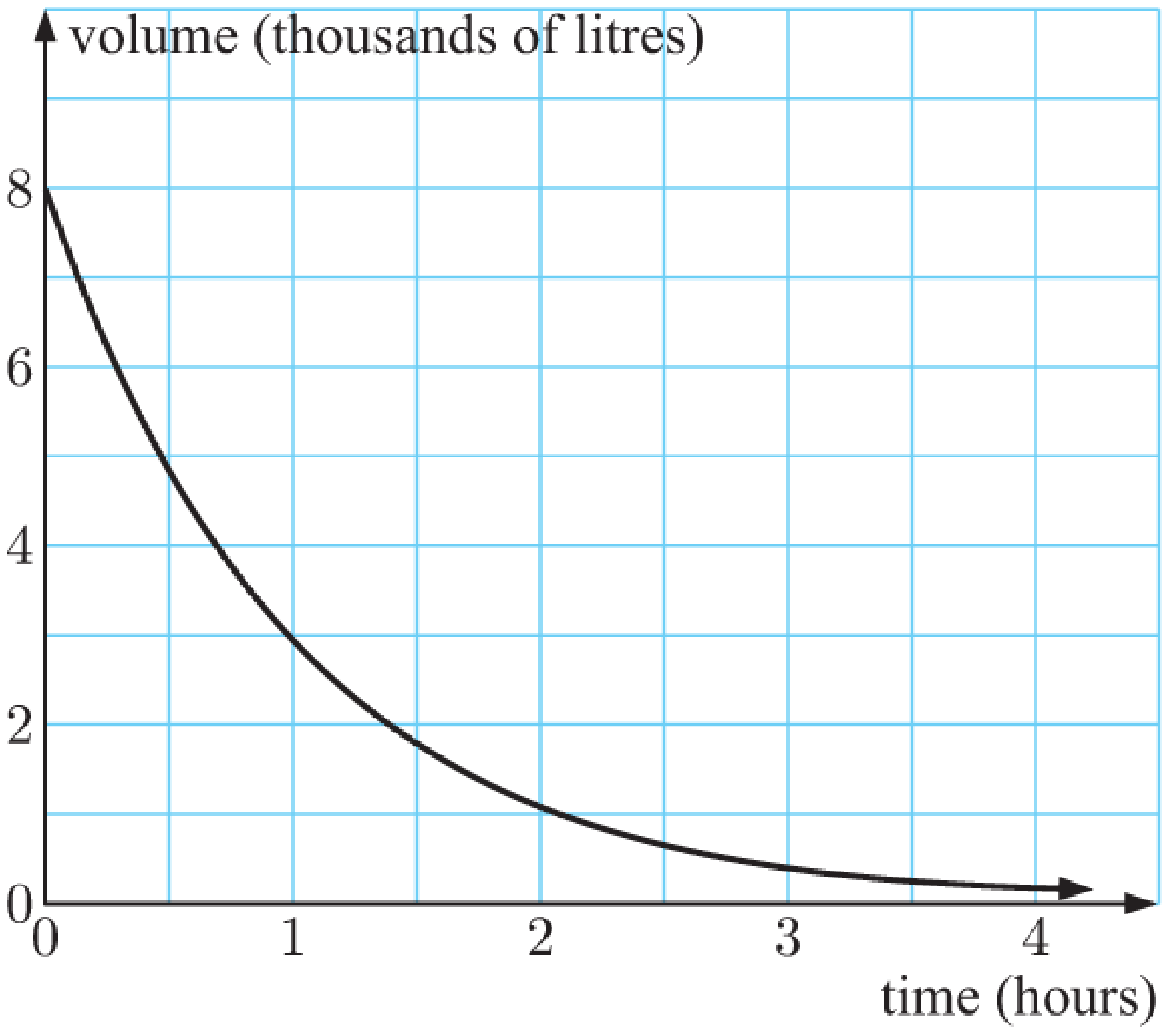


d



4 Water is leaking from a tank. The volume of water left in the tank over time is shown on the graph alongside.

- **a** How much water was in the tank originally?
- **b** How much water was in the tank after 1 hour?
- **c** How quickly was the tank losing water initially?
- **d** How quickly was the tank losing water after 1 hour?
- **e** Describe what happens to the rate at which the water is leaking.



5 **a** Draw an accurate graph of $y = x^2$ on a fine grid.

- **b** Draw, as accurately as possible, the tangent to $y = x^2$ at $x = -1$.
- **c** Hence find the instantaneous rate of change in $y = x^2$ when $x = -1$.

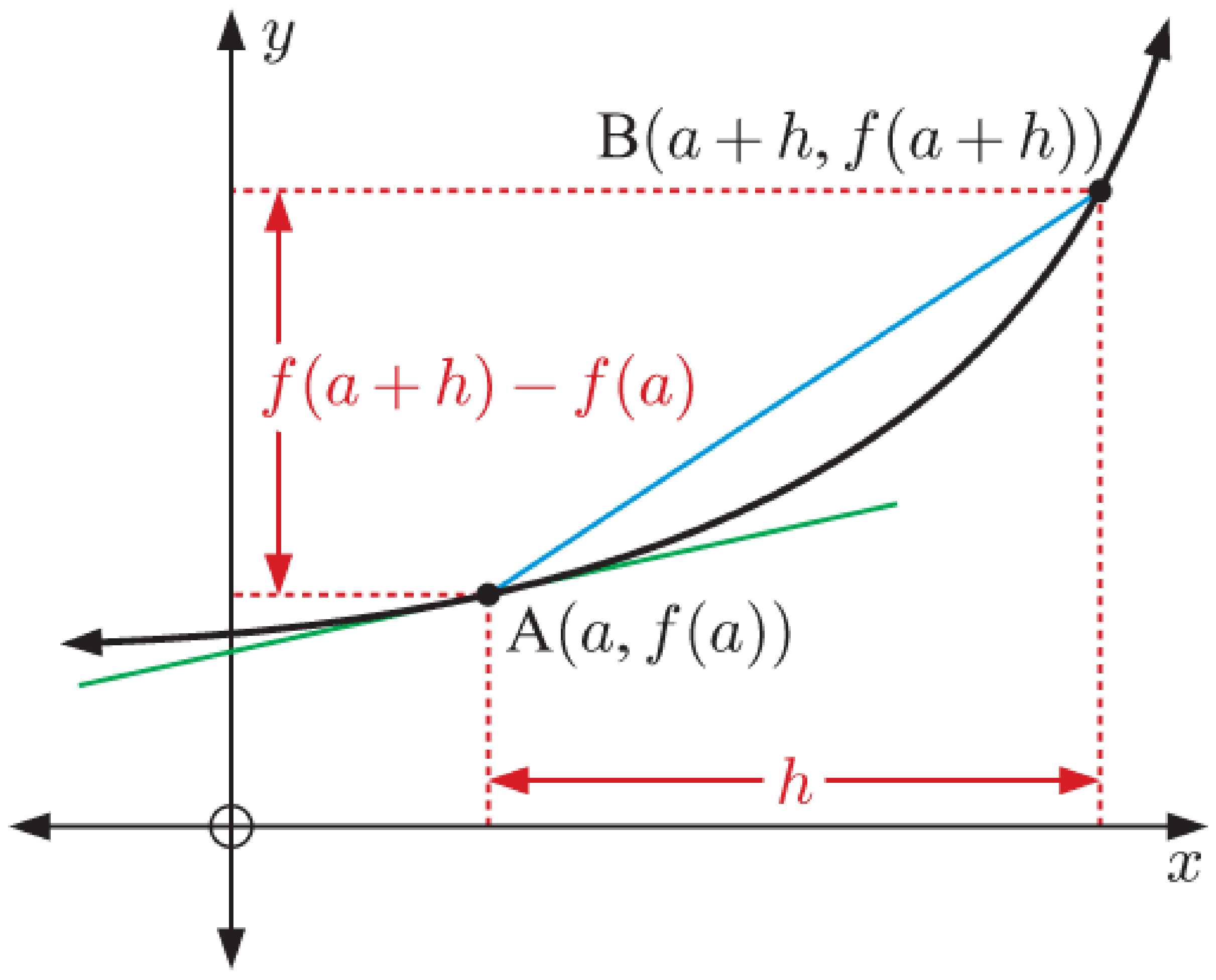
C LIMITS

Drawing a tangent on a graph and measuring its gradient can be time-consuming and inaccurate. We therefore seek a more efficient and accurate method for finding the gradient of a tangent.

We cannot find the gradient of the tangent at point A by direct calculation, because we only know one point on the tangent. However, if B is another point on the function $y = f(x)$, the gradient of the chord [AB] is $\dfrac{f(a+h) - f(a)}{h}$.

To calculate the gradient of the tangent at A, we let the point B get closer and closer to A. This means that the horizontal step h becomes infinitely small.

To understand what happens when this occurs, we use a mathematical principle called **limits**.



The following statement is not a formal definition for a limit, but is sufficient for this course:

> Suppose that by letting x get sufficiently close to (but not equal to) a, we can make the function $f(x)$ be as close as we like to the value A.
>
> We say that the **limit** of f as x approaches a is A, and we write $\lim\limits_{x \to a} f(x) = A$.

Notice that in this statement we have said nothing about the actual value of $f(a)$.

To explain why this is important, consider $\lim\limits_{x \to 0} f(x)$ where $f(x) = \dfrac{5x + x^2}{x}$.

For this function, $f(0) = \frac{0}{0}$ which is undefined.

However, notice that $f(x) = \dfrac{5x + x^2}{x} = \dfrac{x(5+x)}{x}$. So, provided $x \neq 0$, $f(x) = 5 + x$.

As x gets closer and closer (but not equal to) 0, $f(x)$ can be made as close as we like to 5.

$\therefore \lim\limits_{x \to 0} f(x) = 5$, even though $f(0)$ is undefined.

Example 4 Self Tutor

Evaluate: **a** $\lim\limits_{x \to 2} x^2$ **b** $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x}$ **c** $\lim\limits_{x \to 4} \dfrac{2x^2 - 8x}{x - 4}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$\therefore \lim\limits_{x \to 2} x^2 = 4$.

b

$$\begin{aligned}
&\lim_{x \to 0} \frac{x^2 + 3x}{x} \\
&= \lim_{x \to 0} \frac{x(x+3)}{x} \\
&= \lim_{x \to 0} (x + 3) \quad \{\text{since } x \neq 0\} \\
&= 3
\end{aligned}$$

c

$$\begin{aligned}
&\lim_{x \to 4} \frac{2x^2 - 8x}{x - 4} \\
&= \lim_{x \to 4} \frac{2x(x-4)}{x - 4} \\
&= \lim_{x \to 4} 2x \quad \{\text{since } x \neq 4\} \\
&= 8
\end{aligned}$$

EXERCISE 10C

1 Evaluate:

a $\lim\limits_{x\to 3}(x+4)$ **b** $\lim\limits_{x\to -1}(5-2x)$ **c** $\lim\limits_{x\to 4}(3x-1)$

d $\lim\limits_{x\to 2}(5x^2-3x+2)$ **e** $\lim\limits_{h\to 0}h^2(1-h)$ **f** $\lim\limits_{x\to 0}(x^2+5)$

2 Evaluate:

a $\lim\limits_{x\to 0}5$ **b** $\lim\limits_{h\to 2}7$ **c** $\lim\limits_{x\to 0}c$, c a constant

3 Evaluate:

a $\lim\limits_{x\to 1}\dfrac{x^2-3x}{x}$ **b** $\lim\limits_{h\to 2}\dfrac{h^2+5h}{h}$ **c** $\lim\limits_{x\to 0}\dfrac{x-1}{x+1}$

4 Evaluate:

a $\lim\limits_{x\to 0}\dfrac{x}{x}$ **b** $\lim\limits_{x\to 0}\dfrac{x^2-3x}{x}$ **c** $\lim\limits_{x\to 0}\dfrac{x^2+5x}{x}$

d $\lim\limits_{x\to 0}\dfrac{2x^2-x}{x}$ **e** $\lim\limits_{h\to 0}\dfrac{2h^2+6h}{h}$ **f** $\lim\limits_{h\to 0}\dfrac{3h^2-4h}{h}$

g $\lim\limits_{h\to 0}\dfrac{h^3-8h}{h}$ **h** $\lim\limits_{x\to 1}\dfrac{x^2-x}{x-1}$ **i** $\lim\limits_{x\to 2}\dfrac{x^2-2x}{x-2}$

DISCUSSION

1 Do limits always exist?

2 Consider the graph of $f(x)=\dfrac{1}{x}$.

a What happens to the graph as $x\to 0$ from the:

i left **ii** right?

b Does $f(x)=\dfrac{1}{x}$ have a limit as $x\to 0$?

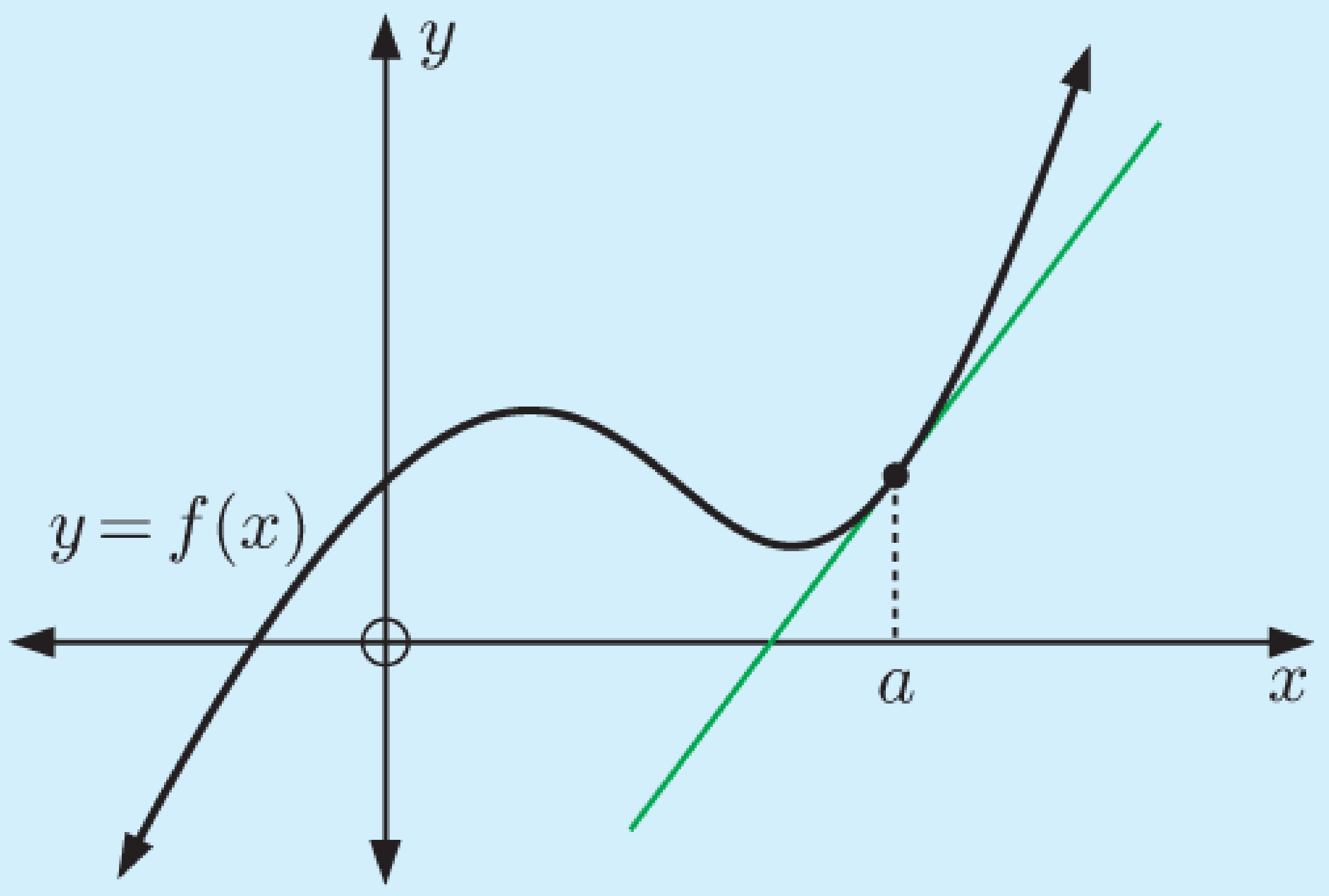
D THE GRADIENT OF A TANGENT

We have seen that for two points $A(a, f(a))$ and $B(a+h, f(a+h))$ on a function, the gradient of the chord [AB] is $\dfrac{f(a+h)-f(a)}{h}$.

Letting B get infinitely close to A is equivalent to taking the limit as $h\to 0$.

The **gradient of the tangent** to the curve $y=f(x)$ at the point where $x=a$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

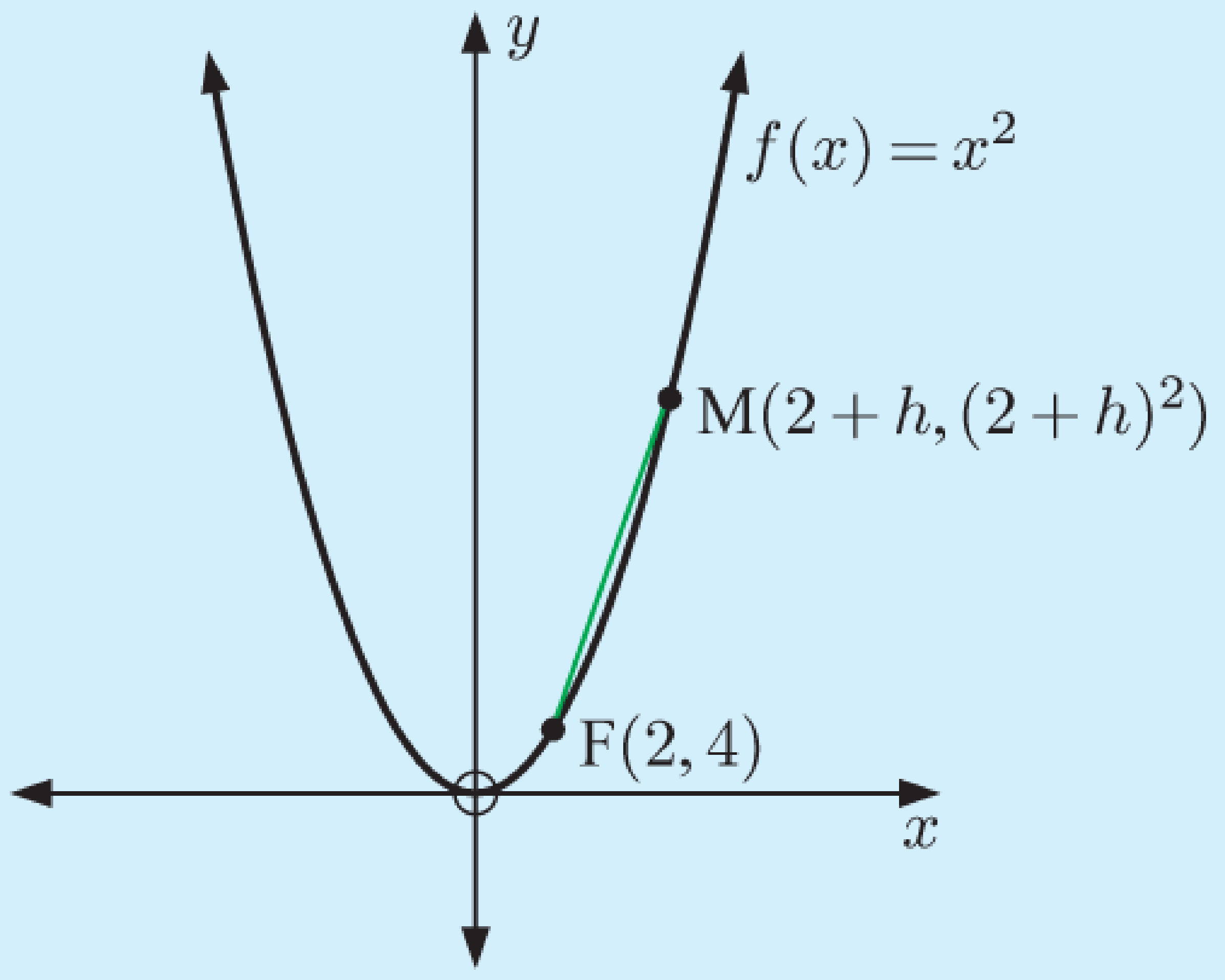


Example 5

Self Tutor

Find the gradient of the tangent to $f(x) = x^2$ at the point $(2,\ 4)$.

Let F be the point $(2,\ 4)$. Suppose M has x-coordinate $2 + h$ and also lies on the graph, so M is $(2+h,\ (2+h)^2)$.



The gradient of the tangent at F

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}\\
&= \lim_{h\to 0} \frac{(2+h)^2 - 4}{h}\\
&= \lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}\\
&= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}\\
&= \lim_{h\to 0} (4+h) \qquad \{\text{as } h \neq 0\}\\
&= 4
\end{aligned}$$

You can use technology to find the gradient of the tangent to a function at a given point, and hence check your answers.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 10D

1 F(3, 9) lies on the graph of $f(x) = x^2$. M also lies on the graph, and has x-coordinate $3 + h$.

- **a** State the y-coordinate of M.
- **b** Show that the gradient of the line segment [FM] is $6 + h$.
- **c** *Hence* find the gradient of [FM] if M has coordinates:
  - **i** $(4,\ 16)$
  - **ii** $(3.5,\ 12.25)$
  - **iii** $(3.1,\ 9.61)$
  - **iv** $(3.01,\ 9.0601)$
- **d** Use limits to find the gradient of the tangent to $f(x) = x^2$ at the point $(3,\ 9)$.

2
- **a** Find the gradient of the tangent to $f(x) = x^2$ at the point where:
  - **i** $x = 1$
  - **ii** $x = 4$
- **b** Use **a** and other results from this Section to complete the table alongside for $f(x) = x^2$.
- **c** Predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$.

x-coordinate	*Gradient of tangent*
1	
2	
3	
4	

3 Find the gradient of the tangent to:

- **a** $f(x) = 2x^2$ at the point $(1,\ 2)$
- **b** $f(x) = x^2 + x$ at the point $(2,\ 6)$
- **c** $f(x) = 2x - x^2$ when $x = 1$
- **d** $f(x) = x^2 - 3x$ when $x = 0$.

4 **a** Use $(1+h)^3 = (1+h)^2(1+h)$ to expand $(1+h)^3$ completely.

b F(1, 1) lies on the graph of $y = x^3$. M also lies on the graph, and has x-coordinate $1+h$.

i Show that the gradient of the chord [FM] is given by $\dfrac{3h+3h^2+h^3}{h}$.

ii Find the gradient of the tangent to $f(x) = x^3$ at the point $(1, 1)$.

5 **a** Show that $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{-h}{x(x+h)}$.

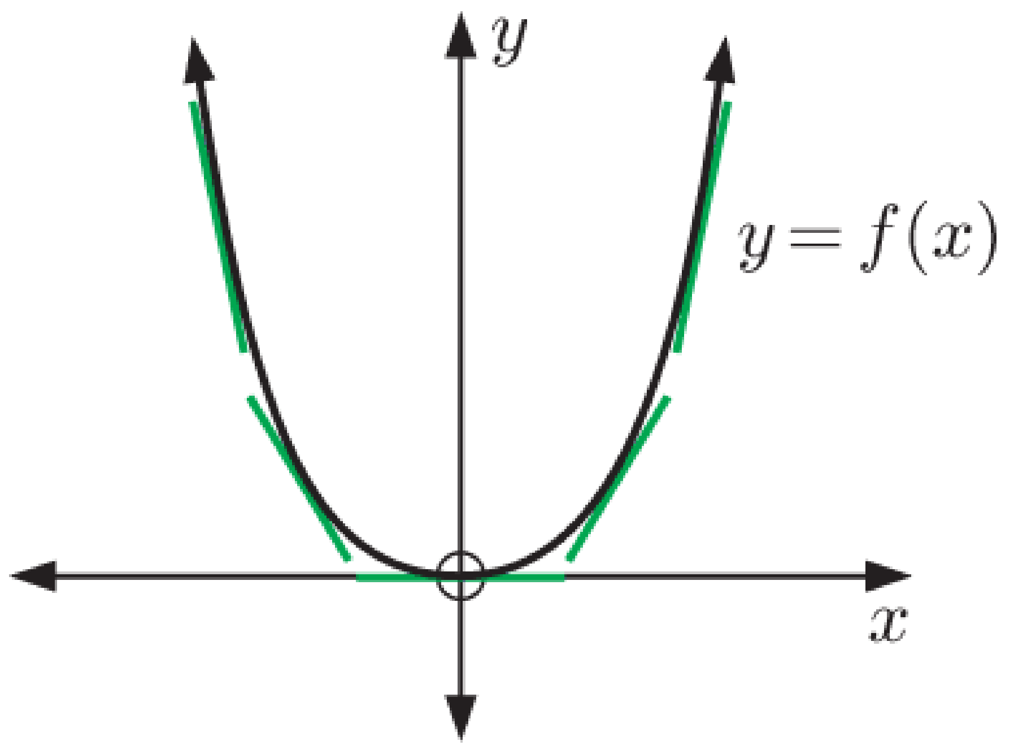
b Find the gradient of the tangent to $y = \dfrac{1}{x}$ at the point where:

i $x = 1$ **ii** $x = 3$

E THE DERIVATIVE FUNCTION

For a non-linear function $y = f(x)$, the gradient of the tangent changes as we move along the curve.

We can therefore write a **gradient function** which gives the gradient of the tangent for any given value of x.

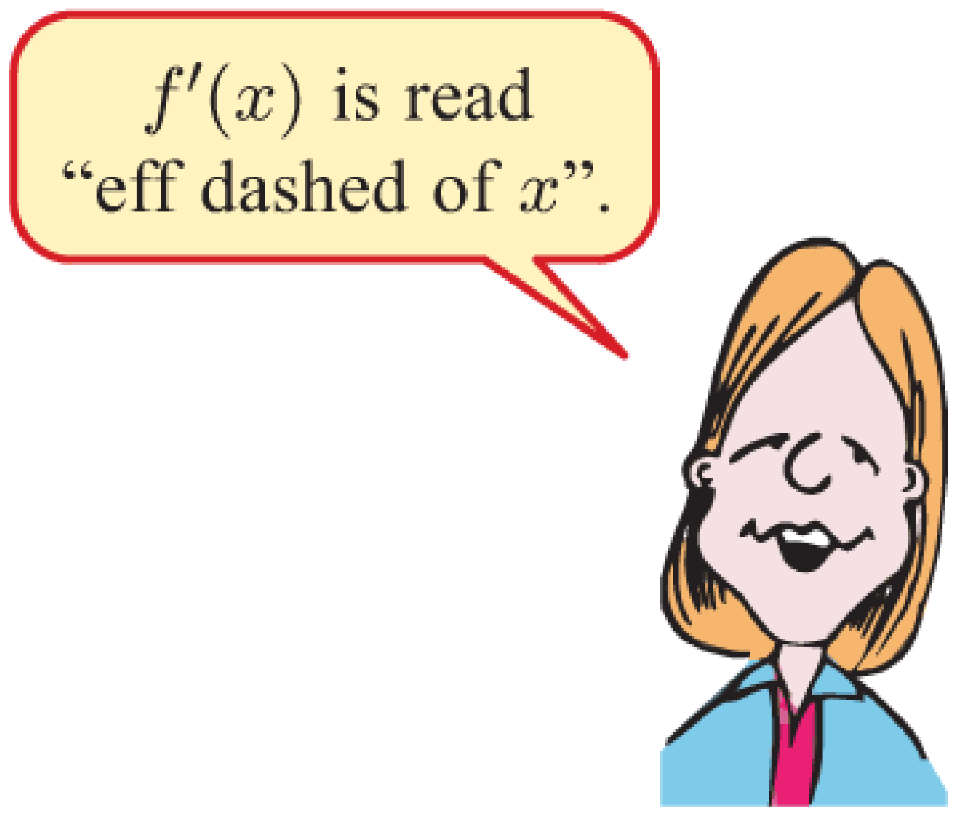


The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

For example, in question **2** of the previous Exercise, you should have observed that for $f(x) = x^2$, the gradient of the tangent is always double the x-coordinate.

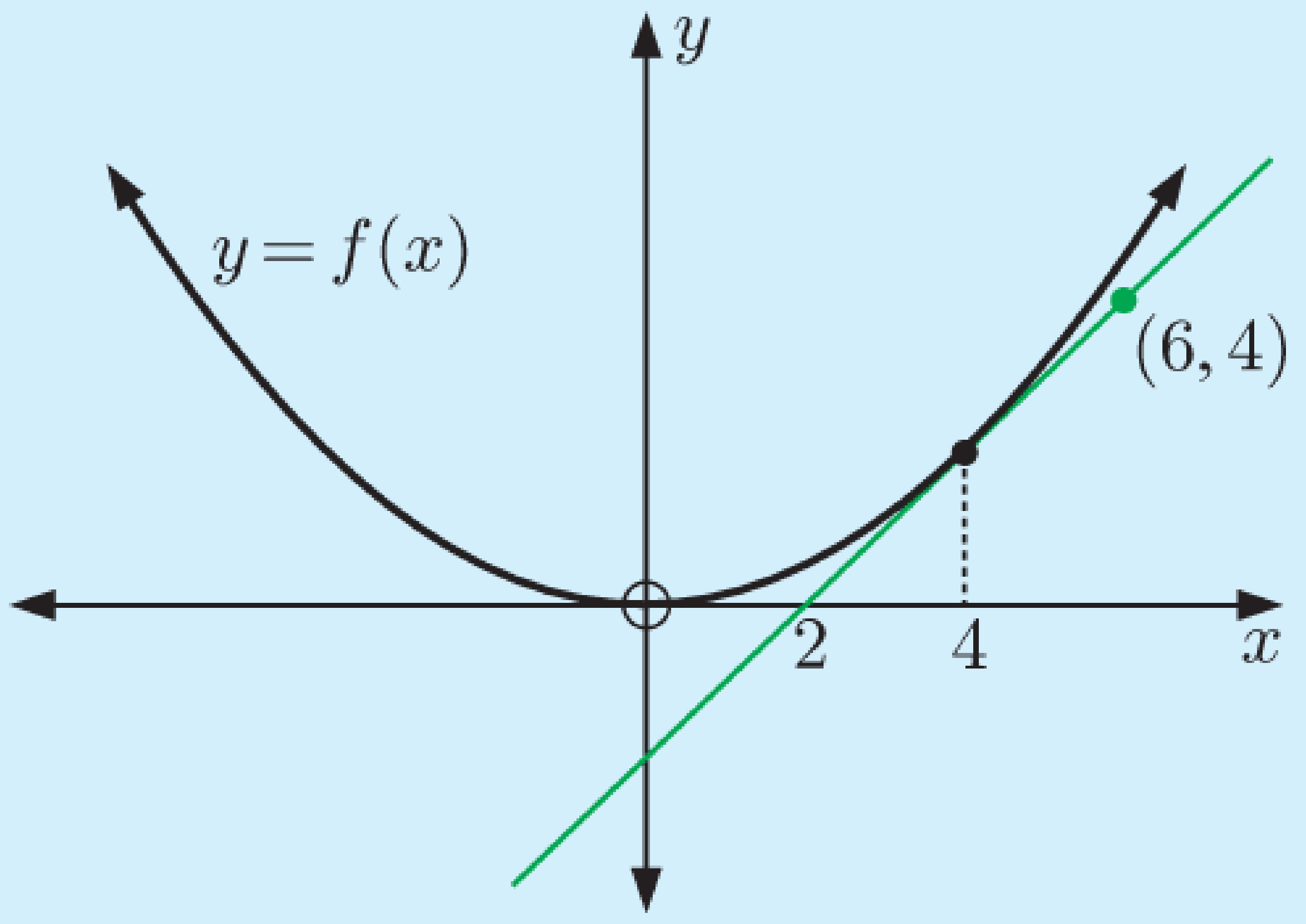
So, for $f(x) = x^2$ we write $f'(x) = 2x$.

Since $f'(3) = 6$, the gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$, is 6.


f'(x) is read "eff dashed of x".

Example 6 **Self Tutor**

For the given graph, find $f'(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$.

$$\begin{aligned}\therefore\ f'(4) &= \text{gradient of the tangent}\\ &= \frac{4-0}{6-2}\\ &= 1\end{aligned}$$

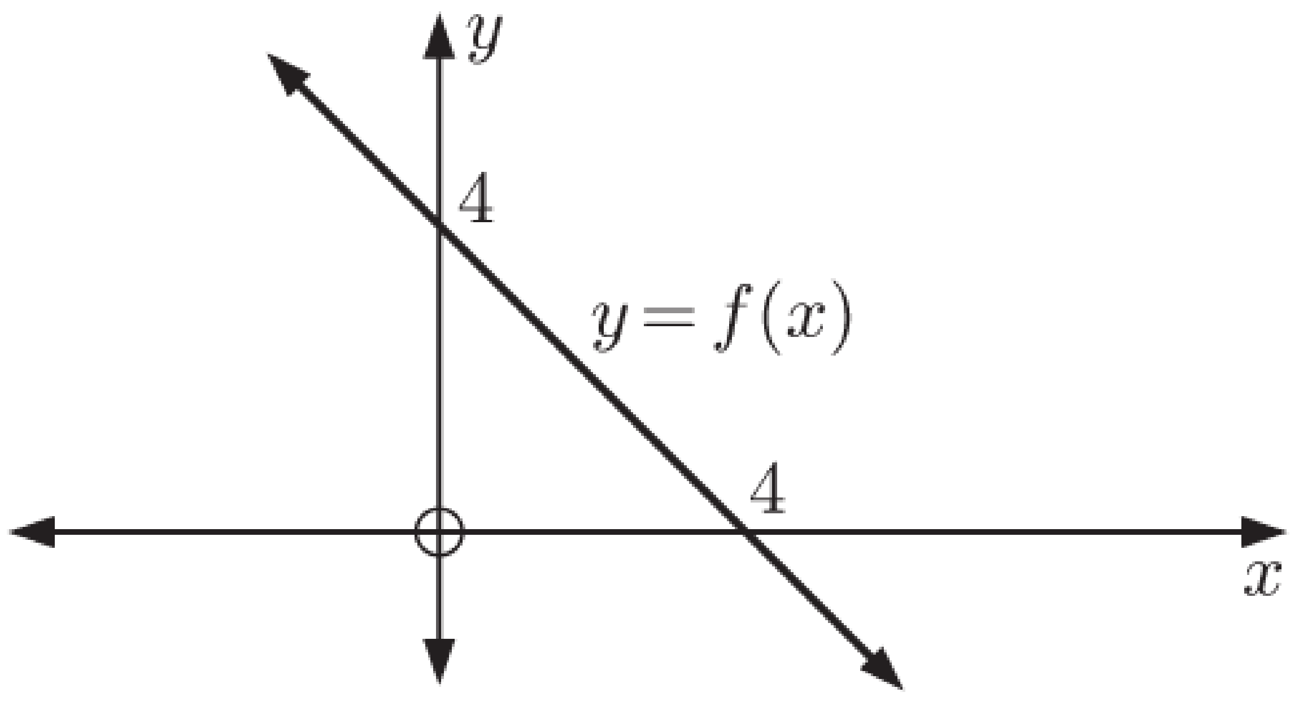
If we are given y written as a function of x, we often write the derivative function as $\frac{dy}{dx}$. This is called "the derivative of y with respect to x", and is read "dee y by dee x".

For example, for $y = x^2$ we have $\frac{dy}{dx} = 2x$.

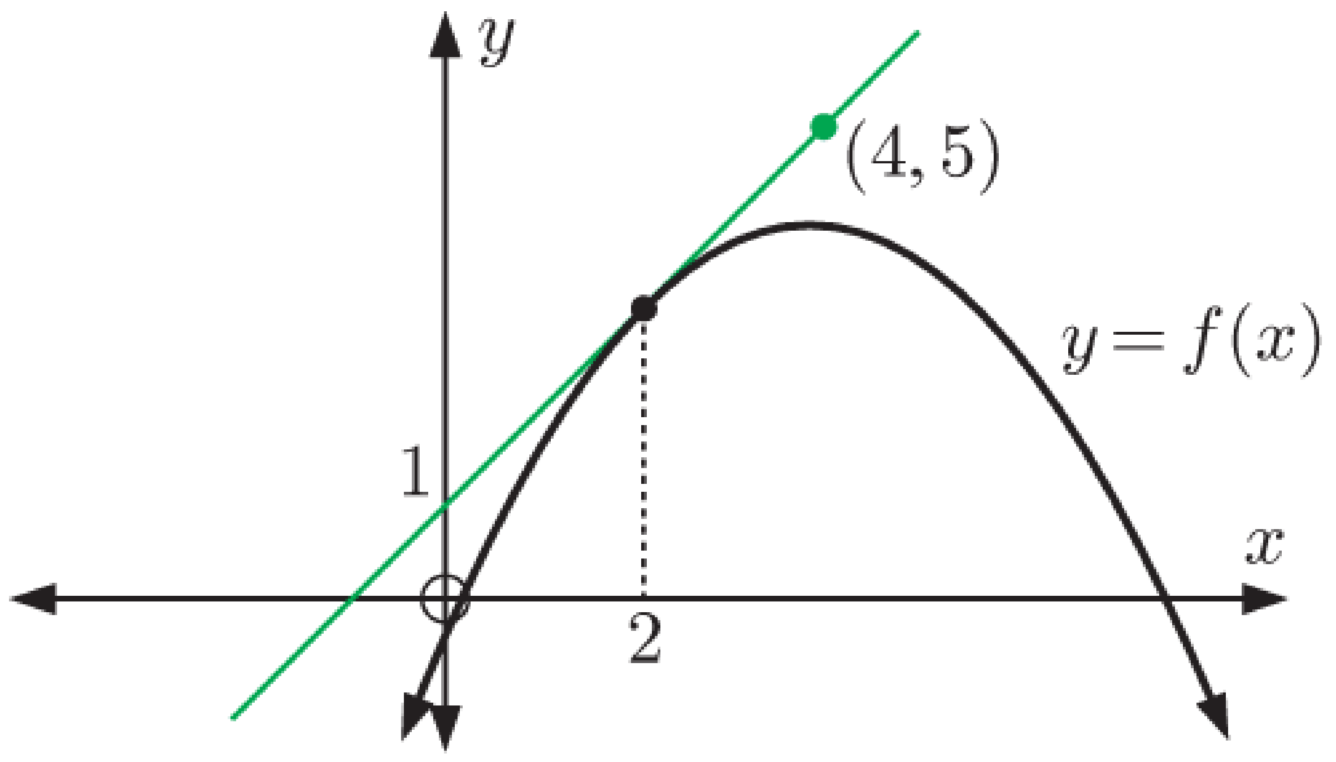
EXERCISE 10E

1 Using the graph below, find:

- **a** $f(0)$
- **b** $f'(0)$

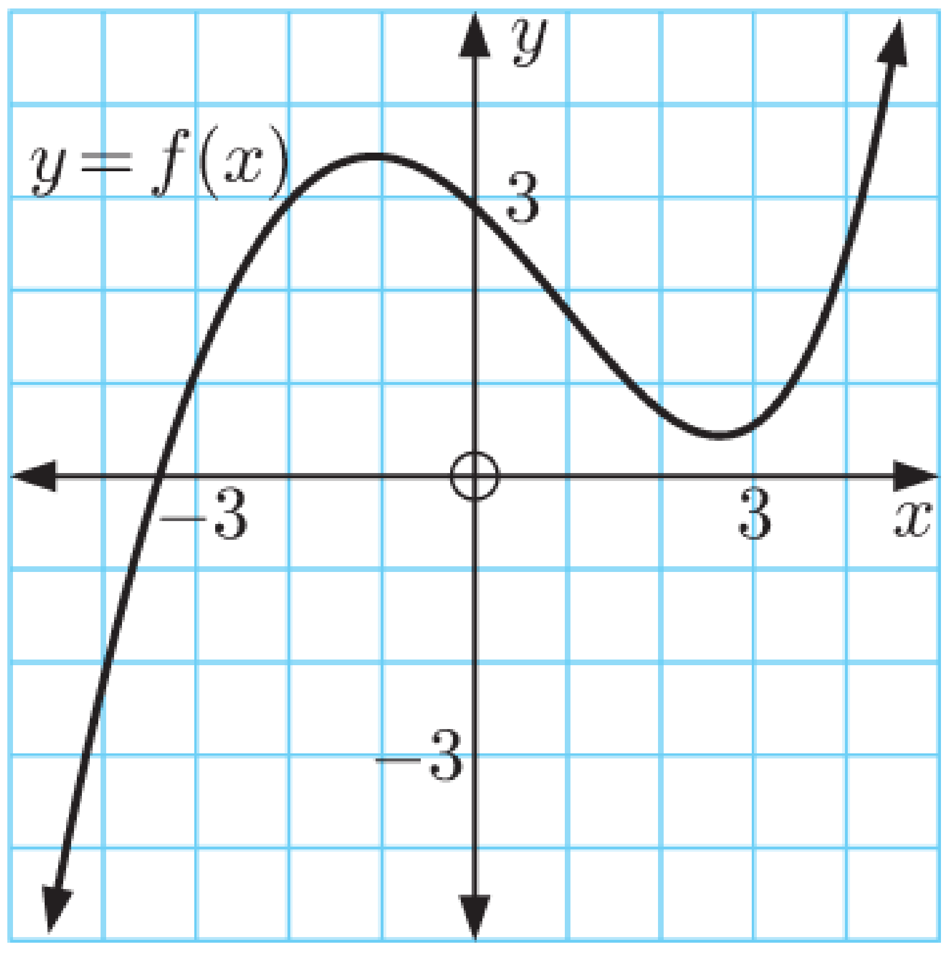


2 Use the graph below to find $f'(2)$.



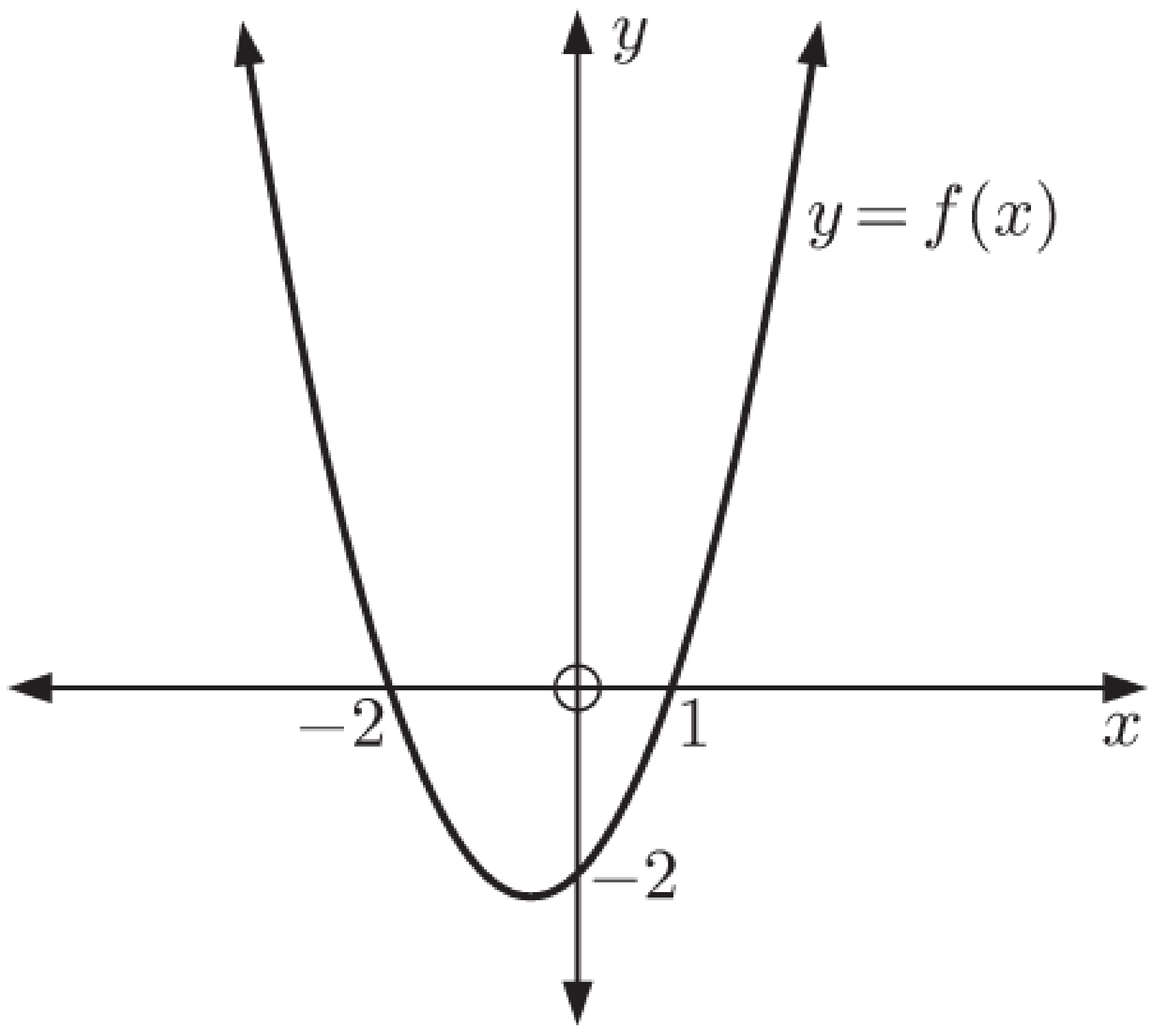
3 For the graph of $y = f(x)$ alongside, decide whether the following are positive or negative:

- **a** $f(3)$
- **b** $f'(1)$
- **c** $f(-4)$
- **d** $f'(-2)$



4 For the graph of $y = f(x)$ alongside, the derivative function is $f'(x) = 2x + 1$.

- **a** Find and interpret:
  - **i** $f'(-2)$
  - **ii** $f'(0)$.
- **b** Copy the graph, and include the information in **a**.



INVESTIGATION 2 GRADIENT FUNCTIONS

The software, accessible using the icon alongside, can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph, we observe the changing gradient of the tangent and hence generate the gradient function $f'(x)$.

GRADIENT FUNCTIONS

What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.

- **a** For each of these functions, what is the gradient?
- **b** Is the gradient constant for all values of x?

2 Consider the *linear* function $f(x) = mx + c$.

- **a** State the gradient of the function.
- **b** Is the gradient constant for all values of x?
- **c** Use the software to graph the following linear functions and observe the gradient function $f'(x)$. Hence verify your answer to **b**.
  - **i** $f(x) = x - 1$
  - **ii** $f(x) = 3x + 2$
  - **iii** $f(x) = -2x + 1$

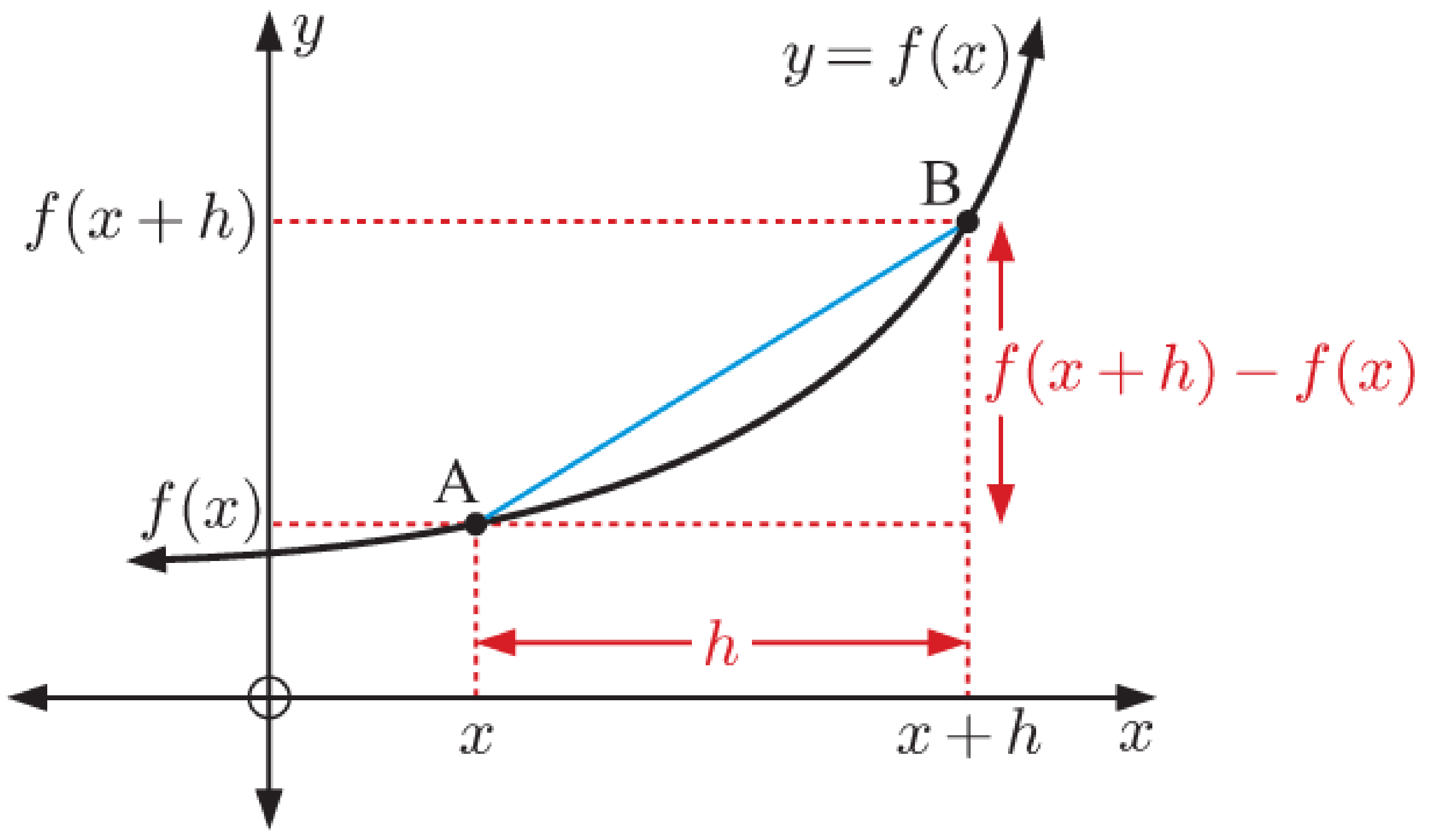
3

- **a** Observe the function $f(x) = x^2$ using the software. What *type* of function is the gradient function $f'(x)$?
- **b** Observe the following *quadratic* functions using the software:
  - **i** $f(x) = x^2 + x - 2$
  - **ii** $f(x) = 2x^2 - 3$
  - **iii** $f(x) = -x^2 + 2x - 1$
  - **iv** $f(x) = -3x^2 - 3x + 6$
- **c** What *type* of function is each of the gradient functions $f'(x)$ in **b**?

F DIFFERENTIATION

To find the derivative function $f'(x)$ for a function $f(x)$, we use limits to find the gradient of the tangent to the curve at a general point $(x, f(x))$.

Let $A(x, f(x))$ and $B(x+h, f(x+h))$ be two points on the curve.



The chord [AB] has gradient

$$= \frac{f(x+h) - f(x)}{x + h - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

So, the gradient of the tangent at the general point $(x, f(x))$ is $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $y = f(x)$ is defined as

$$f'(x) \text{ or } \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of $f'(x)$ is the set of values for which this limit exists.

When we evaluate this limit to find a derivative function, we say we are **differentiating from first principles**.

Example 7 **Self Tutor**

Use first principles to find the gradient function $f'(x)$ of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\ &= 2x \end{aligned}$$

In this Example we prove the result observed in **Exercise 10D**.

EXERCISE 10F

1 For each function $f(x)$, find $f'(x)$ from first principles:

a $f(x) = 1$ **b** $f(x) = x$ **c** $f(x) = 2x - 1$ **d** $f(x) = 3 - x$

2 Find $\frac{dy}{dx}$ from first principles:

a $y = x^2 + 2$ **b** $y = 3 - x^2$ **c** $y = 2x^2 + x$ **d** $y = -x^2 + 5x - 3$

3 Let $f(x) = 3x^2 - 1$.

a Find $f'(x)$ from first principles.

b Hence find $f'(2)$ and interpret your answer.

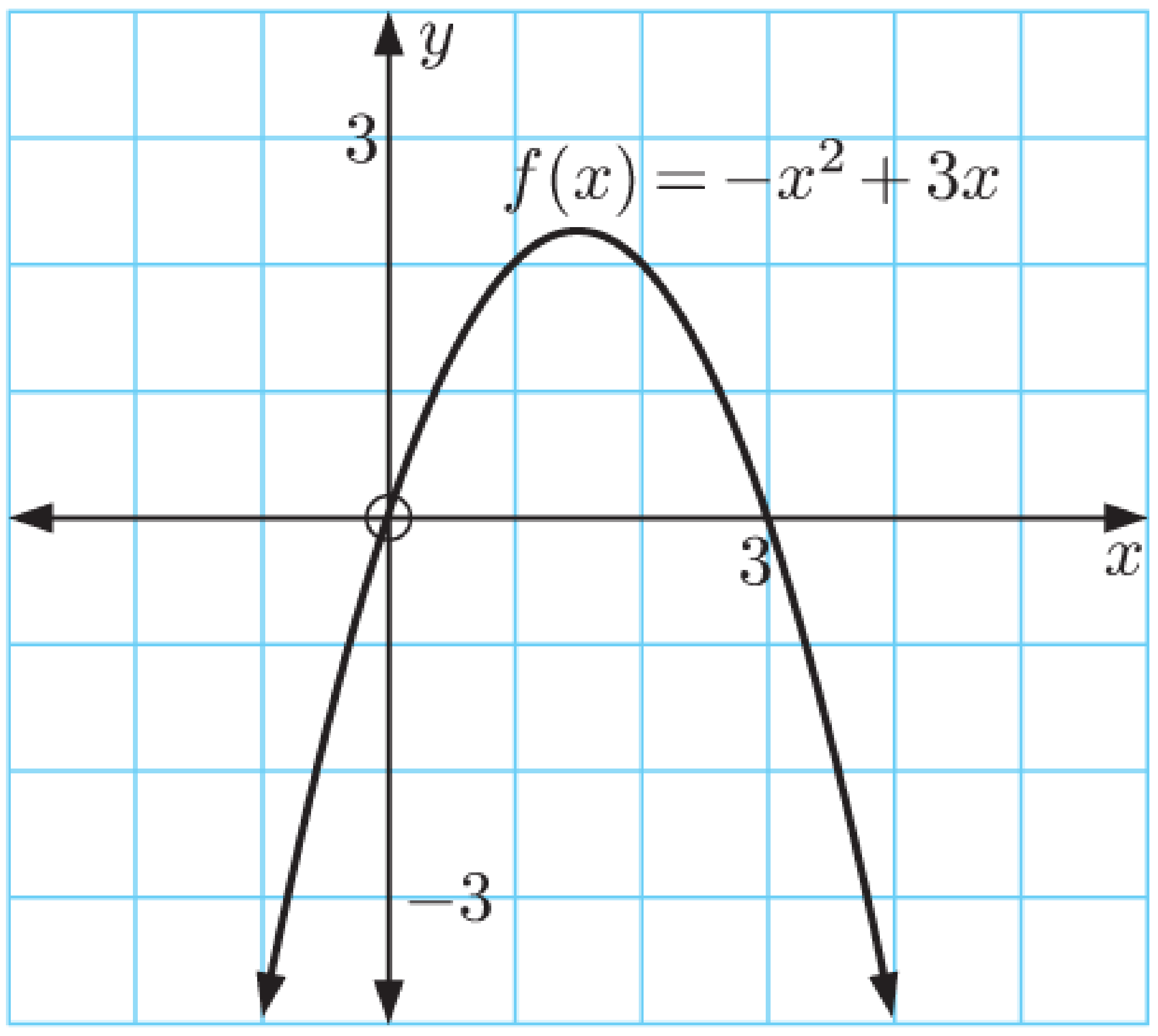
4 The graph of $f(x) = -x^2 + 3x$ is shown alongside.

a Use the graph to estimate the gradient of the tangent to the curve at the point where:

i $x = 0$ **ii** $x = 2$.

b Find $f'(x)$ from first principles.

c Find $f'(0)$ and $f'(2)$, and hence check your estimates in **a**.

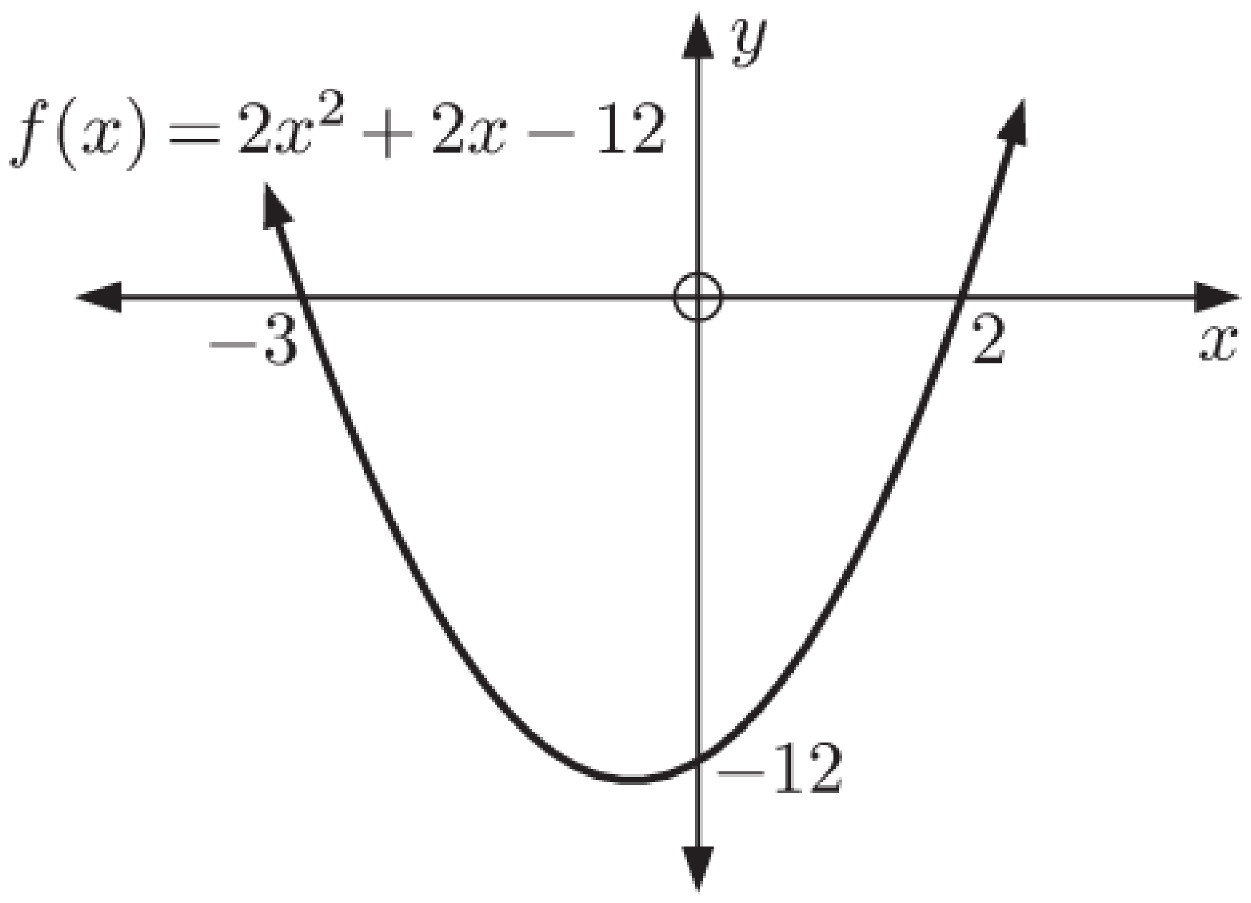


5 The graph of $f(x) = 2x^2 + 2x - 12$ is shown alongside.

a Find $f'(x)$.

b Hence find the point where the tangent has gradient -2.

c Copy the graph, and include the information in **b**.



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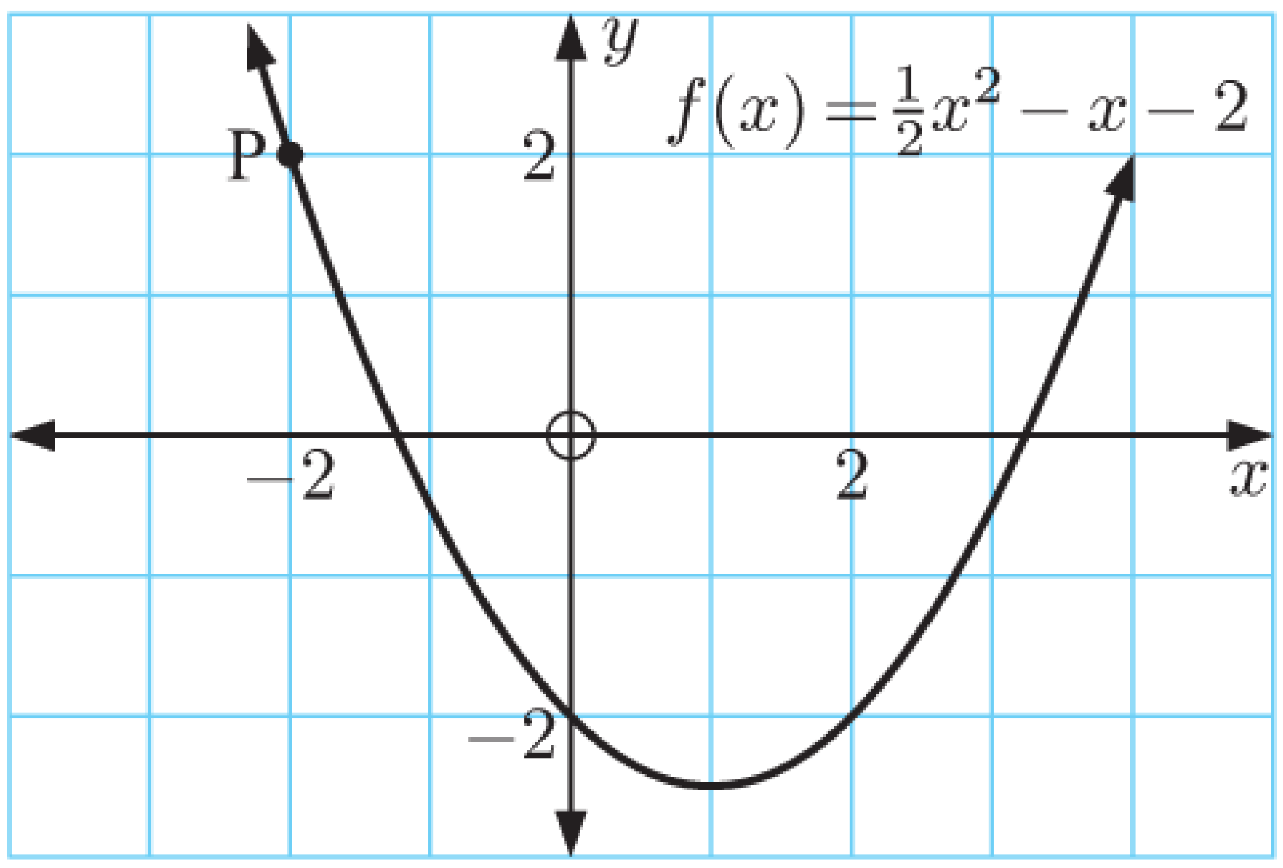


a Find $f'(x)$ from first principles.

b Hence find:

  i the gradient of the tangent to the graph at P

  ii the point at which the tangent has gradient 4.

7 Consider the function $y = x^3$.

a Use the software or your graphics calculator to find the gradient of the tangent when $x = -3, -2, -1, 0, 1, 2, 3$. Record your results in a table:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$							

b Show that $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.
Hint: $(x+h)^3 = (x+h)(x+h)^2$.

c Hence find $\frac{dy}{dx}$ from first principles. Check your answer using the results you found in **a**.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

8 Consider the function $y = \frac{1}{x}$.

a Use technology to complete the following table of derivatives:

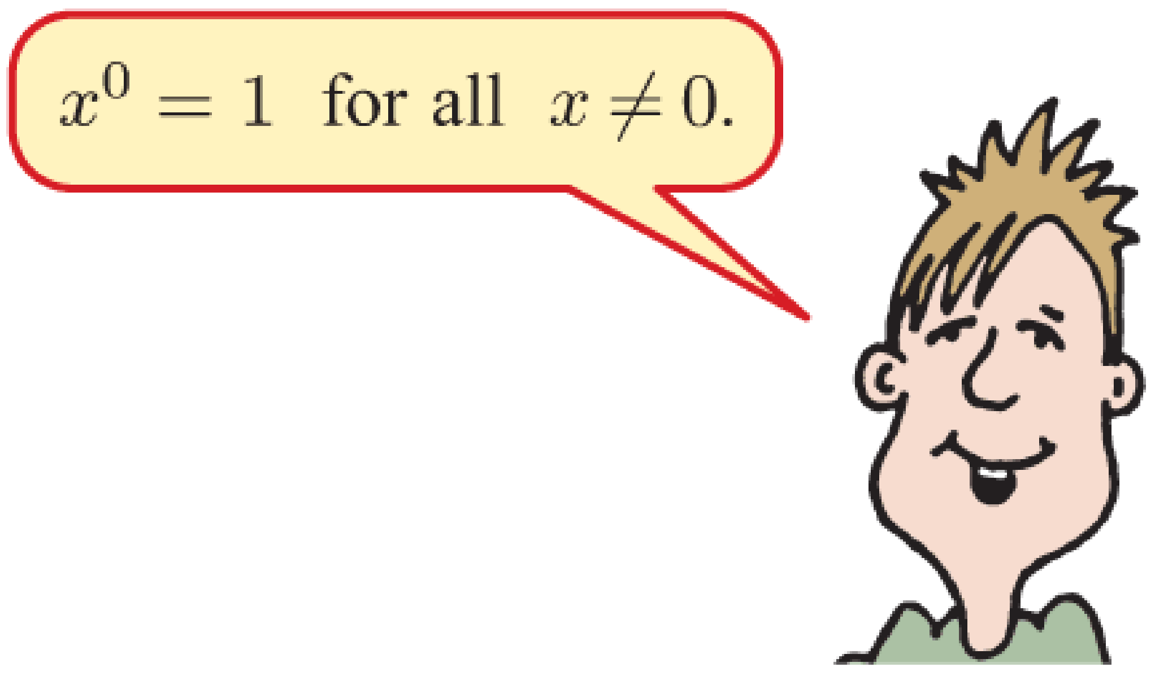
x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$							

b Use the result from question **5 a** in **Exercise 10D** to find $\frac{dy}{dx}$ from first principles. Check your answer using the results you found in **a**.

9 **a** Use the previous results to copy and complete the table:

b Copy and complete:
If $f(x) = x^n$, then $f'(x) = ...$

$f(x)$	$f'(x)$
x^1	
x^2	$2x$
x^3	
x^{-1}	
x^0	


$x^0 = 1$ for all $x \neq 0$.

DISCUSSION

1 Does a function always have a derivative function?

2 Are the domains of a function and its derivative always the same?

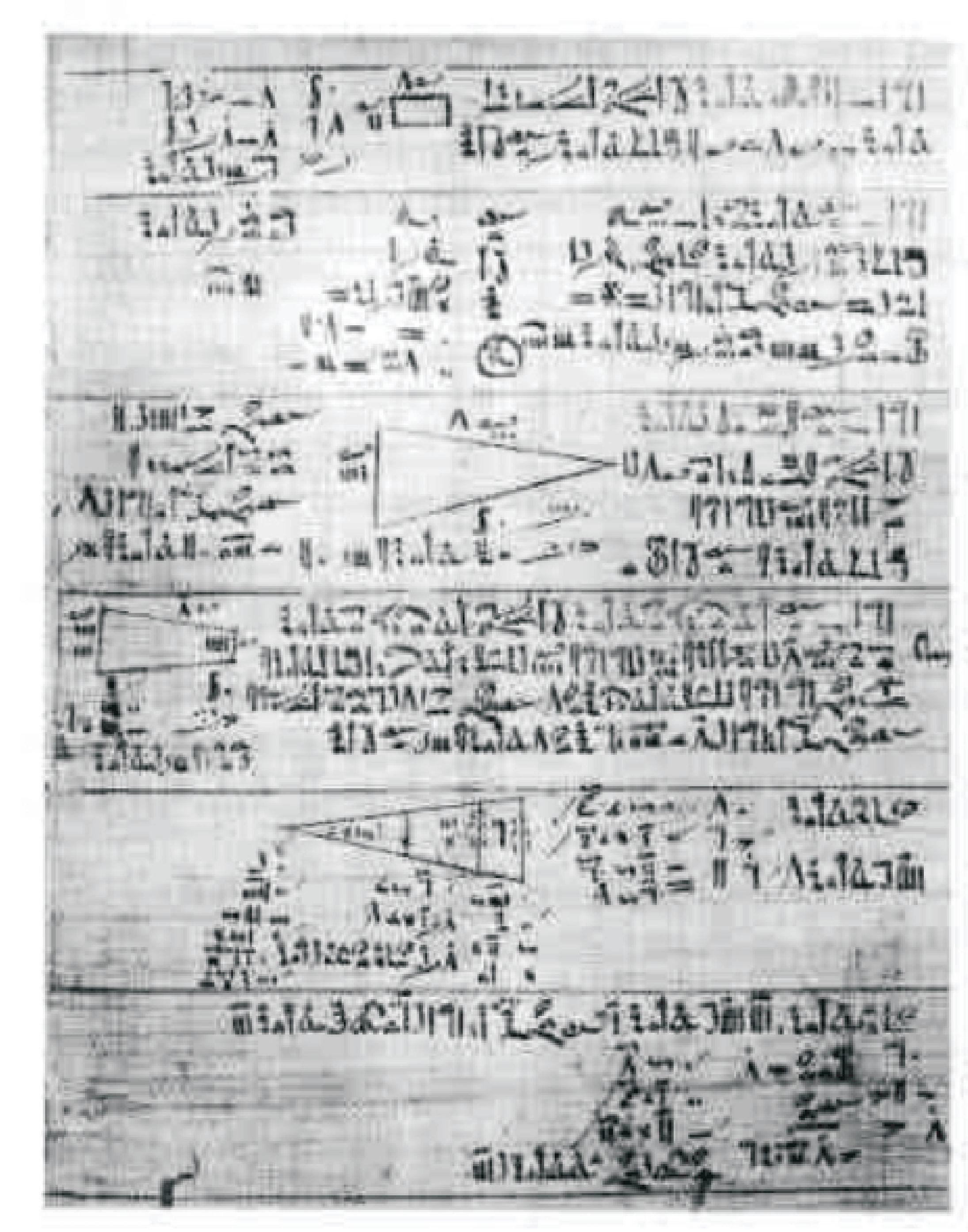
HISTORICAL NOTE

The word "calculus" is a Latin word referring to the small pebbles the ancient Romans used for counting.

The first known description of calculus is found on the **Egyptian Moscow papyrus** from about 1850 BC. Here, it was used to calculate areas and volumes.

Ancient Greek mathematicians such as **Democritus** and **Eudoxus** developed these ideas further by dividing objects into an infinite number of sections. This led to the study of **infinitesimals**, and allowed **Archimedes of Syracuse** to find the tangent to a curve other than a circle.

The methods of Archimedes were the foundation for modern calculus developed almost 2000 years later by mathematicians such as **Johann Bernoulli** and **Isaac Barrow**.



Egyptian Moscow papyrus

G RULES FOR DIFFERENTIATION

There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

For example, in the previous Exercise you should have found that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

INVESTIGATION 3 RULES FOR DIFFERENTIATION

In this Investigation we look for other rules which can help us to differentiate functions without needing first principles.

What to do:

1 **a** Differentiate using first principles:

i $y = 3x^2$ **ii** $y = 5x^2$ **iii** $y = -2x^2$

b Copy and complete: "If $f(x) = cx^n$ where c is a constant, then $f'(x) = ...$"

c Check your answer by using the software to consider the derivatives of:

i $y = 4x^4$ **ii** $y = -\frac{2}{x}$ **iii** $y = \frac{3}{x^2}$

DERIVATIVES

2 **a** Look back at your answers to **Exercise 10F** questions **1** to **3**. In situations where a function is the sum of terms, compare the derivative with the derivatives of the terms.

Hence copy and complete:

"If $f(x) = u(x) + v(x)$, then $f'(x) = ...$"

b Check your answer by using the software to consider the derivatives of x^3, $-2x^2$, and $x^3 - 2x^2$.

We can summarise the following rules:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = c\,u(x)$,
  then $f'(x) = c\,u'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c\,u(x+h)-c\,u(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{u(x+h)-u(x)}{h}\right] \\ &= c\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x)+v(x)$,
  then $f'(x) = u'(x)+v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\ &= u'(x)+v'(x) \end{aligned}$$

Using the rules we have now developed, we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4+2x^3-5x^2+7x+6$ then

$$\begin{aligned} f'(x) &= 3(4x^3)+2(3x^2)-5(2x)+7(1)+0 \\ &= 12x^3+6x^2-10x+7 \end{aligned}$$

Example 8 **Self Tutor**

Find the derivative of $y = 5x^3+6x^2-3x+2$.

$$\begin{aligned} y &= 5x^3+6x^2-3x+2 \\ \therefore\ \frac{dy}{dx} &= 5(3x^2)+6(2x)-3(1) \\ &= 15x^2+12x-3 \end{aligned}$$

EXERCISE 10G

1 Find $f'(x)$ given that $f(x)$ is:

a x^3
b x^8
c x^{11}
d $6x$
e $2x^3$
f $7x^2$
g $3x^5$
h $5x^6$
i x^2+x
j x^2+3x-5
k $5x-2$
l x^2+3
m $2x^2+x-1$
n $3x^2-7x+8$
o $4-2x^2$
p $\frac{1}{2}x^4-6x^2$
q x^3-4x^2+6x
r $7-x-4x^3$
s $\frac{1}{5}x^3-\frac{7}{2}x^2-2$
t $(2x-1)^2$

Example 9 **Self Tutor**

Find $f'(x)$ for $f(x)$ equal to:

a $7x - \frac{4}{x} + \frac{3}{x^3}$ **b** $\frac{x^2+4x-5}{x}$

a
$$\begin{aligned} f(x) &= 7x - \frac{4}{x} + \frac{3}{x^3} \\ &= 7x - 4x^{-1} + 3x^{-3} \\ \therefore\ f'(x) &= 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) \\ &= 7 + 4x^{-2} - 9x^{-4} \\ &= 7 + \frac{4}{x^2} - \frac{9}{x^4} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \frac{x^2+4x-5}{x} \\ &= \frac{x^2}{x} + \frac{4x}{x} - \frac{5}{x} \\ &= x + 4 - 5x^{-1} \\ \therefore\ f'(x) &= 1 - 5(-1x^{-2}) \\ &= 1 + 5x^{-2} \\ &= 1 + \frac{5}{x^2} \end{aligned}$$

2 Differentiate with respect to x:

a $\frac{1}{x^2}$ **b** $\frac{1}{x^5}$ **c** $\frac{1}{x^8}$

d $\frac{3}{x}$ **e** $\frac{4}{x^3}$ **f** $-\frac{7}{x^4}$

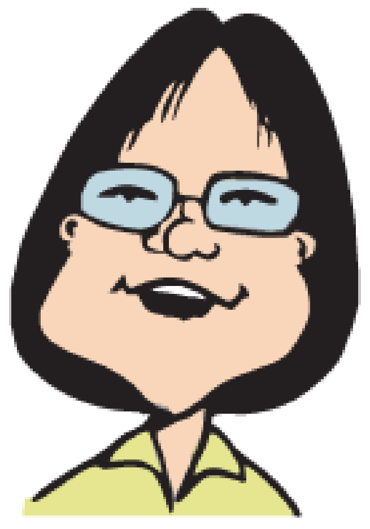
g $2x + \frac{3}{x^2}$ **h** $x^2 - \frac{6}{x}$ **i** $9 - \frac{2}{x^3}$

j $\frac{1}{x} - \frac{5}{x^3}$ **k** $\frac{2}{x^2} + \frac{9}{x^4}$ **l** $3x - \frac{1}{x} + \frac{2}{x^2}$

m $5 - \frac{8}{x^2} + \frac{4}{x^3}$ **n** $\frac{1}{5x^2}$ **o** $4x - \frac{1}{4x}$

p $\frac{x^2-3}{x}$ **q** $\frac{x^3+4}{x}$ **r** $\frac{2x-5}{x^2}$

Remember that $\frac{1}{x^n} = x^{-n}$.



3 Suppose $f(x) = 4x^3 - x$. Find:

a $f'(x)$ **b** $f'(2)$ **c** $f'(0)$

4 Suppose $g(x) = \frac{x^2+1}{x}$. Find:

a $g'(x)$ **b** $g'(3)$ **c** $g'(-2)$

5 Find $\frac{dy}{dx}$ for:

a $y = 100x$ **b** $y = \pi x^2$ **c** $y = 6x + \frac{5}{x}$

d $y = 2.5x^3 - 1.4x^2 - 1.3$ **e** $y = 10(x+1)$ **f** $y = 4\pi x^3$

g $y = (x+1)(x-2)$ **h** $y = (5-x)^2$ **i** $y = (3+x)(2-x)$

6 Find:

a $\frac{dy}{dt}$ for $y = \frac{1}{2}t^4 - \frac{1}{3}t$ **b** $\frac{dy}{dt}$ for $y = 7 - \frac{1}{2t}$ **c** $\frac{dV}{dr}$ for $V = \frac{4}{3}\pi r^3$

Example 10 **Self Tutor**

Find the derivative of $y = x^2 - \frac{4}{x}$, and hence find the gradient of the tangent to the function at the point where $x = 2$.

$$y = x^2 - \frac{4}{x} = x^2 - 4x^{-1}$$

$$\therefore \quad \frac{dy}{dx} = 2x - 4(-1x^{-2})$$

$$= 2x + 4x^{-2}$$

$$= 2x + \frac{4}{x^2}$$

When $x = 2$, $\frac{dy}{dx} = 4 + 1 = 5$. So, the tangent has gradient 5.

7 Find the gradient of the tangent to:

- **a** $y = x^2$ at $x = 2$
- **b** $y = x^3 - 5x + 2$ at the point $(3, 14)$
- **c** $y = \frac{8}{x^2}$ at the point $\left(9, \frac{8}{81}\right)$
- **d** $y = 2x^2 - 3x + 7$ at $x = -1$
- **e** $y = 2x - \frac{5}{x}$ at the point $\left(2, \frac{3}{2}\right)$
- **f** $y = \frac{x^3 - 4x - 8}{x^2}$ at $x = -1$

8 Suppose $f(x) = x^2 + (b+1)x + 2c$, $f(2) = 4$, and $f'(-1) = 2$. Find the constants b and c.

Example 11 **Self Tutor**

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found. It is also the instantaneous rate of change of y with respect to x.

9 If $y = 4x - \frac{3}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

10 The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds.

- **a** Find $\frac{dS}{dt}$ and interpret its meaning.
- **b** Find the value of $\frac{dS}{dt}$ when $t = 3$, and interpret your answer.

11 The cost of producing x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ pounds. Find the value of $\frac{dC}{dx}$ when $x = 1000$, and interpret its meaning.

Example 12

Self Tutor

At what point on the curve $y = 2x^2 + 5x - 3$ does the tangent have gradient 13?

$$y = 2x^2 + 5x - 3$$

$$\therefore \quad \frac{dy}{dx} = 4x + 5$$

$\therefore$ the tangent has gradient 13 when $4x + 5 = 13$

$$\therefore \quad 4x = 8$$

$$\therefore \quad x = 2$$

When $x = 2$, $y = 2(2)^2 + 5(2) - 3 = 15$

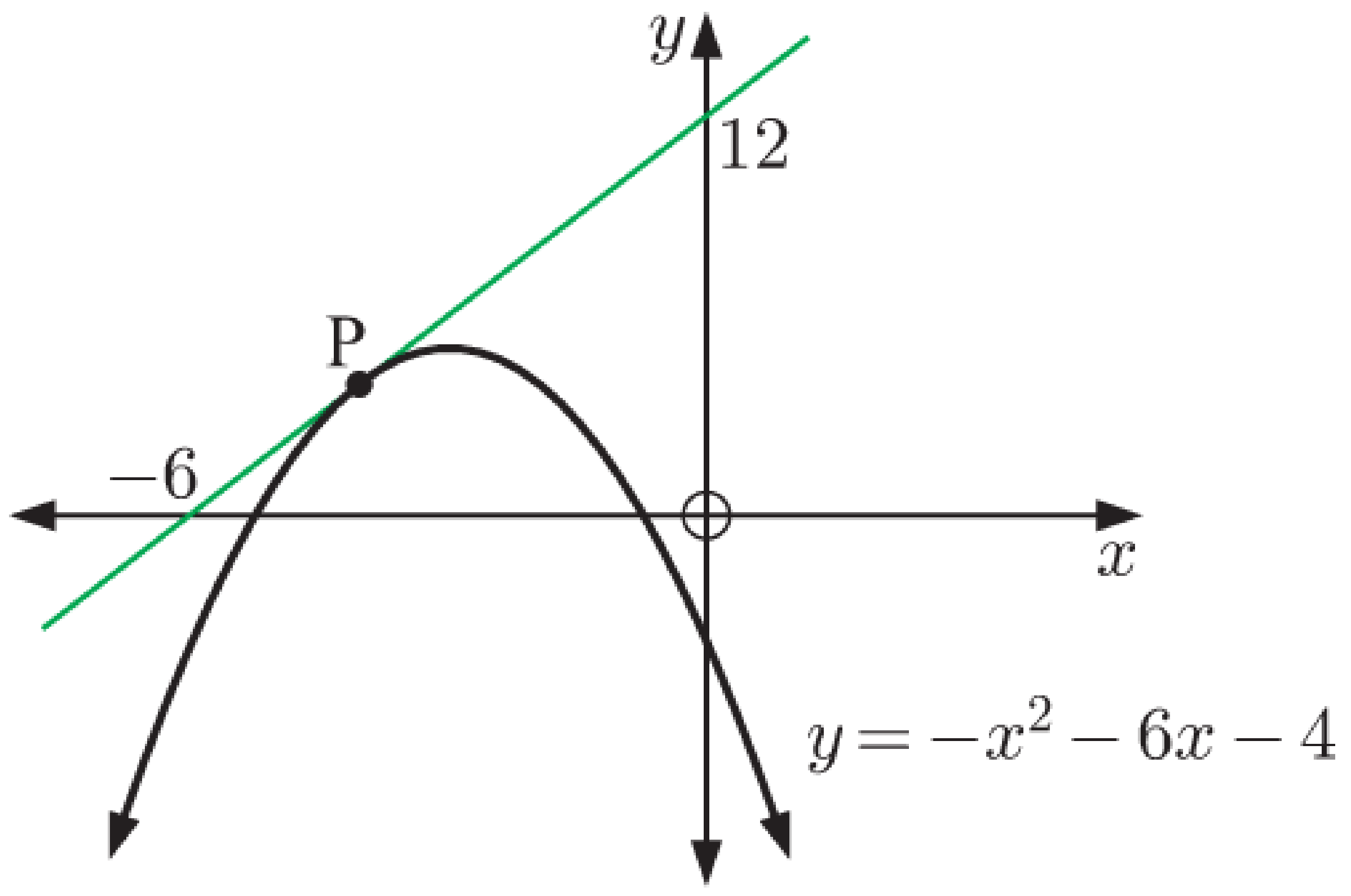
So, the tangent has gradient 13 at the point $(2, 15)$.

12 At what point on the graph of $y = x^2 - 4x + 7$ does the tangent have gradient 2? Draw a diagram to illustrate your answer.

13 Find the coordinates of the point(s) on:

- **a** $y = x^2 + 5x + 1$ where the tangent has gradient 3
- **b** $y = 3x^2 + 11x + 5$ where the tangent has gradient -7
- **c** $f(x) = 2x^{-2} + x$ where the tangent has gradient $\frac{1}{2}$
- **d** $f(x) = 3x^3 - 5x + 2$ where the tangent has gradient 4
- **e** $f(x) = ax^2 + bx + c$ where the tangent is horizontal.

14 Find the coordinates of point P.



Example 13

Self Tutor

The tangent to $f(x) = 2x^2 - ax + b$ at the point $(2, 7)$ has a gradient of 3. Find a and b.

$$f(x) = 2x^2 - ax + b$$

$$\therefore \quad f'(x) = 4x - a$$

Now $f'(2) = 3$, so $4(2) - a = 3$

$$\therefore \quad a = 5$$

Also, $f(2) = 7$, so $2(2)^2 - 5(2) + b = 7$

$$\therefore \quad b = 9$$

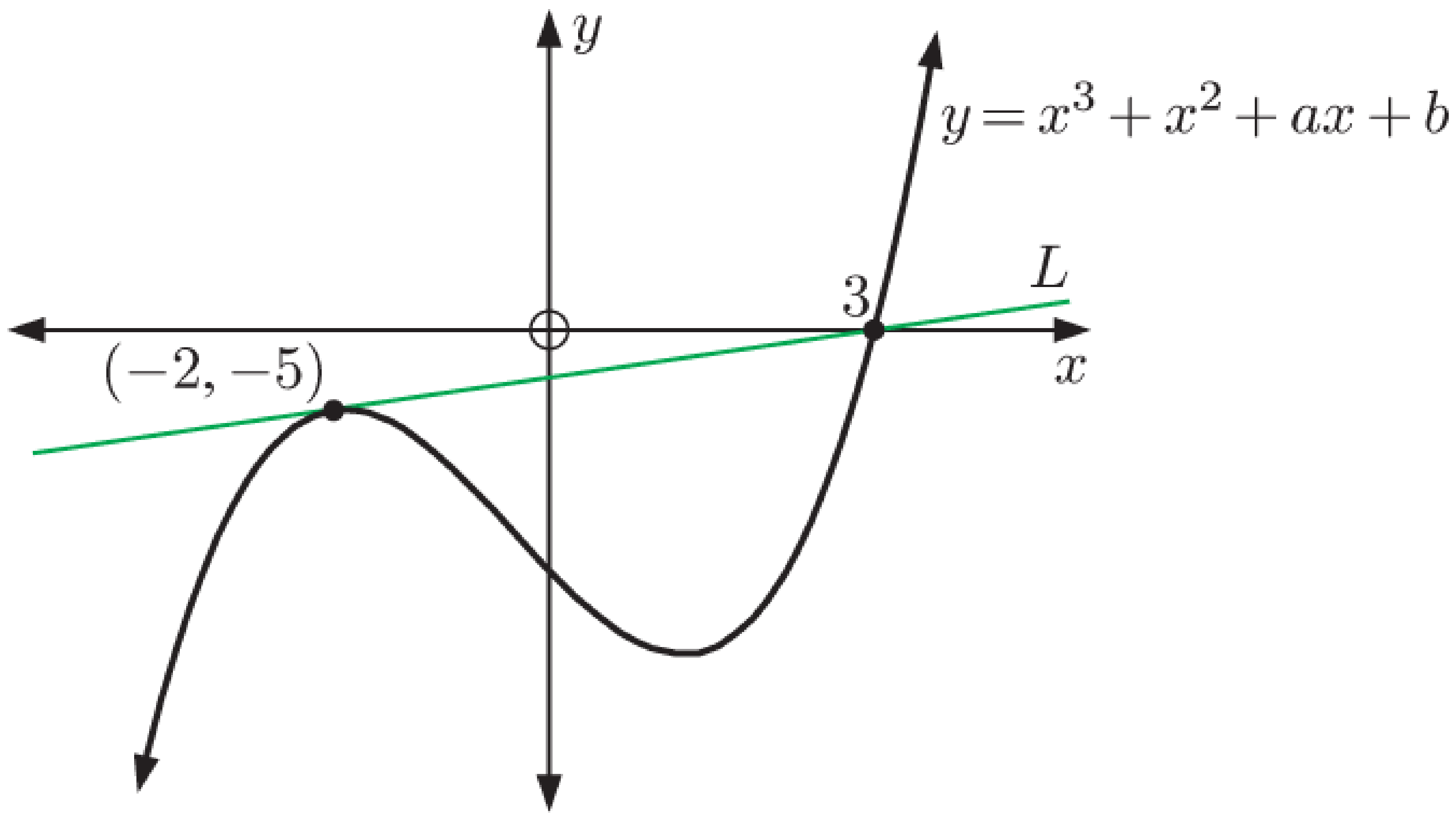
15 The tangent to $f(x) = x^3 + ax + 5$ at the point where $x = 1$, has gradient 10. Find a.

16 The tangent to $f(x) = -3x^2 + ax + b$ at the point $(-3, 8)$ has gradient 9. Find a and b.

17 The tangent to $f(x) = 2x^2 + a + \frac{b}{x}$ at the point $(1, 11)$ has gradient -2. Find a and b.

18 **a** Find the gradient of the tangent line L.

b Hence find a and b.



THEORY OF KNOWLEDGE

The Greek philosopher Zeno of Elea lived in what is now southern Italy, in the 5th century BC. He is most famous for his paradoxes, which were recorded in Aristotle's work *Physics*.

The arrow paradox

> "*If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.*"

This argument says that if we choose any particular instant in time, the arrow is motionless. Therefore, how does the arrow actually move?

The dichotomy paradox

> "*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*"

If an object is to move a fixed distance then it must travel half that distance. Before it can travel a half the distance, it must travel a half *that* distance. With this process continuing indefinitely, motion is impossible.

Achilles and the tortoise

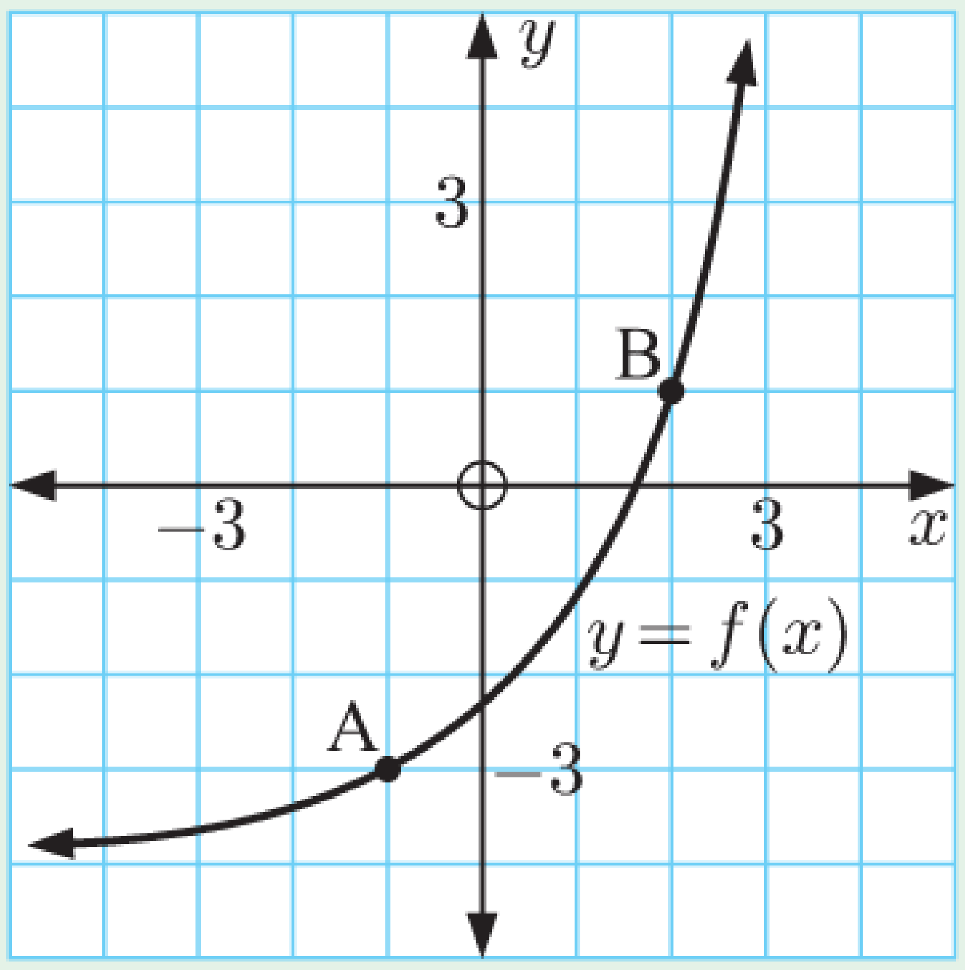
> "*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.*"

According to this principle, the athlete Achilles will never be able to catch the slow tortoise!

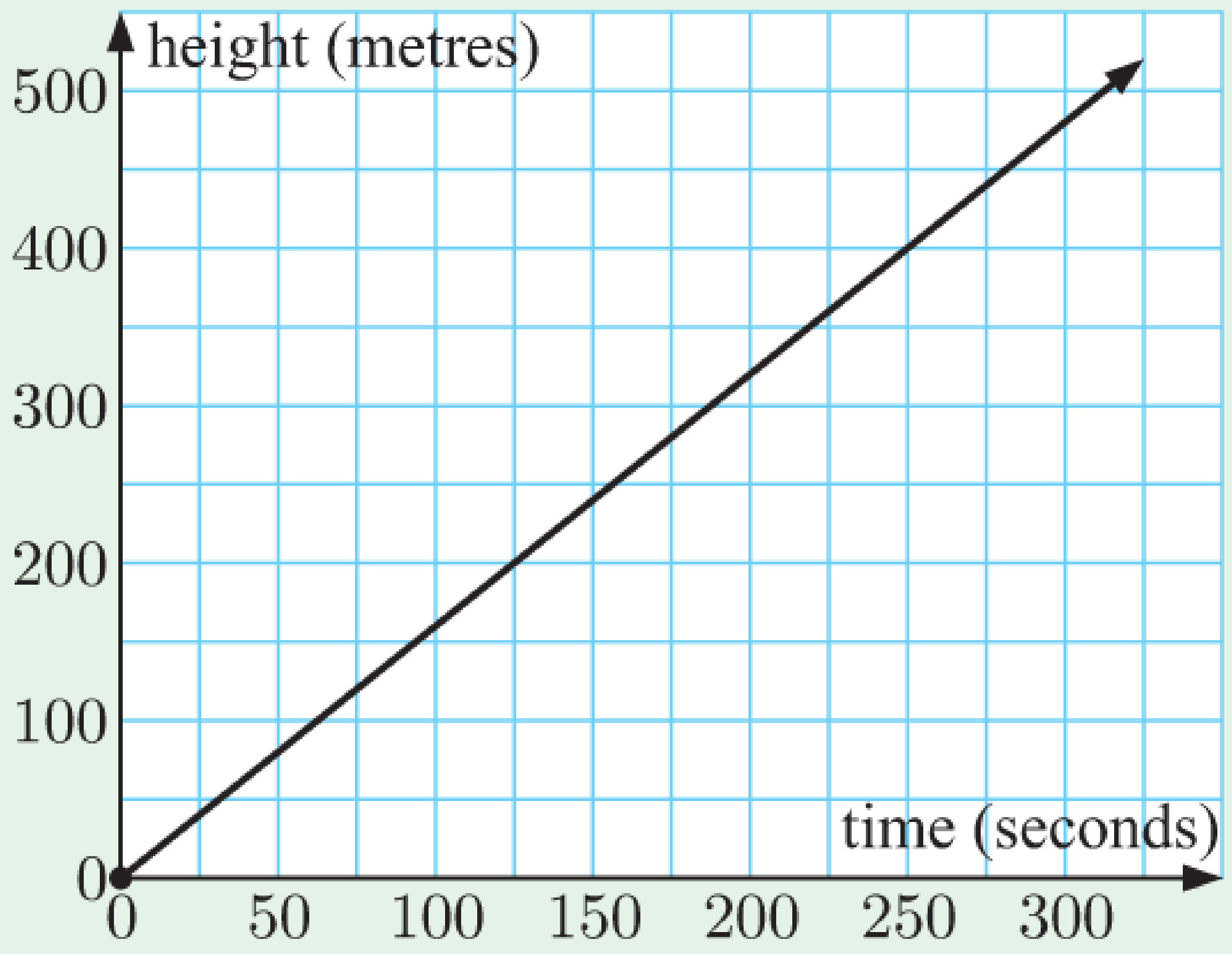
1. A paradox is a logical argument that leads to a contradiction or a situation which defies logic or reason. Can a paradox be the truth?
2. Are Zeno's paradoxes really paradoxes?
3. Are the three paradoxes essentially the same?
4. We know from experience that things *do* move, and that Achilles *would* catch the tortoise. Does that mean that logic has failed?
5. What do Zeno's paradoxes have to do with limits?

REVIEW SET 10A

1 Find the average rate of change in $f(x)$ from A to B.



2 Chantelle is riding in a ski-lift. Her height above the base of the mountain is shown on the graph below.



a Is the ski-lift increasing in height at a constant rate? Explain your answer.

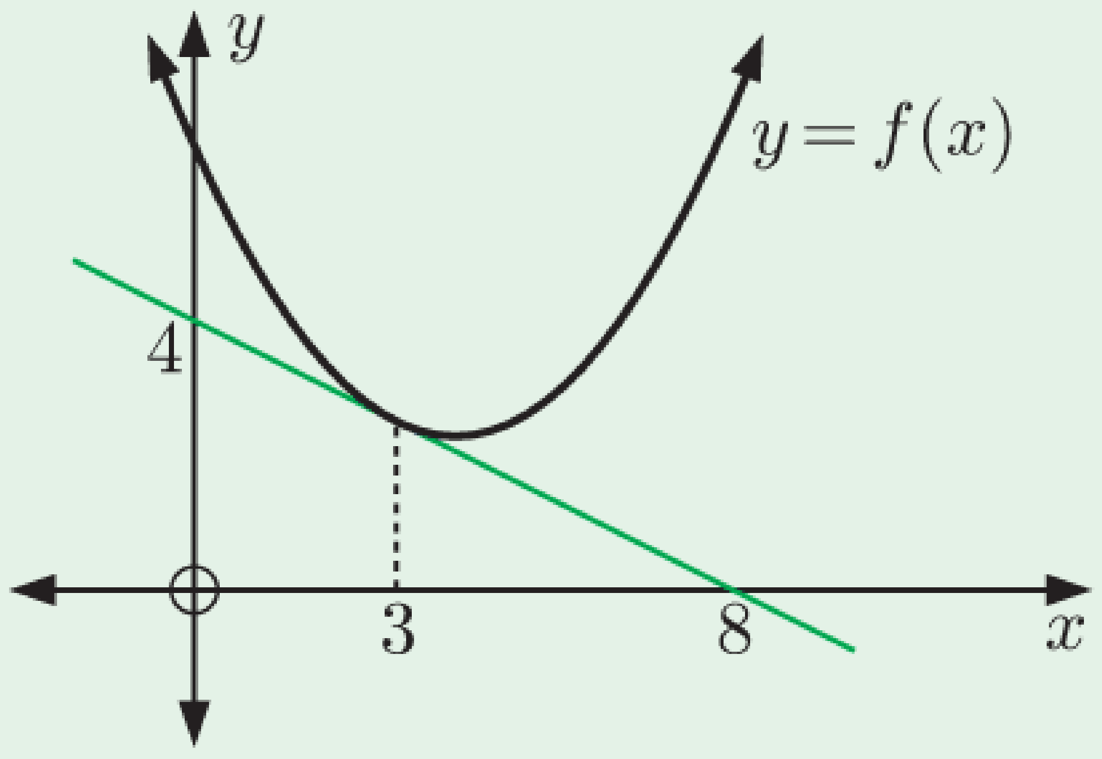
b Find the rate at which the ski-lift is increasing in height.

3 Evaluate:

a $\lim\limits_{x \to 1} (6x - 7)$

b $\lim\limits_{h \to 0} \dfrac{2h^2 - h}{h}$

4 Use the graph alongside to find $f'(3)$.



5 Find, from first principles, the derivative of:

a $f(x) = x^2 + 2x$

b $y = 4 - 3x^2$

6 **a** Given $y = 2x^2 - 1$, find $\dfrac{dy}{dx}$ from first principles.

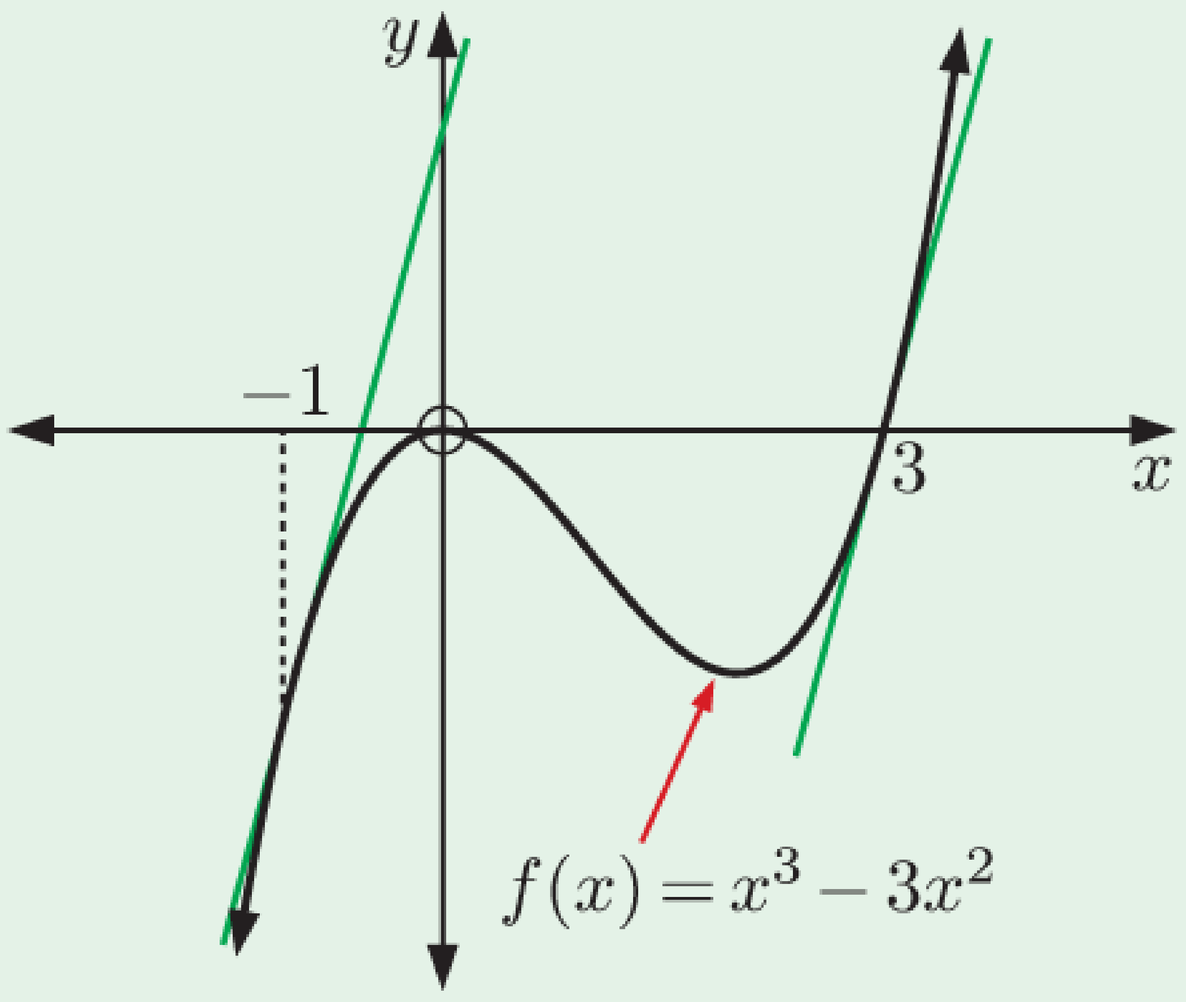
b Hence state the gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$.

c For what value of x is the gradient of the tangent to $y = 2x^2 - 1$ equal to -12?

7 The graph of $f(x) = x^3 - 3x^2$ is shown alongside.

a Find $f'(x)$.

b Hence show that the illustrated tangents are parallel.



8 Find $f'(x)$ given that $f(x)$ is:

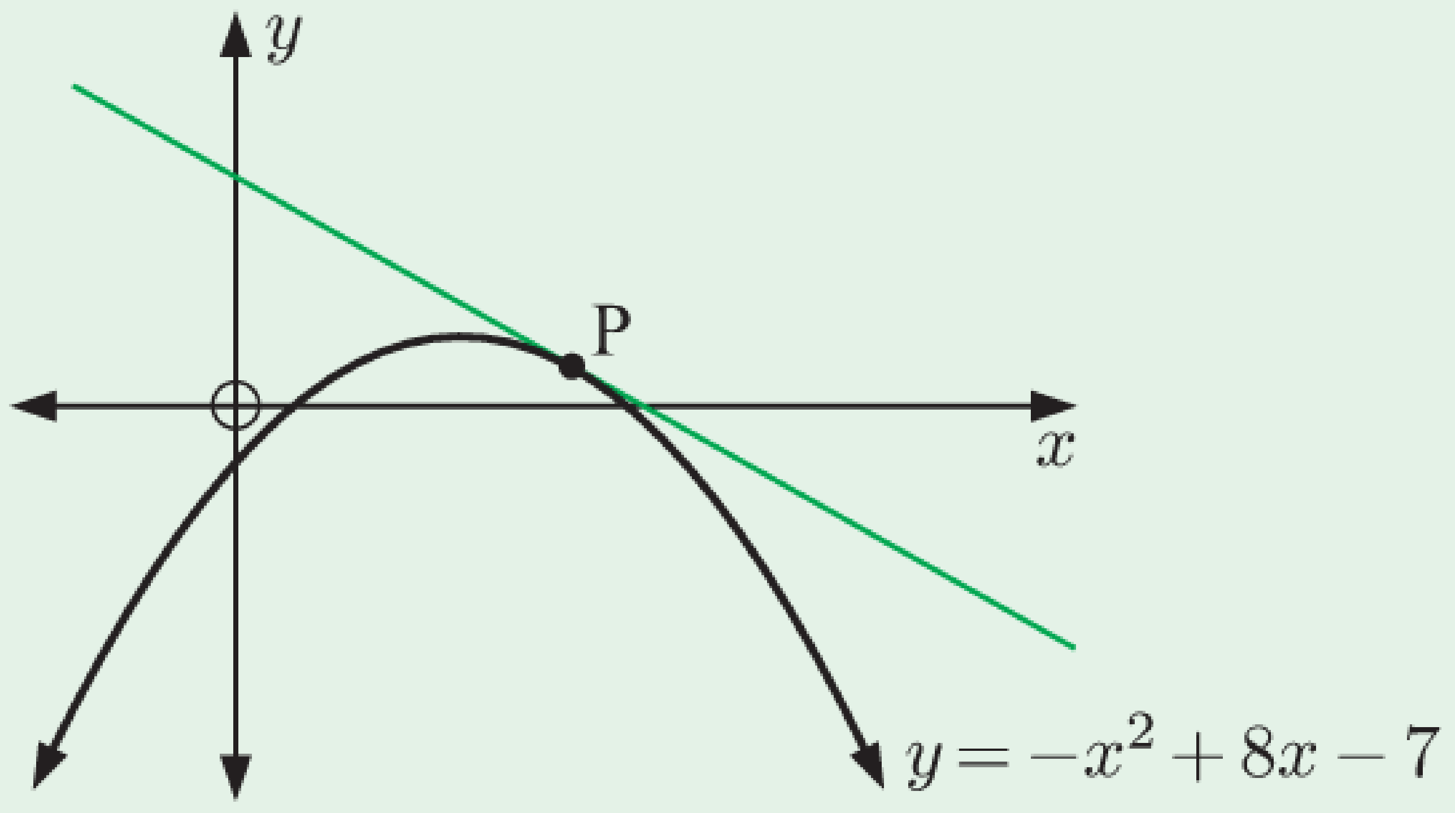
a $5x^3$ **b** $x^6 - 5x$ **c** $7x^2 - \frac{3}{x}$ **d** $3x - \frac{4}{x^2}$

9 Find the gradient of the tangent to $f(x) = -x^2 + 4x - 2$ at $(-3, -23)$.

10 If $f(x) = 7 + x - 3x^2$, find:

a $f(3)$ **b** $f'(3)$

11 The tangent shown has gradient -4.
Find the coordinates of P.



12 The tangent to $y = ax^3 - 3x + 3$ at the point where $x = 2$, has gradient 21. Find a.

13 Find the gradient of $f(x)$ at the given point:

a $f(x) = x^2 - 3x$ at $x = -1$

b $f(x) = -3x^2 + 4$ at $x = 2$

c $f(x) = x + \frac{2}{x}$ at $x = 3$

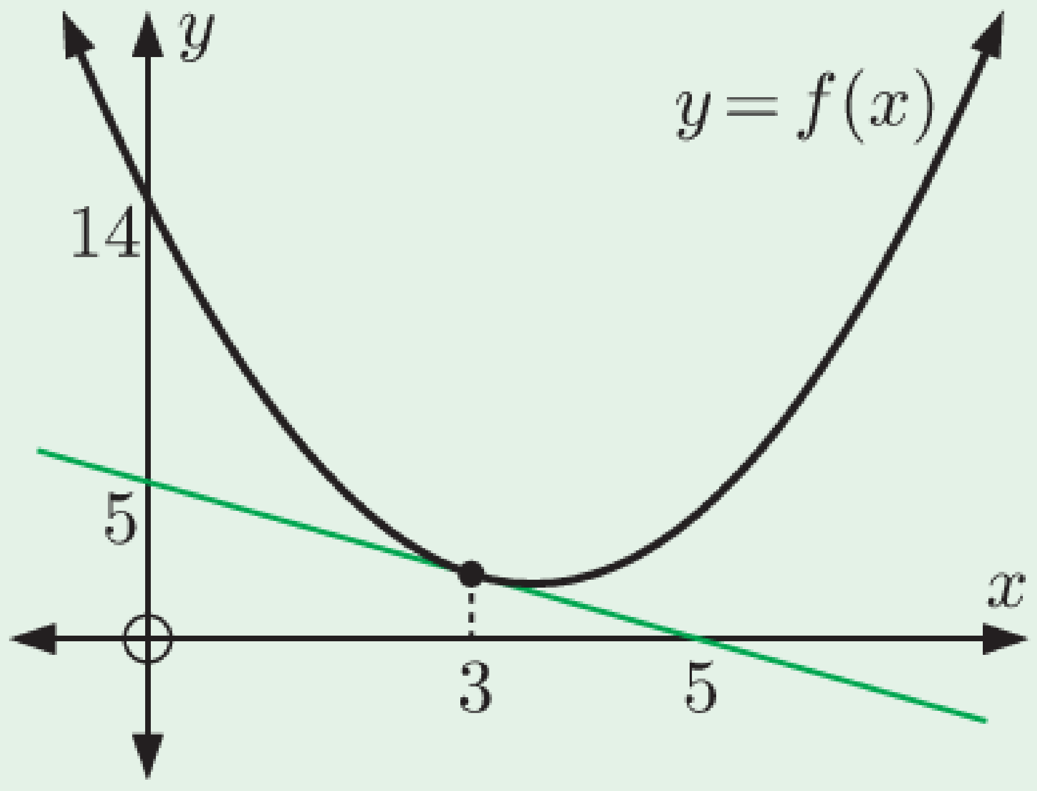
d $f(x) = x^3 - x^2 - x - 2$ at $x = 0$

14 Sand is poured into a bucket for 30 seconds. After t seconds, the weight of sand is given by $S(t) = 0.3t^3 - 18t^2 + 550t$ grams.
Find and interpret $S'(t)$.

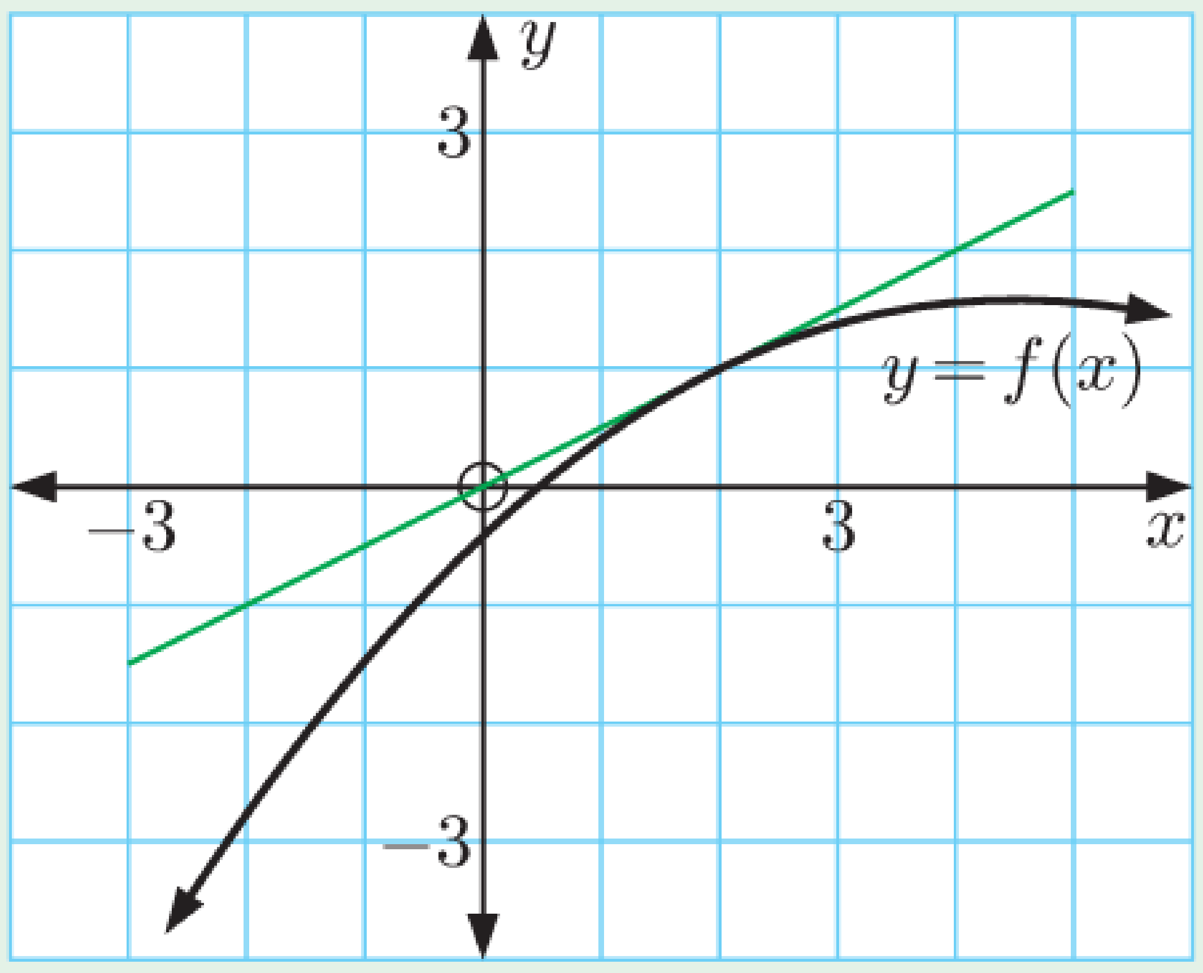
15 $y = f(x)$ is the parabola shown.

a Find $f(3)$ and $f'(3)$.

b Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.



REVIEW SET 10B

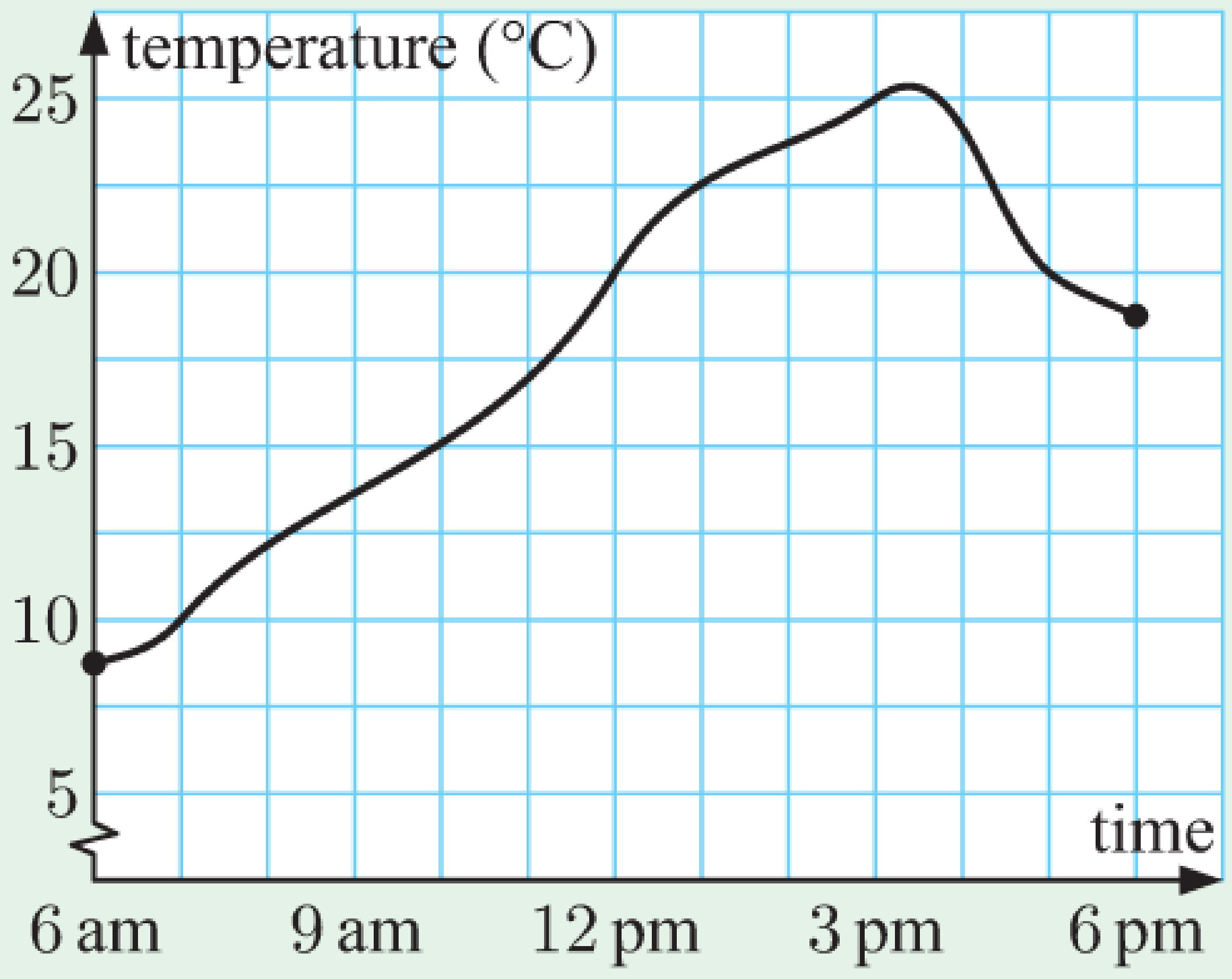
1 

Use the tangent drawn to find the instantaneous rate of change in $f(x)$ at $x = 2$.

2 This graph shows the temperature in Berlin from 6 am to 6 pm on a particular day.
Find the average rate of change in temperature from:

a 7 am to noon

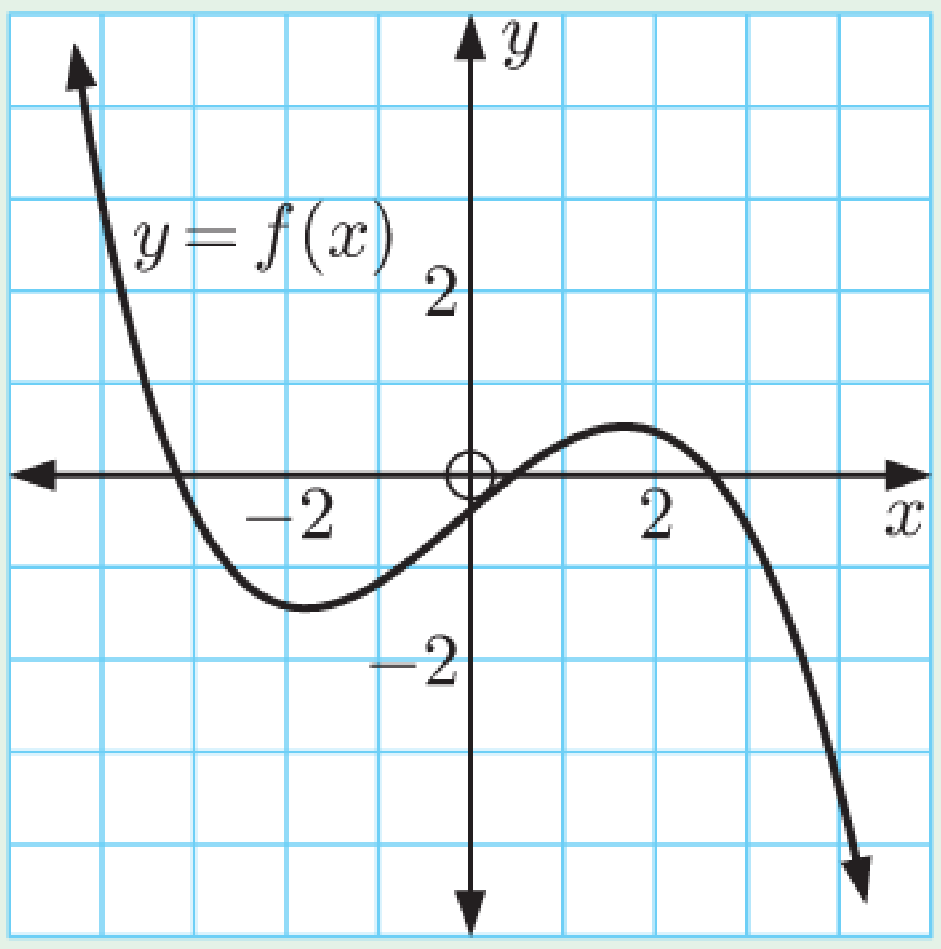
b 3 pm to 5 pm.



3 Evaluate the limits:

a $\lim\limits_{h\to 0} \dfrac{h^3 - 3h}{h}$

b $\lim\limits_{x\to 1} \dfrac{3x^2 - 3x}{x - 1}$

4 

Decide whether the following are positive or negative:

a $f(-1)$

b $f'(0)$

c $f(2)$

d $f'(3)$

5 **a** Draw an accurate graph of $f(x) = x^2 - 2$, and the tangent to the curve at $x = 2$.

b Hence find the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$.

c Check your answer using $f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$.

6 **a** Find $f'(x)$ given $f(x) = x^4 - 2x$.

b Find $f'(-2)$ and interpret your answer.

7 **a** Given $y = x^2 + 5x - 2$, find $\dfrac{dy}{dx}$ from first principles.

b Hence find the point on the graph at which the tangent has gradient -3.

8 In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

- **a** Find the height of the jumper after:
  - **i** 1 second
  - **ii** 2 seconds.
- **b** Find $f'(t)$ from first principles.
- **c** Find the speed of the jumper after:
  - **i** 1 second
  - **ii** 2 seconds.

9 Use the rules of differentiation to find $f'(x)$ for $f(x)$ equal to:

- **a** $7x^3$
- **b** $3x^2 - x^3$
- **c** $(2x-3)^2$
- **d** $\dfrac{7x^3+2x^4}{x^2}$

10 Consider $f(x) = x^4 - 3x - 1$. Find:

- **a** $f'(x)$
- **b** $f'(2)$
- **c** $f'(0)$.

11 Consider the function $f(x) = -2x^2 + 5x + 3$. Find:

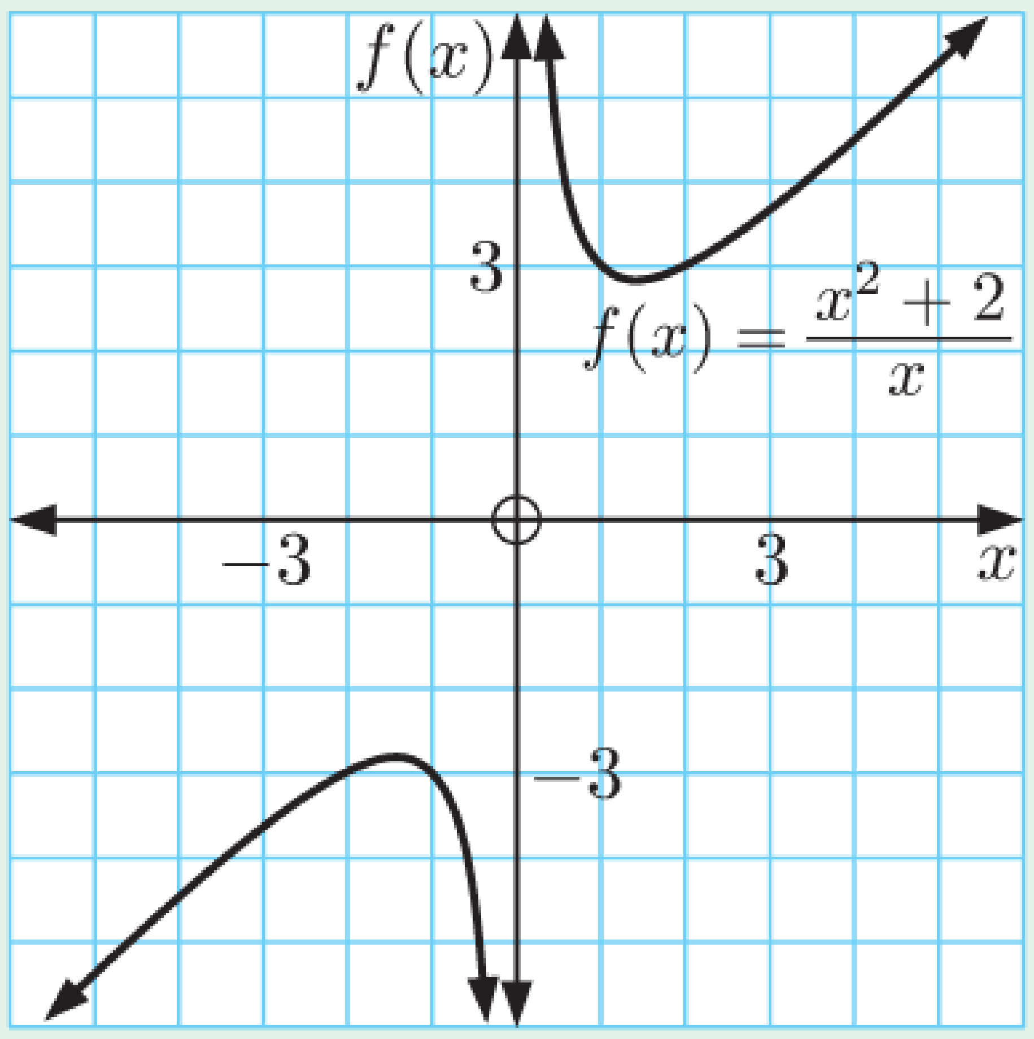
- **a** the average rate of change from $x = 2$ to $x = 4$
- **b** the instantaneous rate of change at $x = 2$.

12 Find $\dfrac{dy}{dx}$ for:

- **a** $y = 3x^2 - 7x + 4$
- **b** $y = 2x^3 - 6x^2 + 7x - 4$
- **c** $y = \dfrac{3}{x} - \dfrac{5}{x^3}$

13 The graph of $f(x) = \dfrac{x^2+2}{x}$ is shown alongside.

- **a** Find $f'(x)$.
- **b** Hence find the gradient of the tangent at:
  - **i** $x = 1$
  - **ii** $x = -2$
- **c** Copy the graph, and include the information from **b**.



PRINTABLE GRAPH

14 The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

Chapter 11

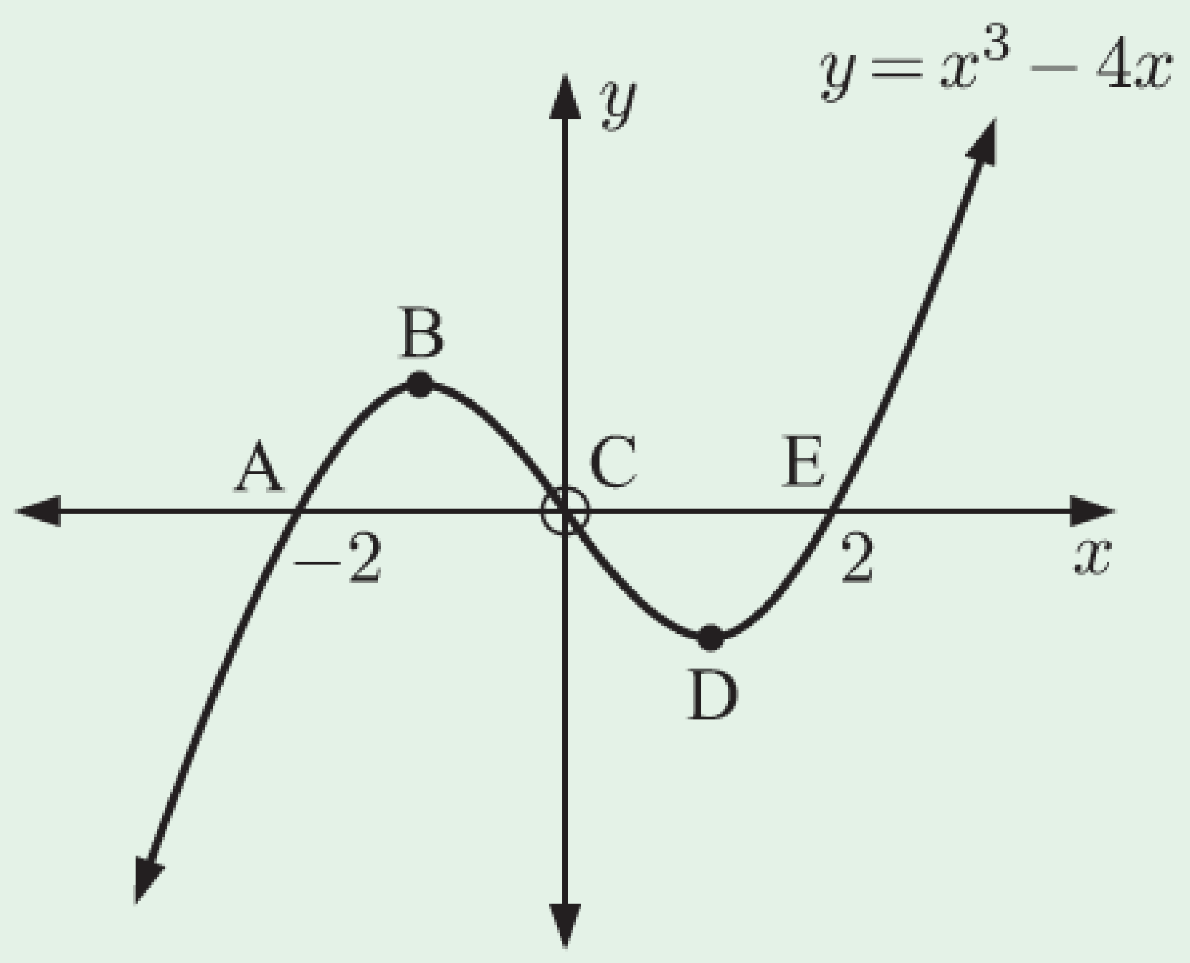
Properties of curves

Contents:

A Tangents
B Normals
C Increasing and decreasing
D Stationary points

OPENING PROBLEM

The curve $y = x^3 - 4x$ is shown alongside.



Things to think about:

a What features of this curve could you describe to someone?

b How can we use calculus to help identify important features?

In the previous Chapter we saw some rules for differentiating functions.

In this Chapter we will use derivatives to find:

- tangents and normals to curves
- intervals where a function is increasing or decreasing
- turning points, which are local minima and maxima.

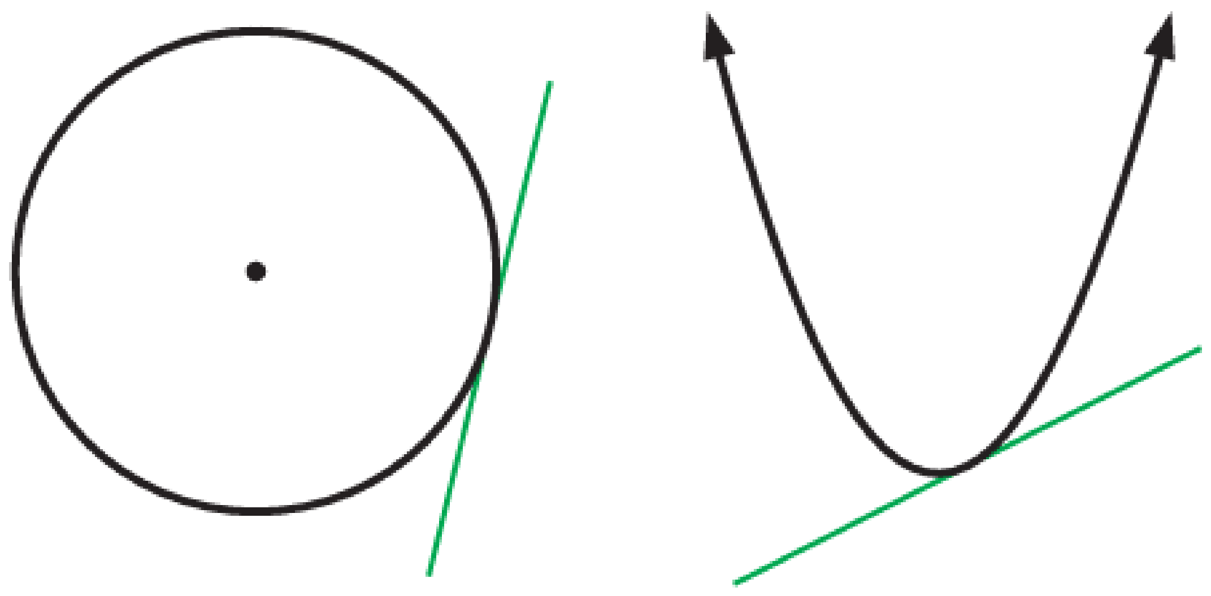
Minima is the plural of minimum.
Maxima is the plural of maximum.

A TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

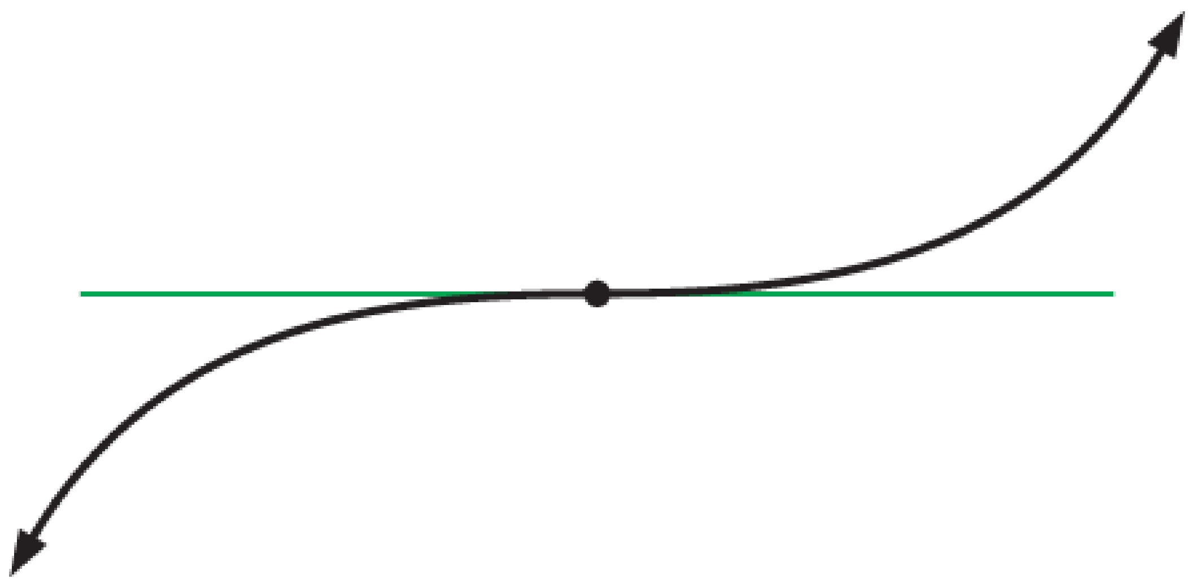
In cases we have seen already, the tangent *touches* the curve.

For example, consider the tangents to the circle and parabola shown.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency.



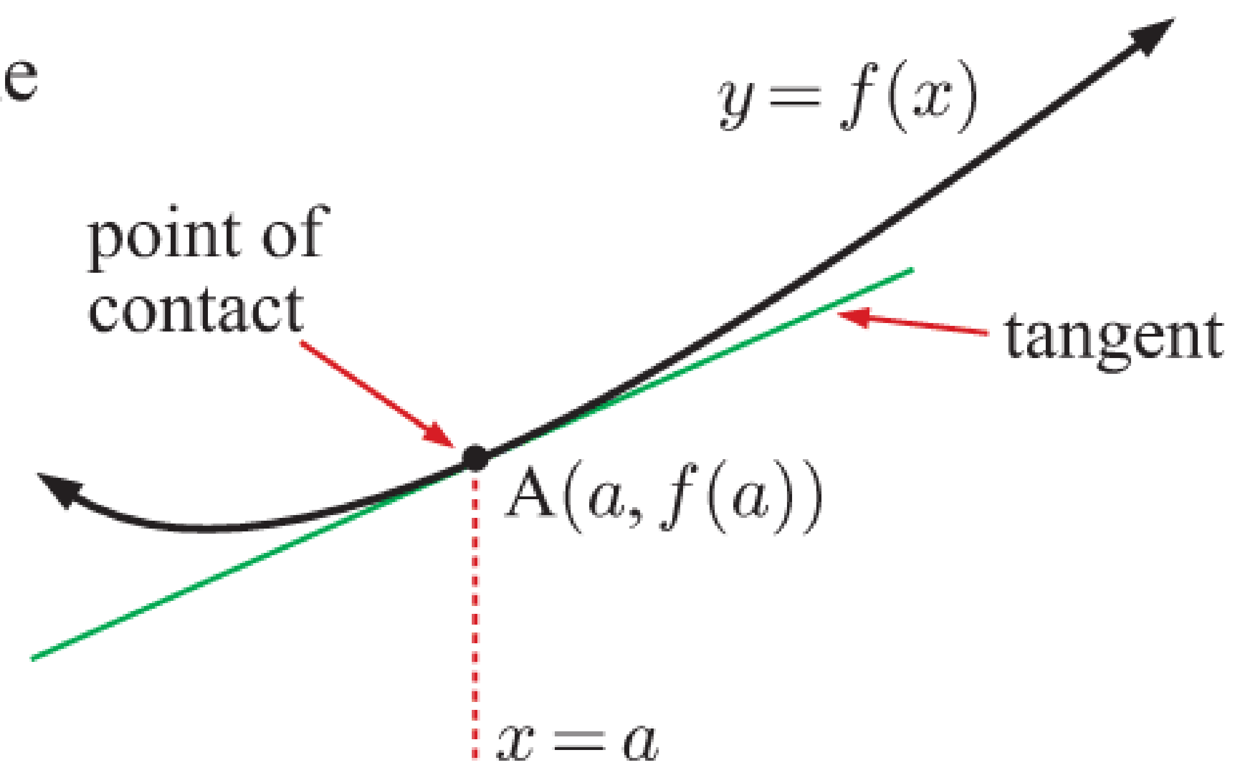
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a)$.

The equation of the tangent is

$$y - f(a) = f'(a)(x - a)$$

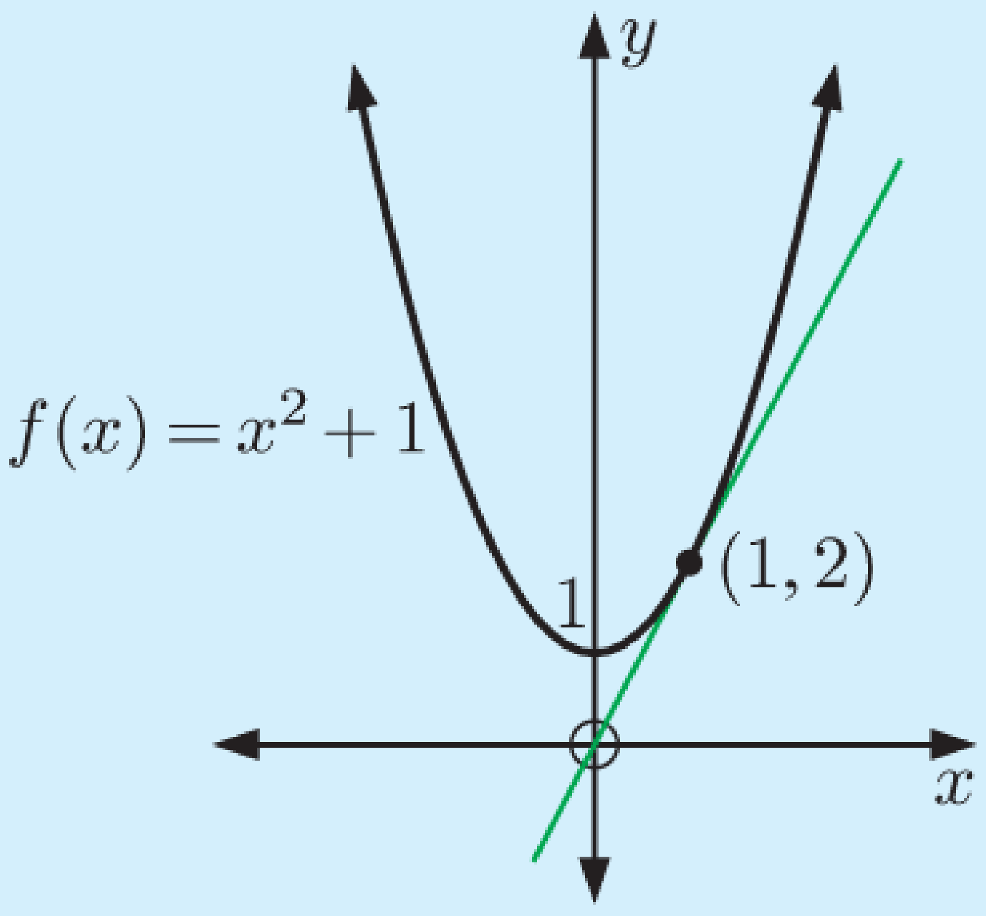
or $\boldsymbol{y = f'(a)(x - a) + f(a)}$.



Example 1

Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1^2 + 1 = 2$, the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so at $x = 1$ the tangent has gradient $f'(1) = 2$.

$\therefore$ the tangent has equation $y = 2(x - 1) + 2$

which is $y = 2x$.

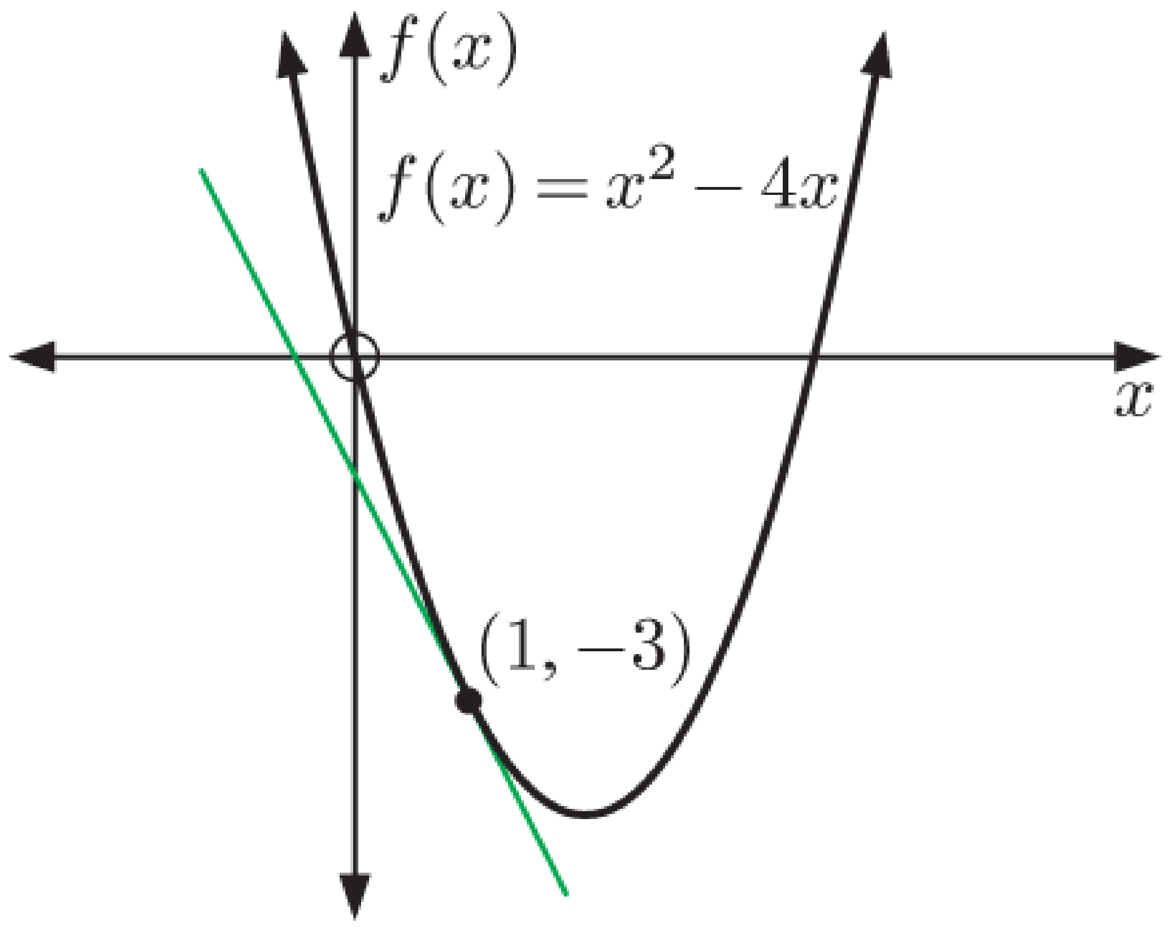
GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 11A

1 The graph of $f(x) = x^2 - 4x$ is shown alongside.

a Find $f'(x)$.

b Hence find the equation of the illustrated tangent.



2 Find the equation of the tangent to:

a $y = x^2$ at $x = 4$

b $y = x^3$ at $x = -2$

c $y = 3x^{-1}$ at $x = -1$

d $y = \dfrac{4}{x^3}$ at $x = 2$

e $y = x^2 + 5x - 4$ at $x = 1$

f $y = 2x^2 + 5x + 3$ at $x = -2$

g $y = x^3 + 2x$ at $x = 0$

h $y = \dfrac{x^2 + 4}{x}$ at $x = -1$

i $y = x - 2x^2 + 3$ at $x = 2$

j $y = x^3 - 5x$ at $x = 1$

k $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

l $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

GRAPHING PACKAGE

Check your answers using technology.

3 Consider the quadratic function $f(x) = -x^2 + 6x + 4$.

a Find $f'(x)$.

b Hence find the value of x for which $f'(x) = 0$.

c Explain the geometric significance of the point where $f'(x) = 0$.

4 The tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

a Find k.

b Hence find the equation of this tangent.

5 Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

6 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

7 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

- **a** Find $f'(x)$.
- **b** Find the values of x at which the tangent to the curve is horizontal.
- **c** Show that the tangents at these points are the same line.

8 Consider the curve $y = x^2 + 1$.

- **a** Find $f'(x)$.
- **b** Find the equation of the tangent to the curve when $x = 1$.
- **c** Show that this tangent passes through the origin.

Example 2 **Self Tutor**

Consider the curve $y = x^3 - 4x^2 - 6x + 8$.

- **a** Find the equation of the tangent to this curve at the point where $x = 0$.
- **b** At what point does this tangent meet the curve again?

a When $x = 0$, $y = 8$. So, the point of contact is $(0, 8)$.

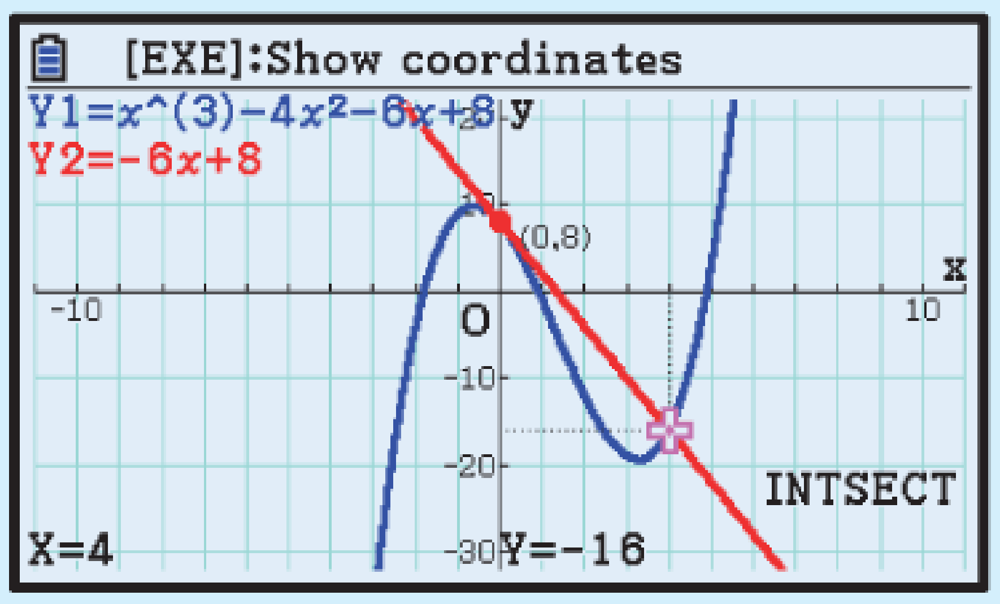
$\dfrac{dy}{dx} = 3x^2 - 8x - 6$, so when $x = 0$, $\dfrac{dy}{dx} = -6$.

$\therefore$ the tangent has equation $y = -6(x - 0) + 8$

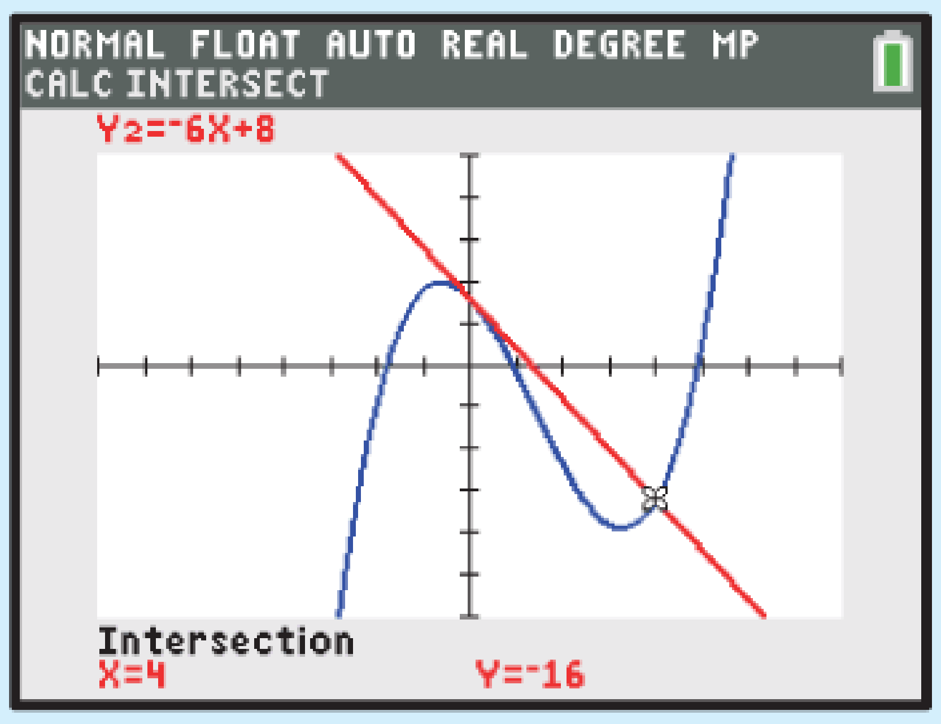
$\therefore$ $y = -6x + 8$

b We use technology to find where the tangent meets the curve again:

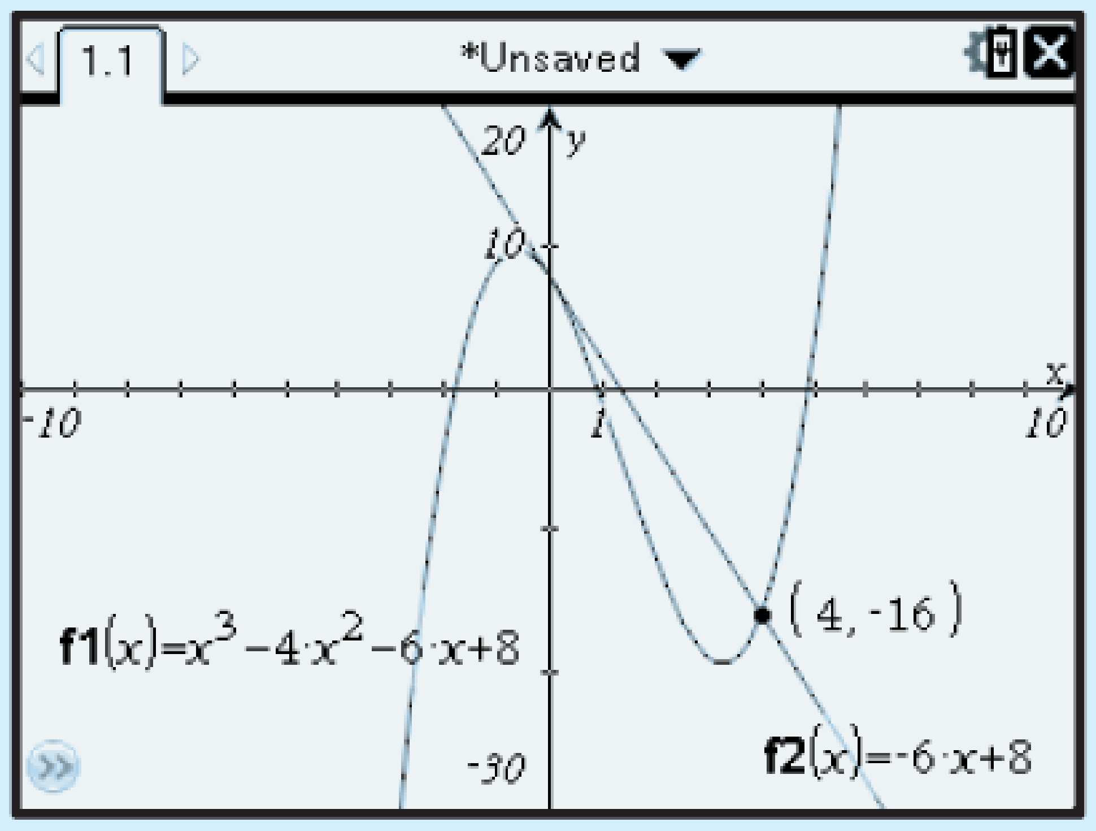
Casio fx-CG50


[EXE]:Show coordinates
Y1=x^(3)−4x²−6x+8
Y2=−6x+8
y
x
(0,8)
O
−10
10
−10
−20
−30
INTSECT
X=4
Y=−16

TI-84 Plus CE


NORMAL FLOAT AUTO REAL DEGREE MP
CALC INTERSECT
Y2=-6X+8
Intersection
X=4
Y=-16

TI-*n*spire


1.1
*Unsaved
20
y
10
-10
1
10
x
f1(x)=x³−4·x²−6·x+8
(4, -16)
-30
f2(x)=-6·x+8

The tangent meets the curve again at $(4, -16)$.

9 For each of the following curves:

- **i** Find the equation of the tangent at the given point.
- **ii** Find the point at which this tangent meets the curve again.

a $f(x) = 2x^3 - 5x + 1$ at $x = -1$

b $y = x^2 + \dfrac{3}{x} + 2$ at $x = 3$

c $f(x) = x^3 + 5$ at $x = 1.5$

d $y = x^3 + \dfrac{1}{x}$ at $x = -1$

e $f(x) = 3x^3 + 2x^2 - x + 2$ at $x = 0.5$

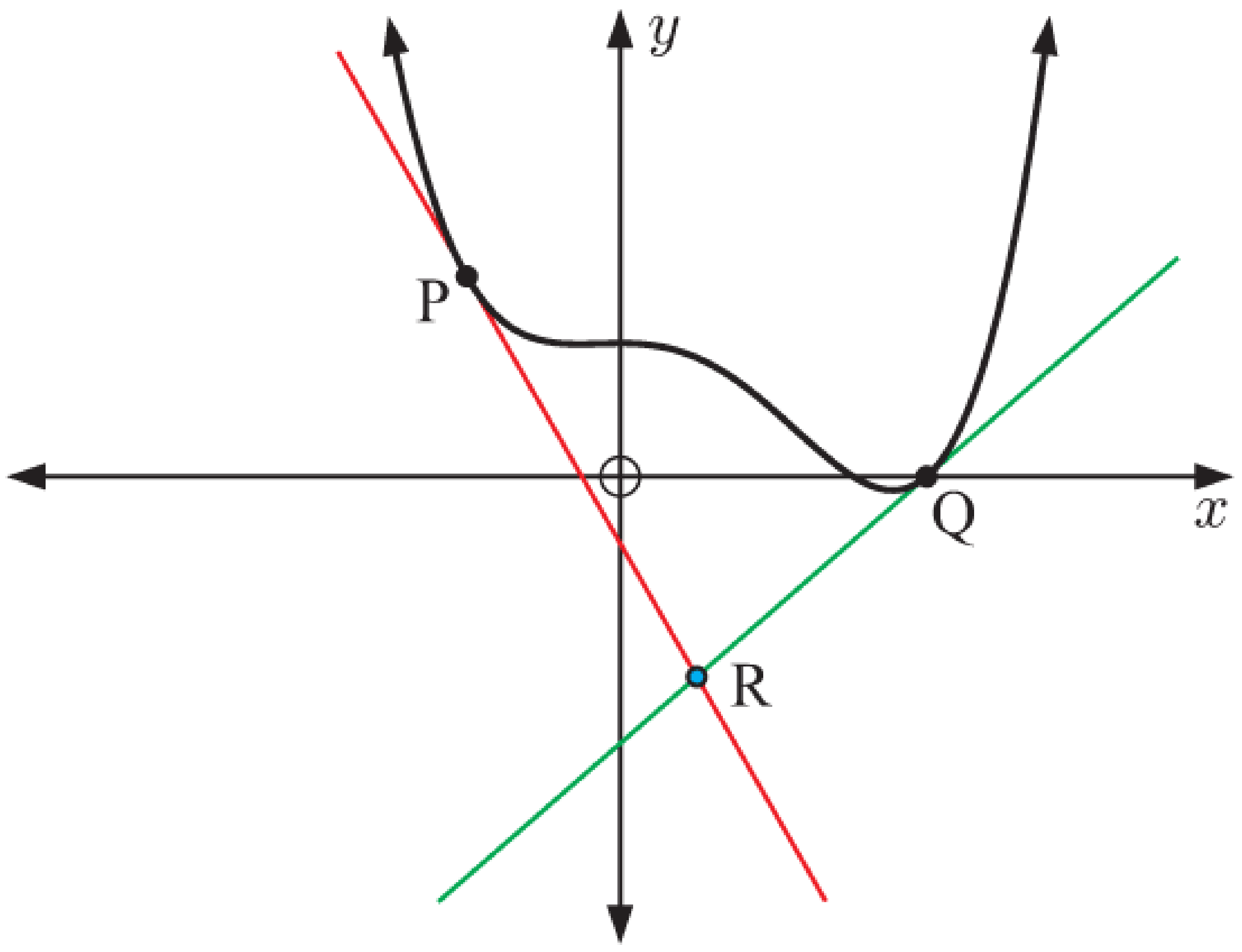
10 Find where the tangent to the curve:

- **a** $y = x^3$ at the point where $x = 2$, meets the curve again
- **b** $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again
- **c** $y = \frac{1}{x} - \frac{1}{x^2}$ at the point where $x = 1$, meets the curve again.

11 Find the point where the tangent to:

- **a** $y = 2x^3 + 3x^2 - x + 4$ at $x = -1$ meets the x-axis
- **b** $y = x^3 + 5$ at $(-2, -3)$ meets the line $y = 2$
- **c** $y = \frac{2}{x} + 1$ at $(-2, 0)$ meets the line $y = 2x - 3$
- **d** $y = 3x^3 - 2x + 1$ at $x = 1$ meets the y-axis.

12 The graph of $f(x) = x^4 - 2x^3 - x^2 + 4$ is shown alongside. The tangents at P$(-1, 6)$ and Q$(2, 0)$ intersect at R.
Find the coordinates of R.



13 Consider the function $f(x) = x^4 - 2x^2 + 2x + 3$.

- **a** Find the equation of the line which is the tangent to the curve at the point where $x = 1$.
- **b** Find the point at which this line meets the curve again.
- **c** Show that the line is the tangent to the curve at this point also.

B NORMALS

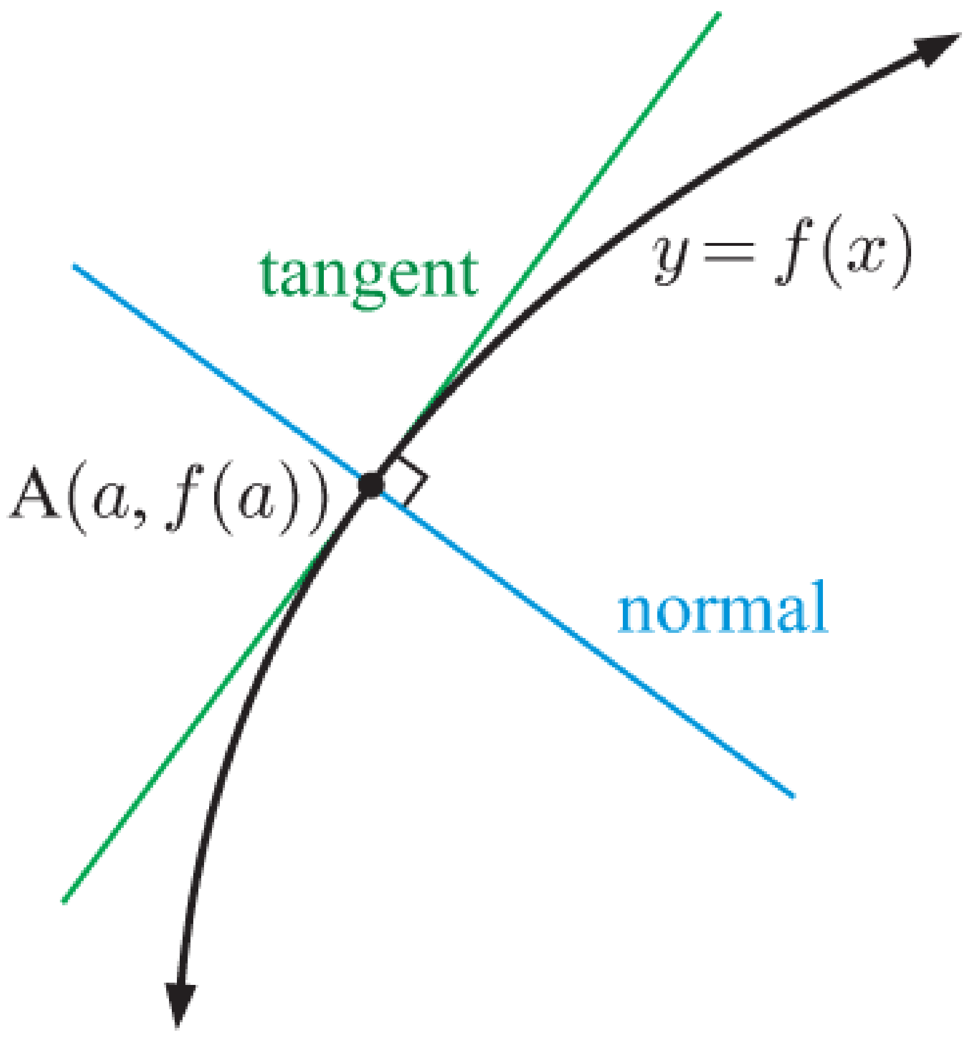
A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

The gradients of perpendicular lines are negative reciprocals of each other, so:

The gradient of the normal to the curve at $x = a$ is $-\frac{1}{f'(a)}$.

The equation of the normal to the curve at $x = a$ is

$$\boldsymbol{y = -\frac{1}{f'(a)}(x - a) + f(a).}$$



Example 3 **Self Tutor**

Find the equation of the normal to $f(x)=x^2-4x+3$ at the point where $x=4$.

Since $f(4)=(4)^2-4(4)+3=3$,
the point of contact is $(4, 3)$.

Now $f(x)=x^2-4x+3$

$\therefore\ f'(x)=2x-4$

$\therefore\ f'(4)=2(4)-4$

$=4$

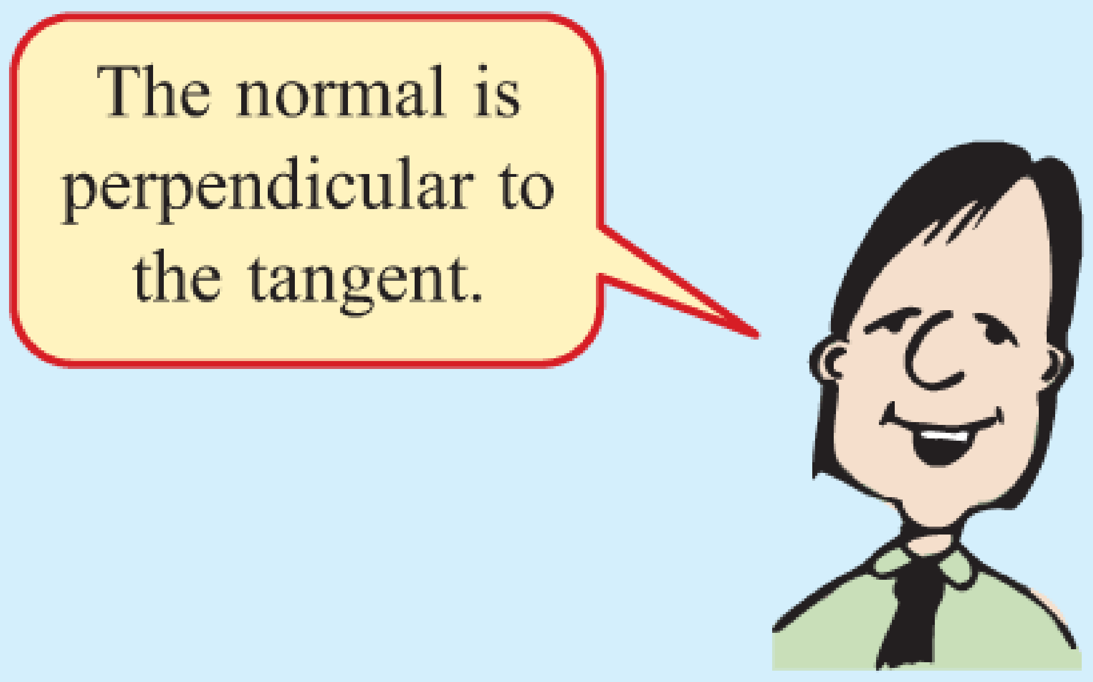
So, the normal at $(4, 3)$ has gradient $-\frac{1}{4}$.

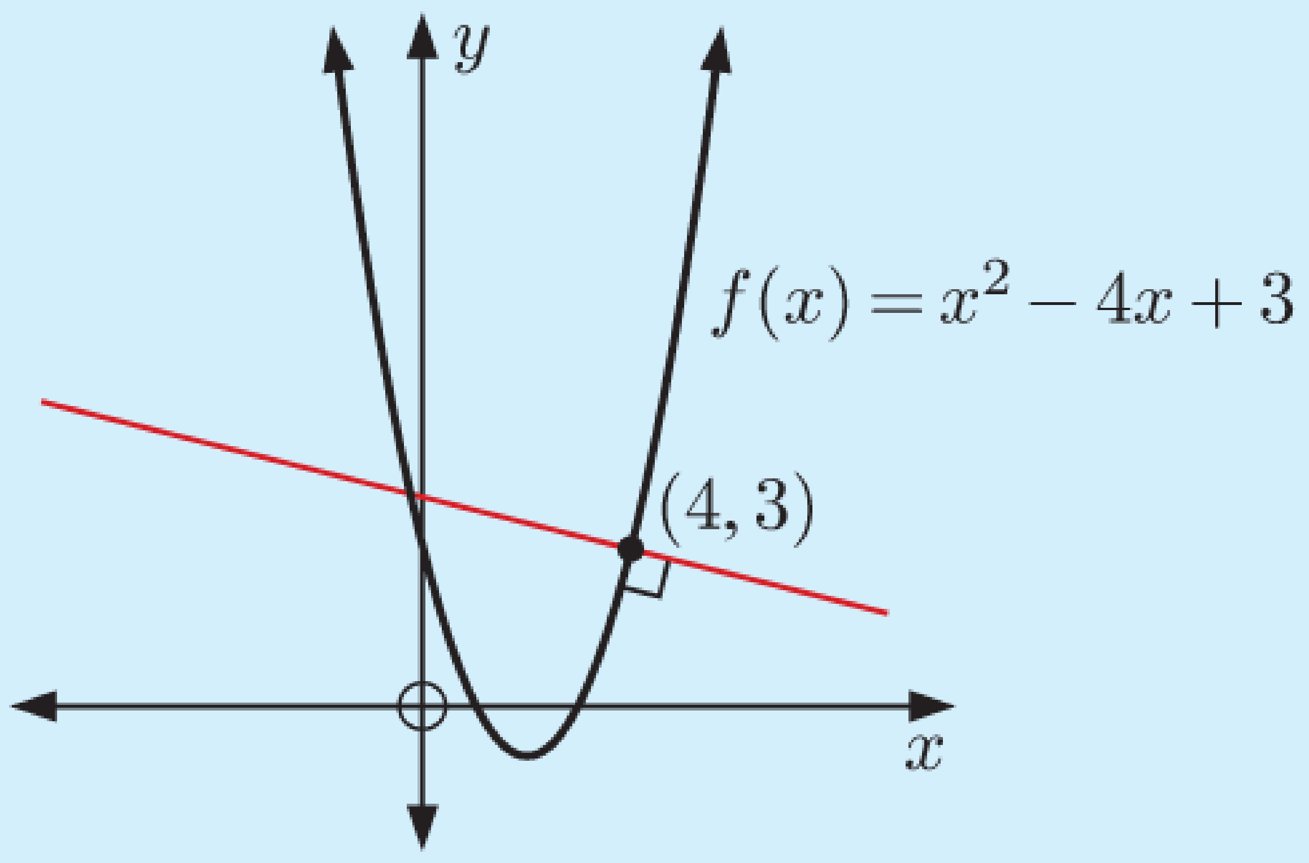
$\therefore$ the normal has equation

$$y=-\tfrac{1}{4}(x-4)+3$$

which is $y=-\frac{1}{4}x+1+3$

or $y=-\frac{1}{4}x+4$


The normal is perpendicular to the tangent.



EXERCISE 11B

1 Find the equation of the normal to:

a $y=x^2$ at the point $(3, 9)$

b $y=x^3-5x+2$ at $x=-2$

c $y=\dfrac{1}{x}+2$ at the point $(-1, 1)$

d $y=2x^3-3x+1$ at $x=1$

e $y=x^2-3x+2$ at $x=3$

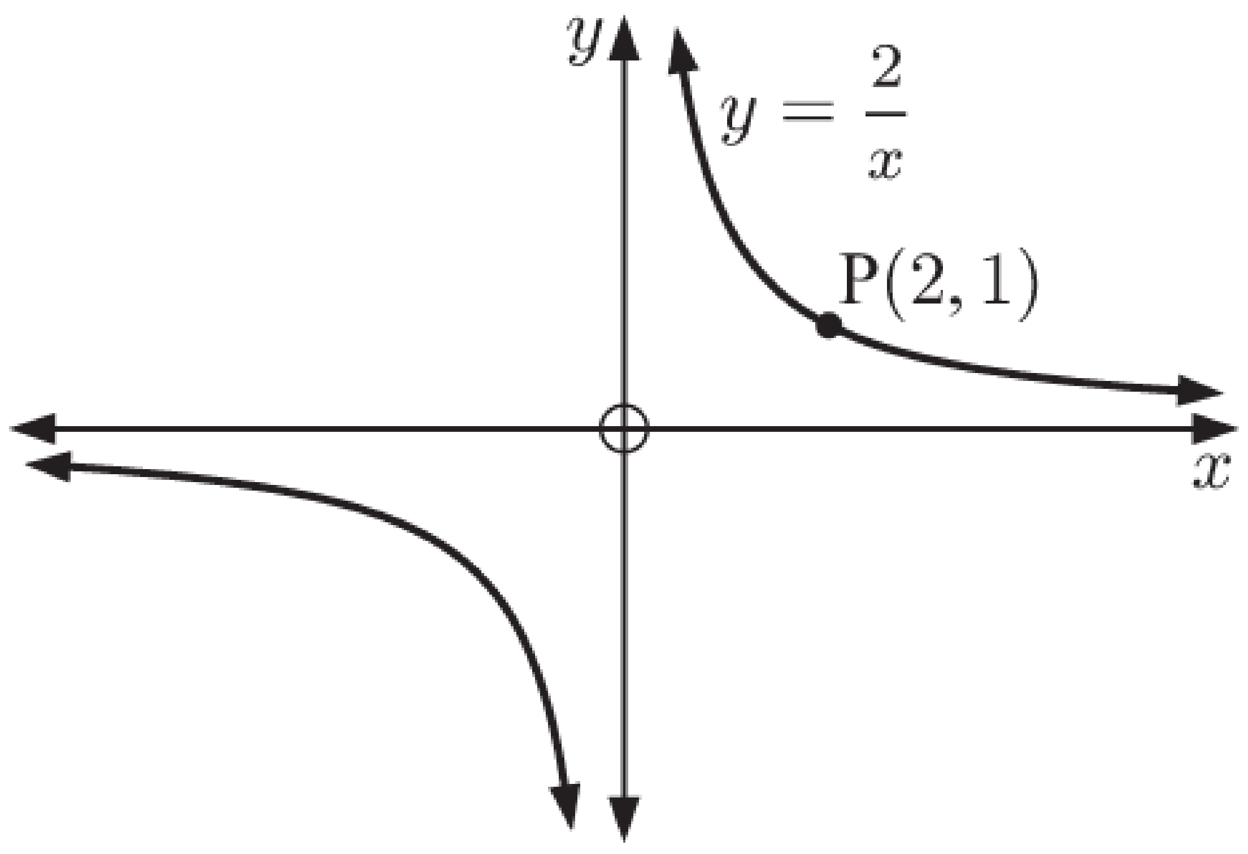
f $y=3x+\dfrac{1}{x}-4$ at $x=1$.

2 Consider the graph of $y=\dfrac{2}{x}$ alongside.

a Find the equation of:

i the tangent at P **ii** the normal at P.

b Sketch the graph of $y=\dfrac{2}{x}$, including the tangent and normal at P.



3 Suppose $f(x)=x^2-\dfrac{8}{x}$. Find the equation of:

a the tangent to $y=f(x)$ at $x=-2$

b the normal to $y=f(x)$ at $x=3$.

4 Let $f(x)=(x-1)(x-4)$.

a Find the axes intercepts of $y=f(x)$.

b Find $f'(x)$.

c Find the equation of the normal to $y=f(x)$ at $x=-1$.

d Sketch $y=f(x)$ and its normal at $x=-1$ on the same set of axes.

Example 4 **Self Tutor**

Find the point where the normal to $y = x^2 - 3$ at $(1, -2)$ meets the curve again.

$\dfrac{dy}{dx} = 2x$ So, when $x = 1$, $\dfrac{dy}{dx} = 2$

So, the normal at $(1, -2)$ has gradient $-\frac{1}{2}$.

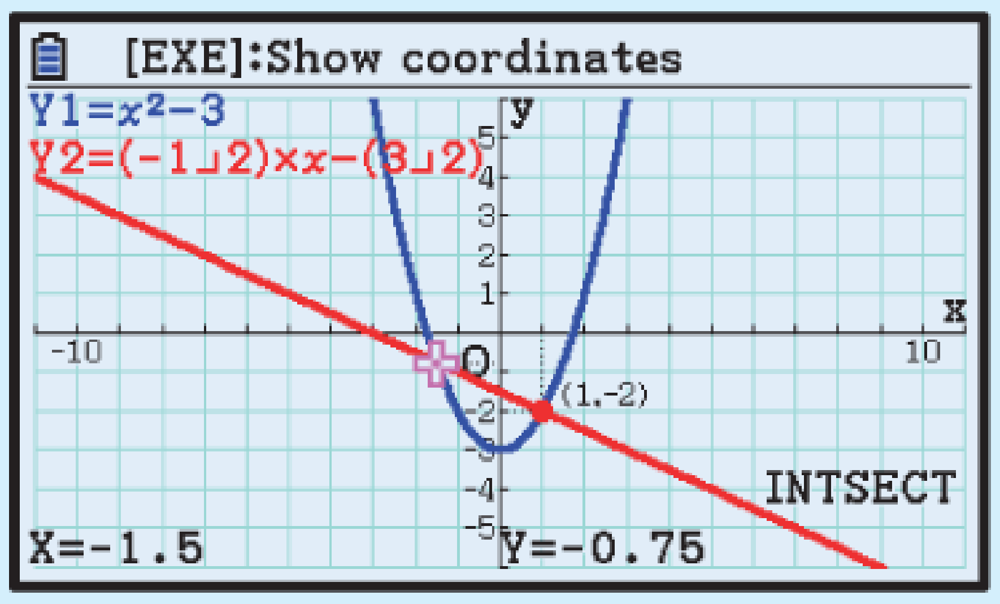
$\therefore$ the normal has equation $y = -\frac{1}{2}(x-1) + (-2)$

which is $y = -\frac{1}{2}x + \frac{1}{2} - 2$

or $y = -\frac{1}{2}x - \frac{3}{2}$

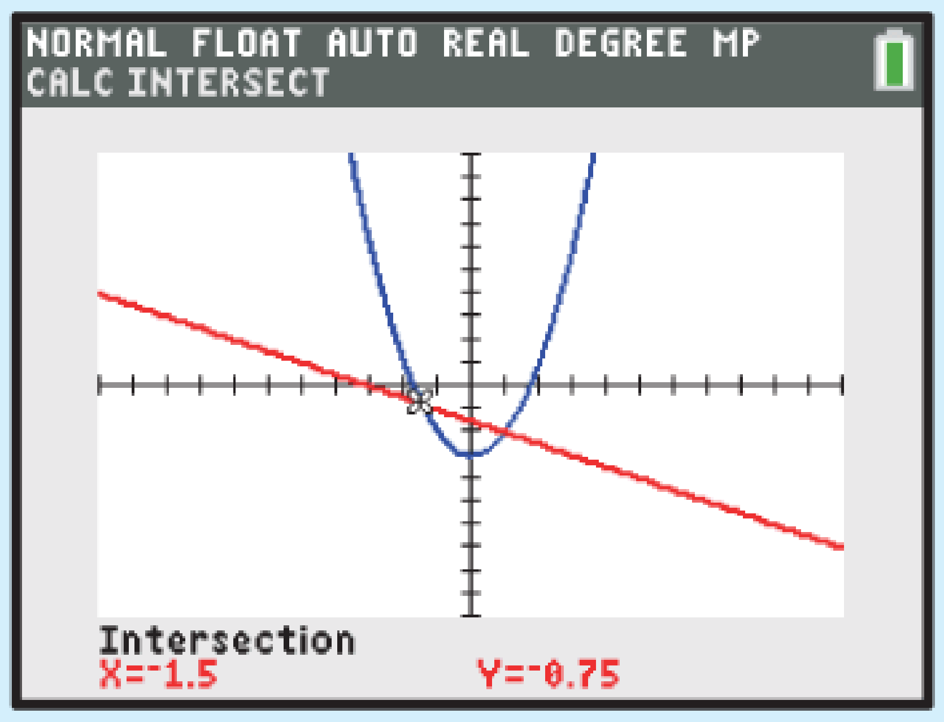
We use technology to find where the normal meets the curve again:

Casio fx-CG50



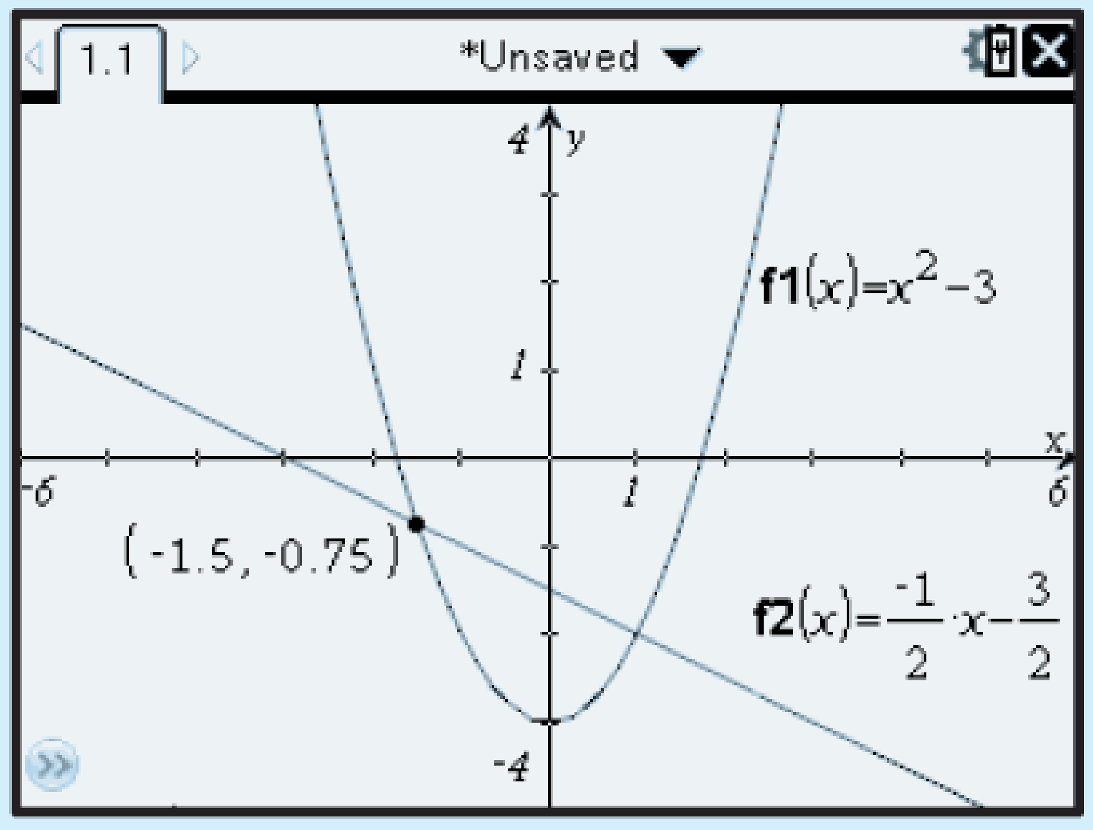
[EXE]:Show coordinates
Y1=x²−3
Y2=(−1⌟2)×x−(3⌟2)
INTSECT
X=-1.5 Y=-0.75

TI-84 Plus CE



NORMAL FLOAT AUTO REAL DEGREE MP
CALC INTERSECT
Intersection
X=-1.5 Y=-0.75

TI-$n$spire



$f1(x)=x^2-3$
$f2(x)=\frac{-1}{2}\cdot x-\frac{3}{2}$
(-1.5, -0.75)

The normal meets the curve again at $(-1.5, -0.75)$.

5 For each curve, find the coordinates of the point where the normal to the curve at the given point, meets the curve again.

a $y = x^2$ at $(2, 4)$

b $y = \dfrac{1}{x} + 2$ at $(-1, 1)$

c $y = x^3$ at $x = -1$

d $y = x^3 - 12x + 2$ at $(3, -7)$.

6 Find where the normal to:

a $y = x^3 - 12x + 2$ at $x = -2$ meets the x-axis

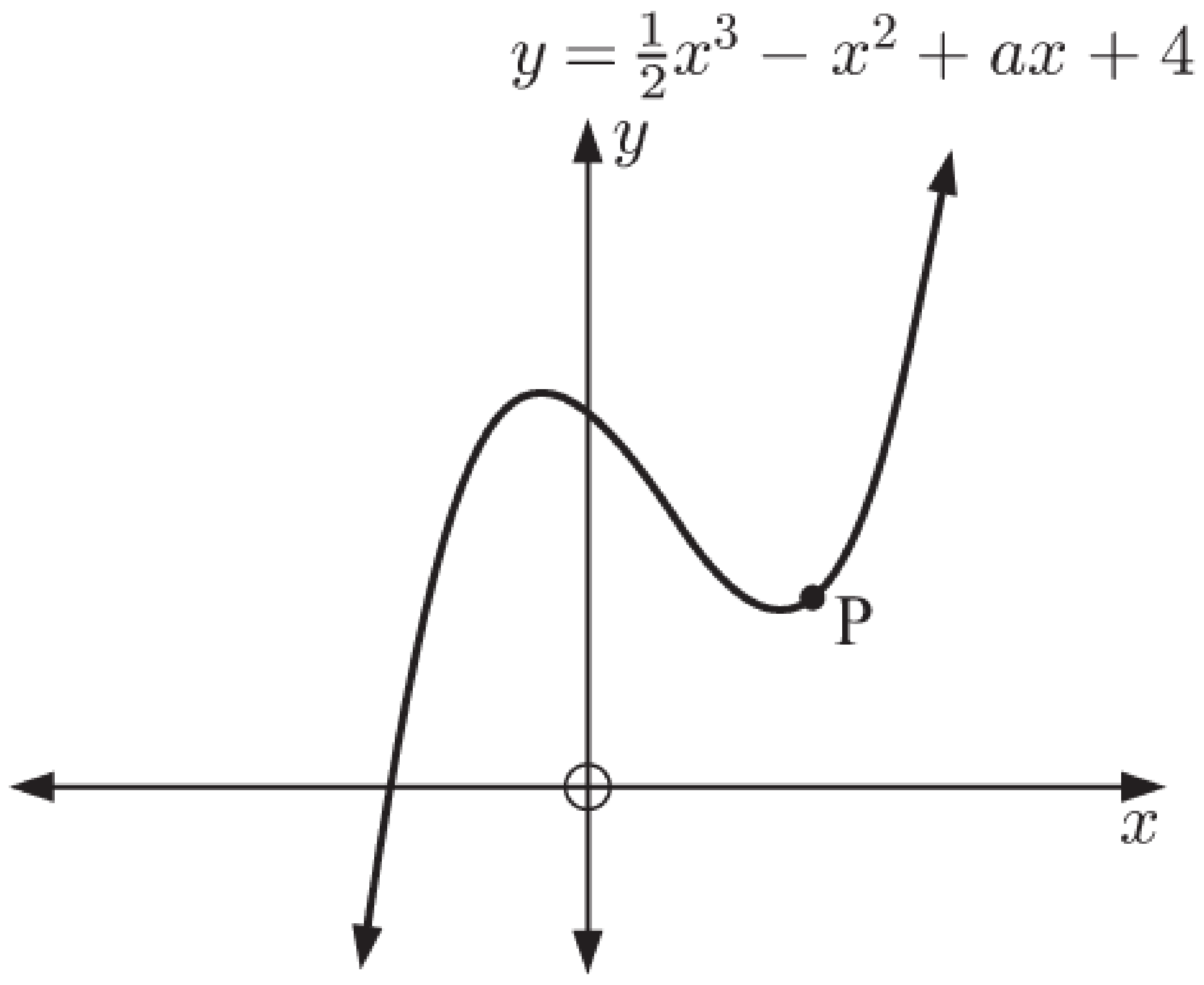
b $y = x^3$ at $(-1, -1)$ meets the line $y = 3$

c $y = \dfrac{1}{x} - 3$ at $(-1, -4)$ meets the line $y = -2x + 1$

d $y = 2x^3 - 3x + 1$ at $x = 1$ meets the y-axis.

7 The normal to $y = 5 - \dfrac{a}{x}$ at the point where $x = -2$, has gradient 1. Find a.

8 In the graph alongside, P is the point with x-coordinate 2.



a The tangent at P has gradient 1. Find:

i a **ii** the coordinates of P.

b Find the equation of the normal at P.

c Find the coordinates of the point Q where the normal at P meets the curve again.

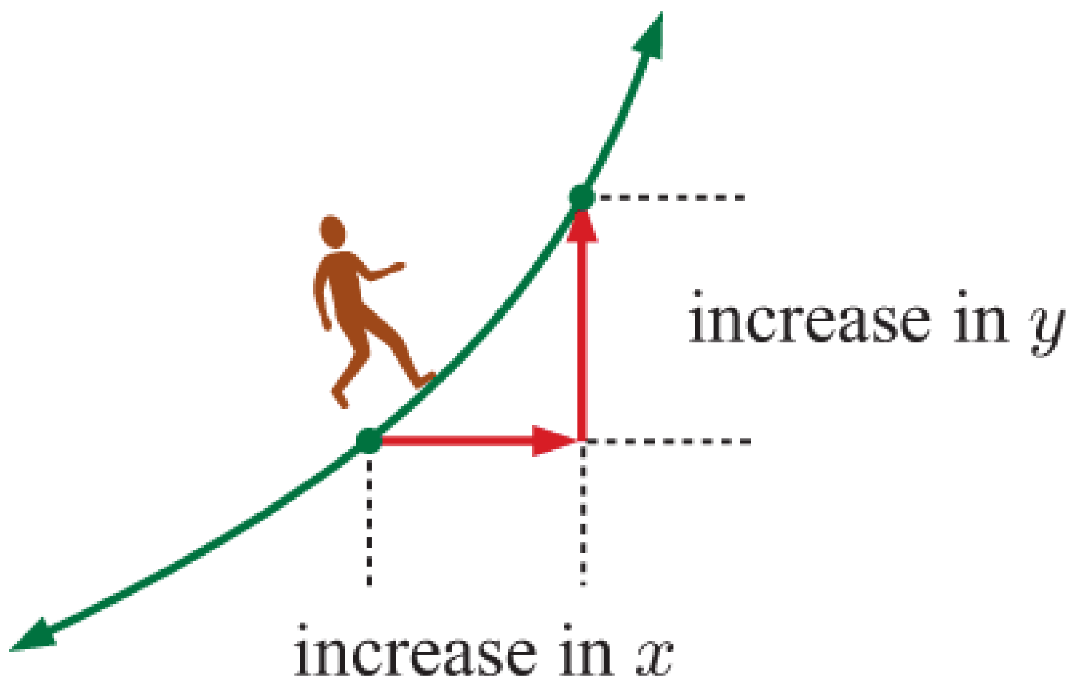
d Find the equation of the tangent at Q. Comment on your answer.

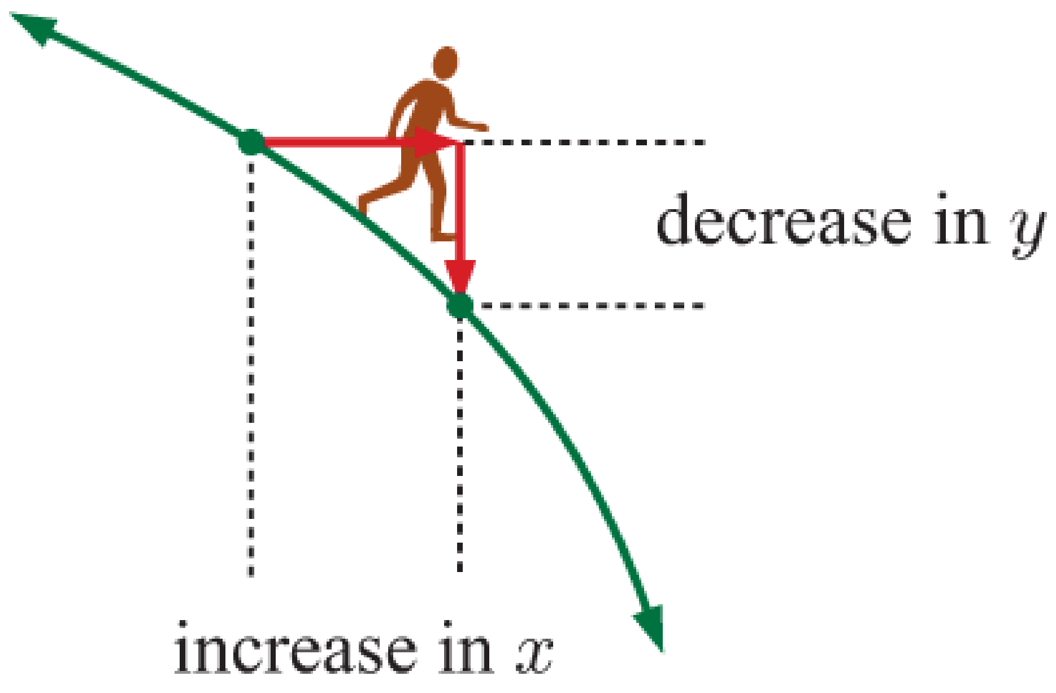
C INCREASING AND DECREASING

When we draw a graph of a function, we may notice that the function is **increasing** or **decreasing** over particular intervals.

On an interval where the function is **increasing**, an increase in x produces an **increase in y**.

On an interval where the function is **decreasing**, an increase in x produces a **decrease in y**.

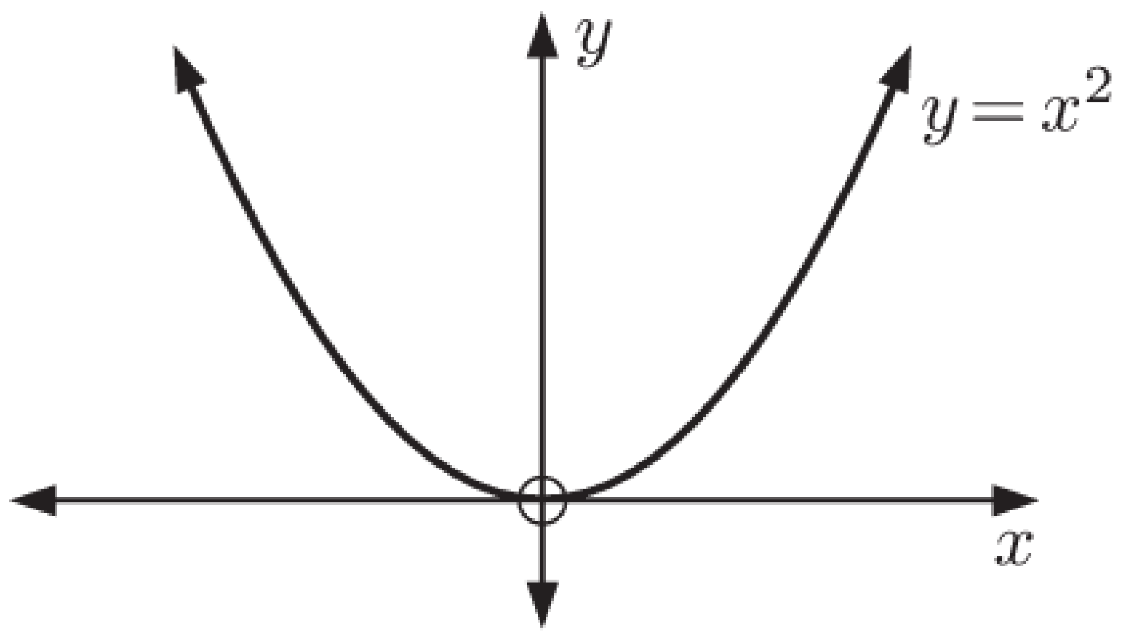




Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f(a) \leqslant f(b)$ for all $a, b \in S$ such that $a < b$.
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f(a) \geqslant f(b)$ for all $a, b \in S$ such that $a < b$.

For example:



$y = x^2$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.

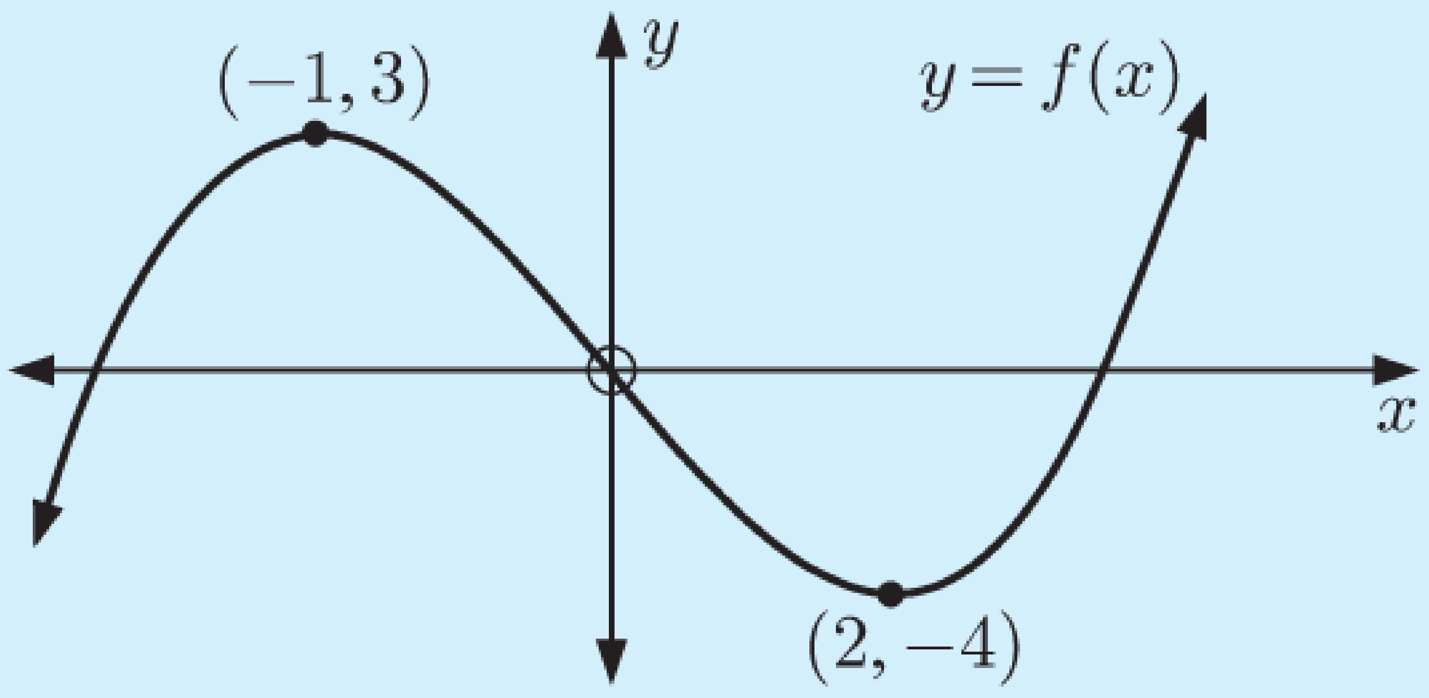
Important: In this example, people often get confused about the point $x = 0$. They wonder how the curve can be both increasing and decreasing at the same point. The answer is that the notions of increasing and decreasing are associated with *intervals*, not particular values for x. We see that $y = x^2$ is decreasing *on the interval* $x \leqslant 0$ and increasing *on the interval* $x \geqslant 0$.

Example 5 **Self Tutor**

Find intervals where $f(x)$ is:

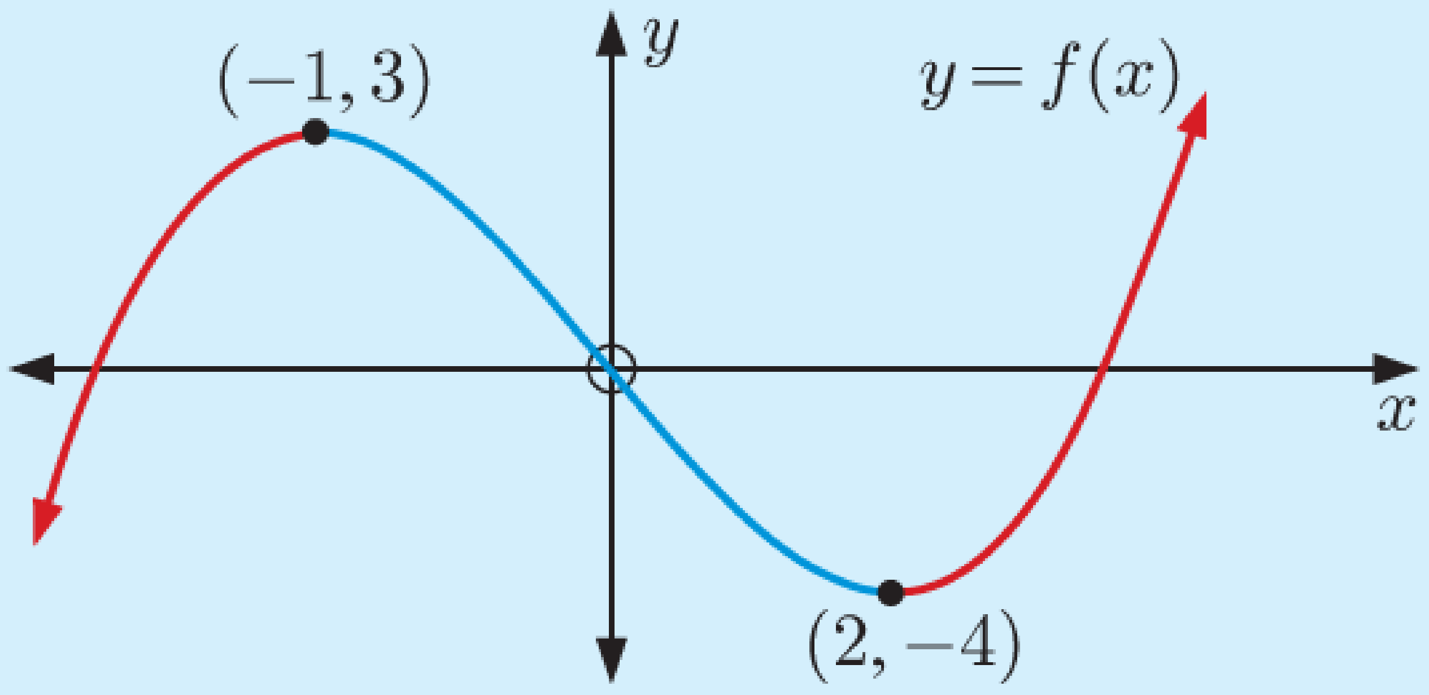
a increasing

b decreasing.



a $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 2$.

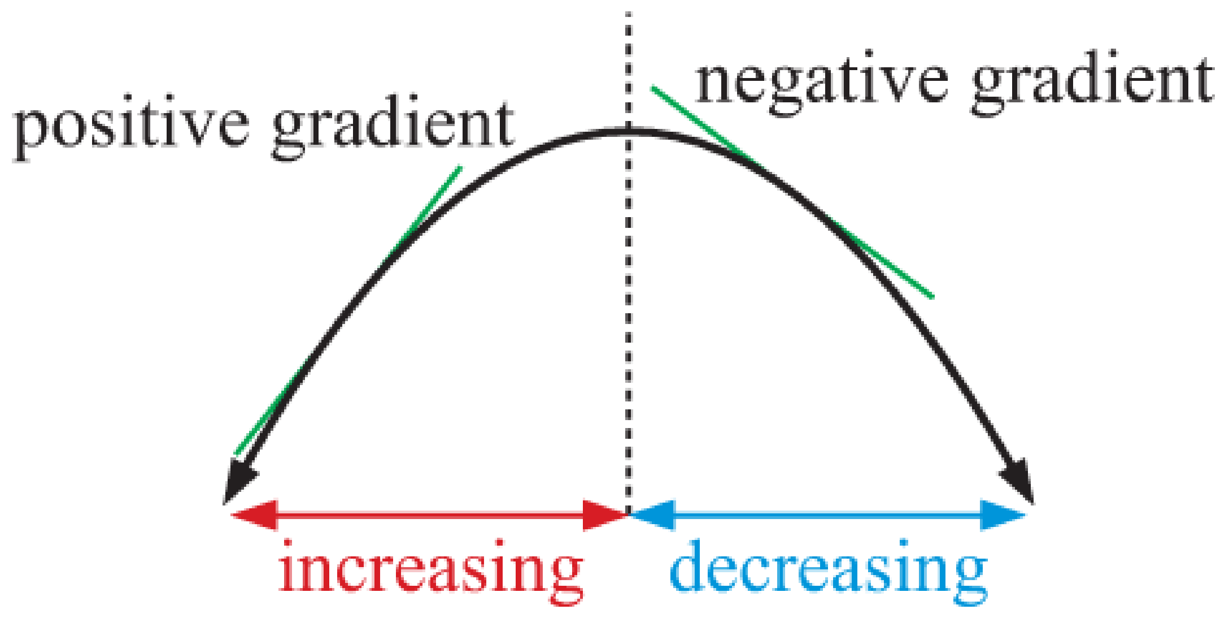
b $f(x)$ is decreasing for $-1 \leqslant x \leqslant 2$.



We can determine intervals where a curve $y = f(x)$ is increasing or decreasing by considering a **sign diagram** of the derivative function $f'(x)$.

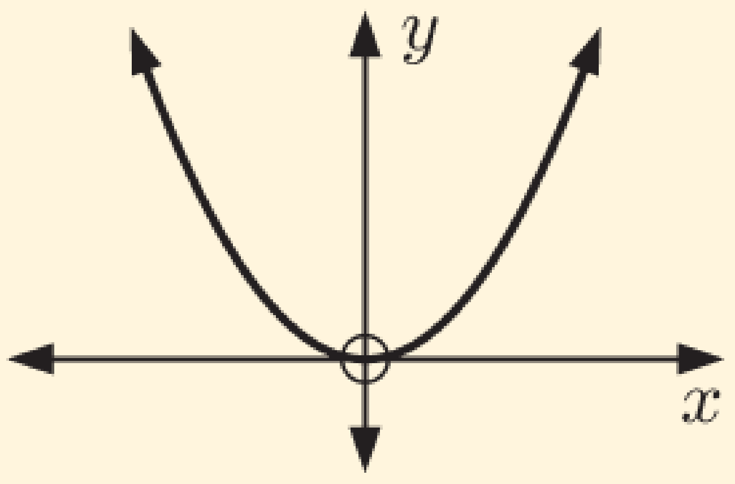
For most functions that we deal with in this course:

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f'(x) \leqslant 0$ for all x in S.

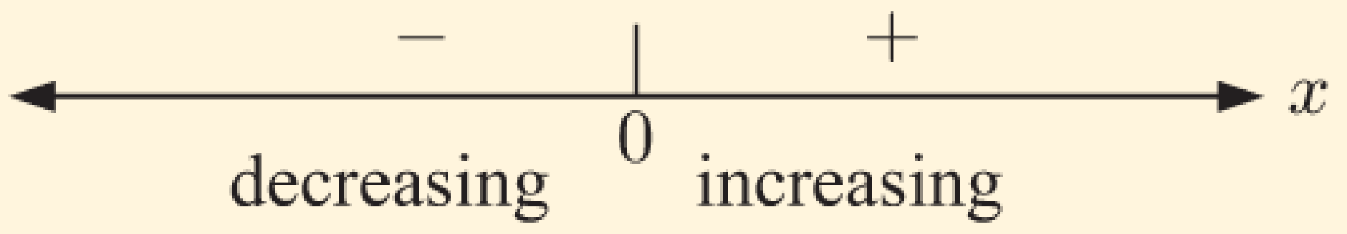

positive gradient
negative gradient
increasing
decreasing

Sign diagrams for the derivative are extremely useful for determining intervals where a function is increasing or decreasing. Consider the following examples:

$f(x) = x^2$

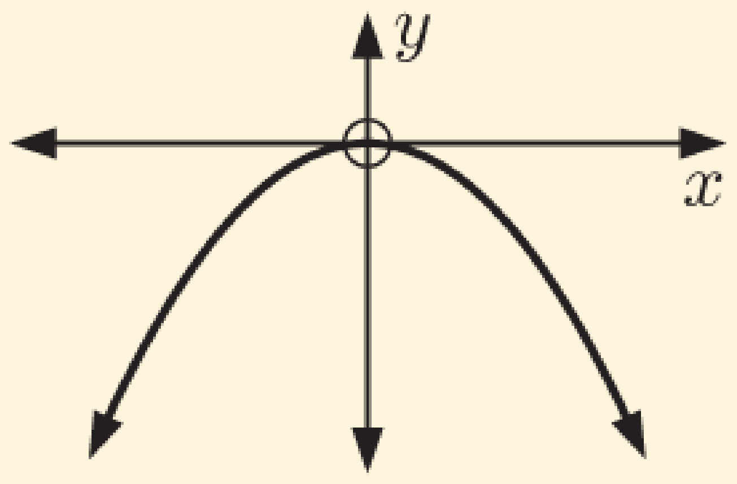


$f'(x) = 2x$ which has sign diagram

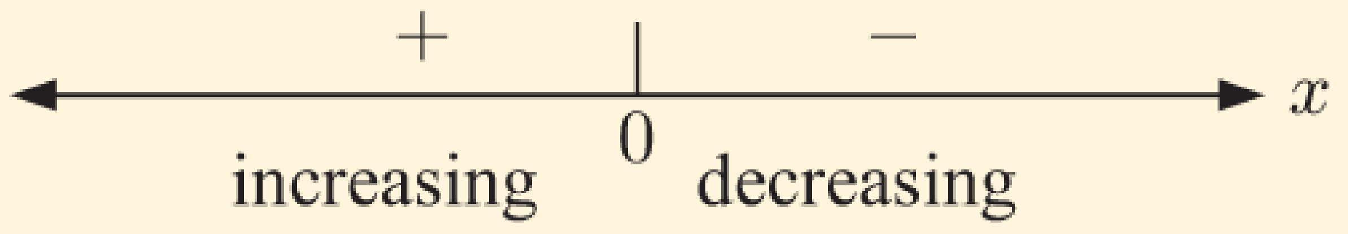

−
+
0
x
decreasing
increasing

$$\therefore \quad f(x) = x^2 \text{ is } \begin{cases} \text{decreasing for } x \leqslant 0 \\ \text{increasing for } x \geqslant 0. \end{cases}$$

$f(x) = -x^2$

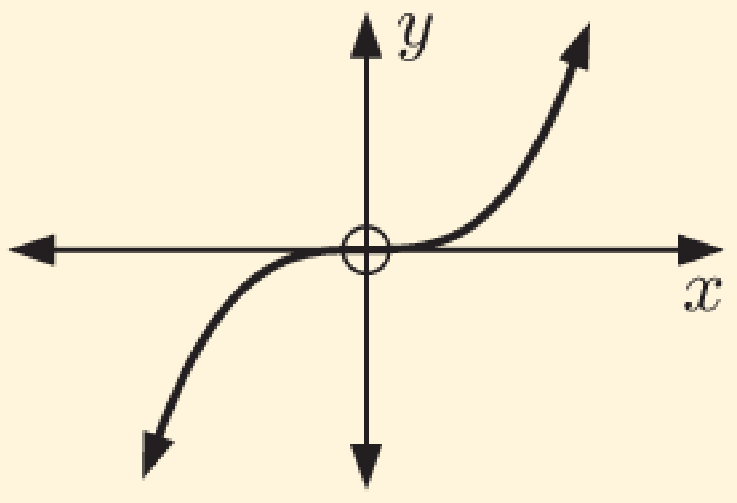


$f'(x) = -2x$ which has sign diagram

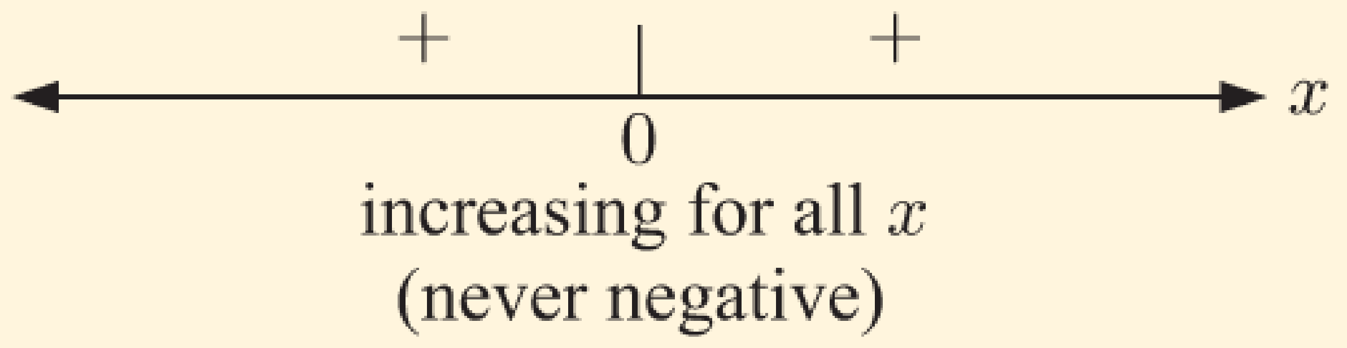

+
−
0
x
increasing
decreasing

$$\therefore \quad f(x) = -x^2 \text{ is } \begin{cases} \text{increasing for } x \leqslant 0 \\ \text{decreasing for } x \geqslant 0. \end{cases}$$

$f(x) = x^3$

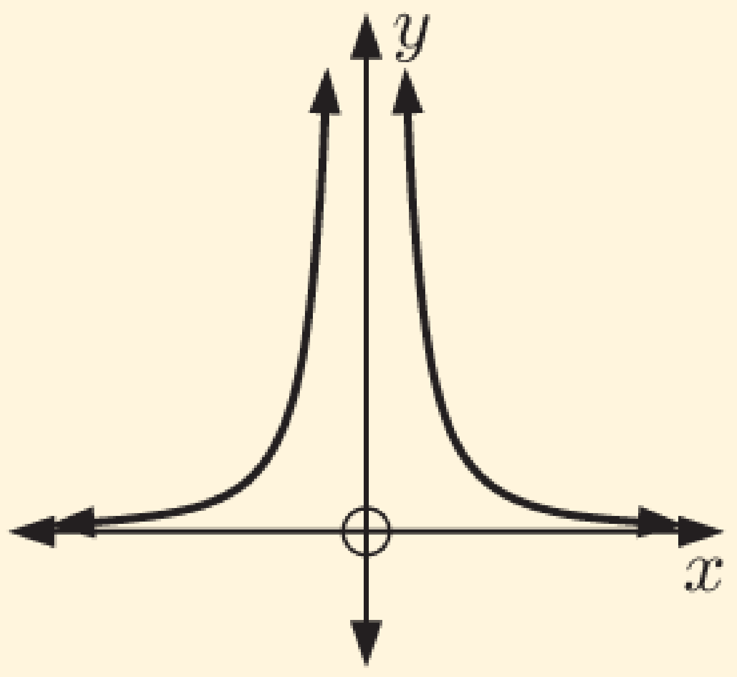


$f'(x) = 3x^2$ which has sign diagram


+
+
0
x
increasing for all x
(never negative)

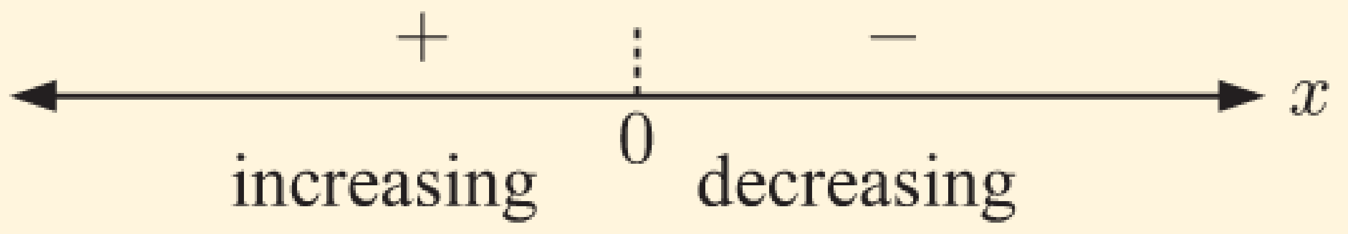
$\therefore \quad f(x) = x^3$ is increasing for all $x \in \mathbb{R}$.

$f(x) = \dfrac{1}{x^2}$



$$f'(x) = -2x^{-3}$$
$$= -\frac{2}{x^3}$$

which has sign diagram


+
−
0
x
increasing
decreasing

$$\therefore \quad f(x) = \frac{1}{x^2} \text{ is } \begin{cases} \text{increasing for } x < 0 \\ \text{decreasing for } x > 0. \end{cases}$$

Example 6 **Self Tutor**

Find the intervals where the following functions are increasing or decreasing:

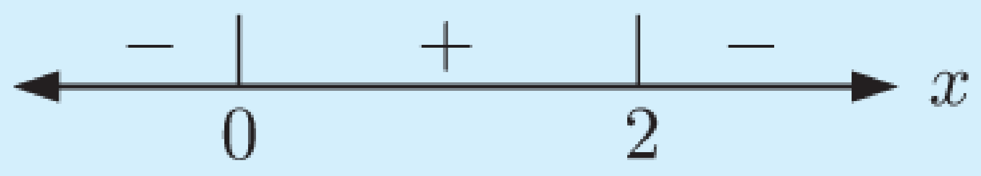
a $f(x) = -x^3 + 3x^2 + 5$ **b** $f(x) = 4x^3 - x^2 - 4x + 3$

a $f(x) = -x^3 + 3x^2 + 5$

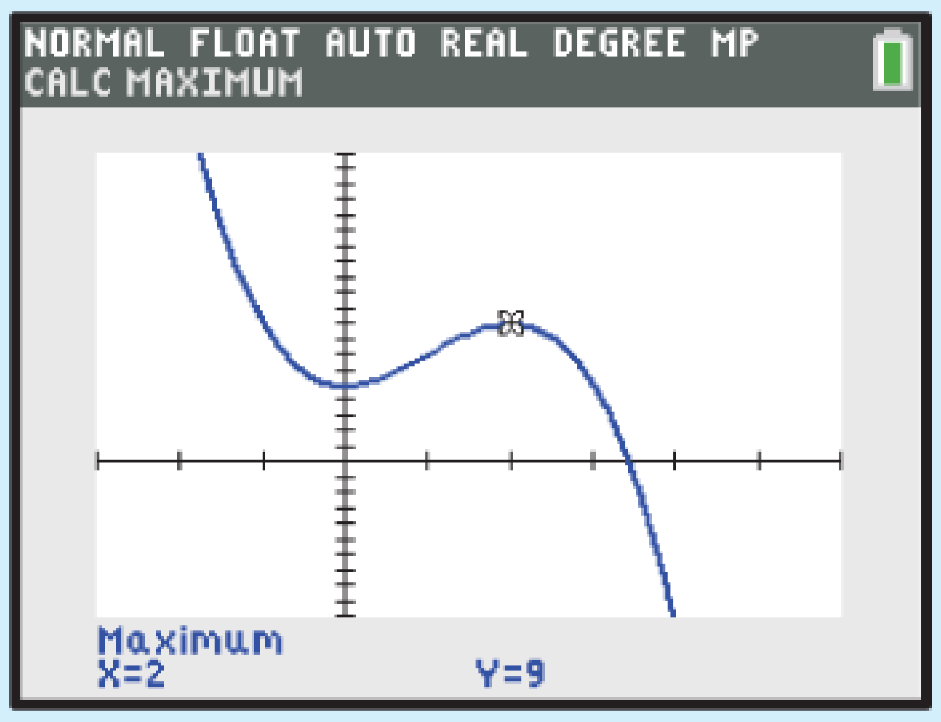
$\therefore\ f'(x) = -3x^2 + 6x$

$= -3x(x-2)$

which has sign diagram



TI-84 Plus CE



NORMAL FLOAT AUTO REAL DEGREE MP
CALC MAXIMUM

```
Maximum
X=2                Y=9
```

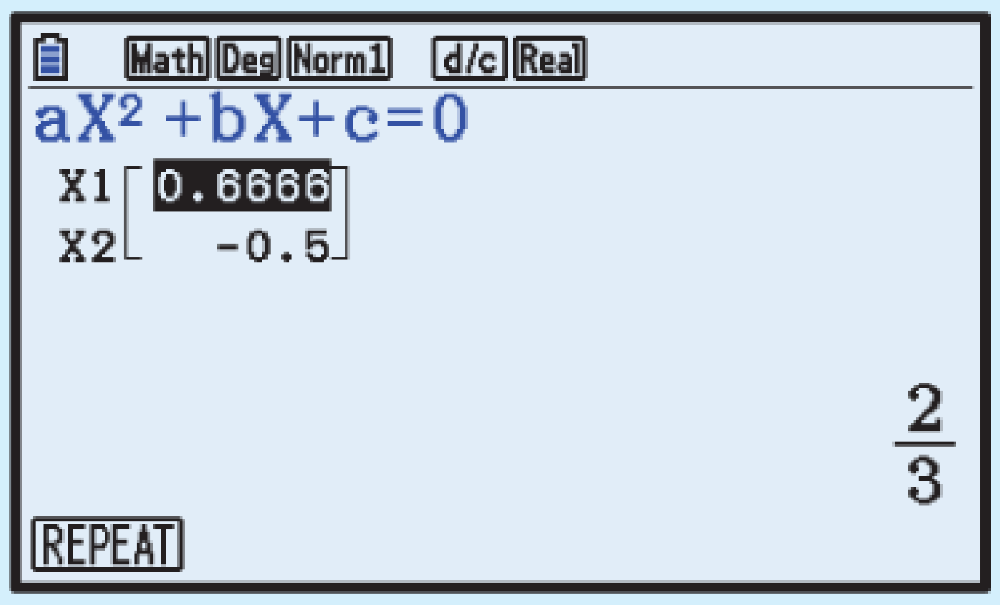
So, $f(x)$ is decreasing for $x \leqslant 0$ and for $x \geqslant 2$, and increasing for $0 \leqslant x \leqslant 2$.

b $f(x) = 4x^3 - x^2 - 4x + 3$

$\therefore\ f'(x) = 12x^2 - 2x - 4$

We find the zeros of $f'(x)$ using technology:

Casio fx-CG50



Math Deg Norm1 d/c Real
REPEAT

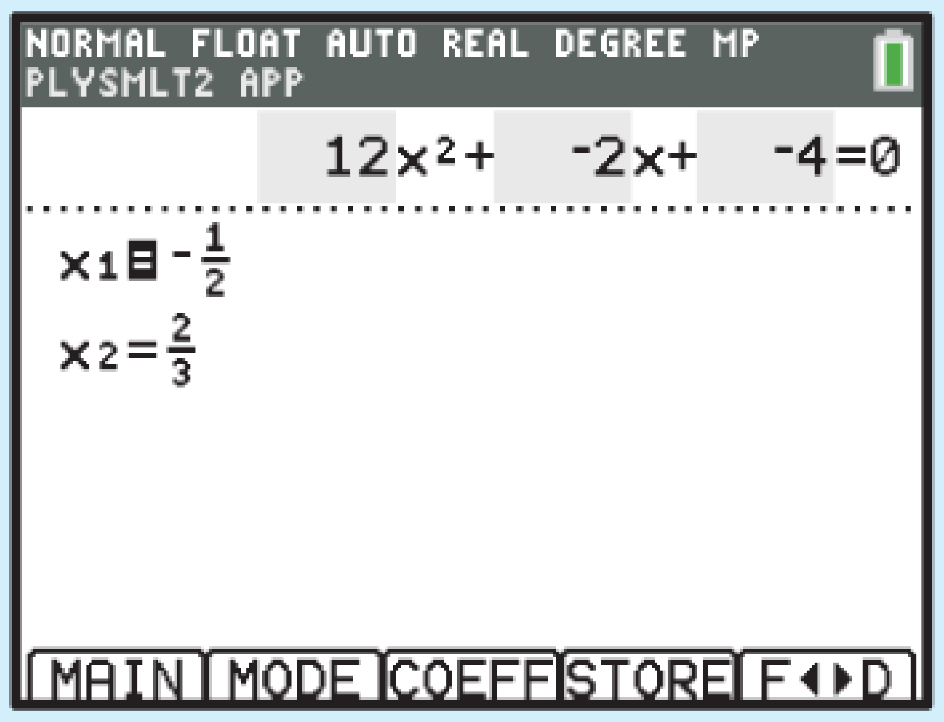
$aX^2+bX+c=0$

$X1\ 0.6666$

$X2\ -0.5$

$\frac{2}{3}$

TI-84 Plus CE



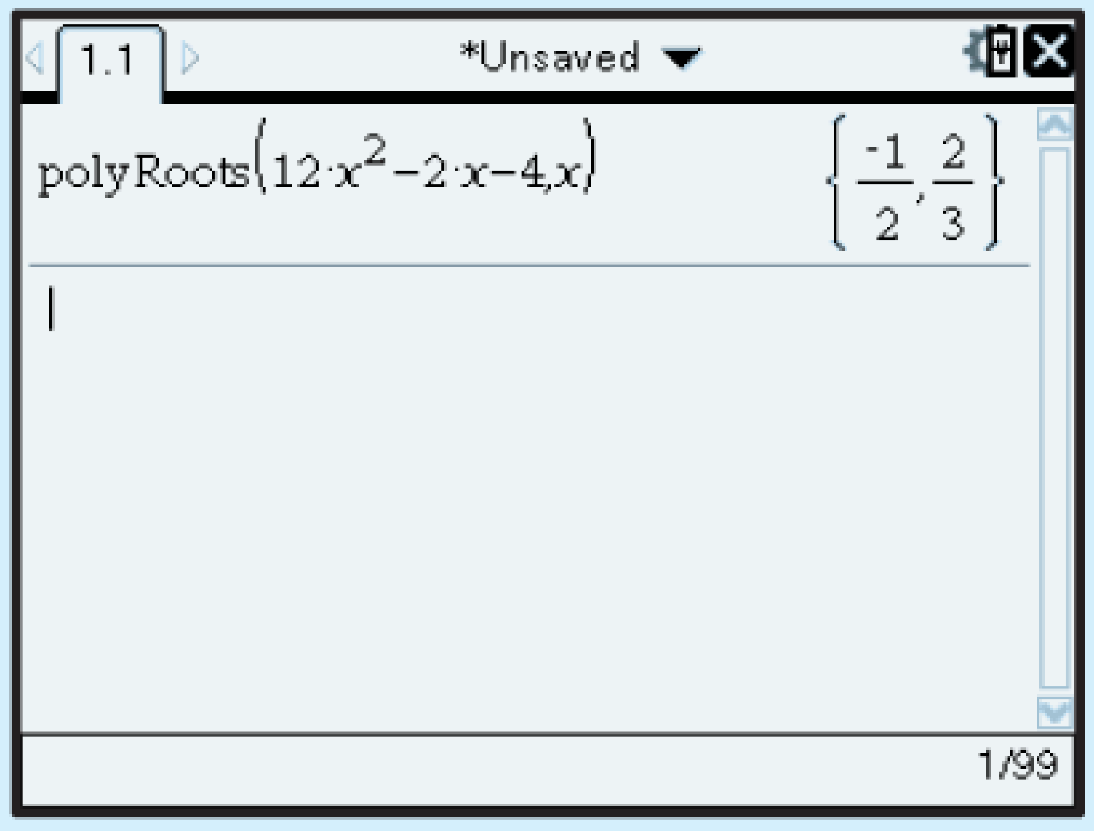
NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
MAIN MODE COEFF STORE F◂▸D

$12x^2+\ {-2}x+\ {-4}=0$

$x_1 = -\frac{1}{2}$

$x_2 = \frac{2}{3}$

TI-*n*spire

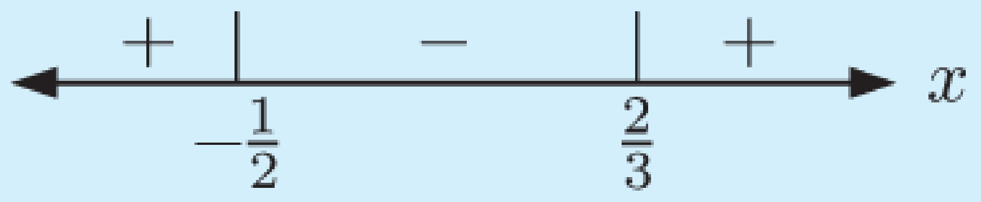


1.1
*Unsaved
1/99

$\text{polyRoots}(12\cdot x^2-2\cdot x-4,x)\quad \left\{\frac{-1}{2},\frac{2}{3}\right\}$

So, $f'(x) = 0$ when $x = -\frac{1}{2}$ or $\frac{2}{3}$

$f'(x)$ has sign diagram



+
–
+
$-\frac{1}{2}$
$\frac{2}{3}$
x

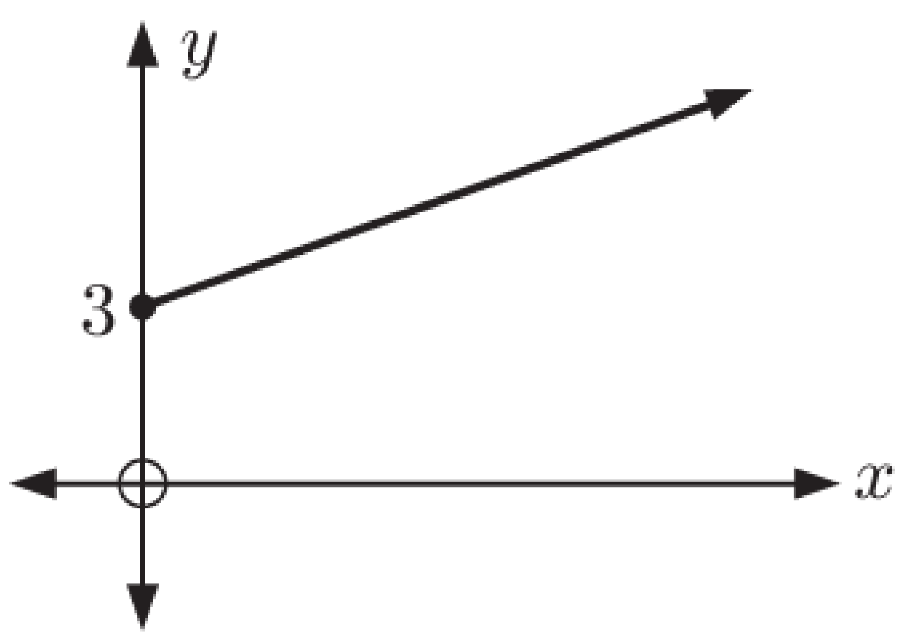
So, $f(x)$ is decreasing for $-\frac{1}{2} \leqslant x \leqslant \frac{2}{3}$, and increasing for $x \leqslant -\frac{1}{2}$ and $x \geqslant \frac{2}{3}$.

Remember that $f(x)$ must be defined for all x on an interval before we can classify the function as increasing or decreasing on that interval. We need to take care with vertical asymptotes and other values for x where the function is not defined.

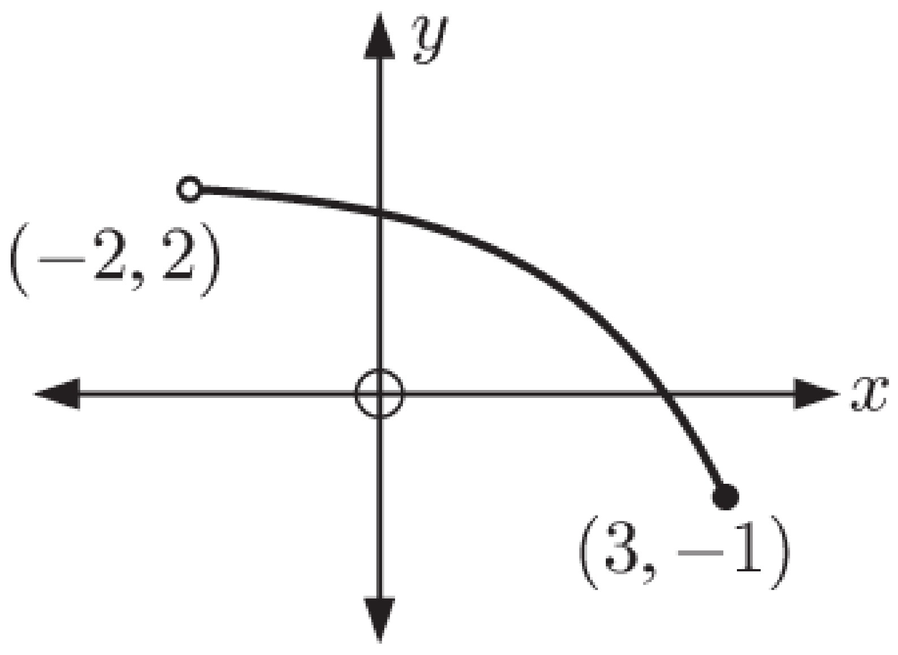
EXERCISE 11C

1 Write down the intervals where the graphs are: **i** increasing **ii** decreasing.

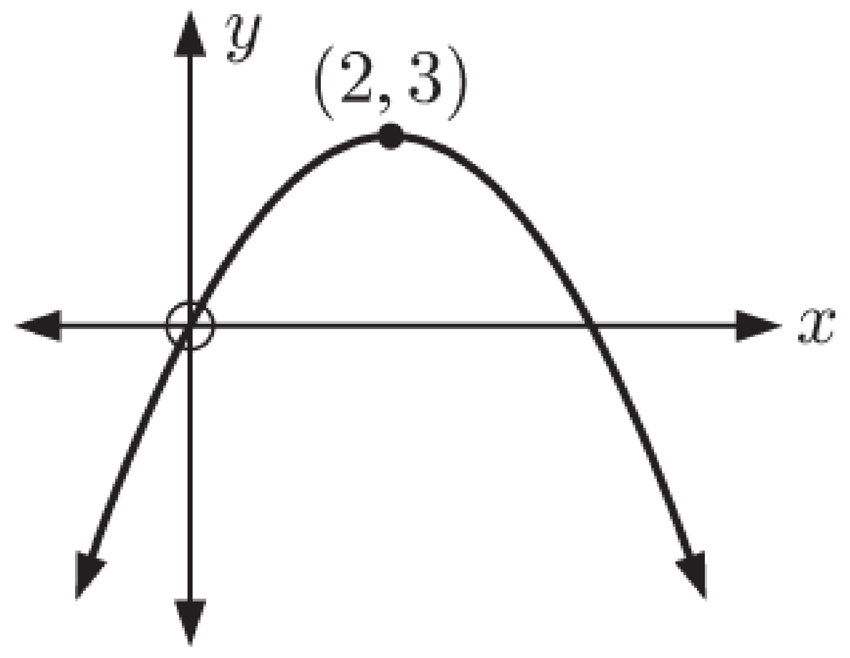
a



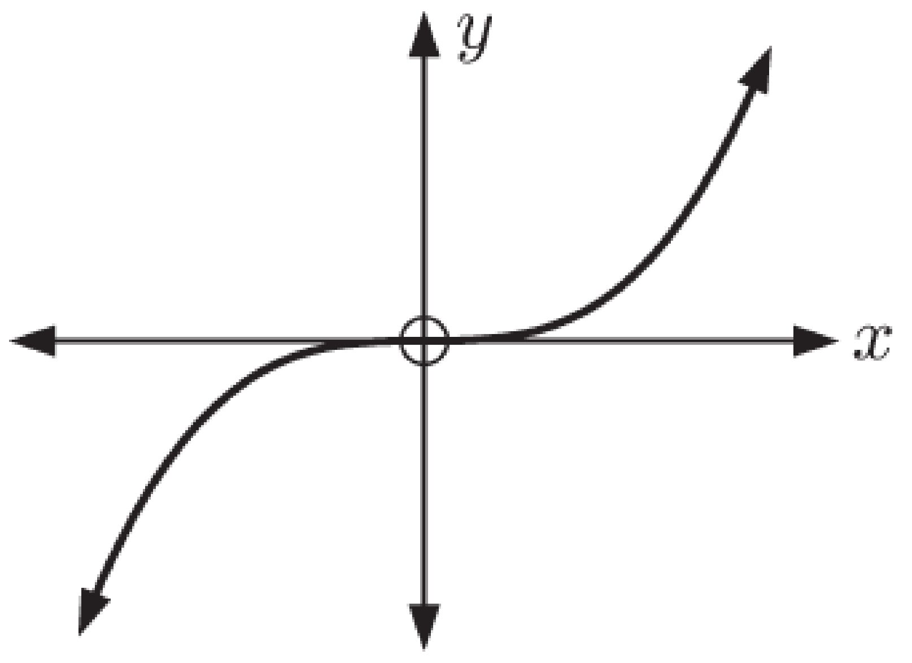
b



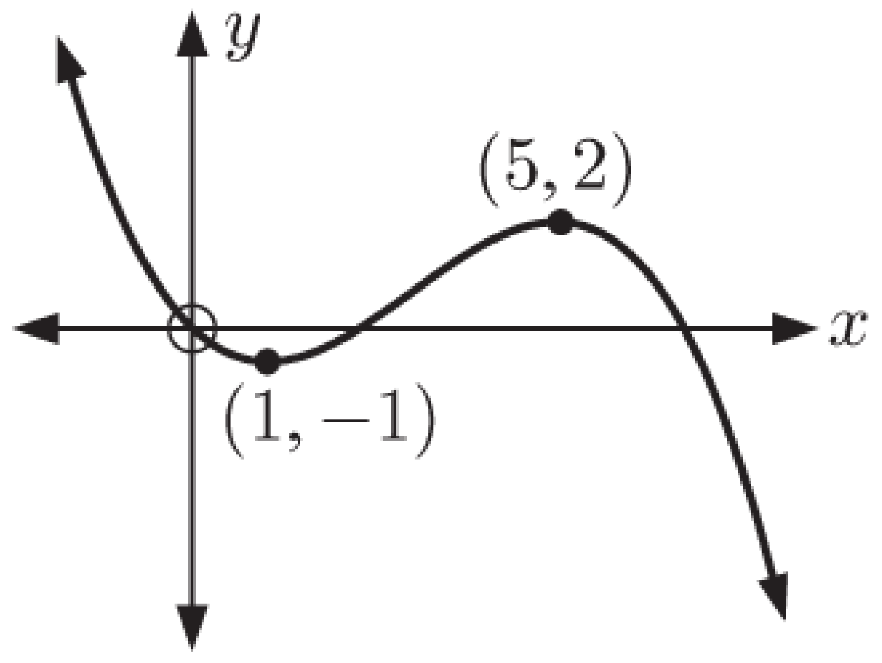
c



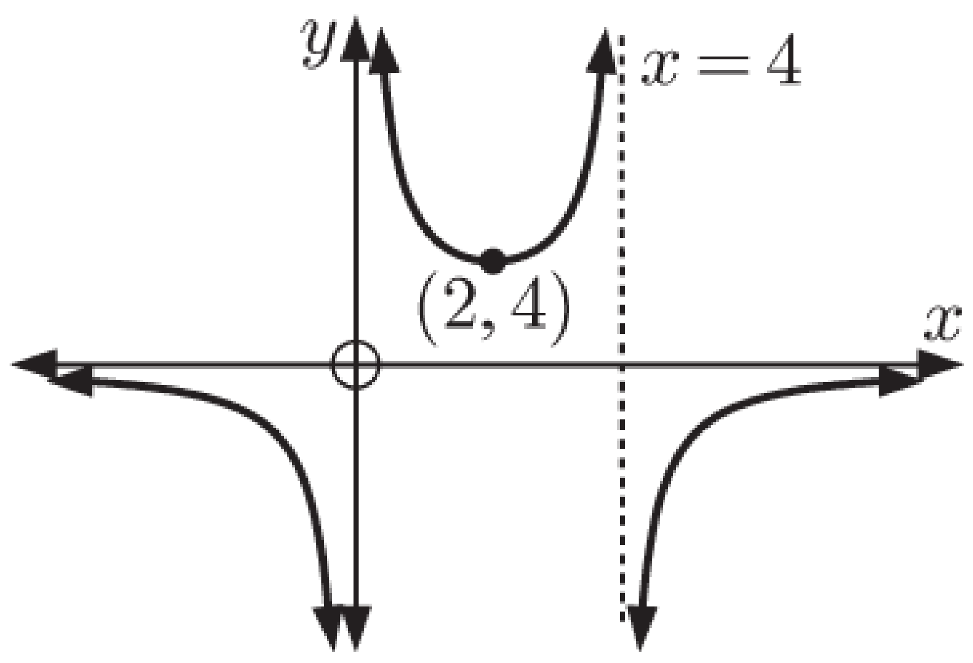
d



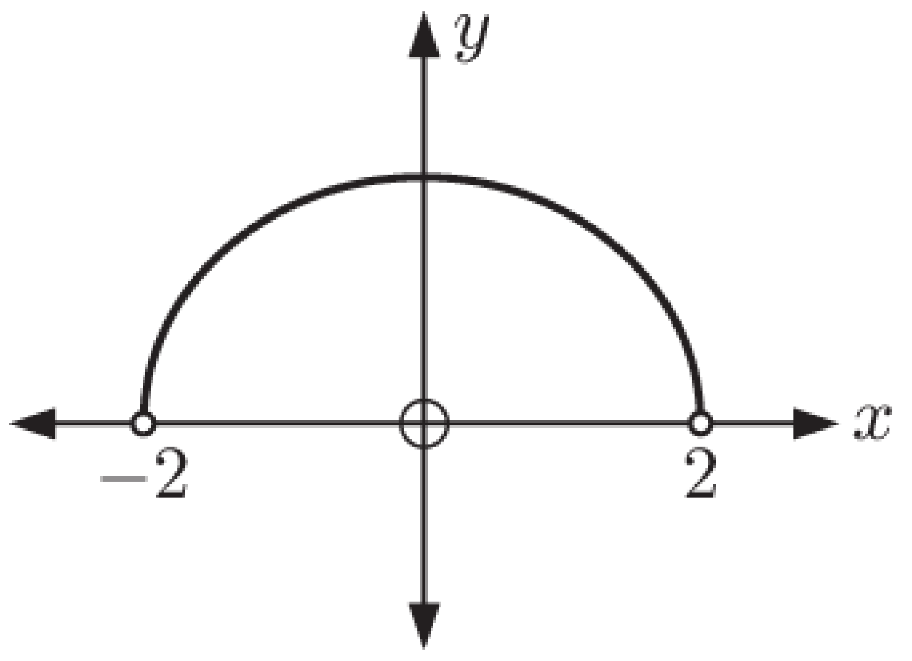
e



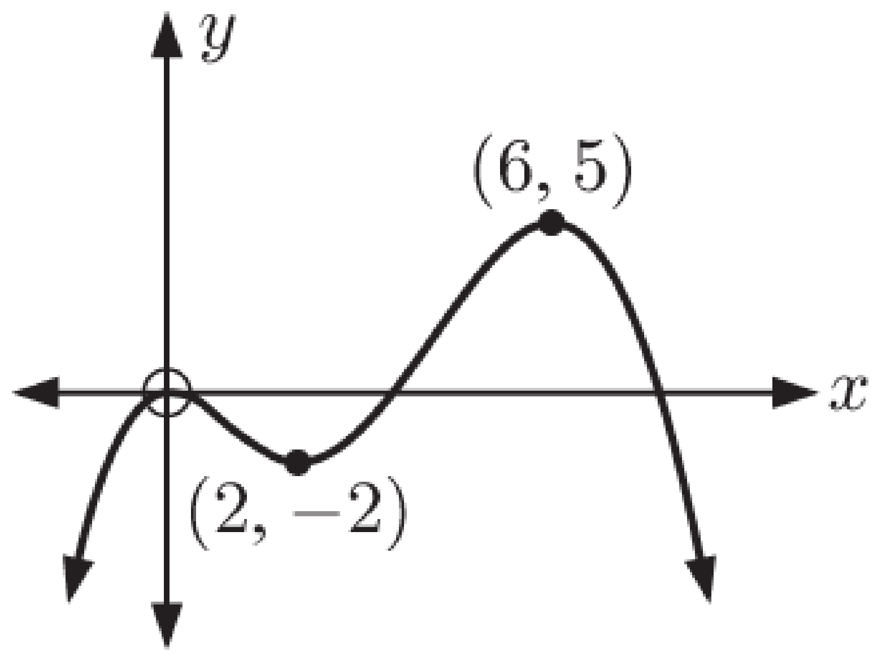
f



g

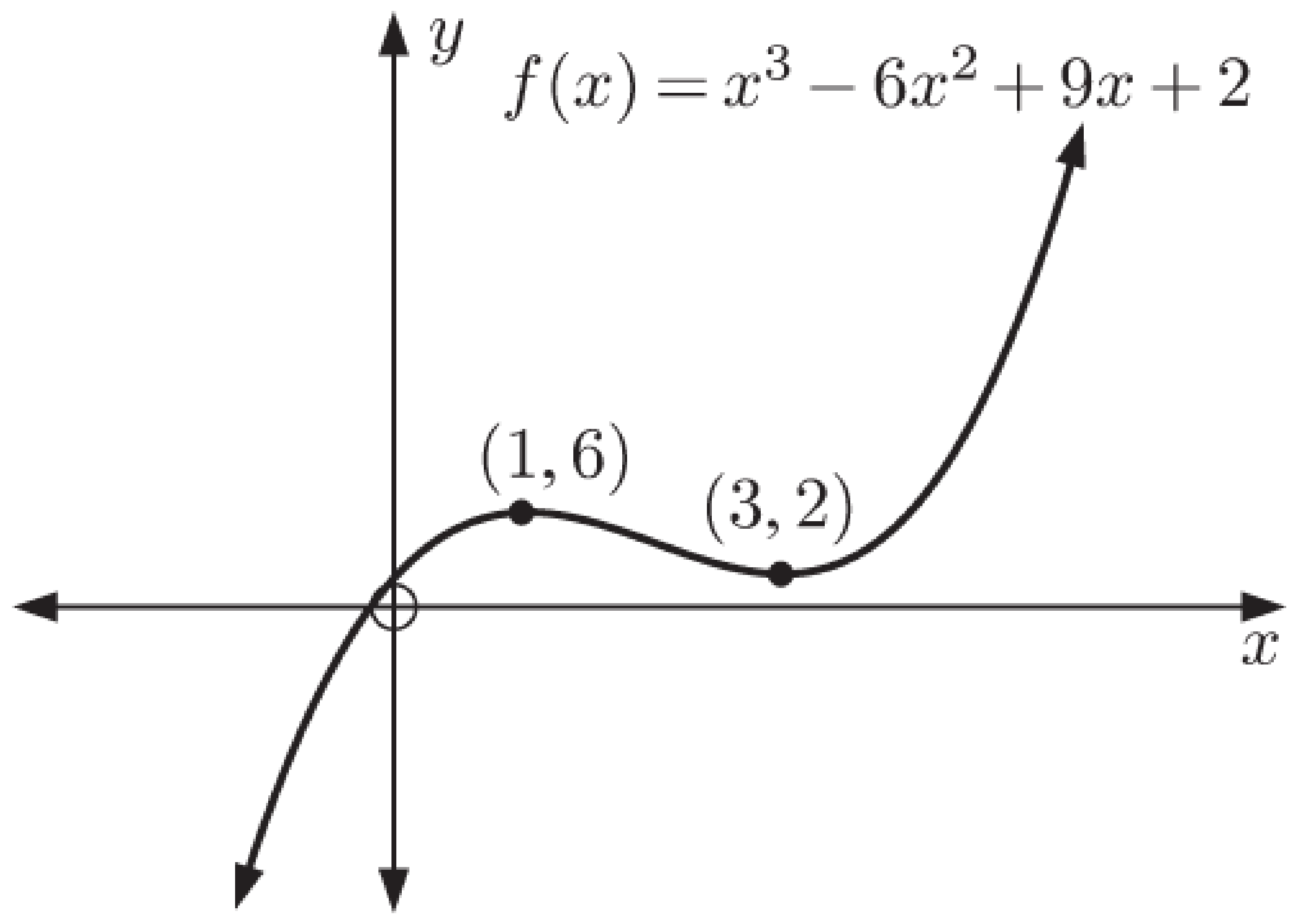


h



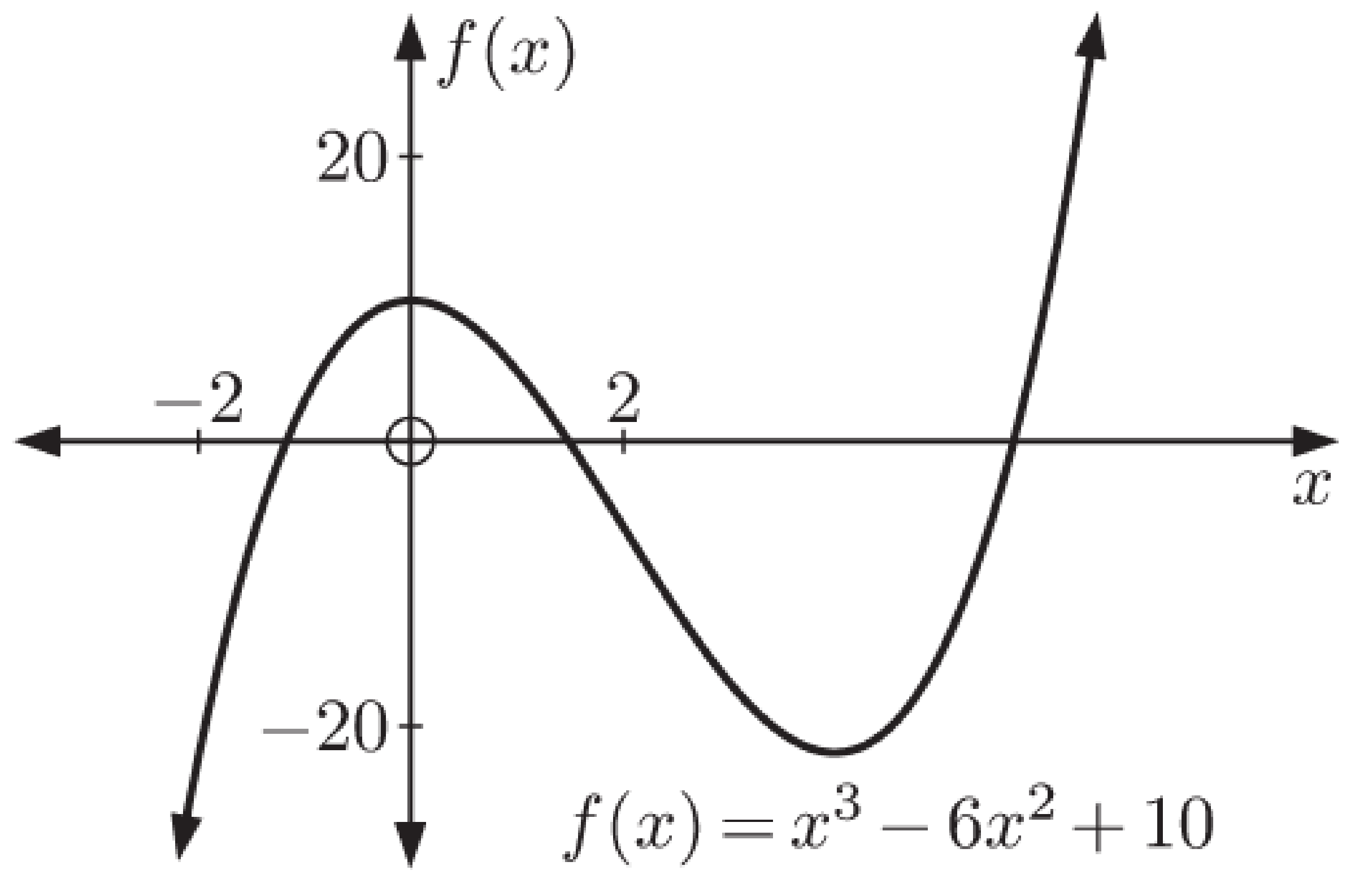
2 The graph of $f(x) = x^3 - 6x^2 + 9x + 2$ is shown alongside.

- **a** Use the graph to write down the intervals where the function is:
  - **i** increasing
  - **ii** decreasing.
- **b** Check your answer by finding $f'(x)$ and constructing its sign diagram.



3 The graph of $f(x) = x^3 - 6x^2 + 10$ is shown alongside.

- **a** Find $f'(x)$, and draw its sign diagram.
- **b** Find the intervals where $f(x)$ is increasing or decreasing.



4 Find the intervals where $f(x)$ is increasing or decreasing:

- **a** $f(x) = 2x + 1$
- **b** $f(x) = -3x + 2$
- **c** $f(x) = x^2$
- **d** $f(x) = -x^3$
- **e** $f(x) = 2x^2 + 3x - 4$
- **f** $f(x) = \frac{1}{x}$
- **g** $f(x) = -\frac{1}{x^2}$
- **h** $f(x) = x^3 - 6x^2$
- **i** $f(x) = -2x^3 + 4x$

5 For each of the following functions:

- **i** Find $f'(x)$.
- **ii** Use technology to find the values of x for which $f'(x) = 0$.
- **iii** Hence find the intervals where $f(x)$ is increasing or decreasing.

- **a** $f(x) = -4x^3 + 15x^2 + 18x + 3$
- **b** $f(x) = x^3 - 6x^2 + 3x - 1$

6 Consider the function $f(x) = x^3 - 3x^2 + 5x + 2$.

- **a** Find $f'(x)$.
- **b** Show that $f'(x) > 0$ for all x, and explain the significance of this result.
- **c** Use technology to sketch $y = f(x)$, and check your answer to **b**.

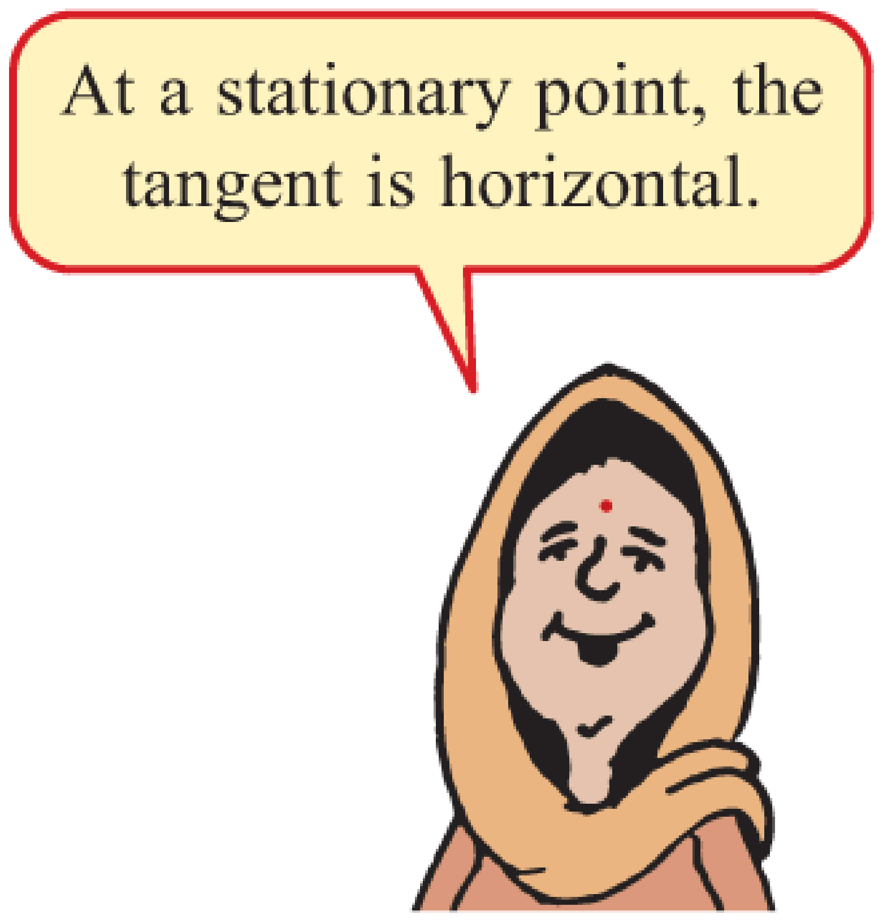
7 Consider the function $f(x) = 2x + \frac{8}{x}$.

- **a** Show that $f'(x) = \frac{2(x+2)(x-2)}{x^2}$.
- **b** Draw the sign diagram for $f'(x)$.
- **c** Hence find intervals where $y = f(x)$ is increasing or decreasing.

D STATIONARY POINTS

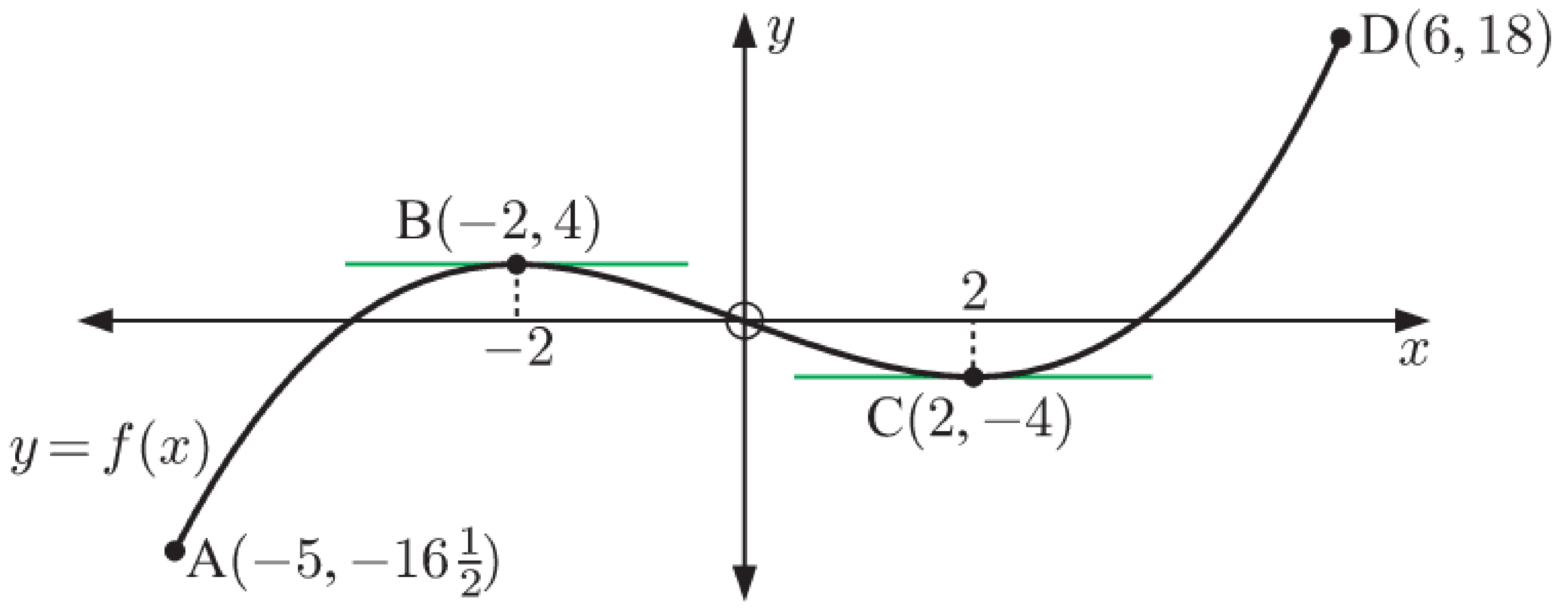
A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a **local maximum** or **local minimum**, or else a **stationary inflection**.

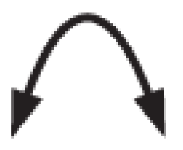

At a stationary point, the tangent is horizontal.

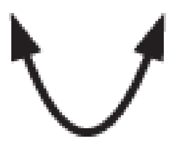
TURNING POINTS (MAXIMA AND MINIMA)

The graph shown has the restricted domain $-5 \leqslant x \leqslant 6$.



A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.

For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

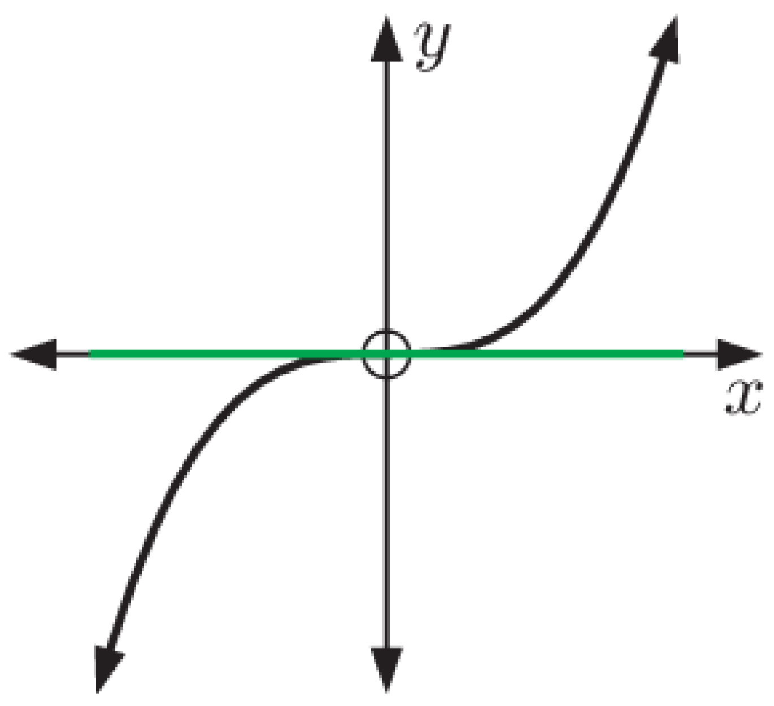
STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

For example, $f(x) = x^3$ has $f'(x) = 3x^2$, so $f'(x) = 0$ when $x = 0$.

The tangent to the curve crosses over the curve at O(0, 0). This tangent is horizontal, but O(0, 0) is neither a local maximum nor a local minimum.

Rather, this point is called a **stationary inflection** (or **inflexion**) as the curve changes its curvature or shape.



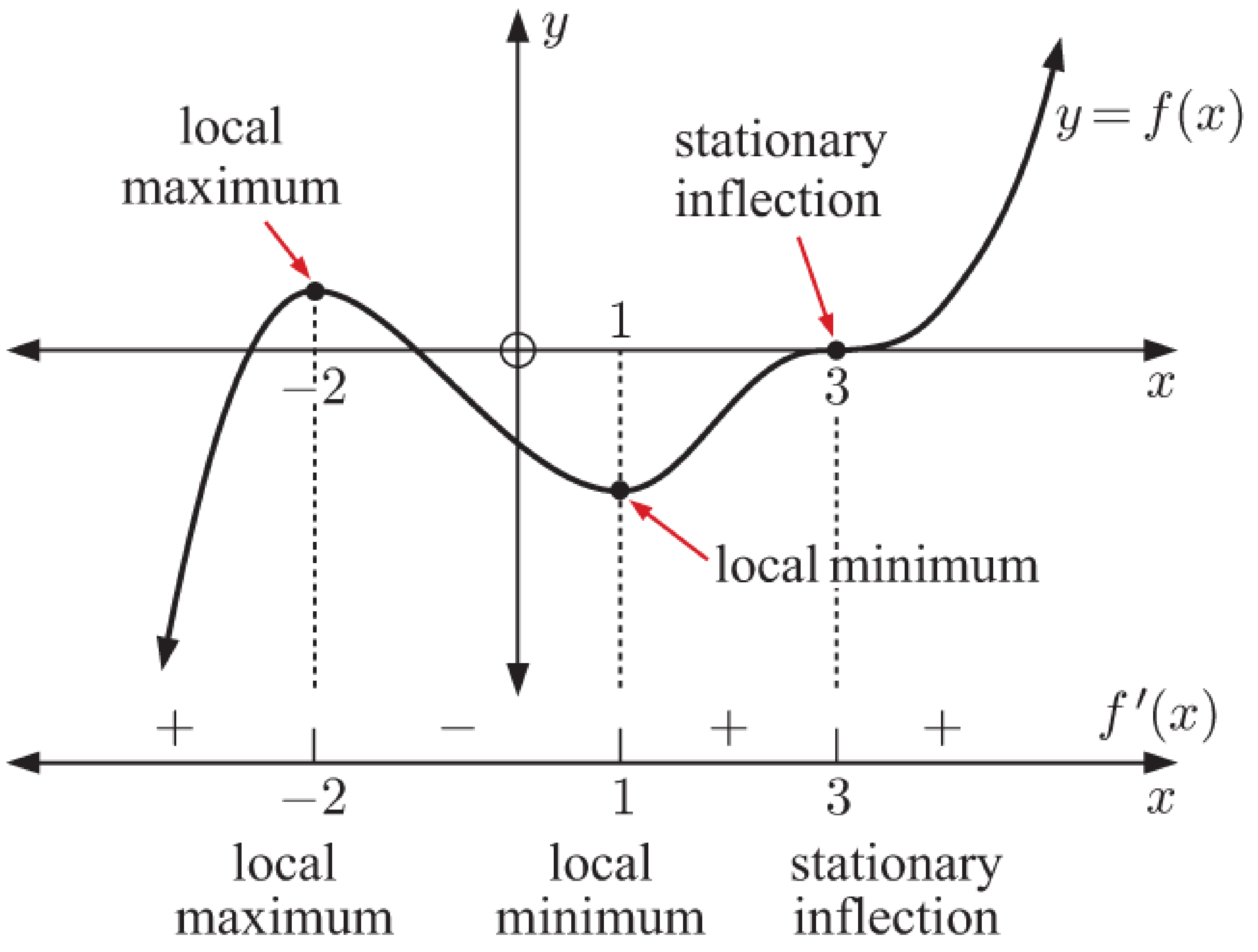
SIGN DIAGRAMS

In calculus we commonly use sign diagrams of the derivative function $f'(x)$ to determine the nature of a stationary point.

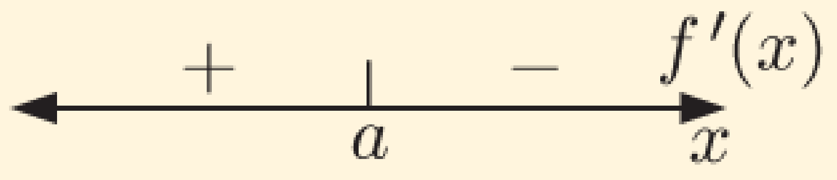
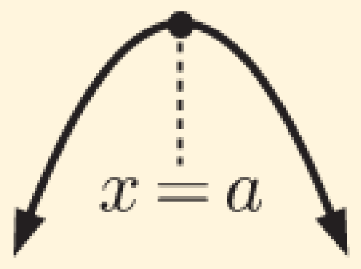
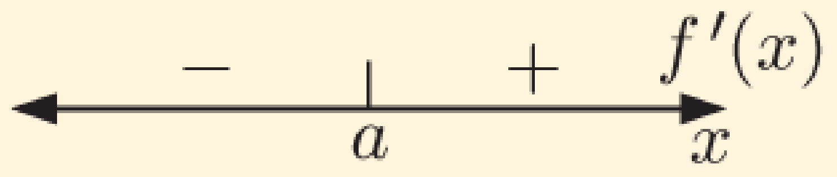
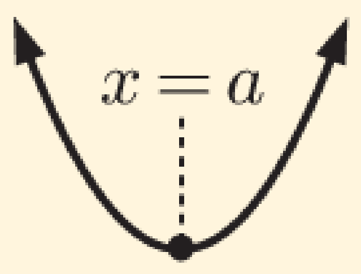
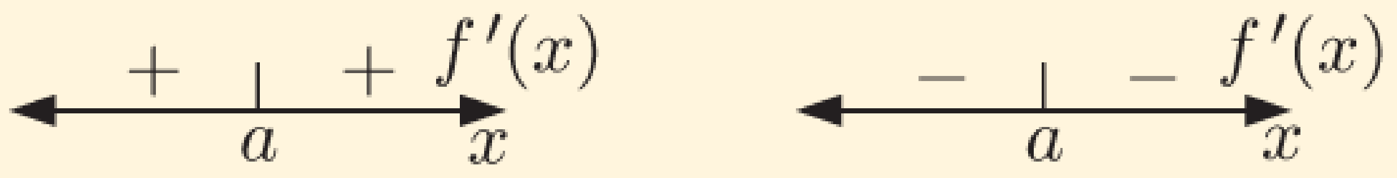
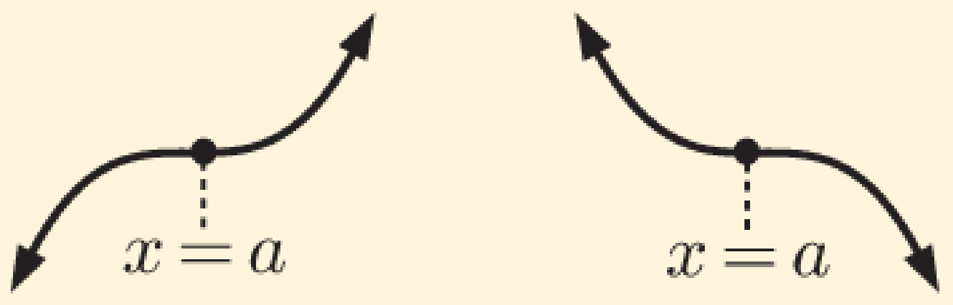
Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative on that interval.



We observe the following properties:

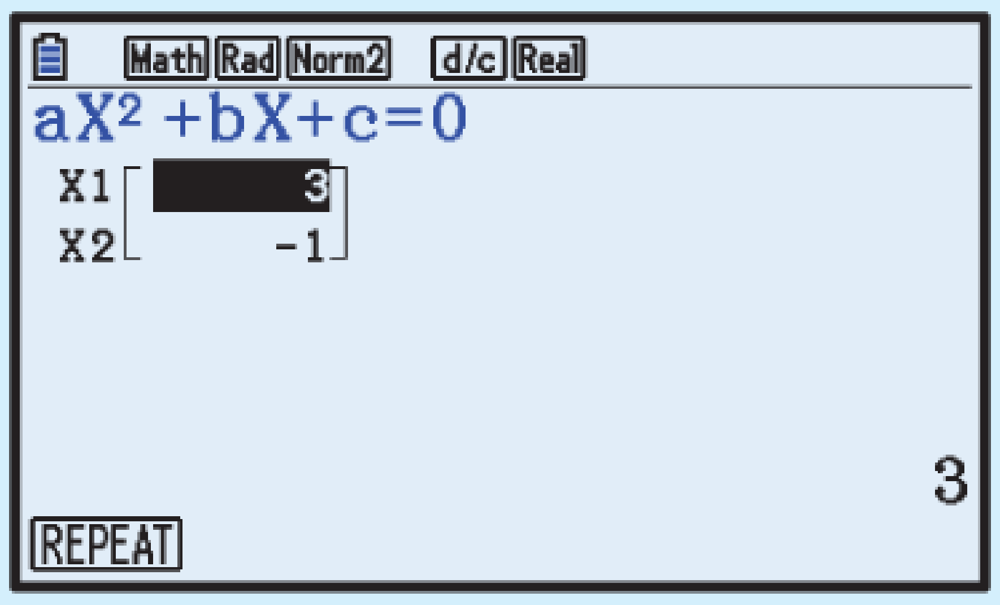
Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection		

To find the stationary points of a function $f(x)$, we find the values of x for which $f'(x) = 0$. The sign diagram of $f'(x)$ tells us whether each stationary point is a local maximum, local minimum, or stationary inflection.

Example 7 **Self Tutor**

Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

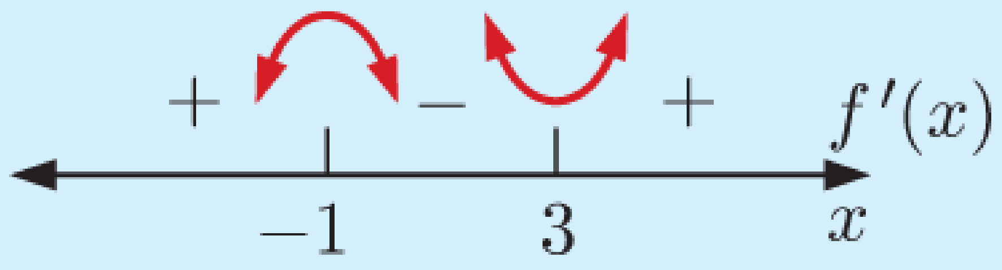
$$f(x) = x^3 - 3x^2 - 9x + 5$$
$$\therefore\ f'(x) = 3x^2 - 6x - 9$$


Math Rad Norm2 d/c Real
REPEAT

```
aX²+bX+c=0
X1 [ 3
X2   -1 ]
                3
```

Using technology, $f'(x) = 0$ when $x = -1$ or 3.

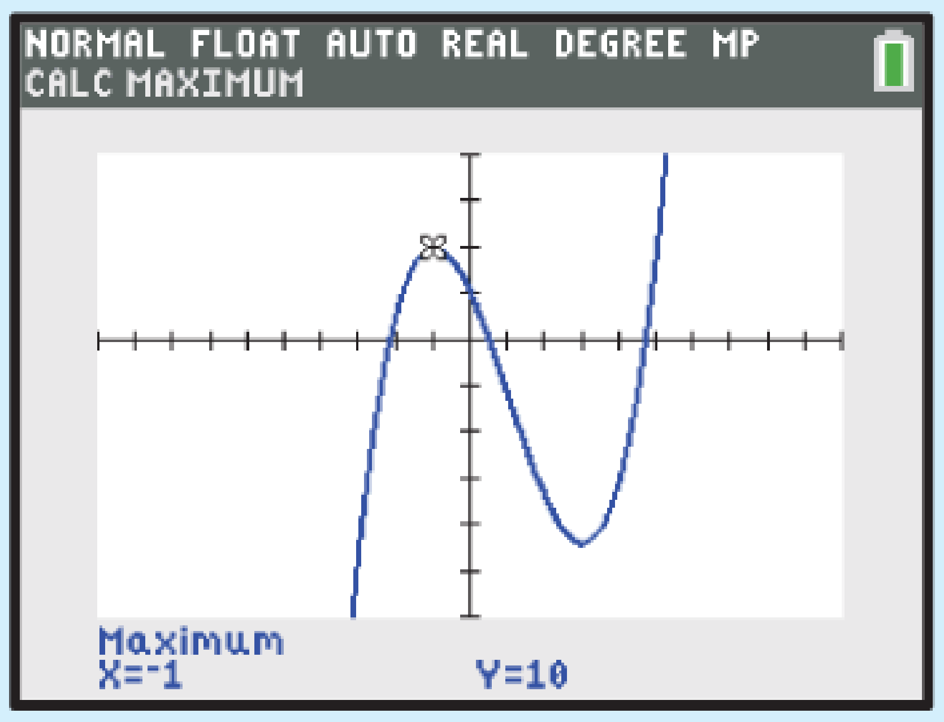
$\therefore$ the sign diagram for $f'(x)$ is



$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5$$
$$= 10$$

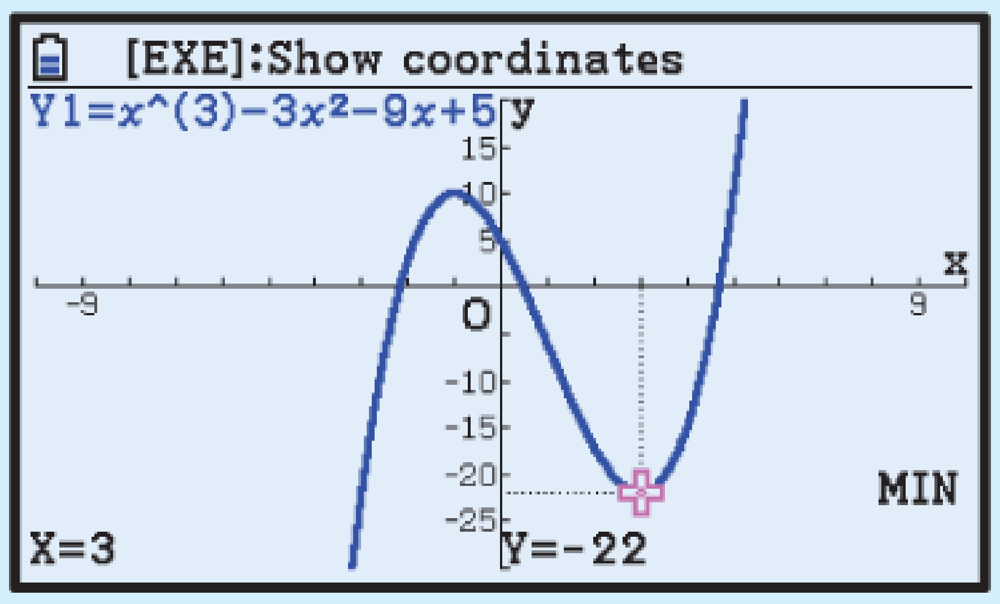
$$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5$$
$$= -22$$

TI-84 Plus CE


NORMAL FLOAT AUTO REAL DEGREE MP
CALC MAXIMUM

```
Maximum
X=-1          Y=10
```

Casio fx-CG50

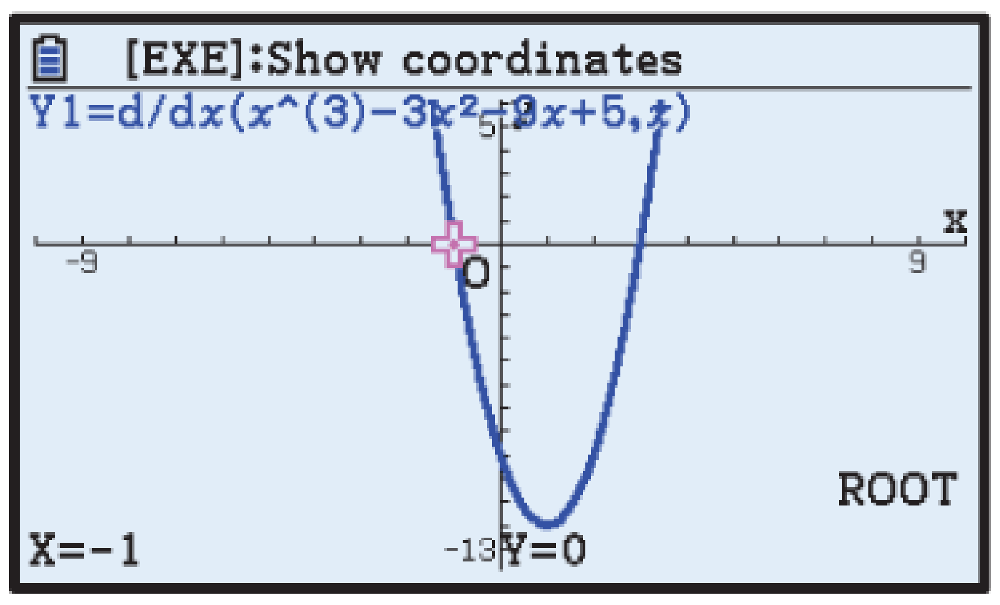

[EXE]:Show coordinates
y
15
10
5
-9
O
9
x
-10
-15
-20
-25
MIN

```
Y1=x^(3)-3x²-9x+5
X=3          Y=-22
```

So, there is a local maximum at $(-1, 10)$ and a local minimum at $(3, -22)$.

To help construct the sign diagram for $f'(x)$, you can use technology to draw the graph of $f'(x)$, and solve $f'(x) = 0$.

GRAPHICS CALCULATOR INSTRUCTIONS


[EXE]:Show coordinates
5
-9
O
9
x
-13
ROOT

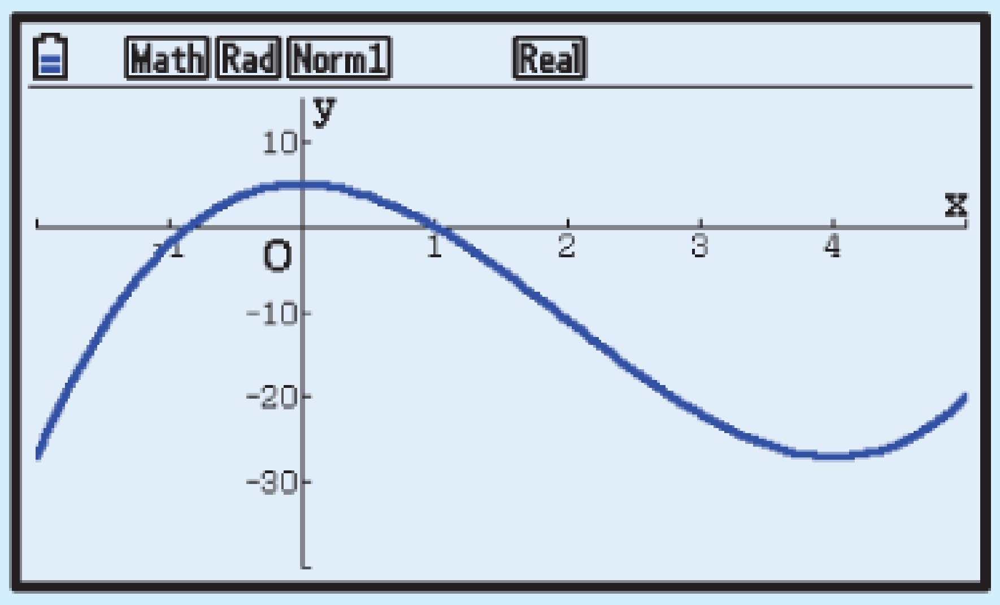
```
Y1=d/dx(x^(3)-3x²-9x+5,x)
X=-1          Y=0
```

If we are asked to find the greatest or least value of a function on an interval, we must also check the value of the function at the end points. We seek the *global* maximum or minimum on the given domain.

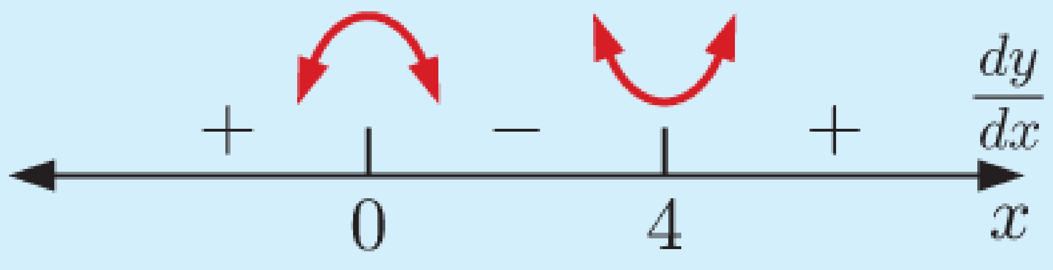
Example 8 **Self Tutor**

Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

$$\text{Now}\quad \frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x-4)$$
$$\therefore\ \frac{dy}{dx} = 0 \quad\text{when}\quad x = 0 \text{ or } 4.$$


Math Rad Norm1 Real
y
10
x
-1
O
1
2
3
4
-10
-20
-30

The sign diagram of $\frac{dy}{dx}$ is:



$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

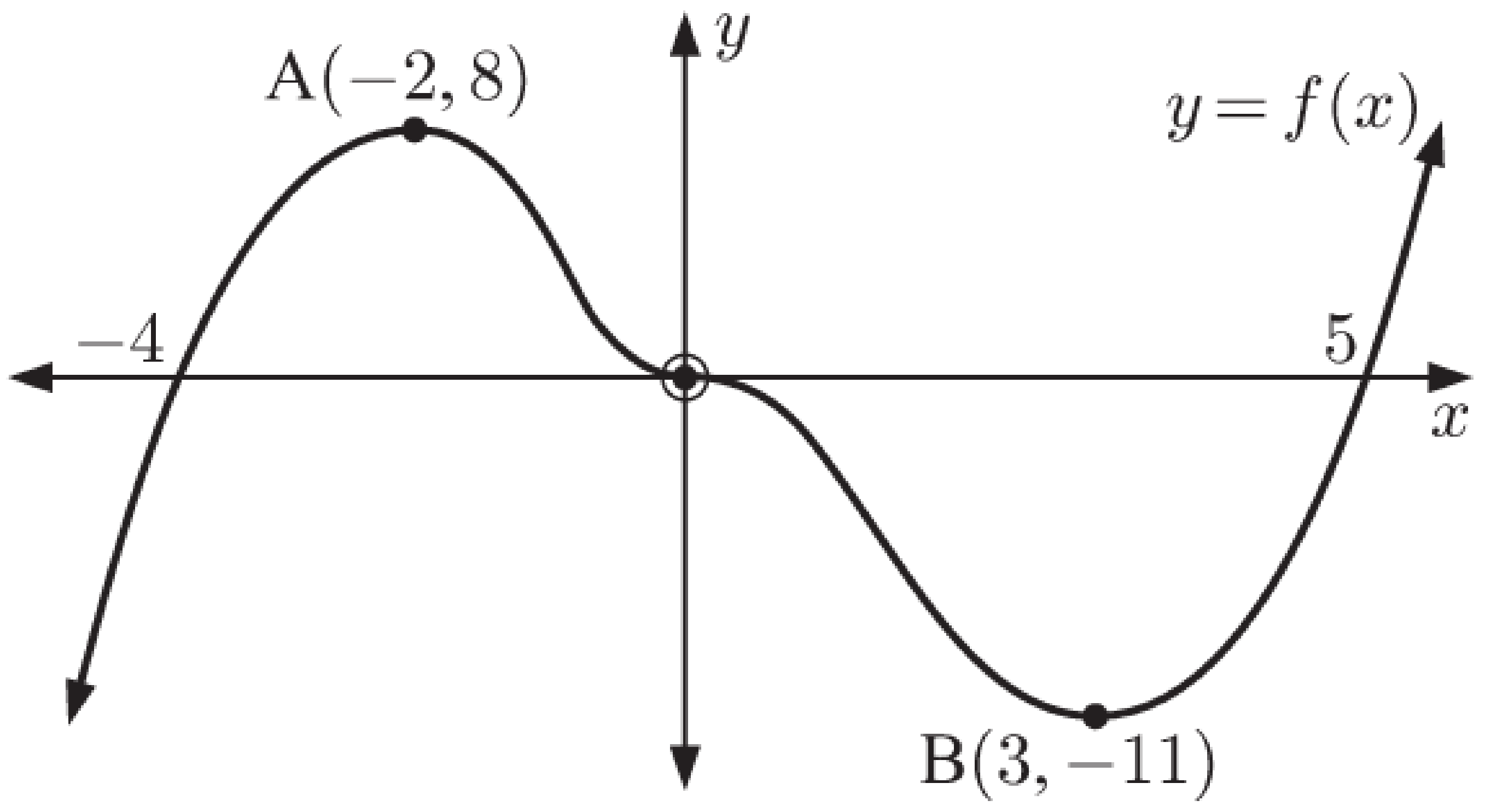
Critical value (x)	y
-2 (end point)	-27
0 (local maximum)	5
4 (local minimum)	-27
5 (end point)	-20

The greatest value is 5 when $x = 0$.

The least value is -27 when $x = -2$ and when $x = 4$.

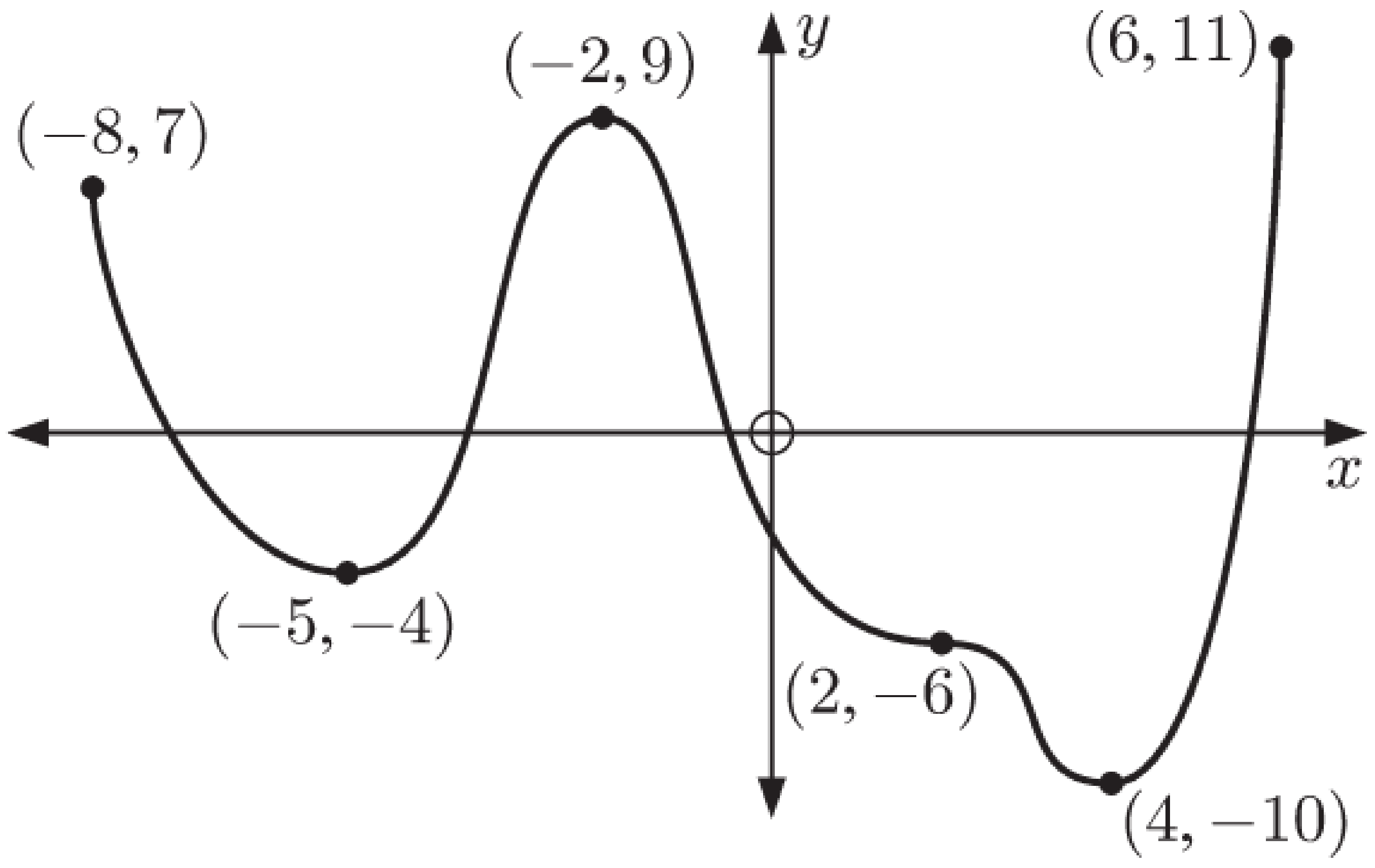
EXERCISE 11D

1 The tangents at points A, O, and B are horizontal.



- **a** Classify points A, O, and B.
- **b** Draw a sign diagram for the gradient function $f'(x)$ for all x.
- **c** State intervals where $y = f(x)$ is:
  - **i** increasing
  - **ii** decreasing.
- **d** Draw a sign diagram for $f(x)$ for all x.

2 Consider the graph of $y = f(x)$ on the domain $-8 \leqslant x \leqslant 6$.

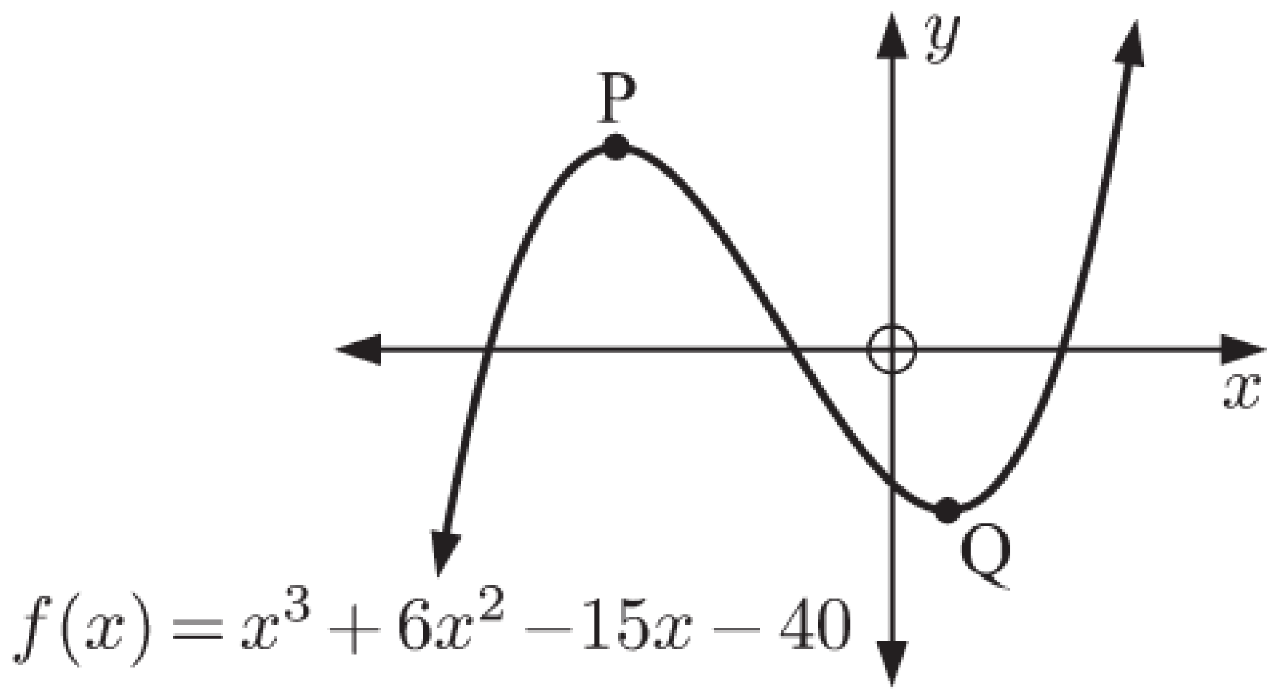


- **a** How many stationary points does the graph have?
- **b** Write down the coordinates of the:
  - **i** local maximum
  - **ii** horizontal inflection.
- **c** Find the **i** greatest **ii** least value of $f(x)$ on $-8 \leqslant x \leqslant 6$.
- **d** Find the greatest value of $f(x)$ on $-8 \leqslant x \leqslant 4$.
- **e** Find the least value of $f(x)$ on $-5 \leqslant x \leqslant 2$.

3 Consider the quadratic function $f(x) = 2x^2 - 5x + 1$.

- **a** Use quadratic theory to find the equation of the axis of symmetry.
- **b** Find $f'(x)$, and hence find x such that $f'(x) = 0$. Explain your result.

4



The graph of $f(x) = x^3 + 6x^2 - 15x - 40$ is shown alongside. P and Q are stationary points.

- **a** Classify points P and Q.
- **b** Find $f'(x)$.
- **c** Find the coordinates of P and Q.

5

- **a** Using technology to help, sketch a graph of $f(x) = x + \frac{1}{x}$.
- **b** Find $f'(x)$.
- **c** Draw a sign diagram for $f'(x)$.
- **d** Determine the position and nature of any stationary points.

6 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

a $f(x) = x^2 - 2$ **b** $f(x) = x^3 + 1$ **c** $f(x) = x^3 - 3x + 2$

d $f(x) = x^4 - 2x^2$ **e** $f(x) = x^3 - 6x^2 + 12x + 1$ **f** $f(x) = 4x - x^3$

g $f(x) = 2x + \dfrac{1}{x^2}$ **h** $f(x) = -x - \dfrac{9}{x}$ **i** $f(x) = x^2 + \dfrac{16}{x}$

7 **a** At what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point?

b Under what conditions is the stationary point a local maximum or a local minimum?

8 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$.

a Find a. **b** Find the coordinates of the local maximum.

9 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b. **b** Find $f'(x)$.

c Hence find the position and nature of all stationary points.

10 $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

11 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

c $x^2 + \dfrac{16}{x}$ for $1 \leqslant x \leqslant 4$ **d** $-2x^3 - 2x^2 + 8x + 3$ for $-2 \leqslant x \leqslant 2$.

REVIEW SET 11A

1 Find the equation of the tangent to:

a $y = -2x^2$ at the point where $x = -1$ **b** $y = x^3 - 5x + 2$ at $(2, 0)$

c $y = \dfrac{1 - 2x}{x^2}$ at $(1, -1)$.

2 The tangent to $f(x) = a - \dfrac{b}{x^2}$ at $(-1, -1)$ has equation $y = -6x - 7$. Find the values of a and b.

3 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ meets the curve again.

4 Find the equation of the normal to:

a $y = x^3 + 3x - 2$ when $x = 2$ **b** $y = 2 + \dfrac{1}{x} + 3x$ when $x = 1$.

5 Find the point where the normal to $y = x^2 - 7x - 44$ at $x = -3$ meets the curve again.

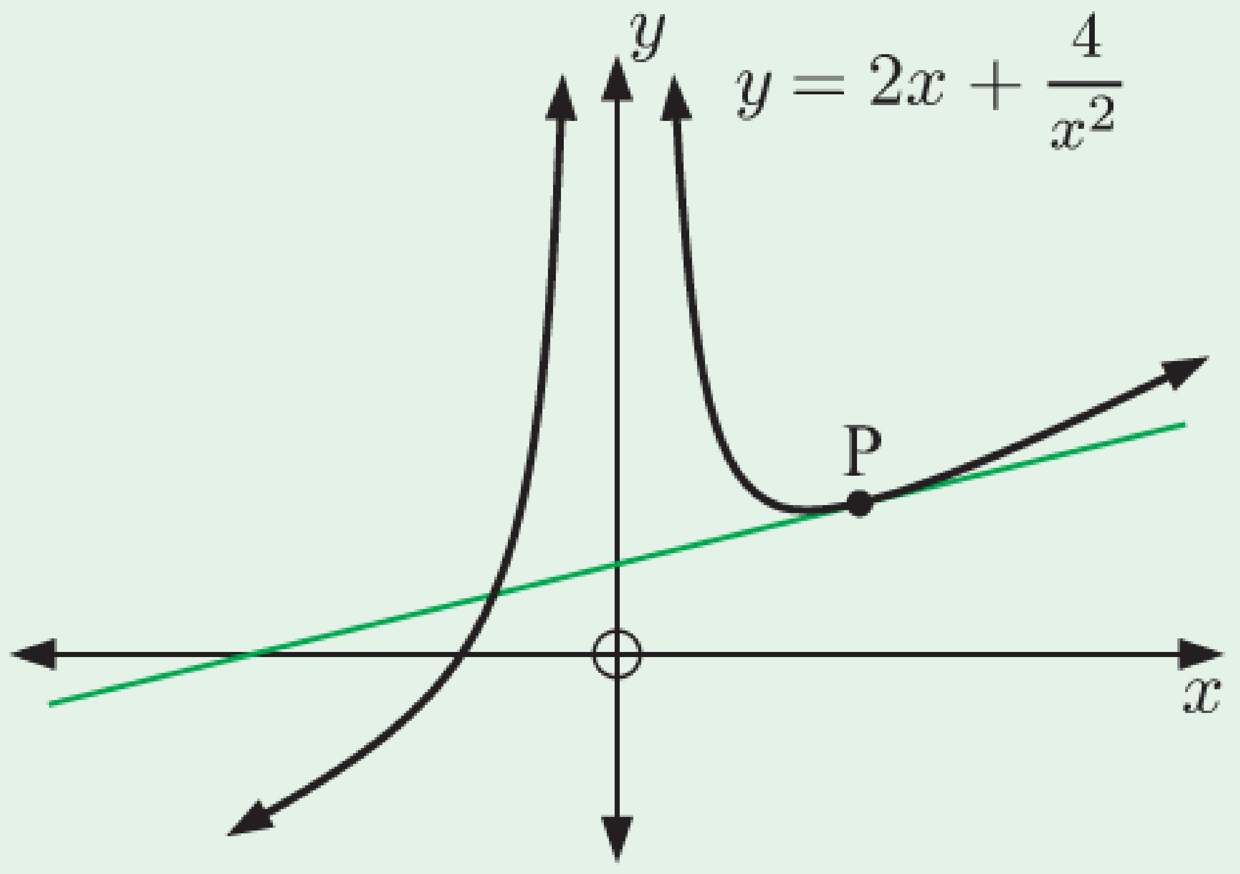
6 The tangent shown has gradient 1.

a Find the coordinates of P.

b Find the equation of the tangent.

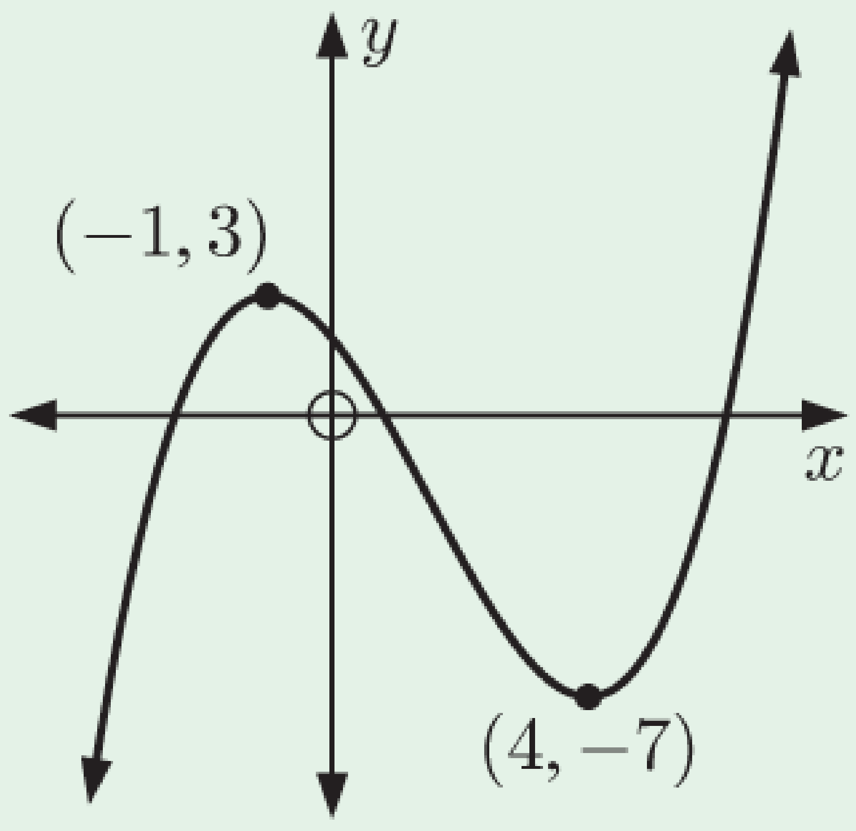
c Find the point where the tangent cuts the x-axis.

d Find the equation of the normal at P.

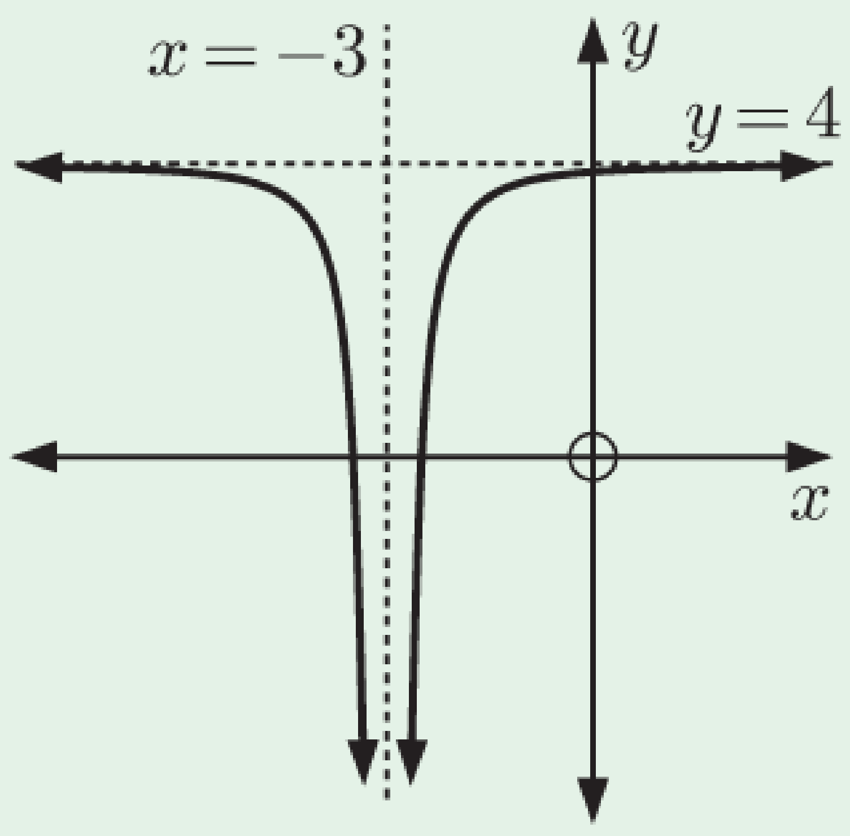


7 Find intervals where the given function is increasing or decreasing.

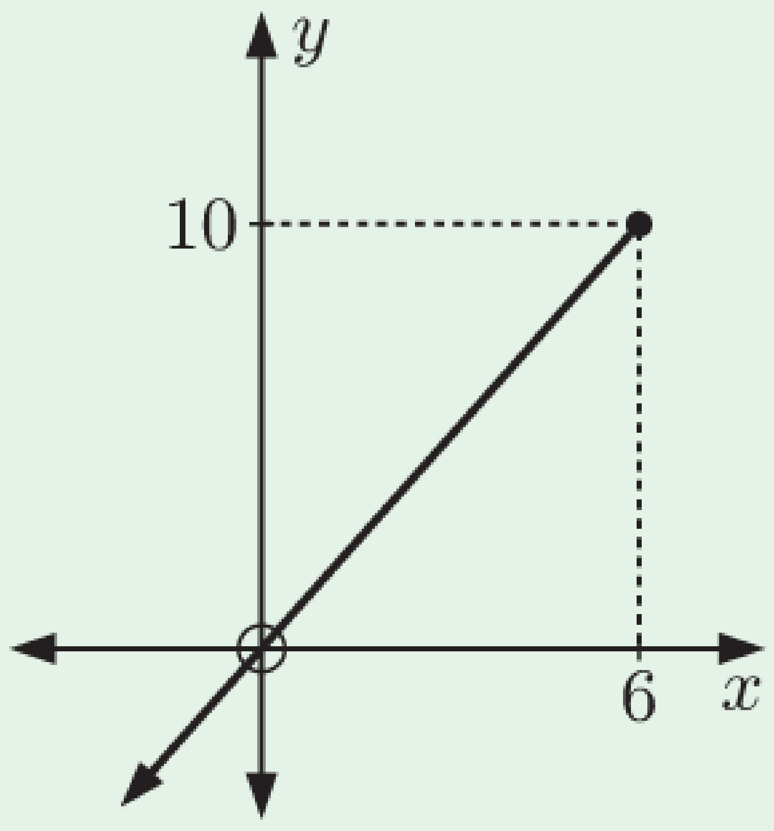
a



b



c

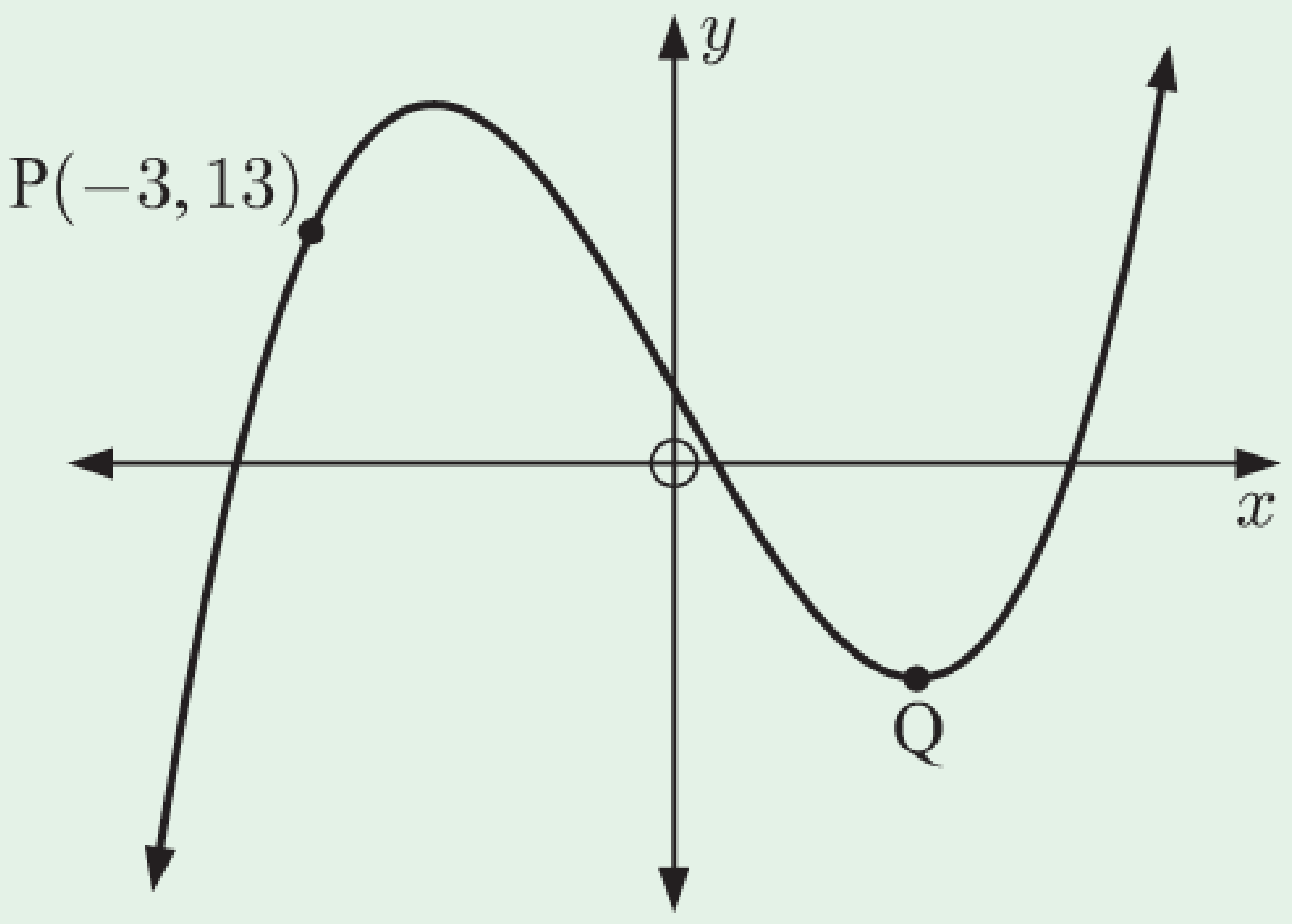


8 Consider the function $f(x) = x^3 - 3x$.

- **a** Determine the y-intercept of the function.
- **b** Find $f'(x)$.
- **c** Hence find the position and nature of any stationary points.
- **d** Sketch the graph of the function, showing the features you have found.

9 The graph of $f(x) = x^3 - 12x + 4$ is shown.

- **a** Find $f'(x)$.
- **b** Find the gradient of the tangent to the graph at P.
- **c** Find the coordinates of the local minimum Q.



10 Find intervals where $f(x) = -x^3 - 6x^2 + 36x - 17$ is:

- **a** increasing
- **b** decreasing.

11 Find and classify the stationary points of $f(x) = -x^3 + 2x^2 - x + 3$.

12 Consider the function $f(x) = 3x + 2 + \frac{48}{x}$.

- **a** Find $f'(x)$ and draw its sign diagram.
- **b** Find and classify the stationary points of the function.
- **c** Sketch the graph of $y = f(x)$.

13 Find the maximum and minimum values of:

- **a** $y = \frac{1}{3}x^3 + x^2 - 3x$ for $-4 \leqslant x \leqslant 4$
- **b** $x + \frac{32}{x^2}$ for $2 \leqslant x \leqslant 10$.

REVIEW SET 11B

1 Find the equation of the tangent to:

- **a** $y = x^3 - 3x + 5$ at the point where $x = 2$
- **b** $f(x) = x^4 - 2x^2 + 7x - 3$ at $(2, 19)$
- **c** $y = \frac{12}{x^2}$ at the point $(1, 12)$.

2 Find the equation of the normal to:

a $y = 3 + 2x - x^2$ at the point $(2,\ 3)$ **b** $y = \frac{1}{x^2} - \frac{2}{x}$ at the point where $x = 1$.

3 The tangent to $y = x^3 + ax^2 - 4x + 3$ at $x = 1$ is parallel to the line $y = 3x$.

a Find a.

b Find the equation of the tangent at $x = 1$.

c Where does the tangent cut the curve again?

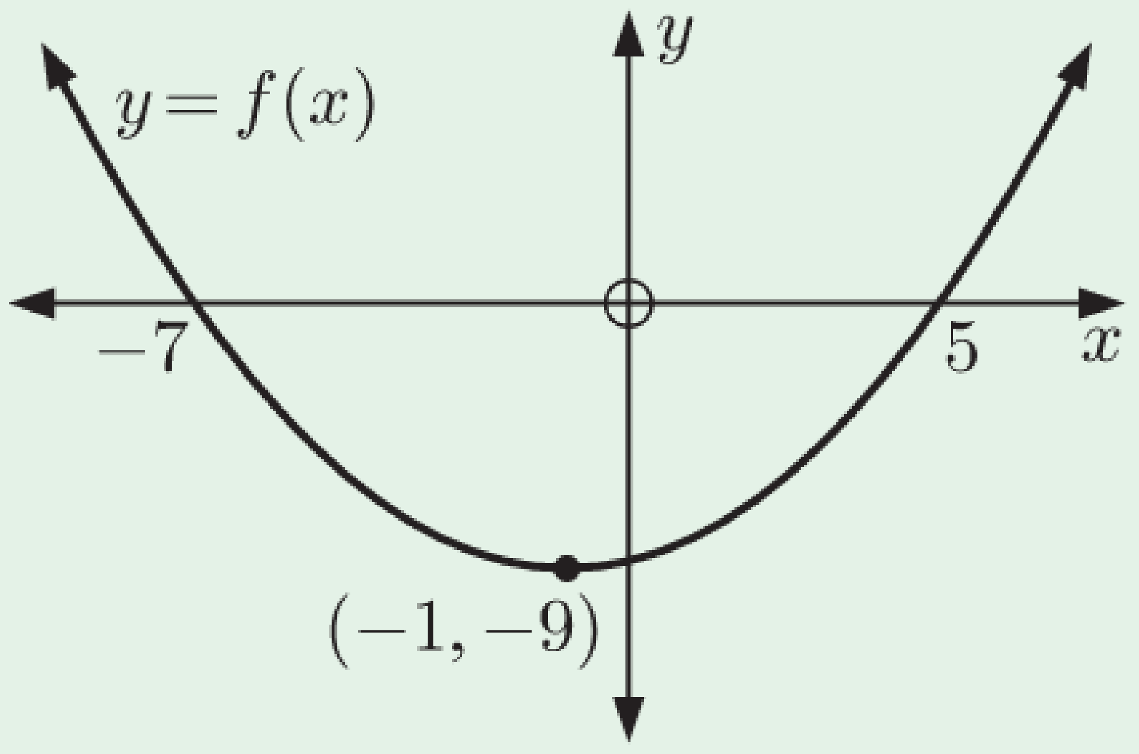
4 Find the point where the normal to $y = x^2 - 4x + 2$ at $x = 3$ meets the curve again.

5 The tangent to $y = x^3 - 2x^2 + ax - b$ at $(2,\ -1)$ has equation $y = 7x - 15$.
Find the values of a and b.

6 Find the point(s) where the normal to $y = -3x^3 + 5x - 1$ at $x = 0$ meets the curve again.

7 For the function $y = f(x)$ shown, draw a sign diagram of:

a $f(x)$ **b** $f'(x)$.



8 Find intervals where $f(x) = x^4 - 4x^3 - 8x^2 + 5$ is:

a increasing **b** decreasing.

9 $f(x) = x^3 - 3x^2 + ax + 50$ has a stationary point at $x = 3$.

a Find a. **b** Find the position and nature of all stationary points.

10 $f(x) = x^3 + Ax + B$ has a stationary point at $(1,\ 5)$.

a Find A and B. **b** Find the nature of all the stationary points of $f(x)$.

11 Consider the function $f(x) = x^3 - 4x^2 + 4x$.

a State the y-intercept.

b Find $f'(x)$ and draw its sign diagram.

c State intervals where the function is increasing or decreasing.

d Find the position and nature of any stationary points.

e Sketch the function, showing the features you have found.

12 Find the maximum and minimum values of $x^3 - 3x^2 + 5$ for $-1 \leqslant x \leqslant 4$.

13 Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 7$.

a Find $f'(x)$.

b Find and classify all stationary points.

c Find intervals where the function is increasing or decreasing.

d Sketch the graph of $y = f(x)$, showing the features you have found.

Chapter 12

Applications of differentiation

Contents:

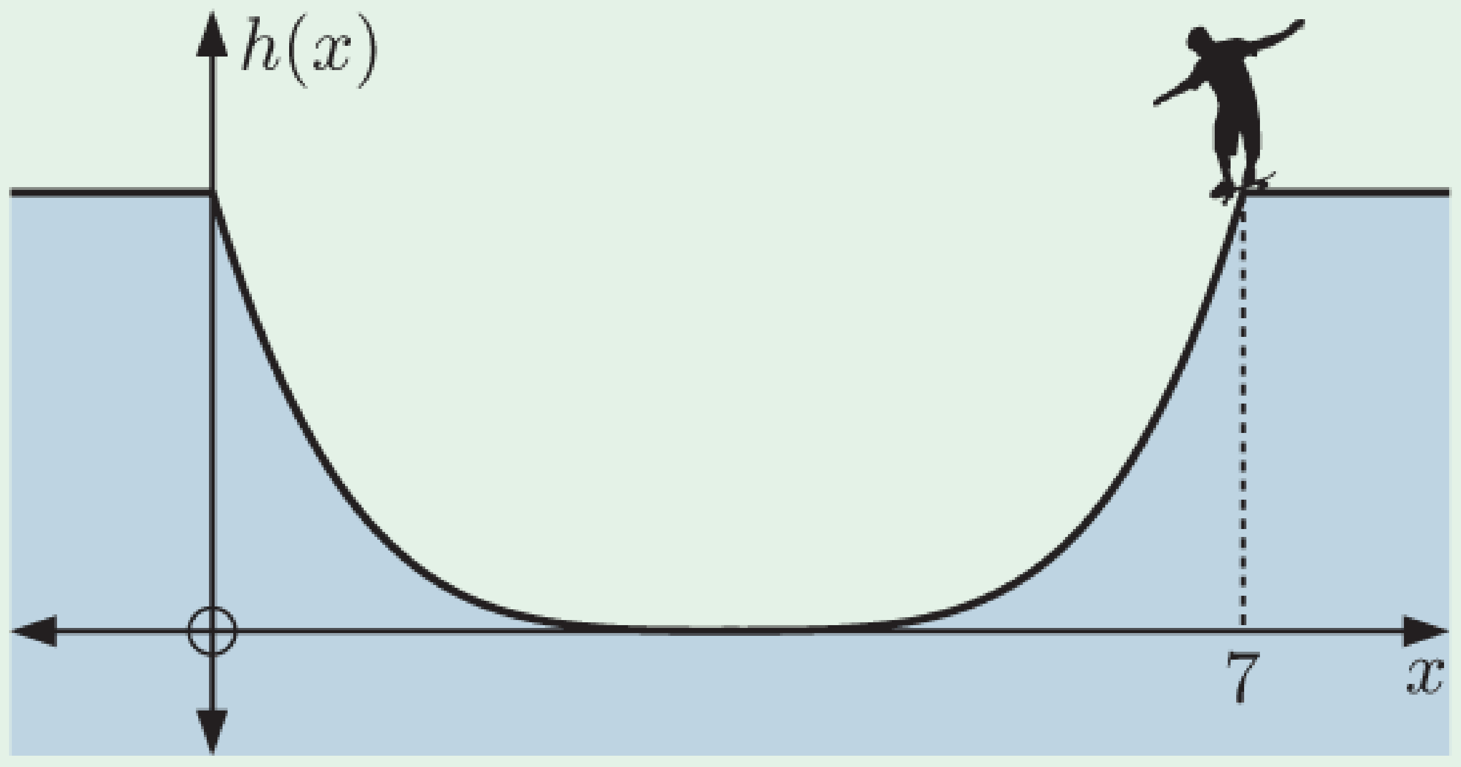
A Rates of change
B Optimisation
C Modelling with calculus

OPENING PROBLEM

A skatepark designer proposes to build a half-pipe with cross-section given by
$h(x) = 0.014x^4 - 0.196x^3 + 1.039x^2 - 2.471x + 2.225$ metres, $0 \leqslant x \leqslant 7$.

Things to think about:

- **a** How high is the wall of the half-pipe?
- **b** Is the lowest point exactly in the middle of the half-pipe?
- **c** How steep are the end points of the half-pipe? What units could we use to measure this?



We have seen how differential calculus can be used to study curves and their tangents. In this Chapter we apply the techniques we have learnt to real-world problems of:

- rates of change
- optimisation.

A RATES OF CHANGE

When we first introduced derivative functions, we discussed how

$\boldsymbol{\dfrac{dy}{dx}}$ gives the **rate of change in $\boldsymbol{y}$ with respect to $\boldsymbol{x}$**.

- If y increases as x increases, then $\dfrac{dy}{dx}$ will be positive.
- If y decreases as x increases, then $\dfrac{dy}{dx}$ will be negative.

TIME RATES OF CHANGE

There are countless examples in the real world of quantities that vary with time.

For example:
- temperature varies continuously
- the height of a tree increases as it grows
- the prices of stocks and shares vary with each day's trading.

We can model these quantities using functions of time. The derivative of a function tells us the **rate** at which that quantity is varying.

For example:

- If $H(t)$ models the height above the ground of a person riding on a Ferris wheel, then $\dfrac{dH}{dt}$ or $H'(t)$ is the person's instantaneous rate of ascent.
  It might have units metres per second or $\text{m}\,\text{s}^{-1}$.
- If $C(t)$ models the capacity of a person's lungs, then $\dfrac{dC}{dt}$ or $C'(t)$ is the instantaneous rate of change in lung capacity as the person breathes.
  It might have units litres per second or $\text{L}\,\text{s}^{-1}$.

Example 1 **Self Tutor**

As a hot air balloon is inflated, the volume of air inside it after t minutes is given by $V = 2t^3 - 3t^2 + 10t + 2$ m^3 where $0 \leqslant t \leqslant 8$.

Find:

a the initial volume of air in the balloon

b the volume at $t = 8$ minutes

c $\frac{dV}{dt}$ and explain what it means

d the rate of increase in volume at $t = 4$ minutes.

a When $t = 0$, $V = 2$ m^3

Initially there were 2 m^3 of air in the balloon.

b When $t = 8$,
$$V = 2(8)^3 - 3(8)^2 + 10(8) + 2$$
$$= 914 \text{ m}^3$$

After 8 minutes there were 914 m^3 of air in the balloon.

c $\frac{dV}{dt} = 6t^2 - 6t + 10$ m^3 min^{-1}

This function tells us the rate at which the volume of air in the balloon is increasing after t minutes.

d When $t = 4$,
$$\frac{dV}{dt} = 6(4)^2 - 6(4) + 10$$
$$= 82 \text{ m}^3 \text{ min}^{-1}$$

Since $\frac{dV}{dt} > 0$, V is increasing.

The volume of air in the balloon is increasing at 82 m^3 min^{-1} at $t = 4$ minutes.

EXERCISE 12A.1

1 Find:

a $\frac{dM}{dt}$ if $M = t^3 - 3t^2 + 1$

b $\frac{dR}{dt}$ if $R = (2t + 1)^2$

2 **a** An area A is measured in cm^2, and time t is measured in seconds.

i What are the units for $\frac{dA}{dt}$?

ii What does $\frac{dA}{dt}$ mean?

b A volume V is measured in m^3, and time t is measured in minutes.

i What are the units for $\frac{dV}{dt}$?

ii What does $\frac{dV}{dt}$ mean?

3 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

a What is the current annual profit?

b Find $\frac{dP}{dt}$ and state its units.

c Find $\frac{dP}{dt}$ when $t = 8$. Explain what this value means.

4 In a hot, dry summer, water is evaporating from a desert oasis. The volume of water remaining after t days is $V = 2(50 - t)^2$ m^3. Find:

- **a** the average rate at which the water evaporates in the first 5 days
- **b** the instantaneous rate at which the water is evaporating at $t = 5$ days.

5 The number of bacteria in a dish is modelled by $B(t) = 0.3t^3 + 30t + 150$ million, where t is in hours, and $0 \leqslant t \leqslant 10$.

- **a** Find $B'(t)$ and state its meaning.
- **b** Find $B'(3)$ and state its meaning.
- **c** How do we know that $B(t)$ is increasing over the first 10 hours?

6 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres, where t is the time in seconds.

- **a** From what distance above the ground was the ball released?
- **b** Find $s'(t)$ and explain what it means.
- **c** Find t when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed:
  - **i** when released
  - **ii** at $t = 2$ s
  - **iii** at $t = 5$ s.

  State the significance of the sign of the derivative $s'(t)$ for each of these values.
- **f** How long will it take for the ball to hit the ground?

7 Joseph starts to pump air in his bicycle tyre. From the time 1 second after he starts pumping, the volume of air in the tyre is given by $V = -\dfrac{1200}{t} + 1380$ cm^3.

- **a** Find $\dfrac{dV}{dt}$ and state its units.
- **b** At what rate is air being pumped into the tyre after:
  - **i** 2 seconds
  - **ii** 6 seconds?
- **c** Graph $V(t)$.
- **d** Discuss what happens to $\dfrac{dV}{dt}$ as time increases.

8
- **a** A palm tree is grown from a seed in a pot. Its height after t years is given by $h = 0.1t^2 + 0.15t$ metres, where $0 \leqslant t \leqslant 4$.
  - **i** Find $\dfrac{dh}{dt}$ and state its units.
  - **ii** How fast was the tree growing after 1 year?
  - **iii** How tall was the tree after 4 years?
- **b** After 4 years, the palm tree is taken out of its pot and planted in the ground. Its height over time can now be modelled by $H = 20 - \dfrac{k}{t}$ metres, where k is a constant at $t \geqslant 4$.
  - **i** Find the value of k.
  - **ii** Find $\dfrac{dH}{dt}$.
  - **iii** Does the rate at which the tree is growing change when it is taken out of its pot and planted in the ground? Explain your answer.
  - **iv** Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 4$ and explain what this means.
  - **v** Find the height of the tree after 10, 20, and 50 years. Comment on your answer.
- **c** Graph the height of the tree for $0 \leqslant t \leqslant 10$ years.

9 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

a Find $\frac{dV}{dt}$, and draw the graph of $\frac{dV}{dt}$ against t.

b Find $\frac{dV}{dt}$ at $t = 40$ minutes and explain what this means.

c At what time was the outflow fastest?

GENERAL RATES OF CHANGE

Other rate problems can be treated in the same way as those involving time. However, we must always pay careful attention to the *units* of the quantities involved.

For example:

- If the cost of manufacturing x items is given by the **cost function** $C(x)$ dollars, then $\frac{dC}{dx}$ or $C'(x)$ is the **instantaneous rate of change in cost** with respect to the number of items made. In this case $\frac{dC}{dx}$ has the units dollars per item.
- The **profit** $P(x)$ in making and selling x items is given by $P(x) = R(x) - C(x)$ where $R(x)$ is the **revenue function** and $C(x)$ is the **cost function**.

  $\frac{dP}{dx}$ or $P'(x)$ represents the rate of change in profit with respect to the number of items sold.

Example 2 **Self Tutor**

The cost of producing x items in a factory each day is given by
$C(x) = -0.000\,013x^3 + 0.002x^2 + 15x + 2200$ dollars, where $0 \leqslant x \leqslant 600$.

a Find $C(0)$ and explain what it represents.

b Find $C'(x)$.

c Find $C'(150)$. Interpret this result.

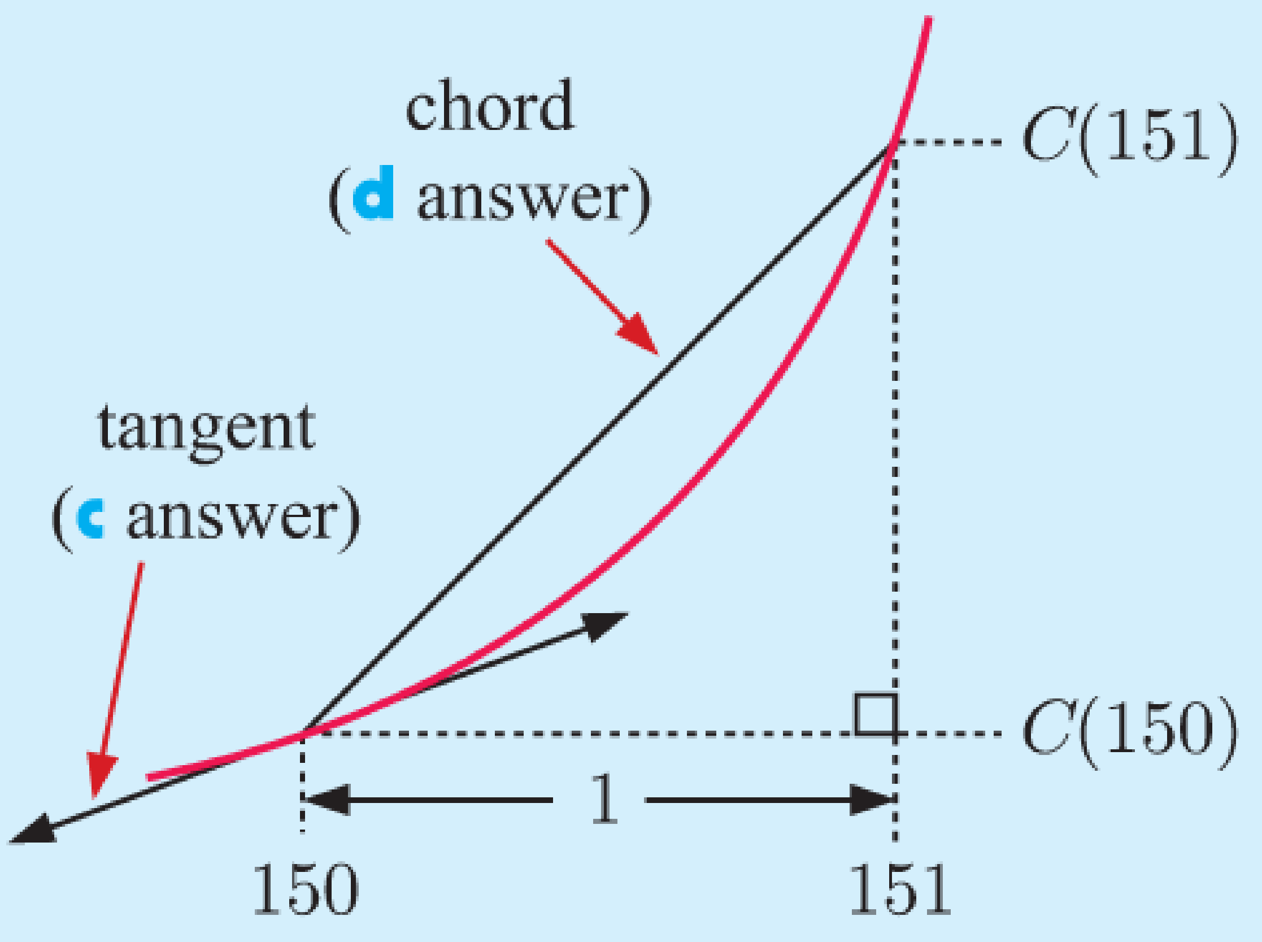
d Find $C(151) - C(150)$. Compare this with the answer in **c**.

a $C(0) = \$2200$
This value represents the fixed costs for the company to operate, even without manufacturing anything.

b $C'(x) = -0.000\,039x^2 + 0.004x + 15$

c $C'(150) = \$14.72$ per item
This is the rate at which the costs are increasing with respect to the production level x, when 150 items are made per day.
It gives an estimate of the cost of making the 151st item each day.

d $C(151) - C(150) \approx \$4465.84 - \$4451.13$
$\approx \$14.71$
This is the actual cost of making the 151st item each day, and is similar to the answer from **c**.



EXERCISE 12A.2

1 Find:

a $\dfrac{dT}{dr}$ if $T = r^2 - \dfrac{100}{r}$ **b** $\dfrac{dA}{dh}$ if $A = 2\pi h + \frac{1}{4}h^2$

2 If C is measured in pounds and x is the number of items produced, what are the units for $\dfrac{dC}{dx}$?

3 The cost function for producing x items each day is
$C(x) = -0.000\,007\,2x^3 + 0.0061x^2 + 18x + 14\,230$ dollars, where $0 \leqslant x \leqslant 1200$.

a Find $C(0)$ and explain what it represents.

b Find $C'(x)$ and explain what it represents.

c Find $C'(300)$ and explain what it estimates.

d Find the actual cost of producing the 301st item.

4 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 7800 + 7x - 0.0001x^2 \text{ dollars.}$$

a Find the marginal cost function $C'(x)$. **b** Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. Discuss your answer.

5 The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \dfrac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.

a Find the total cost of the journey if the average speed is:

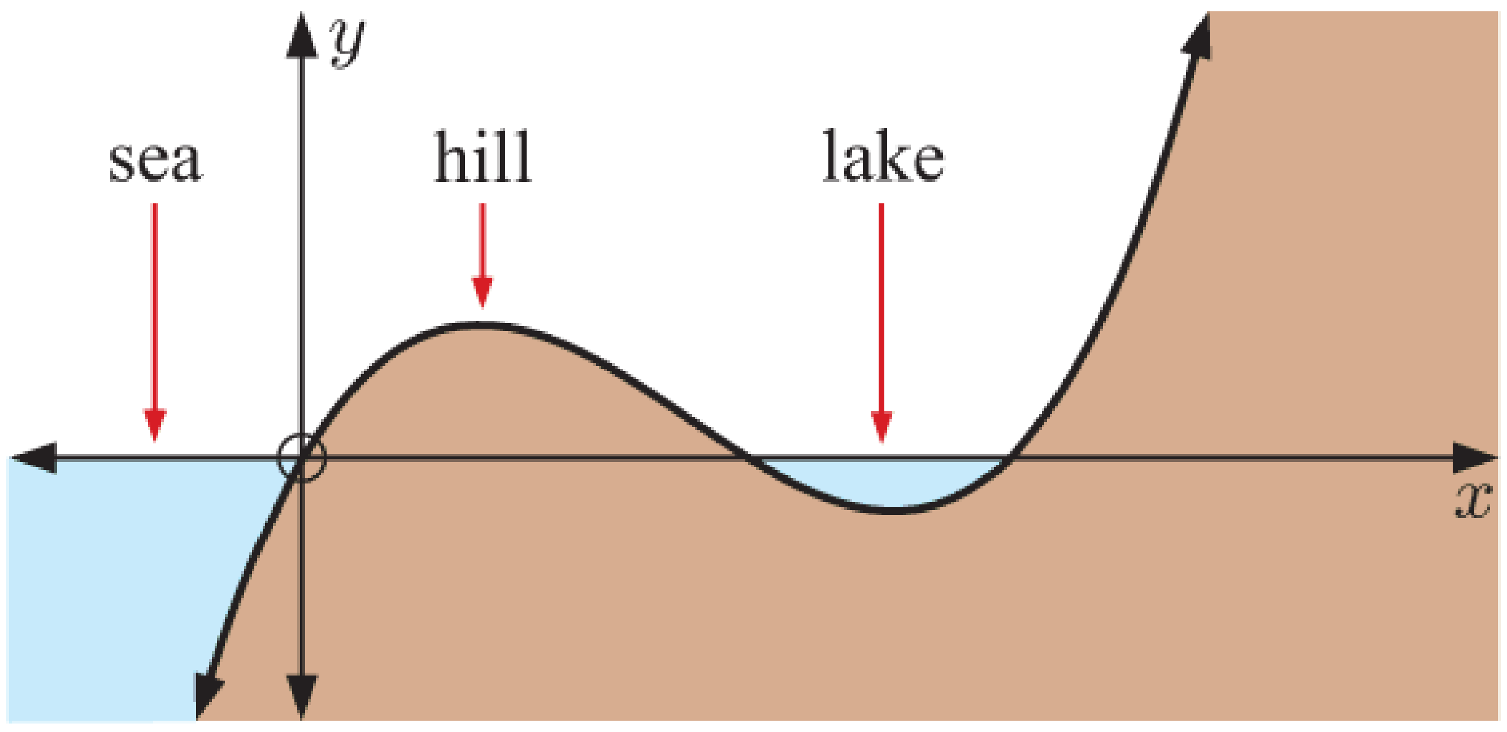
i 50 km h^{-1} **ii** 100 km h^{-1}.

b Find the rate of change in the cost of running the train for the average speed:

i 30 km h^{-1} **ii** 90 km h^{-1}.

c At what speed will the cost be a minimum?

6 Alongside is a land and sea profile where the x-axis is sea level.
The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.



a Find where the lake is located relative to the shore line of the sea.

b Find $\dfrac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake, and the depth at this point.

7 If a company produces x items per year, its profit is given by $P(x) = 5x - 2000 - \dfrac{x^2}{10\,000}$ dollars.

a Graph $P(x)$ using technology.

b Determine the production levels which create a profit.

c Find $P'(x)$ and explain what it means.

d Hence find x such that the profit is increasing.

8 The cost of producing x items is given by $C(x) = -0.000\,004x^3 + 10x + 3000$ dollars where $0 \leqslant x \leqslant 600$. Each item sells for \$24.

- **a** Find the revenue function $R(x)$.
- **b** Plot $C(x)$ and $R(x)$ on the same set of axes for $0 \leqslant x \leqslant 600$.
- **c** Find where $C(x) = R(x)$ and explain the significance of this result.
- **d** Find the profit function $P(x)$.
- **e** Find $P'(x)$.
- **f** Find $P'(120)$, and explain the significance of this result.

DISCUSSION

- Consider a sphere with radius r cm, surface area A cm^2, and volume V cm^3.
  - What does $\dfrac{dV}{dr}$ mean?
  - Find $\dfrac{dV}{dr}$.
  - Discuss why it is reasonable that $\dfrac{dV}{dr} \propto A$.
- For a cube with side length x cm, surface area A cm^2, and volume V cm^3, is there a similar relationship between $\dfrac{dV}{dx}$ and A?

B OPTIMISATION

Optimisation is the process of finding the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimal solution**.

We can find optimal solutions in several ways:

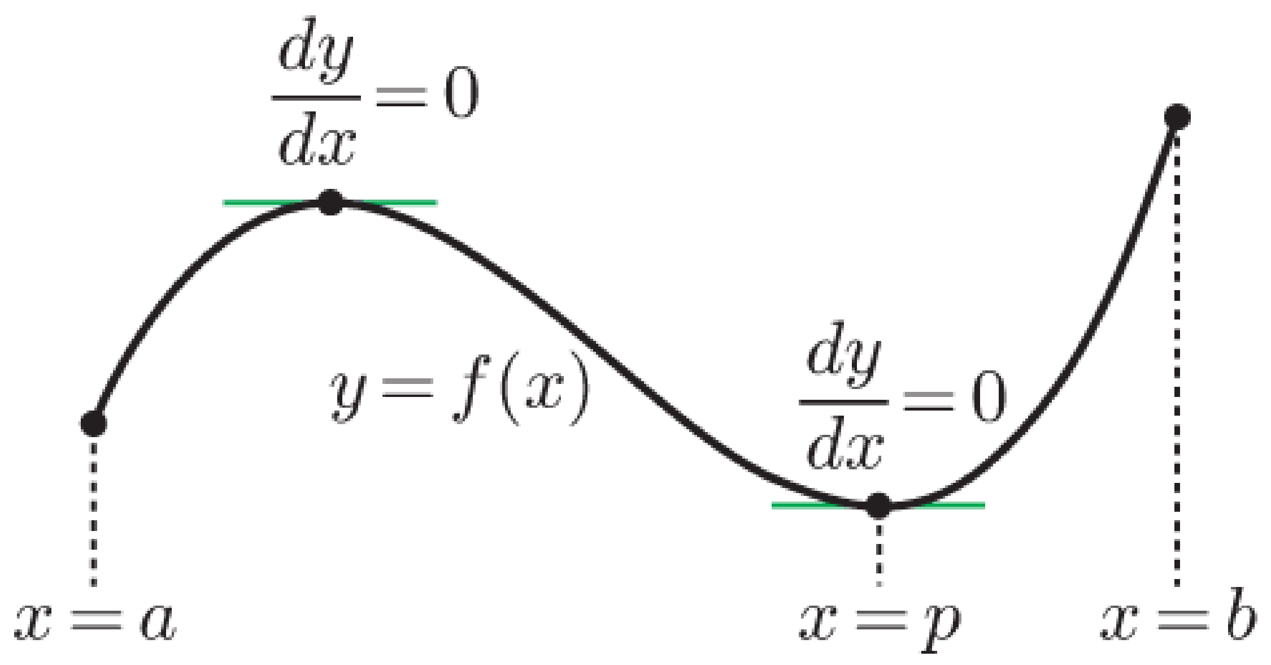
- using technology to graph the function and search for the maximum or minimum value
- using analytical methods such as the formula $x = -\dfrac{b}{2a}$ for the vertex of a parabola
- using differential calculus to locate the turning points of a function.

These last two methods are useful especially when exact solutions are required.

You should always be aware that:

> The maximum or minimum value does not always occur when the first derivative is zero.
>
> It is essential to also examine the values of the function at the end point(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the end point $x = b$.

The minimum value of y occurs at the local minimum $x = p$.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

Step 5: Identify the optimal solution, also considering end points where appropriate.

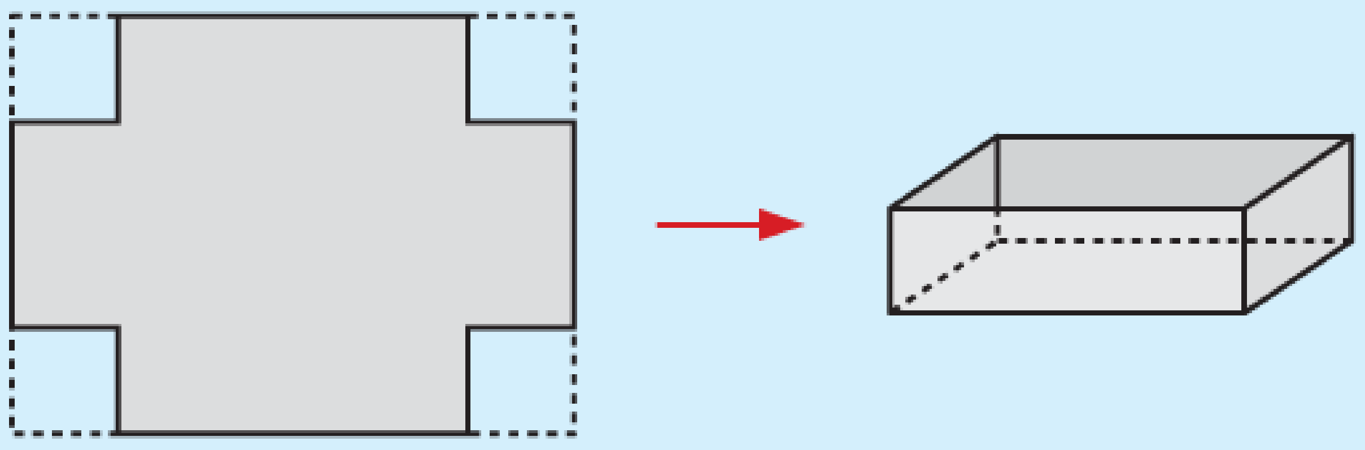
Step 6: Write your answer in a sentence, making sure you specifically answer the question.

Example 3

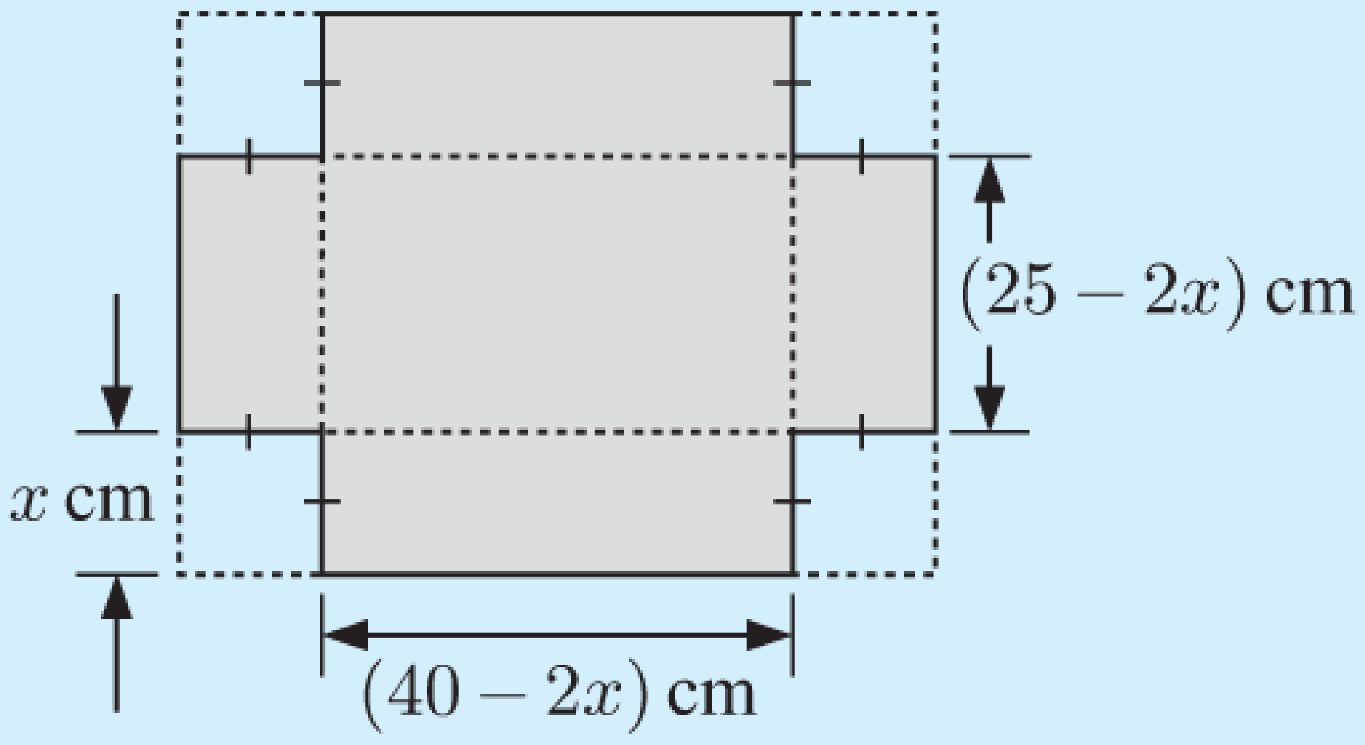
Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2: Volume = length × width × depth

$$\begin{aligned} &= (40-2x)(25-2x)x \\ &= (1000-80x-50x+4x^2)x \\ &= 1000x-130x^2+4x^3 \text{ cm}^3 \end{aligned}$$

Since the side lengths must be positive, $x > 0$ and $25-2x > 0$.

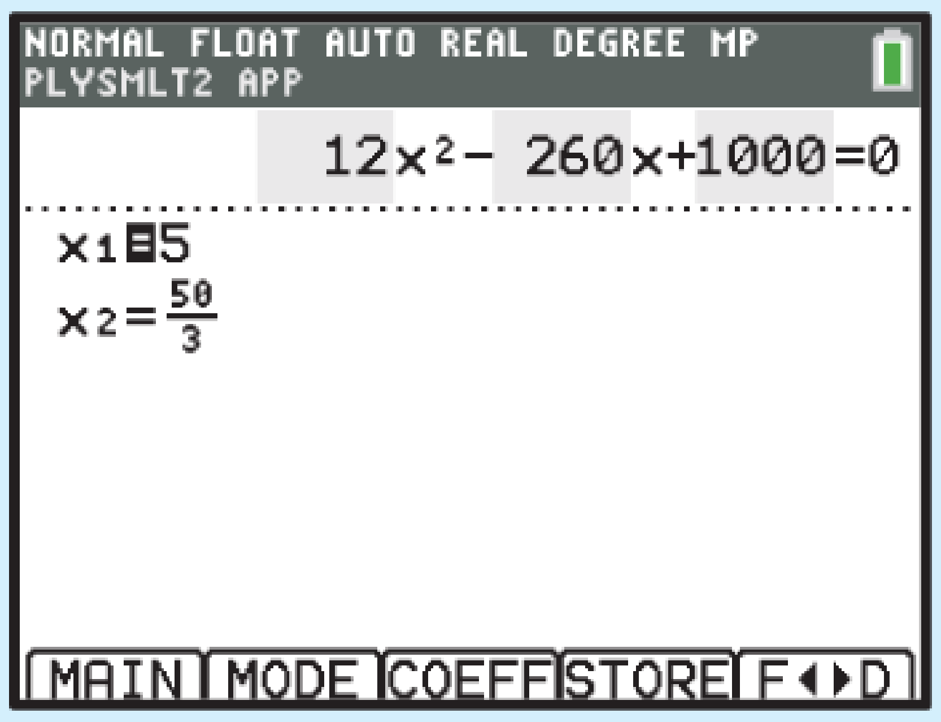
$\therefore \quad 0 < x < 12.5$

Step 3: $\dfrac{dV}{dx} = 12x^2-260x+1000$

Using technology, $\dfrac{dV}{dx} = 0$ when

$x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

$\therefore \quad x = 5$ as $0 < x < 12.5$

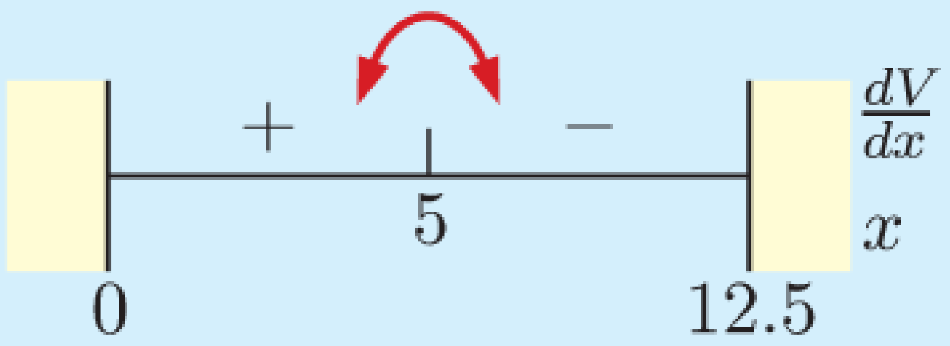


$12x^2-260x+1000=0$

$x_1=5$

$x_2=\frac{50}{3}$

Step 4: $\dfrac{dV}{dx}$ has sign diagram:

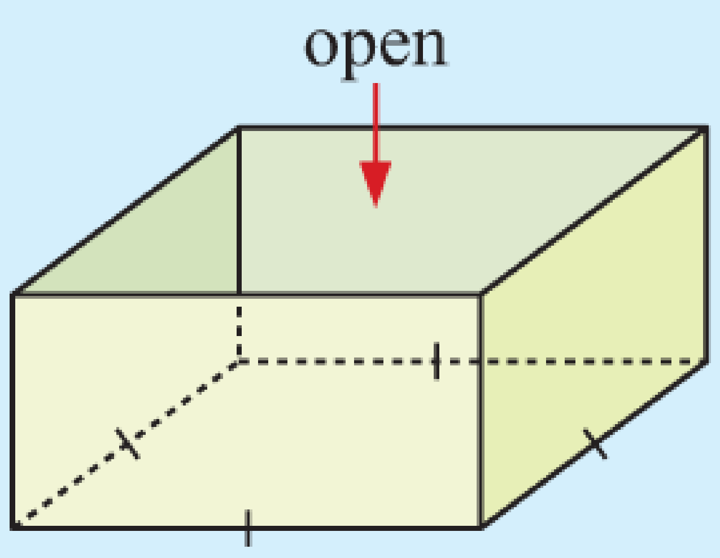


Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

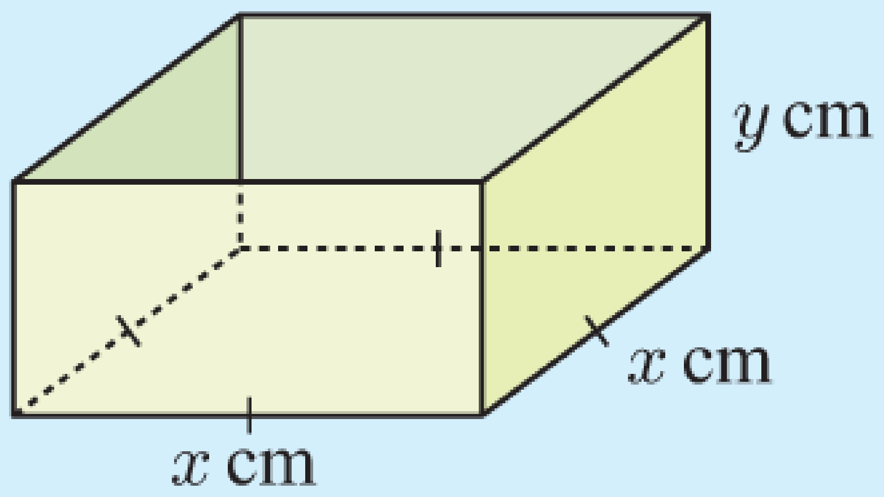
Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

Example 4 **Self Tutor**

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:



Let the base lengths be x cm and the depth be y cm.

The volume $V = \text{length} \times \text{width} \times \text{depth}$

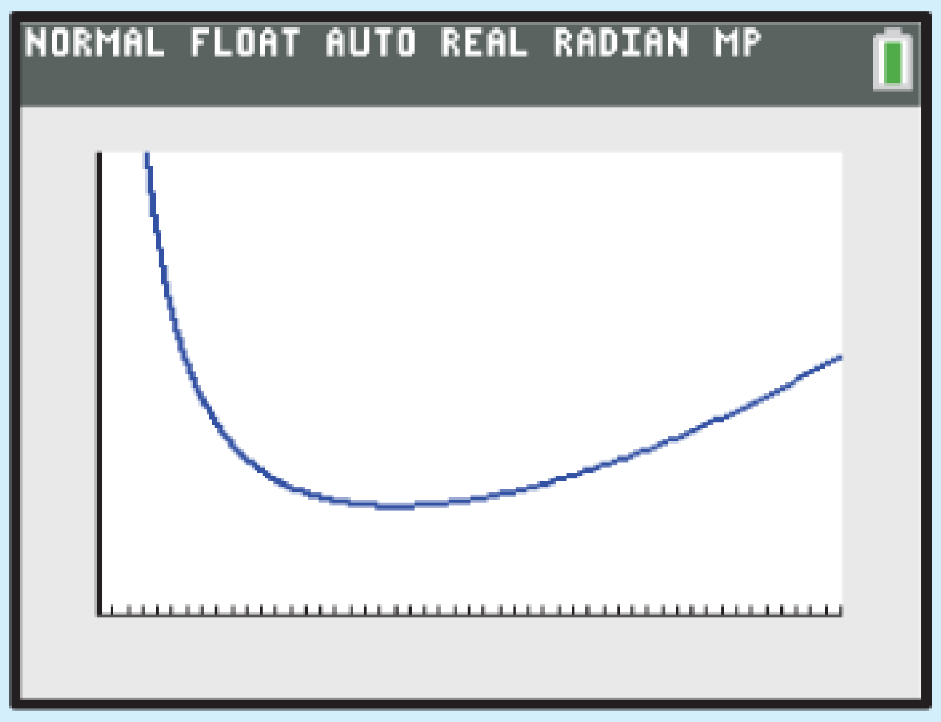
$\therefore\ V = x^2y$

$\therefore\ 4000 = x^2y$ .... (1) $\{1 \text{ litre} \equiv 1000 \text{ cm}^3\}$

Step 2: The inner surface area

$$\begin{aligned} A &= \text{area of base} + 4(\text{area of one side}) \\ &= x^2 + 4xy \\ &= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using (1)}\} \end{aligned}$$

$\therefore\ A(x) = x^2 + 16\,000x^{-1}$ where $x > 0$



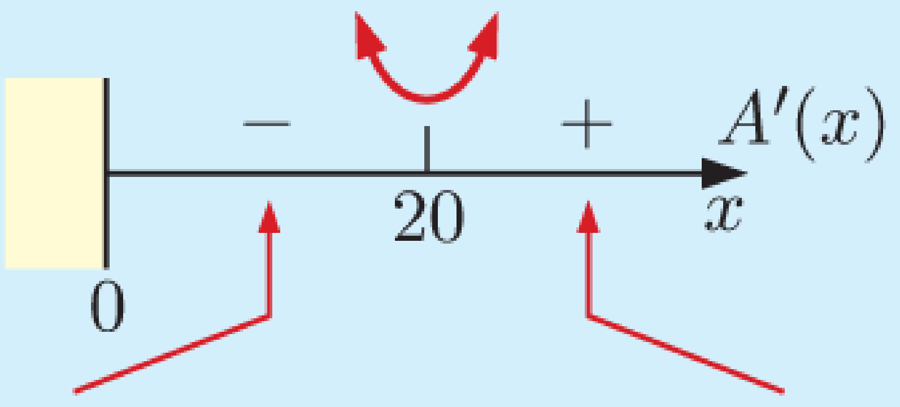
Step 3: $\therefore\ A'(x) = 2x - 16\,000x^{-2}$

$\therefore\ A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

$\therefore\ 2x^3 = 16\,000$

$\therefore\ x = \sqrt[3]{8000} = 20$

Step 4: $A'(x)$ has sign diagram:



If $x = 10$,

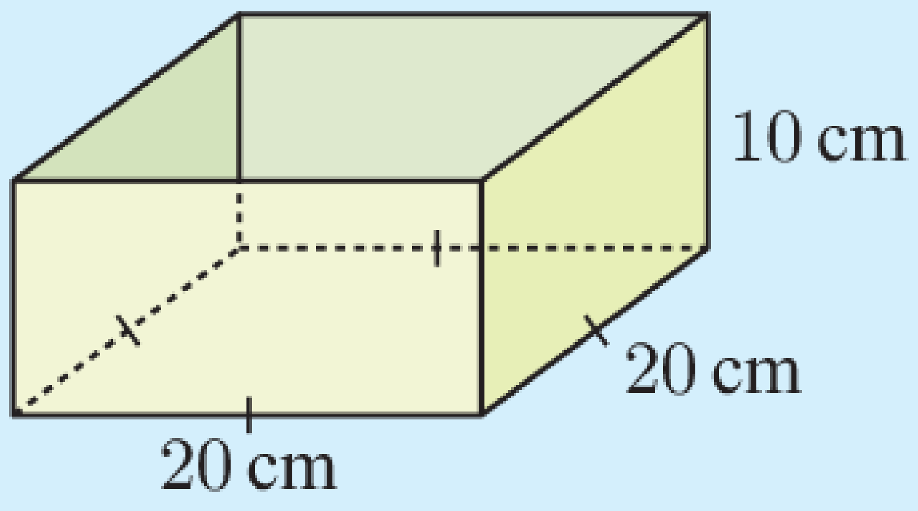
$$\begin{aligned} A'(10) &= 20 - \tfrac{16\,000}{100} \\ &= 20 - 160 \\ &= -140 \end{aligned}$$

If $x = 30$,

$$\begin{aligned} A'(30) &= 60 - \tfrac{16\,000}{900} \\ &\approx 60 - 17.8 \\ &\approx 42.2 \end{aligned}$$

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.

Step 6: The most economical shape has a square base 20 cm × 20 cm, and height 10 cm.

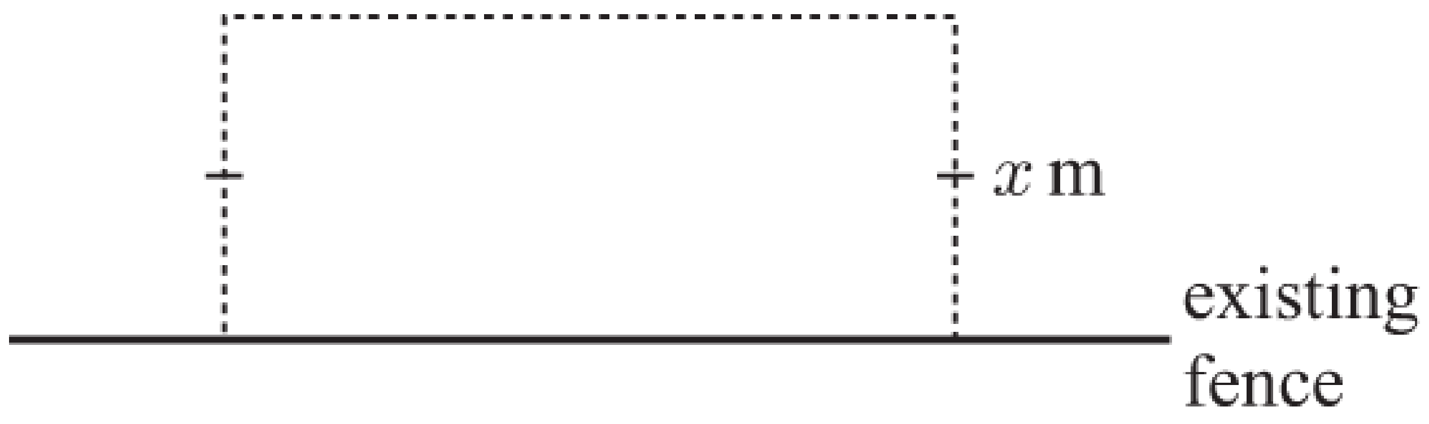


EXERCISE 12B.1

1 When a manufacturer makes x items per day, the profit function is $P(x) = -0.022x^2 + 11x - 720$ pounds. Find the production level that will maximise profits.

2 When a stone is thrown into the air, its height above the ground is given by $h(t) = 1.6 + 32t - 4.9t^2$ metres. Find the maximum height reached by the stone.

3 60 metres of fencing is used to build a rectangular enclosure along an existing fence. Suppose the sides adjacent to the existing fence are x m long.



a Show that the area A of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b Find the dimensions which maximise the area of the enclosure.

4 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

CONDITIONS OF EMPLOYMENT

a If the sides adjacent to the existing fence have length x m, show that the required length of wire netting to be purchased is $L = 2x + \frac{100}{x}$.

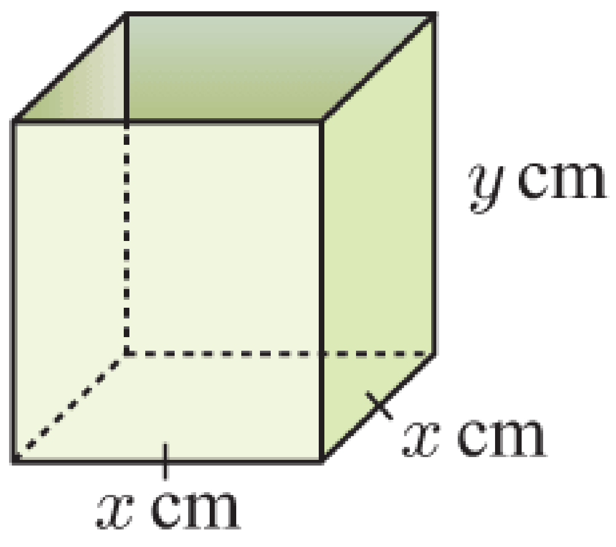
b Find the minimum value of L and the corresponding value of x when this occurs.

c Sketch the optimal situation, showing all dimensions.

5 The open rectangular box shown has a square base, and a fixed inner surface area of 108 cm^2.

a Explain why $x^2 + 4xy = 108$. **b** Hence show that $y = \frac{108 - x^2}{4x}$.

c Find a formula for the capacity C of the container, in terms of x only.

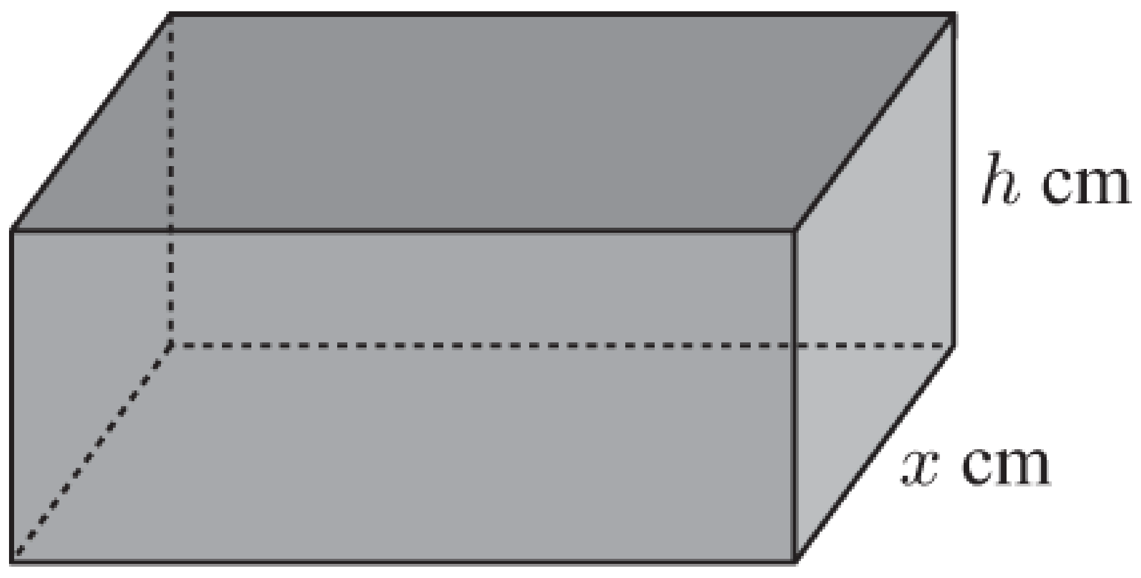


d Find $\frac{dC}{dx}$.

e Find the value of x such that $\frac{dC}{dx} = 0$.

f What size must the base be, in order to maximise the capacity of the box?

6 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of a box has dimensions in the ratio 2 : 1.



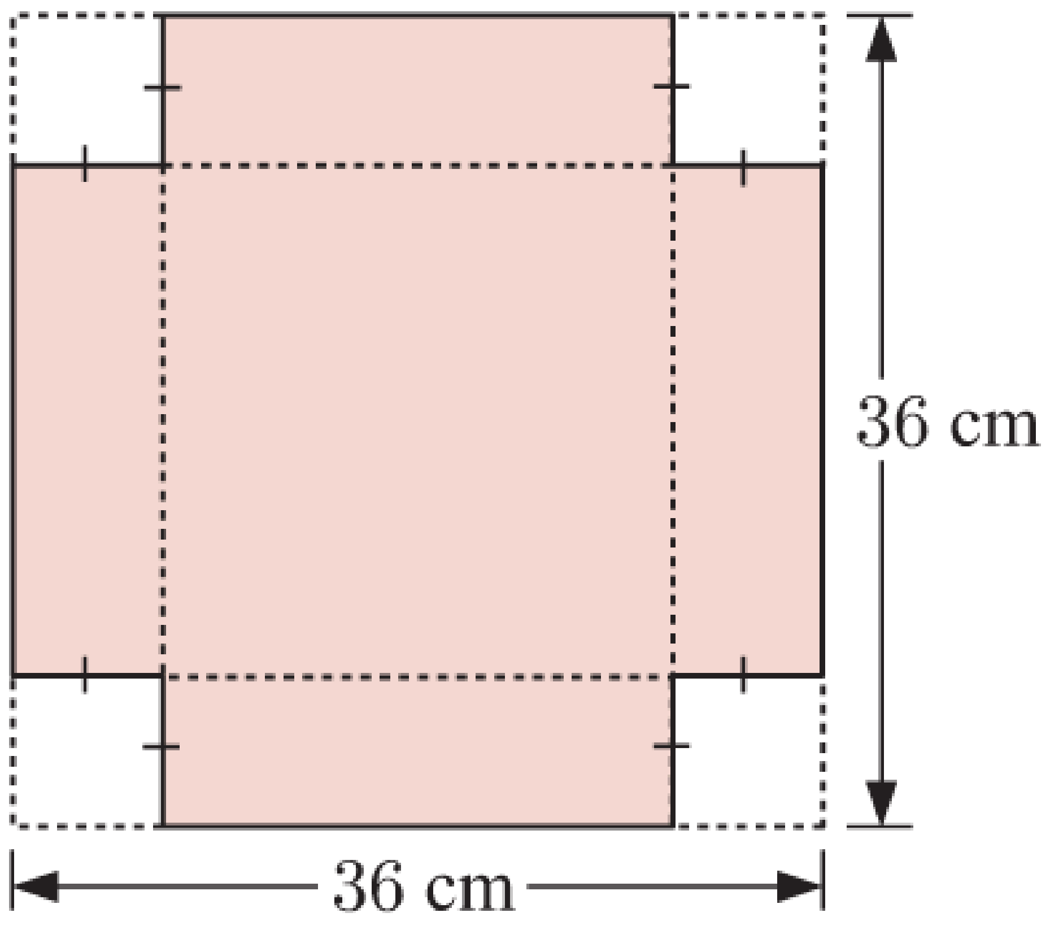
a Show that $x^2 h = 100$.

b Show that the inner surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2.

c Find the minimum inner surface area of the box and the corresponding value of x.

d Sketch the optimal box shape, showing all dimensions.

7 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.

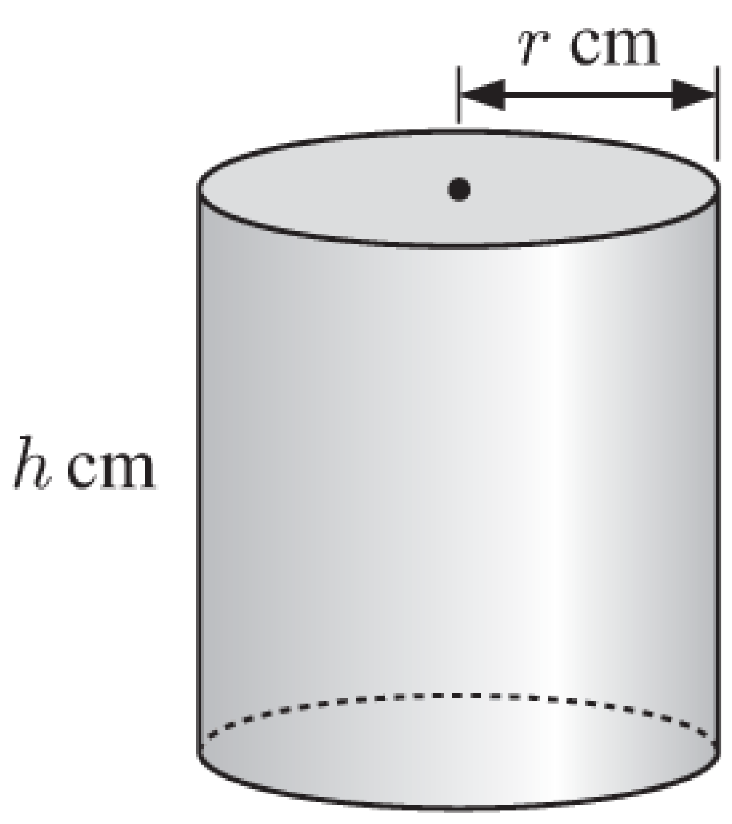


a Show that the volume of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

8 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ pounds, and for this production level each blanket may be sold for $\left(23 - \frac{1}{2}x\right)$ pounds.
How many blankets should be produced per day to maximise the total profit?

9 Brenda is designing a cylindrical tin can for a canned fruit company. The cans must have capacity 1 litre, and they must use as little metal as possible.

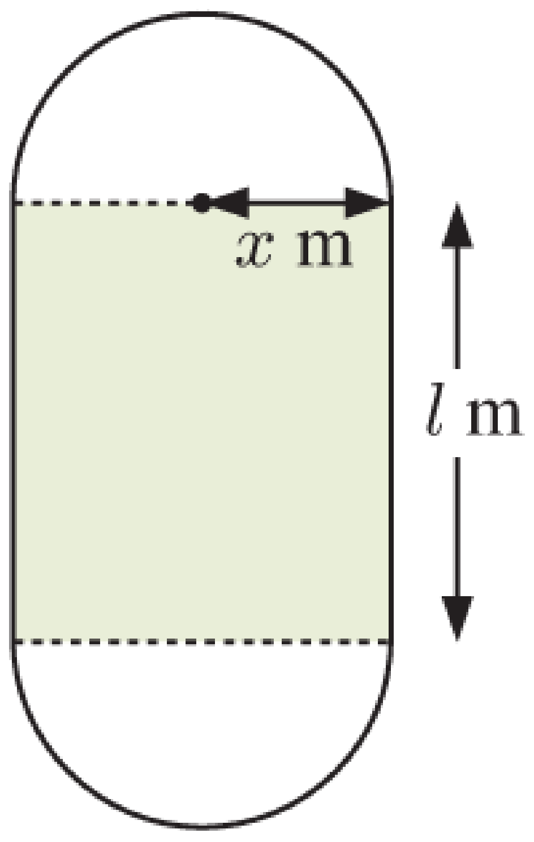


a Explain why the height h is given by $h = \dfrac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by
$A = 2\pi r^2 + \dfrac{2000}{r}$ cm^2.

c Find the dimensions of the can which make A as small as possible.

10 An athletics track has two "straights" of length l m, and two semi-circular ends of radius x m. The perimeter of the track is 400 m.



a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?

11 A 60 cm length of wire is bent into a rectangle with length x cm and width y cm.

a Write an expression for y in terms of x.

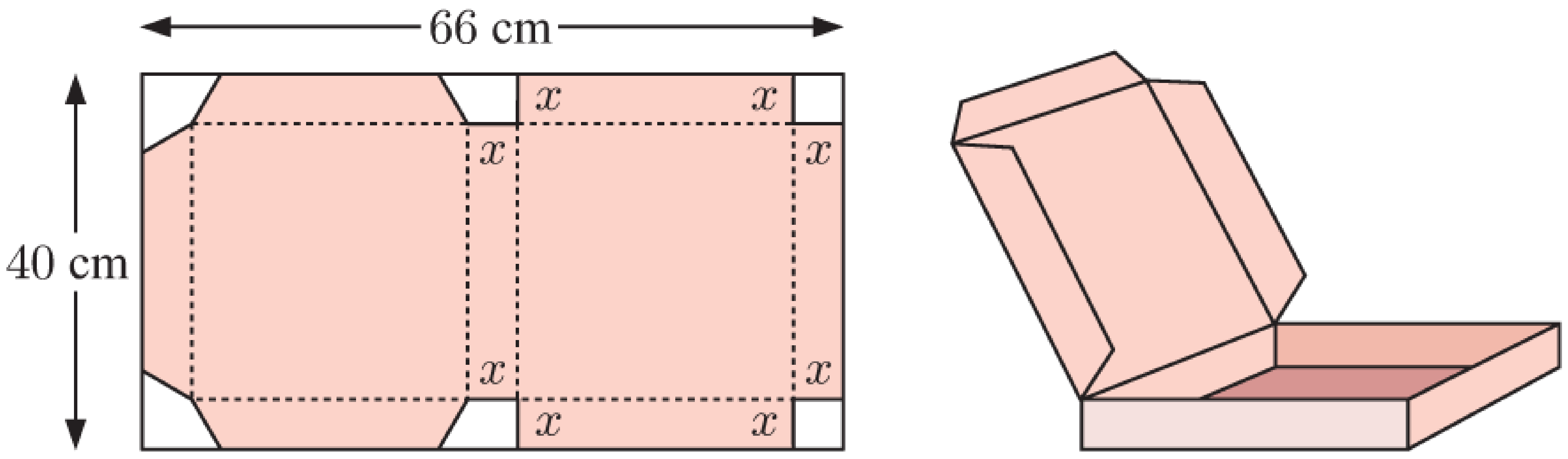
b Write an expression for the area $A(x)$ of the rectangle enclosed by the wire.

c Find $A'(x)$.

d Hence determine the value of x which maximises the area. What are the dimensions of the rectangle in this case?

12 Answer the **Opening Problem** on page **288**.

13 A closed pizza box is folded from a sheet of cardboard 66 cm by 40 cm. To achieve this, the cardboard is cut as shown in the diagram.



a Find the dimensions of the closed box in terms of x.

b Show that the volume of the box is given by $V = 3x^3 - 126x^2 + 1320x$ cm^3.

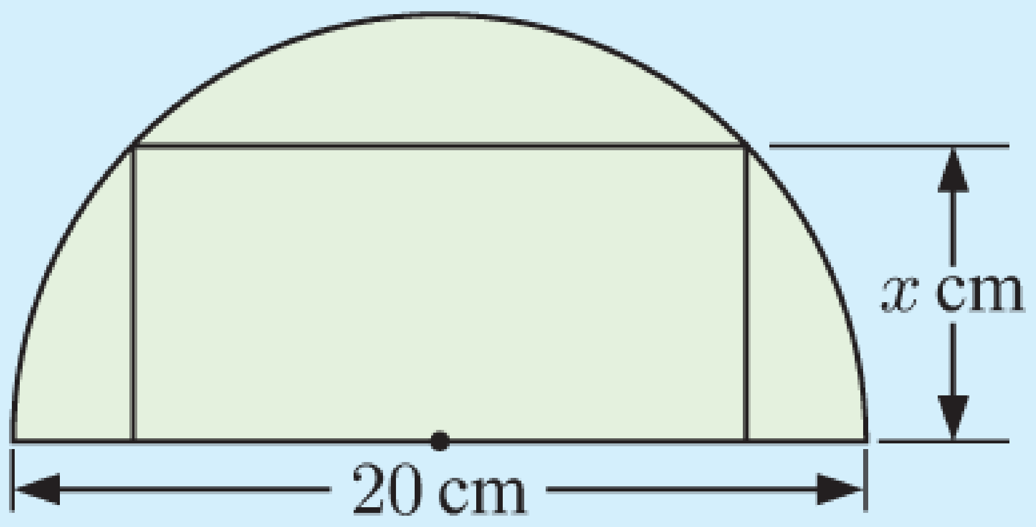
c Find $V'(x)$.

d Hence find the maximum possible volume of the box, and the dimensions of the box which give this result.

Example 5

Self Tutor

Consider a rectangle inscribed in a semi-circle of diameter 20 cm. Suppose it has height x cm.

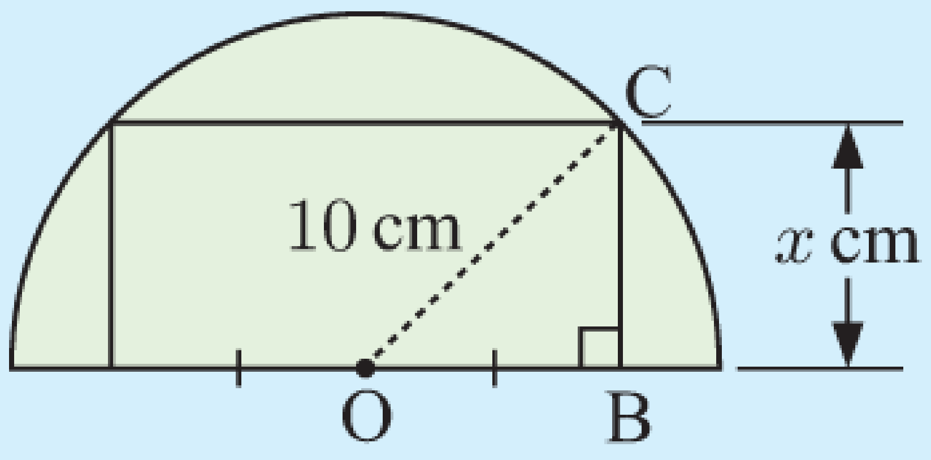


a Show that if the area of the rectangle is A, then $A^2 = 400x^2 - 4x^4$.

b Find $\frac{d}{dx}(A^2)$ and hence find the value of x which maximises A^2.

c Hence find the shape of the largest rectangle which can be inscribed in the semi-circle.

a



In $\triangle$OBC, $\text{OB}^2 + x^2 = 10^2$ {Pythagoras}

$\therefore$ $\text{OB} = \sqrt{100 - x^2}$ {as OB > 0 and $0 < x < 10$}

The rectangle has area $A = \text{length} \times \text{height}$

$$\therefore \; A = 2\sqrt{100 - x^2} \times x$$

$$\therefore \; A^2 = 4x^2(100 - x^2)$$

$$= 400x^2 - 4x^4$$

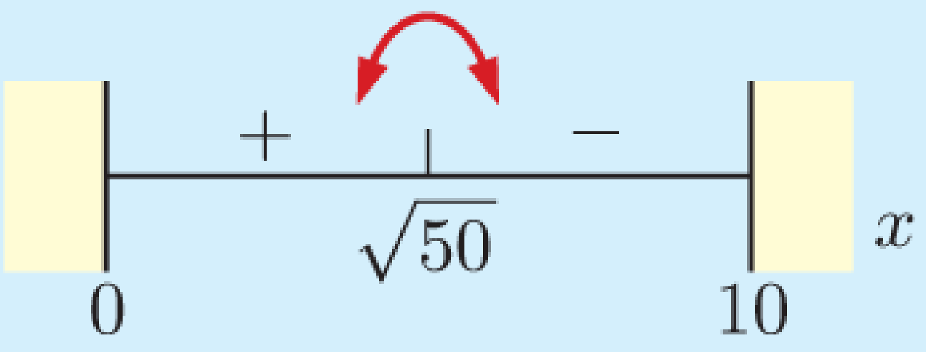
Since $A > 0$, we can maximise A by maximising A^2.

b $\frac{d}{dx}(A^2) = 800x - 16x^3$

$$= 16x(50 - x^2)$$

So, $\frac{d}{dx}(A^2) = 0$ when $x = 0$ or $\pm\sqrt{50}$.

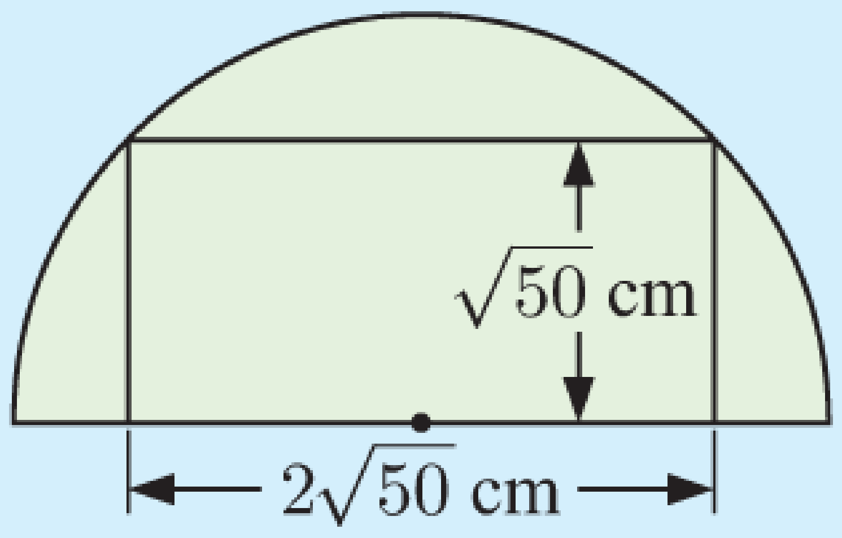
$\frac{d}{dx}(A^2)$ has sign diagram:



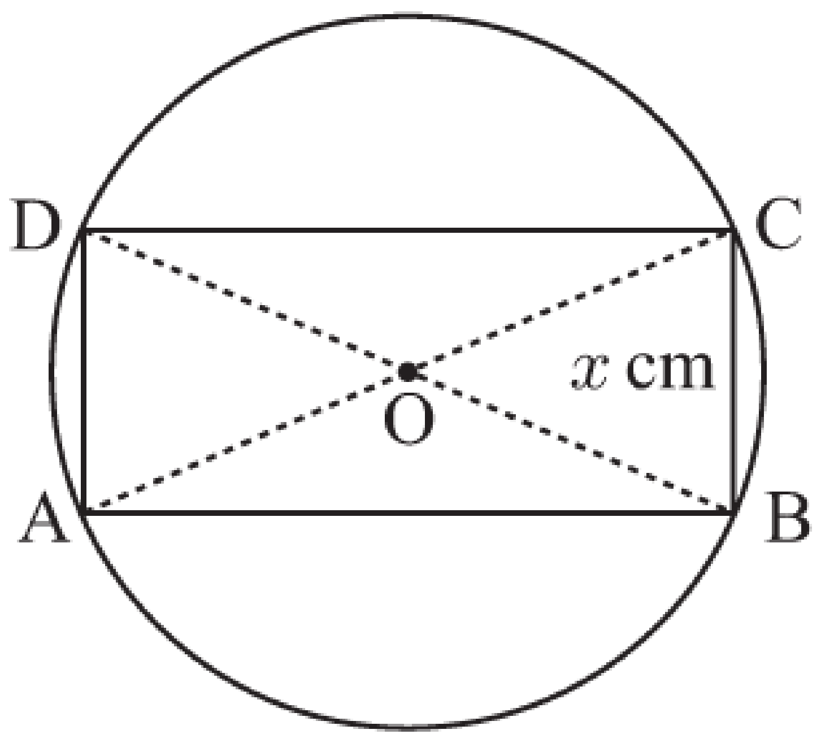
$\therefore$ A^2 is maximised when $x = \sqrt{50}$.

c When $x = \sqrt{50}$, $\text{OB} = \sqrt{100 - 50} = \sqrt{50}$ cm

So, the largest rectangle which can be inscribed is $2\sqrt{50}$ cm long and $\sqrt{50}$ cm high.



14 Consider a rectangle inscribed in a circle of diameter 10 cm. In the diagram alongside, suppose BC $= x$ cm.

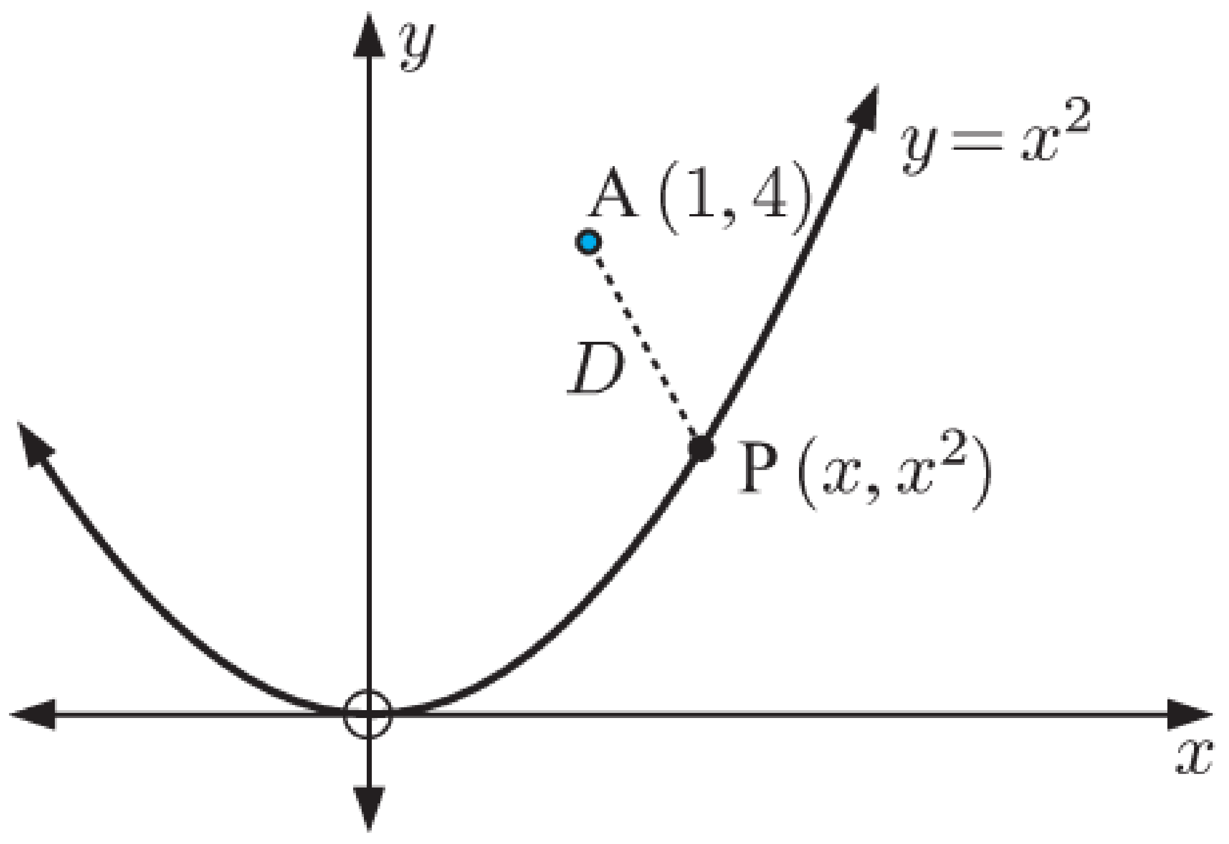


a Show that if the area of the rectangle is A, then $A^2 = 100x^2 - x^4$.

b Find $\frac{d}{dx}(A^2)$ and hence find the value of x which maximises A^2.

c Hence find the dimensions of the largest rectangle which can be inscribed in the circle.

15 Suppose the point A$(1, 4)$ is distance D from a general point P on the curve $y = x^2$.



a Explain why $D^2 = (x-1)^2 + (x^2-4)^2$.

b Find $\dfrac{d}{dx}(D^2)$ and hence find the value of x which minimises D^2.

c Hence find the closest point on $y = x^2$ to A, and its distance from A.

d Check your result by graphing $D = \sqrt{(x-1)^2 + (x^2-4)^2}$ on your calculator and finding its minimum turning point.

16 At 1:00 pm ship A leaves port P. It sails in the direction 30° east of north at 12 km h^{-1}. At the same time, ship B is 100 km due east of P, and is sailing at 8 km h^{-1} towards P.

a Show that the distance D between the two ships is given by
$D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ km, where t is the number of hours after 1:00 pm.

b Find the minimum value of D^2 for all $t \geqslant 0$.

c At what time, to the nearest minute, are the ships closest?

17 A mosquito flying with position M$(x,\, y,\, z)$ is repelled by scent emitted from the origin O. At time t seconds, the coordinates of the mosquito are given by $x(t) = 3 - t^2$, $y(t) = 2 + \sqrt{t}$, and $z(t) = 2 - \sqrt{t}$, where all distance units are metres.

a Show that if the mosquito is D m from the origin at time t, then $D^2 = t^4 - 6t^2 + 2t + 17$.

b *Hence* find the closest distance that the mosquito came to the source of the repellent.

OPTIMISATION USING TECHNOLOGY

We have seen how to use calculus to solve optimisation problems. However, we only know how to differentiate a limited number of functions.

If we do not know how to differentiate a function, we can use technology to find the optimal solution.

GRAPHING PACKAGE

EXERCISE 12B.2

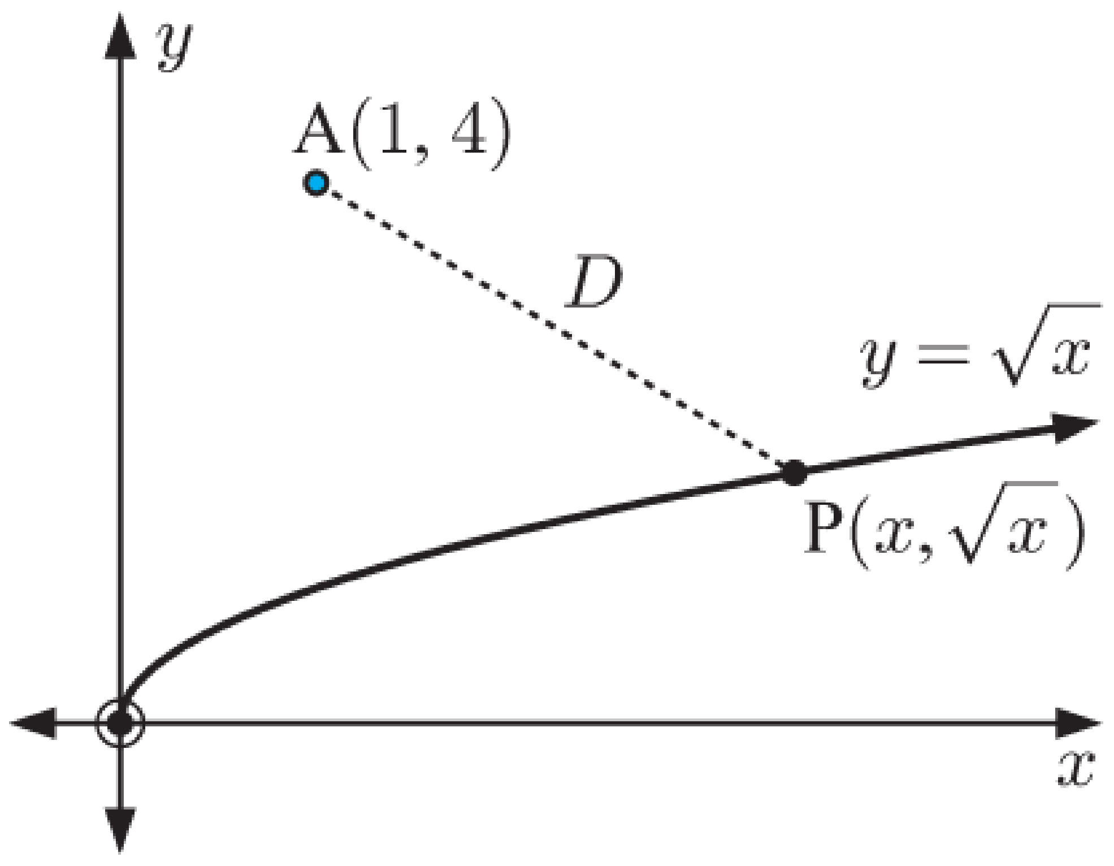
1 When a new anaesthetic is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.

a Graph $E(t)$ using technology.

b *Hence* draw a sign diagram for $E'(t)$.

c At what time is the anaesthetic most effective?

2 Suppose the point A$(1, 4)$ is distance D from a general point P on the curve $y = \sqrt{x}$.



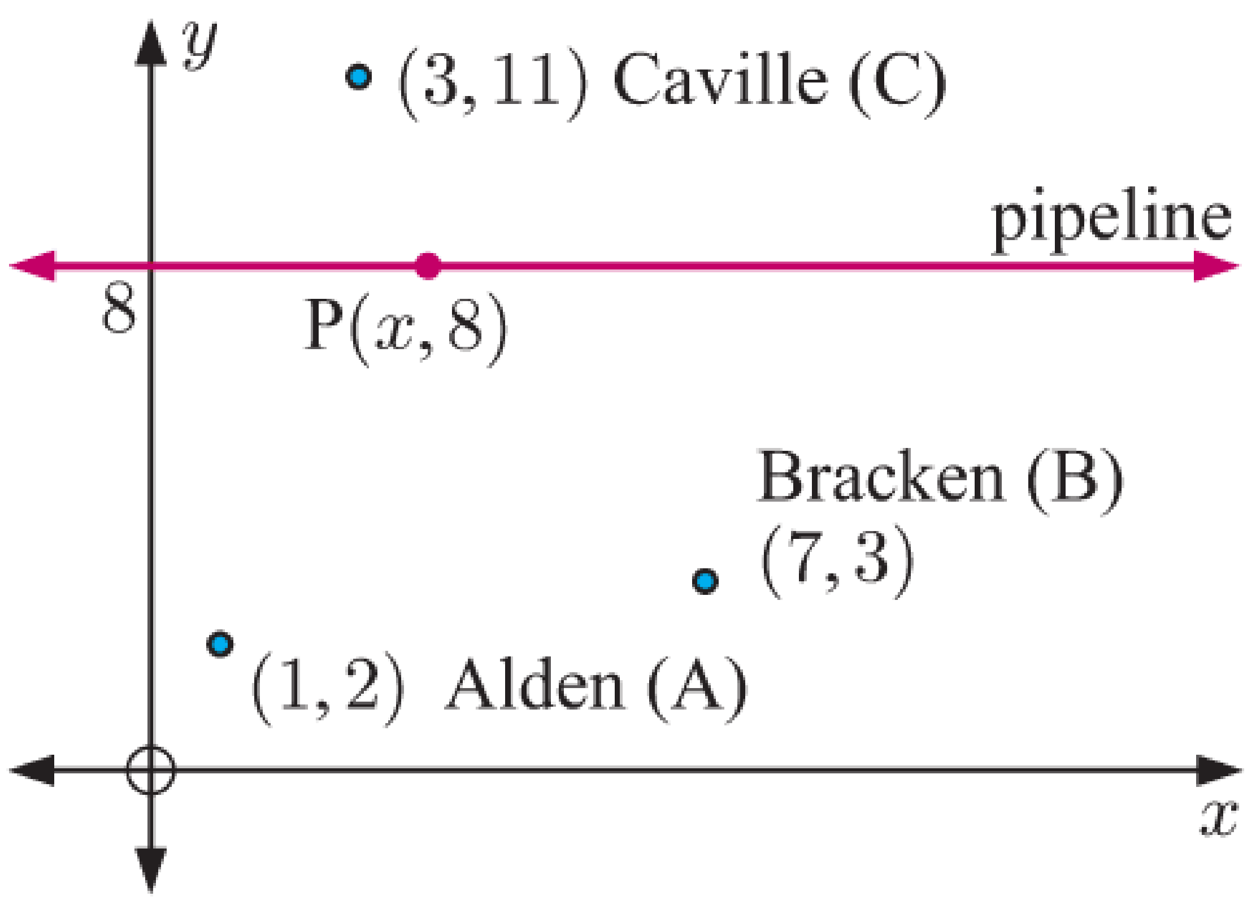
a Explain why $D = \sqrt{(x-1)^2 + (\sqrt{x}-4)^2}$ units.

b Sketch the graph of D against x for $0 \leqslant x \leqslant 8$.

c Find the smallest value of D and the value of x where it occurs.

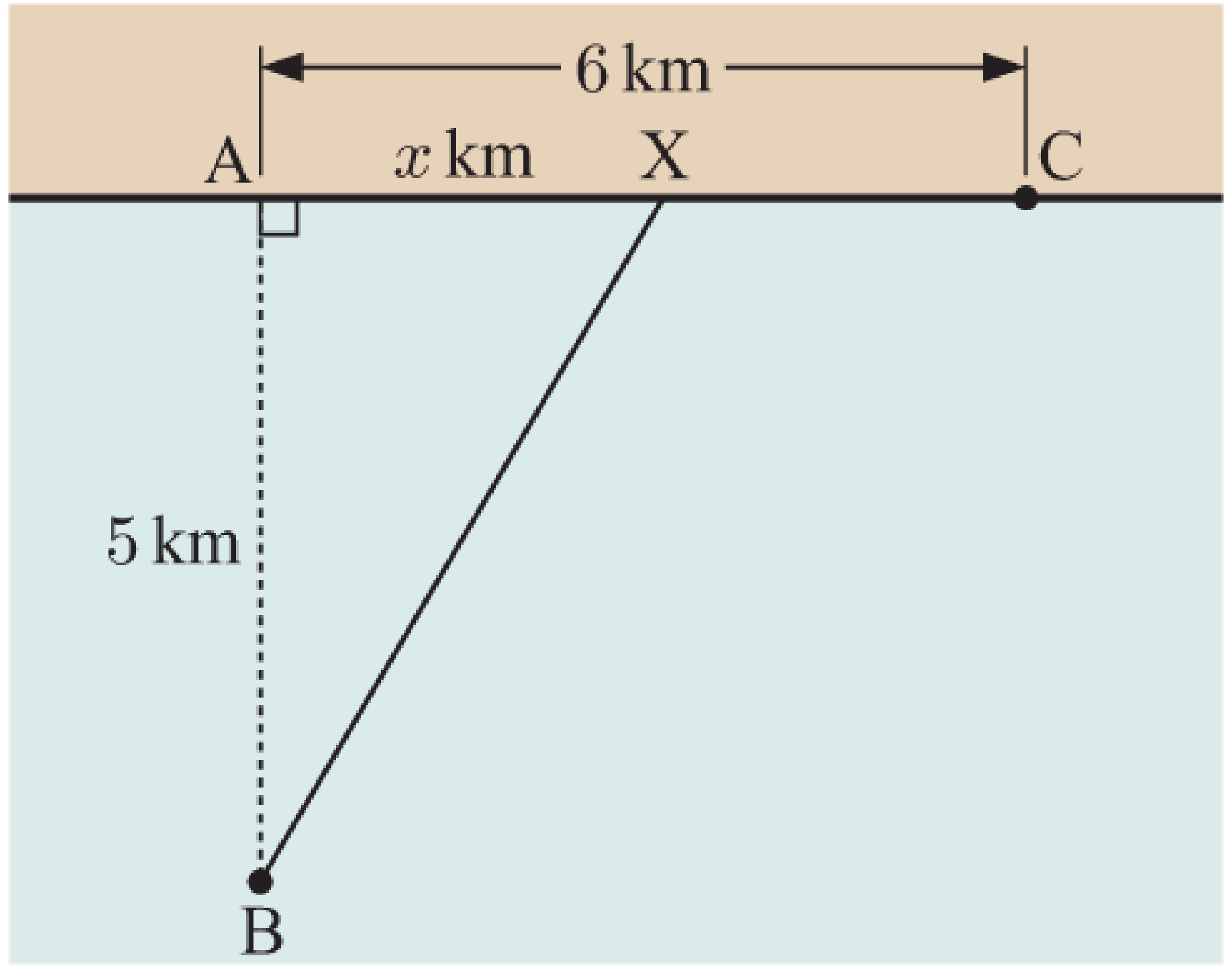
d What does this tell you about $\dfrac{dD}{dx}$?

3 Three towns and their grid references are marked on the diagram. A pumping station is to be located at P on the pipeline, to pump water to the three towns. The grid units are kilometres.



- **a** Show that the distance PC is given by PC $= \sqrt{(x-3)^2+9}$.
- **b** Find formulae for the distances PA and PB.
- **c** Write a formula for the sum of the distances $S = \text{PA} + \text{PB} + \text{PC}$ in terms of x.
- **d** Draw the graph of S against x.
- **e** *Hence* draw a sign diagram for $S'(x)$.
- **f** Where should P be located to minimise the total length of connecting pipe needed?

4 B is a boat 5 km out at sea from A. [AC] is a straight sandy beach, 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that AX $= x$ km.

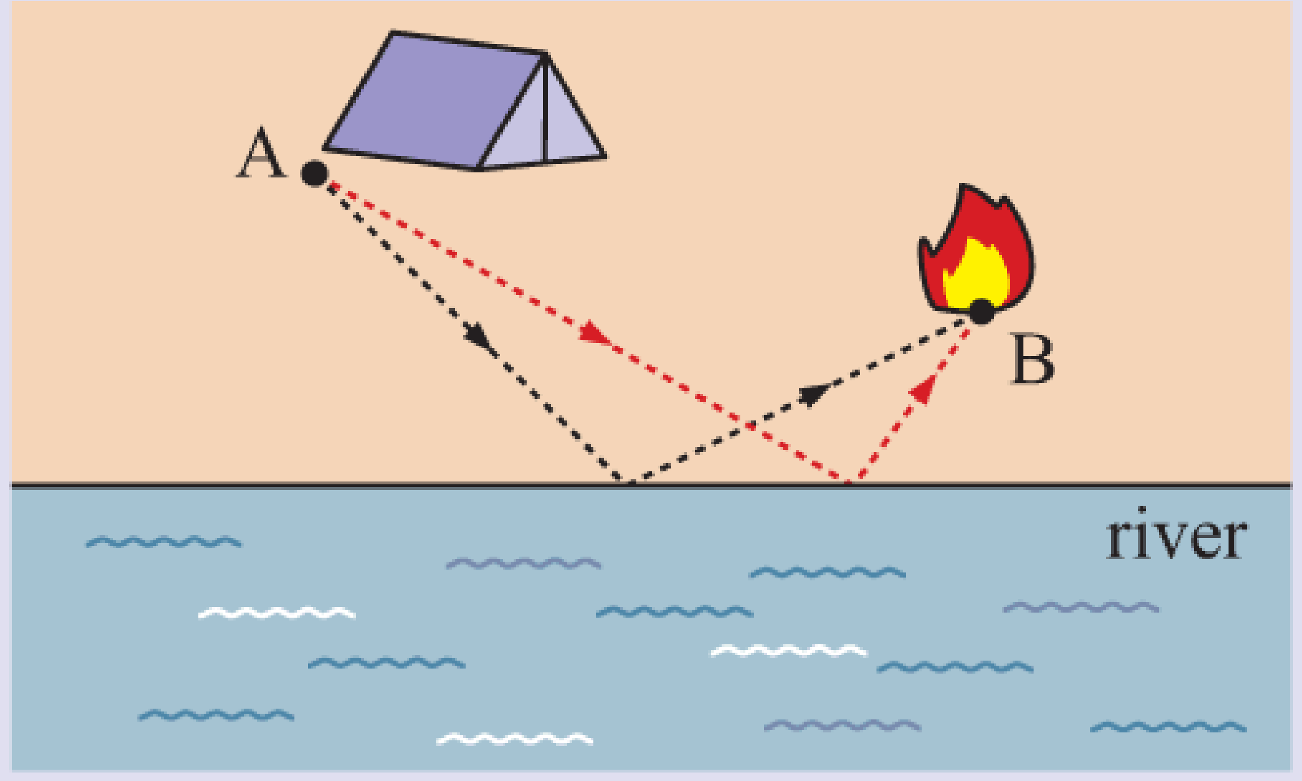


- **a** Explain why $0 \leqslant x \leqslant 6$.
- **b** Show that the *total time* Peter takes to row to X and then run along the beach to C, is given by
  $$T(x) = \frac{\sqrt{x^2+25}}{8} + \frac{6-x}{17} \text{ hours}, \quad 0 \leqslant x \leqslant 6.$$
- **c** Use technology to draw the graph of $T(x)$.
- **d** *Hence* find x such that $\dfrac{dT}{dx} = 0$. Explain the significance of this value.

THEORY OF KNOWLEDGE

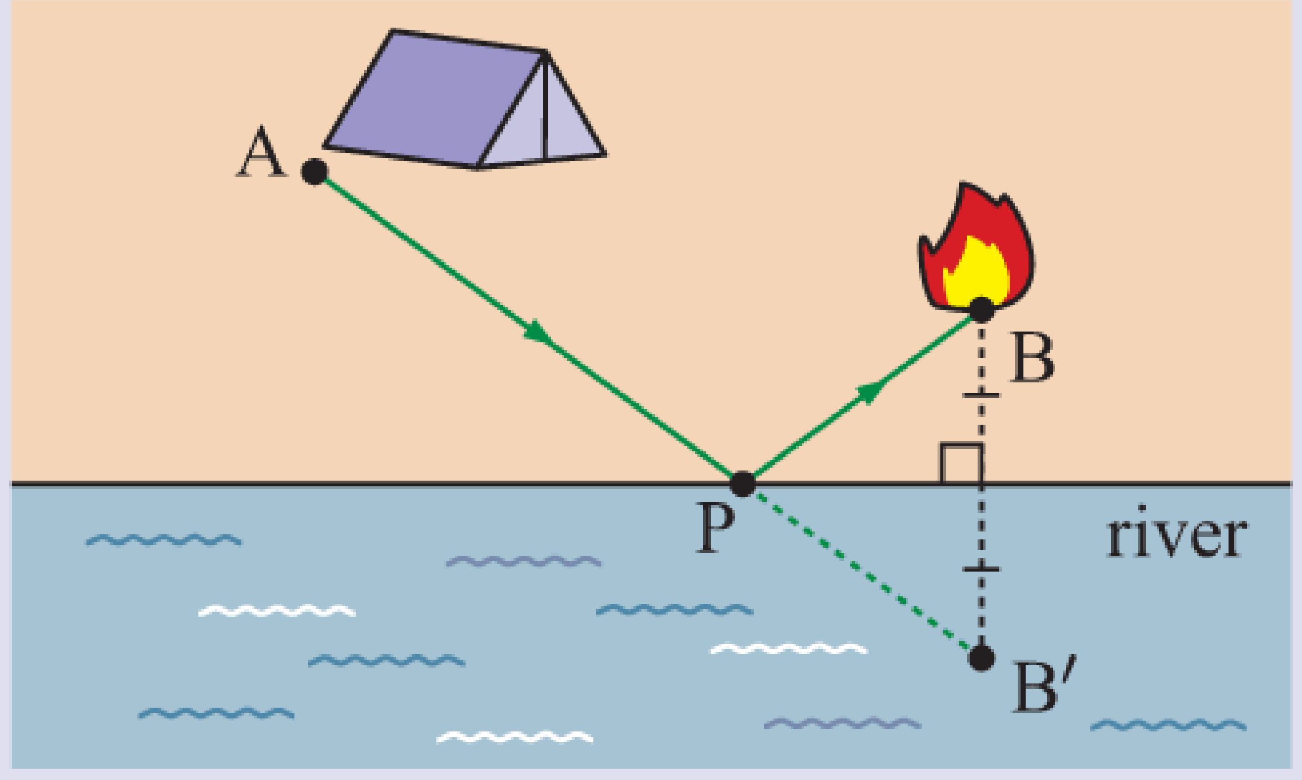
Suppose you are camping next to a river, and you pitch your tent at A. Suddenly you notice your campfire at B is out of control. You pick up your bucket and run to the river to get some water, then head to the fire to put it out.

Time is critical so you must take the shortest path. Where, on the river bank, should you fill your bucket?



To solve this problem, we reflect the point B in the river, creating the point B$'$. Each point on the river is equidistant from B and B$'$.

The shortest path from A to B$'$ is clearly the line [AB$'$]. Therefore, you should fill your bucket at the point P where the line [AB$'$] intersects the river.



1 Is optimisation a mathematical principle?

2 Is mathematics an intrinsic or natural part of other subjects?

C MODELLING WITH CALCULUS

We have seen how derivatives can be interpreted as rates of change, and how zeros of a derivative correspond to stationary points of a function. We can use knowledge of rates and turning points to help construct mathematical models.

Example 6 **Self Tutor**

The cost of manufacturing the parts for x robots each day is given by $C(x) = ax^3 + bx^2 + cx + d$ dollars, where the maximum output for the factory each day is 200 robots. The fixed costs for keeping the factory running amount to \$3400 each day. The parts for the first robot each day cost \$449. The cost of manufacturing parts for 100 robots each day is \$38 300, and the cost of the 101st set of parts is \$246.

a Find the cost function for producing the robot parts, using $C'(0) \approx C(1) - C(0)$ and $C'(100) \approx C(101) - C(100)$.

b Hence find the cost for producing the parts for 150 robots each day.

a The maximum output is 200 sets of parts, so the domain of $C(x)$ is $0 \leqslant x \leqslant 200$.

The fixed costs are $C(0) = \$3400$, so $d = 3400$.

Differentiating with respect to x, $C'(x) = 3ax^2 + 2bx + c$.

Now $C'(0) \approx 449$, so $c \approx 449$

and $C'(100) \approx 246$, so $30\,000a + 200b + 449 = 246$

$$\therefore \ 30\,000a + 200b = -203 \quad .... \ (1)$$

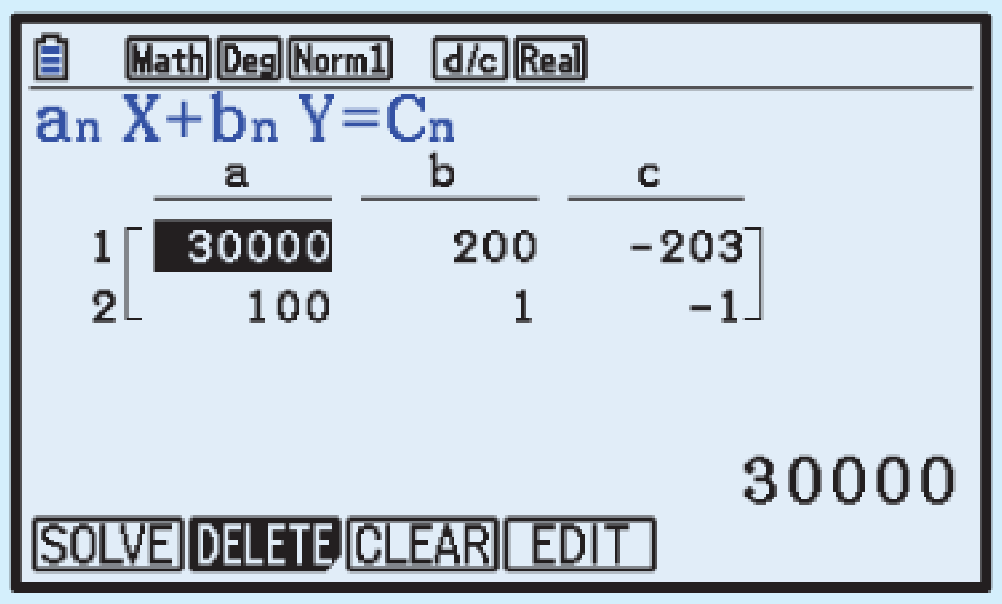
$C(100) = 38\,300$, so $1\,000\,000a + 10\,000b + 100c + 3400 = 38\,300$

$$\therefore \ 1\,000\,000a + 10\,000b = -10\,000$$

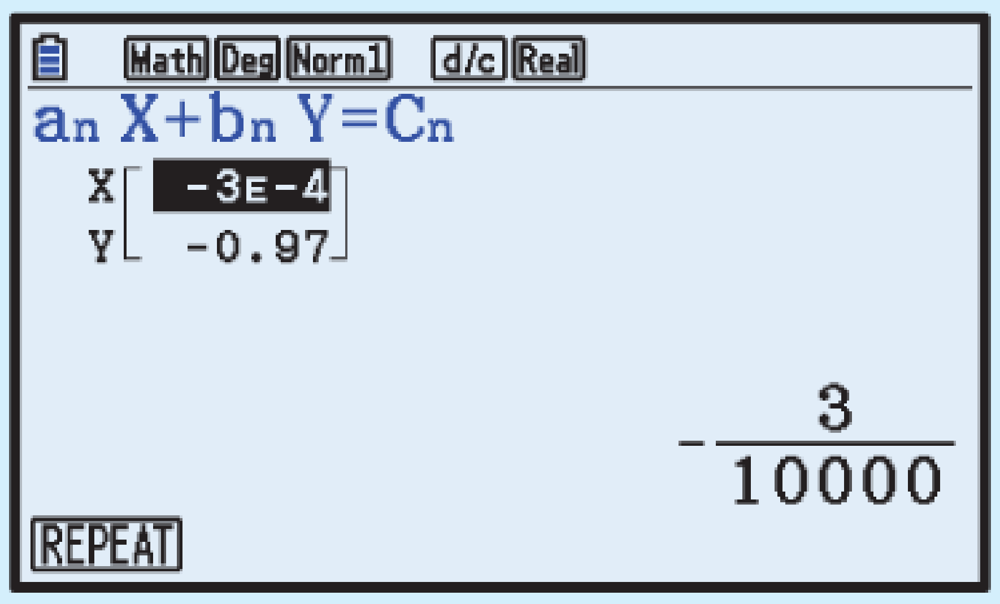
$$\therefore \ 100a + b = -1 \quad .... \ (2)$$

We solve (1) and (2) simultaneously using technology.

$\therefore$ $a \approx -0.0003$ and $b \approx -0.97$



```
     a       b      c
1 [ 30000   200   -203 ]
2 [   100     1     -1 ]
                    30000
```



```
X [ -3E-4 ]
Y [ -0.97 ]
                     3
                 - -----
                   10000
```

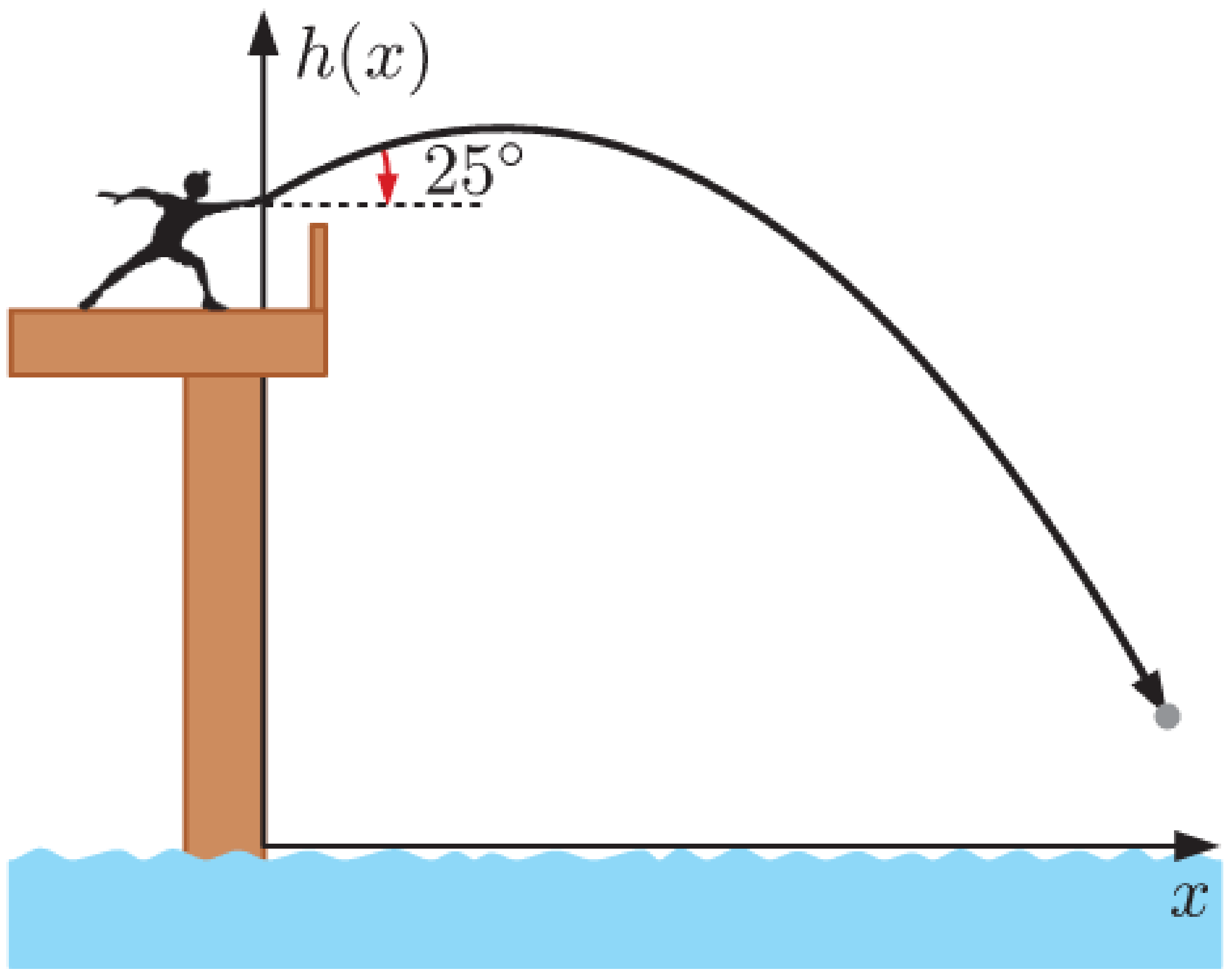
So, $C(x) = -0.0003x^3 - 0.97x^2 + 449x + 3400$, $0 \leqslant x \leqslant 200$.

b $C(150) = -0.0003(150)^3 - 0.97(150)^2 + 449(150) + 3400 = 47\,912.5$

$\therefore$ it costs about \$47 900 to produce the parts for 150 robots each day.

EXERCISE 12C

1 A stone is thrown from a bridge with initial trajectory $25°$ above horizontal. At the time when it has travelled x m horizontally, the height of the stone above the water under the bridge is given by $h(x) = ax^2 + bx + c$ metres. The stone is thrown from 3 m above the water, and it reaches its maximum height when $x = 5$.



- **a** State $h(0)$ and hence find c.
- **b** Find $h'(x)$.
- **c** Explain why $h'(0) = \tan 25°$, and hence find b.
- **d** Find the value of x when the stone lands in the water.

2 The cost of manufacturing the parts for x cars each day is given by $C(x) = ax^3 + bx^2 + cx + d$ euros. The maximum output for the factory each day is 140 cars. The fixed costs for keeping the factory running total €24 500 each day. The parts for the first car each day cost €3280. The cost of manufacturing parts for 80 cars each day is €294 000, and the cost of the 81st set of car parts is €2320.

- **a** Find the cost function for producing the car parts, using $C'(0) \approx C(1) - C(0)$ and $C'(80) \approx C(81) - C(80)$.
- **b** Hence find the cost of producing the parts for 120 cars each day.

3 The introduction of the automobile in the early 20th century had a dramatic effect on the population of domesticated horses, which had been used extensively for agriculture and transportation. In the United States of America there were 21.5 million horses in 1900, and the population peaked at 26.5 million in 1915. By 1950, the population had decreased to only 7.6 million.

Let $P(t) = at^3 + bt^2 + ct + d$ describe the population in millions of horses, in the United States of America t years after 1900.

- **a** State $P(0)$ and hence find d.
- **b** Find $P'(t)$.
- **c** State the value of t for which $P'(t) = 0$.
- **d** Construct three linear equations for a, b, and c. Hence find $P(t)$.
- **e** Estimate the population of horses in:
  - **i** 1930
  - **ii** 1960.
- **f** Discuss the limitations of your model.

4 Concerned with parking, transportation, catering, and all aspects of the local fair, the town mayor decided to model the attendance at the fair using the function

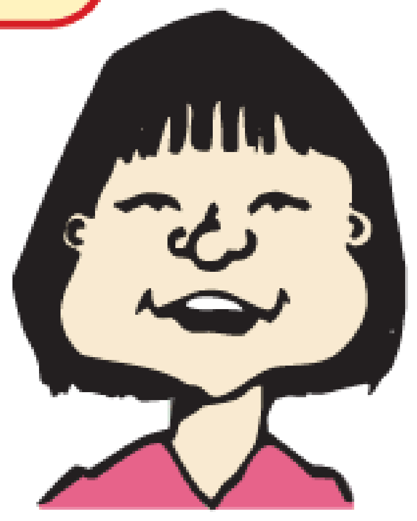
$A(t) = at^3 + bt^2 + ct + d$, where t is the number of hours after 8 am.

The fair is open from 8 am to 6 pm. At 10 am, the attendance was increasing at the rate of 1770 people per hour. At 4 pm, the attendance was decreasing at the rate of 1800 people per hour.

- **a** State *four* conditions for $A(t)$ and $A'(t)$ you can deduce from the information given.
- **b** Hence find the function $A(t)$.
- **c** Graph $A(t)$ and discuss whether the mayor's model is reasonable.
- **d** Predict the maximum attendance at the fair, and when this occurred.

5 Tia wanted to test how much light is in her office. She took measurements of the illuminance at different distances away from her light, as shown in the table:

Distance (r cm)	20	40	80
Illuminance (I lx)	450	112.3	28.1



Illuminance is measured in *lux* (lx).

a Which of the following models do you think will best fit the data? Explain your answer.

A $I = \dfrac{k}{r}$ **B** $I = \dfrac{k}{r^2}$ **C** $I = \dfrac{k}{r^3}$

b Use technology to find the power model which best fits the data, and hence check your answer to **a**.

c Find $\dfrac{dI}{dr}$.

d Hence find the rate at which the illuminance is decreasing at the distance 1 m from the light.

e Can you explain how this model tells us that the *total* illuminance from the light at any distance r cm will be the same, no matter what the value of r is?

REVIEW SET 12A

1 A factory makes x thousand pairs of chopsticks per day with a cost of $C(x) = -0.0002x^3 + 24x + 5400$ dollars where $0 \leqslant x \leqslant 200$.

a Find $C(0)$ and explain what it represents.

b Find $C'(x)$ and explain what it represents.

c Find $C'(100)$ and explain what it estimates.

d Find the actual cost of producing the 101st lot of one thousand pairs of chopsticks.

CHOPSTICKS

2 The cost per hour of running a barge up the Rhein is given by $C(v) = 10v + \dfrac{90}{v}$ euros, where v is the average speed of the barge.

a Find the cost of running the barge for:

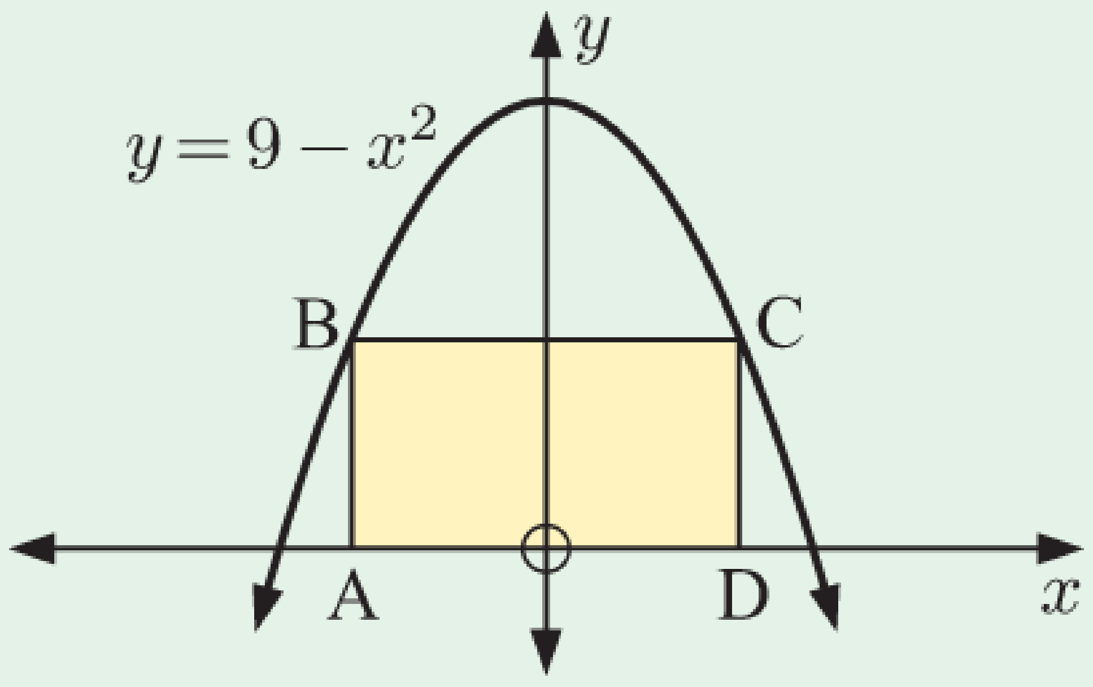
i two hours at 15 $\text{km}\,\text{h}^{-1}$ **ii** 5 hours at 24 $\text{km}\,\text{h}^{-1}$.

b Find the rate of change in the cost of running the barge at speeds of:

i 10 $\text{km}\,\text{h}^{-1}$ **ii** 6 $\text{km}\,\text{h}^{-1}$.

c At what speed will the cost per hour be a minimum?

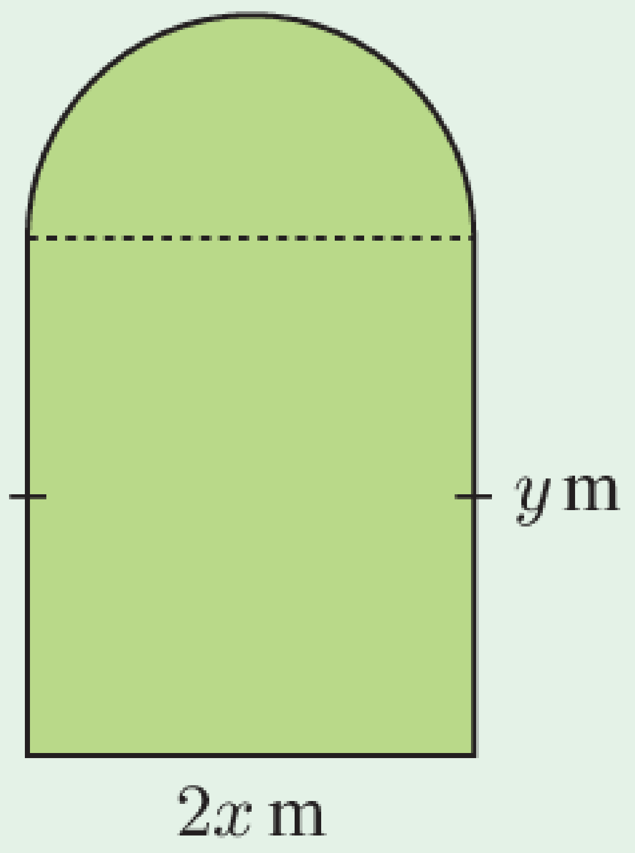
3



Rectangle ABCD is inscribed within the parabola $y = 9 - x^2$ and the x-axis, as shown.

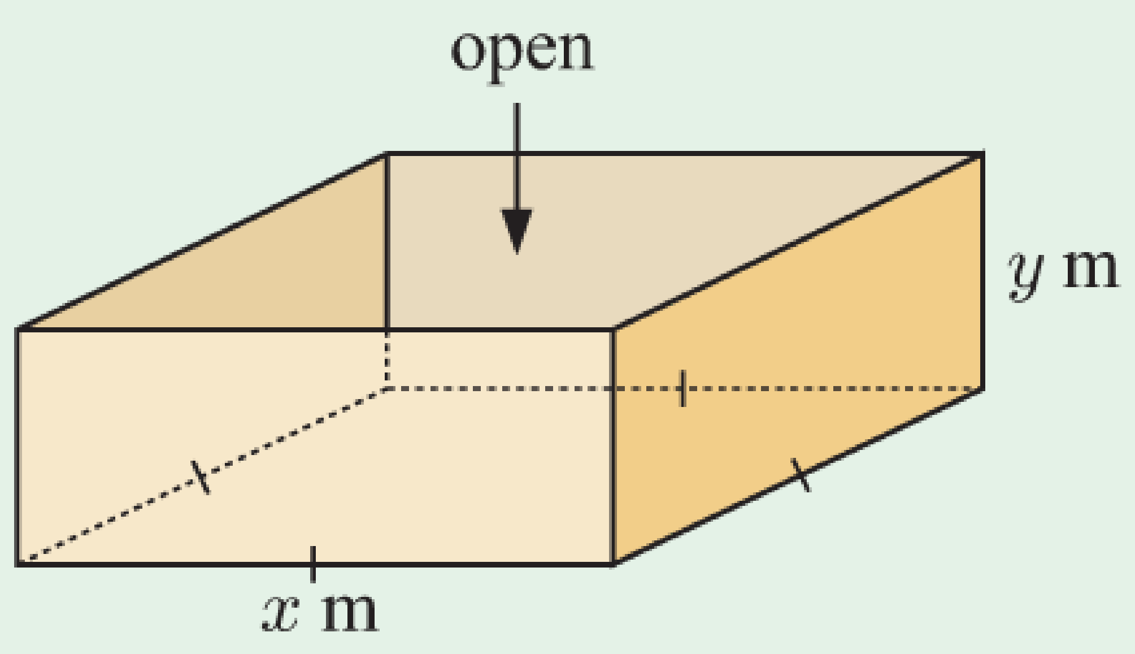
a If OD $= x$, show that the rectangle ABCD has area function $A(x) = 18x - 2x^3$.

b Find the coordinates of C when rectangle ABCD has maximum area.



4 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

- **a** Use the perimeter to explain why $y = 100 - x - \frac{\pi}{2}x$.
- **b** Show that the area of the lawn A can be written as $A = 200x - x^2\left(2 + \frac{\pi}{2}\right)$.
- **c** Use calculus to find the dimensions of the lawn of maximum area.



5 A manufacturer of open steel boxes has to make a box with a square base and a capacity of 1 kL. The steel costs \$24 per square metre.

- **a** If the base measures x m by x m and the height is y m, find y in terms of x.
- **b** Hence show that the total cost of the steel is $C(x) = 24x^2 + \dfrac{96}{x}$ dollars.
- **c** Find $C'(x)$.
- **d** Hence find the dimensions which minimise the cost of the box, and the cost of the steel in this case.

6 When Max throws a stone into the air, it flies for 5.9 seconds before landing. Over the course of its flight, the distance between the stone and Max after t seconds is given by

$$D(t) = \sqrt{24.01t^4 - 294t^3 + 936t^2} \text{ metres.}$$

- **a** State the domain of $D(t)$.
- **b** Find the time at which D^2 is maximised.
- **c** Hence find the maximum distance between the stone and Max.
- **d** How far away from Max does the stone land?

7 A ball is thrown into the air from height 1.6 m above the ground. It initially gains height at 16.4 m s^{-1}. After 2 seconds, it is falling at 3.2 m s^{-1}. The height of the ball over time can be modelled by the quadratic function $h(t) = at^2 + bt + c$ metres, where t is the time in seconds.

- **a** State $h(0)$ and hence find c.
- **b** State $h'(0)$ and hence find b.
- **c** Use the remaining information to find a.
- **d** What was the maximum height reached by the ball, and when did it reach this height?

REVIEW SET 12B

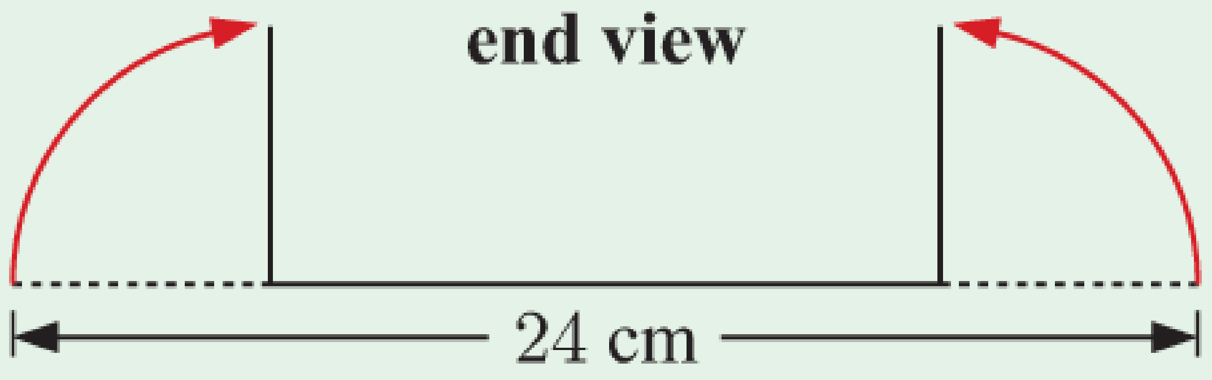
1 An astronaut standing on the moon throws a ball directly upwards. The ball's height above the surface of the moon is given by $H(t) = 1.5 + 19t - 0.8t^2$ metres, where t is the time in seconds after the ball is released.

- **a** Find $H'(t)$ and state its units.
- **b** Calculate $H'(0)$, $H'(10)$, and $H'(20)$. Interpret these values, including their sign.
- **c** How long does it take for the ball to return to the ground?

2 The cost per hour of running a freight train is given by $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

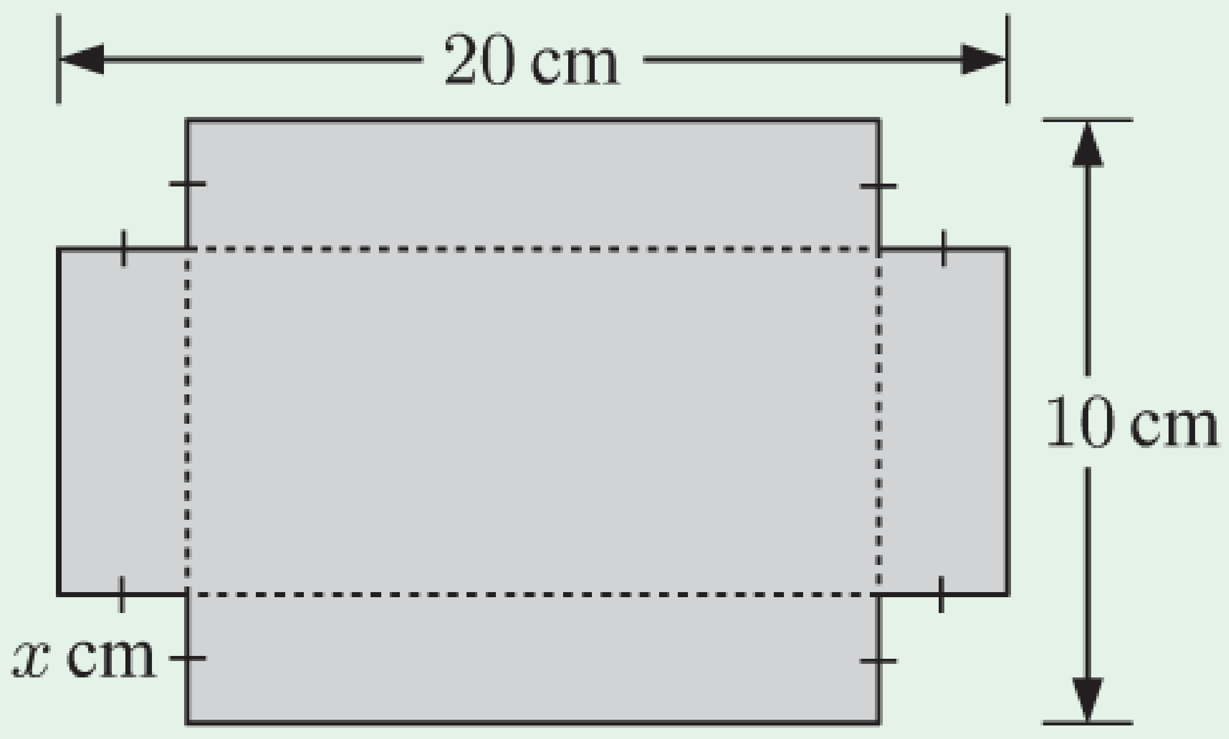
- **a** Find the cost of running the train for 5 hours at 64 km h^{-1}.
- **b** Find the rate of change in the hourly cost of running the train at speeds of:
  - **i** 75 km h^{-1}
  - **ii** 90 km h^{-1}.
- **c** At what speed will the cost per hour be a minimum?

3 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.



Where must the bends be made in order to maximise the capacity of the gutter?

4



A rectangular sheet of tin-plate is 20 cm by 10 cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.

5 At time t years after mining begins on a mountain of iron ore, the rate of iron ore output is given by $R(t) = \frac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}}$ million tonnes per year, $t \geqslant 0$.

- **a** Graph $R(t)$ against t for $0 \leqslant t \leqslant 100$.
- **b** *Hence* draw a sign diagram for $R'(t)$.
- **c** At what rate will the ore be mined after $t = 20$ years?
- **d** When will the rate of mining be 10 million tonnes per year?
- **e** What will be the maximum rate of mining, and at what time will it occur?

6 The cost of running an advertising campaign for x days is given by $C(x) = 16\,900 + 950x$ pounds.

Research suggests that after x days, $7500 - \frac{15\,000}{x}$ people will have responded, bringing post-production cost revenue of £20 per person.

- **a** Write a function $P(x)$ for the *profit* from running the campaign for x days.
- **b** Use calculus to find how long the campaign should last, in order to maximise the profit.

THE GREATEST GIZMO ON EARTH
BUY IT!

7 The share price for a business listed on the New York Stock Exchange is given by $S(t) = at^3 + bt^2 + ct + d$ dollars, where t is the number of hours after 9:30 am, $0 \leqslant t \leqslant 6.5$. When the market opened at 9:30 am, the share price was \$22.81, but it immediately began to fall with instantaneous rate 16 cents per hour. At 12:30 pm the share price was \$22.49, and by 2:30 pm it had increased to \$22.60.

a State $S(0)$ and hence find d.

b State $S'(0)$ and hence find c.

c Use the remaining information to find a and b.

d Graph $S(t)$.

e Predict the share price at the 4 pm market close.

f Find the minimum and maximum values of the share price during the day.

Chapter 13

Integration

Contents:

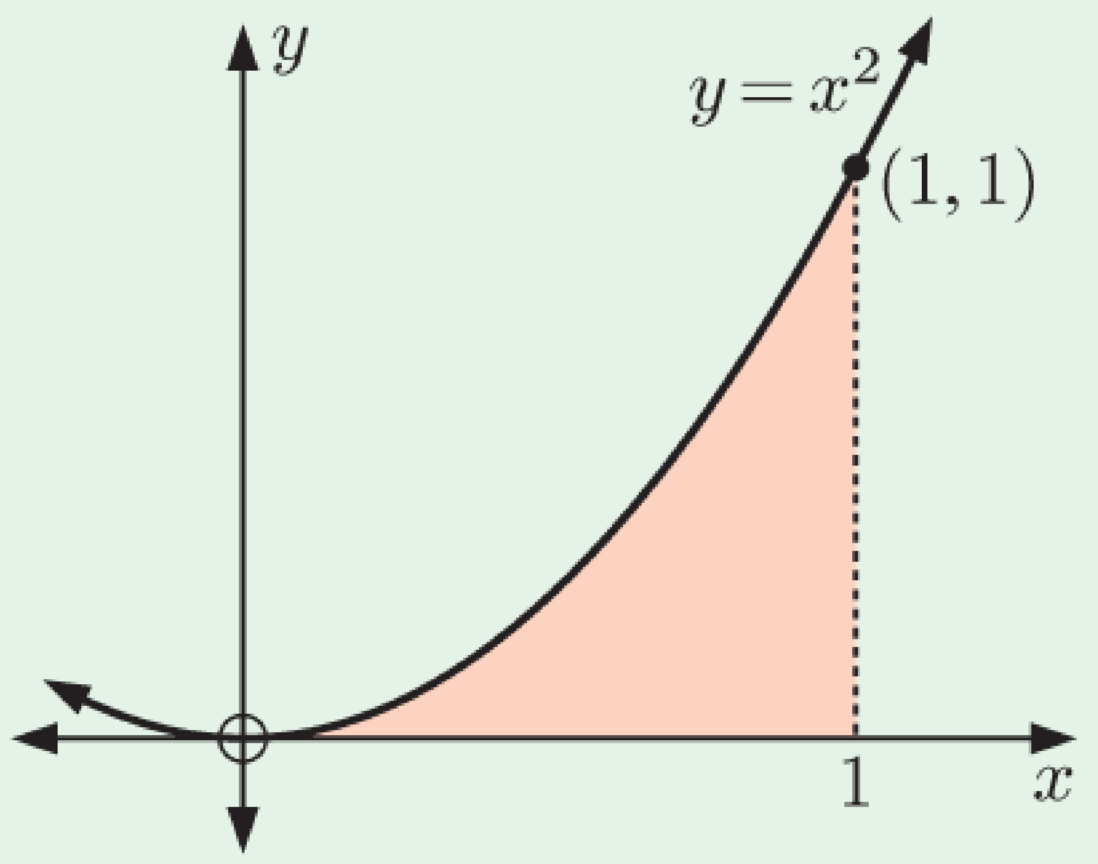
A Approximating the area under a curve
B The Riemann integral
C The Fundamental Theorem of Calculus
D Antidifferentiation and indefinite integrals
E Rules for integration
F Particular values
G Definite integrals
H The area under a curve

OPENING PROBLEM

Another of **Archimedes'** achievements was devising a method for calculating the area under a curve.

In an article containing 24 propositions, he provided essential theory for what, over 1800 years later, would be developed into **integral calculus**.

In the process, Archimedes found the exact area A between the curve $y = x^2$ and the x-axis, on the interval $0 \leqslant x \leqslant 1$.

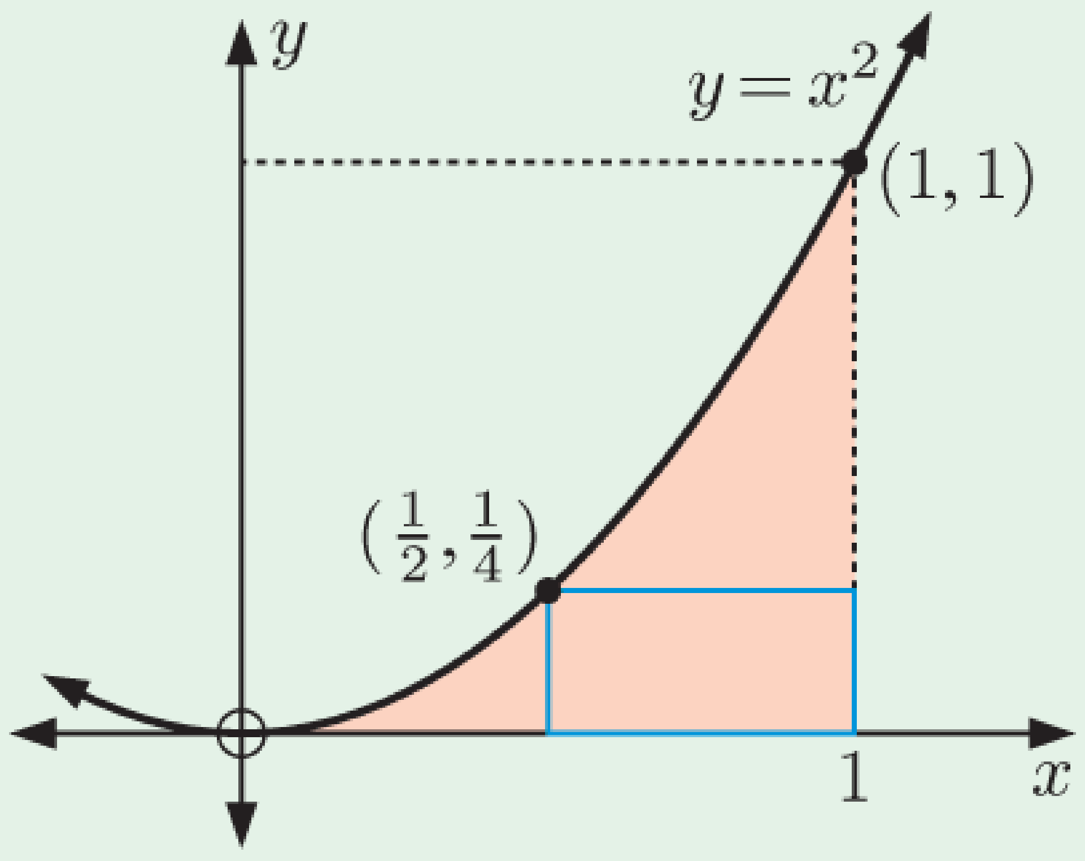


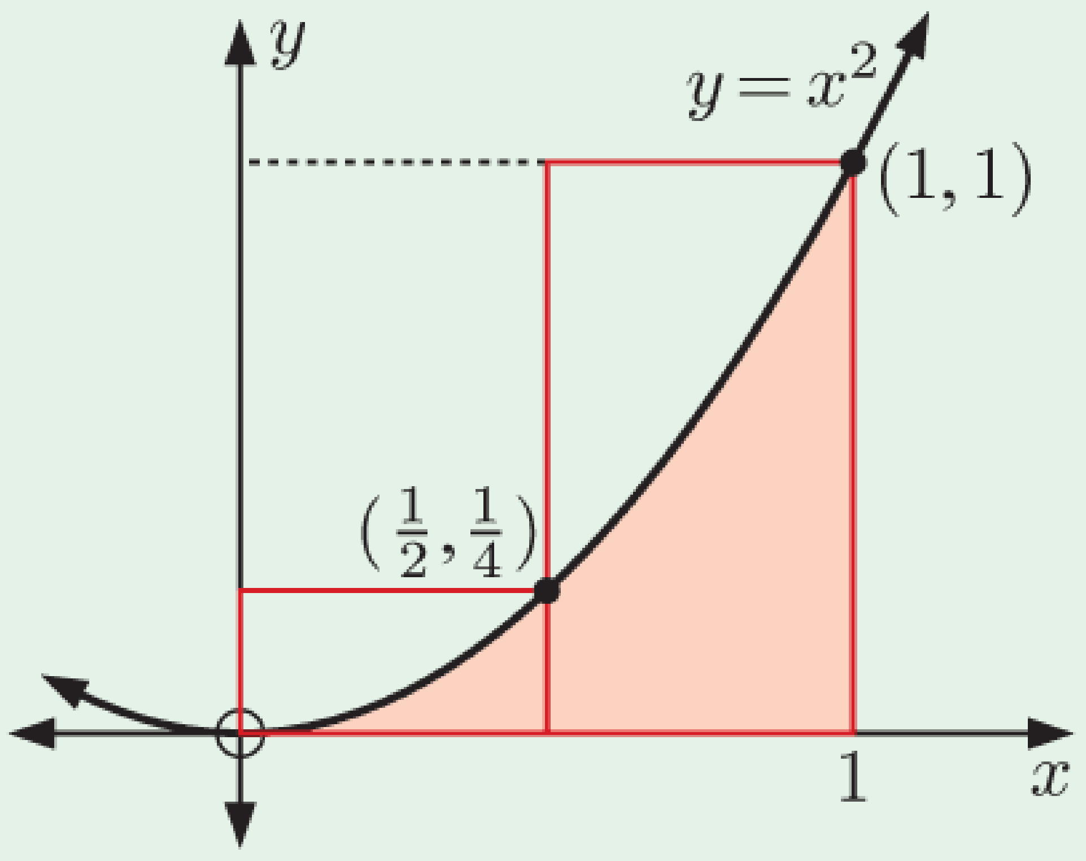
Things to think about:

a Can you use the:

i blue rectangle to explain why $A > \frac{1}{8}$

ii red rectangles to explain why $A < \frac{5}{8}$?





b How can we obtain a better estimate for A?

c What function has x^2 as its derivative?

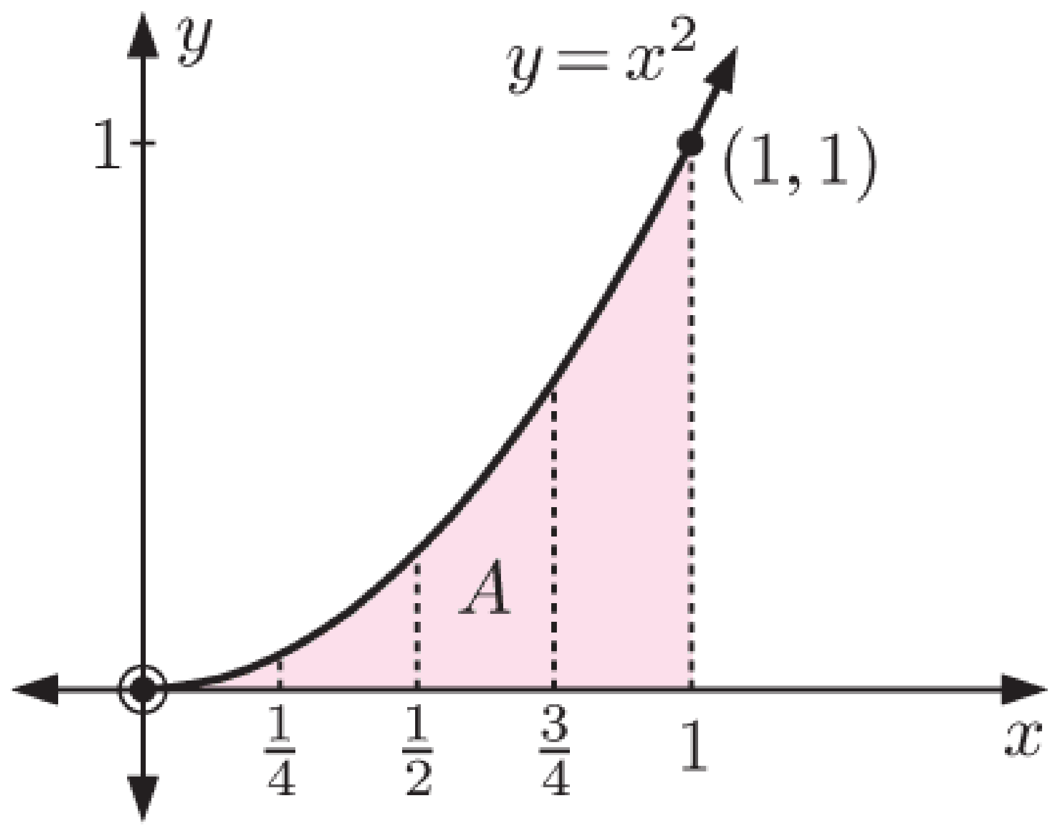
In this Chapter we consider **integral calculus**. This involves **antidifferentiation**, which is the reverse process of differentiation.

A APPROXIMATING THE AREA UNDER A CURVE

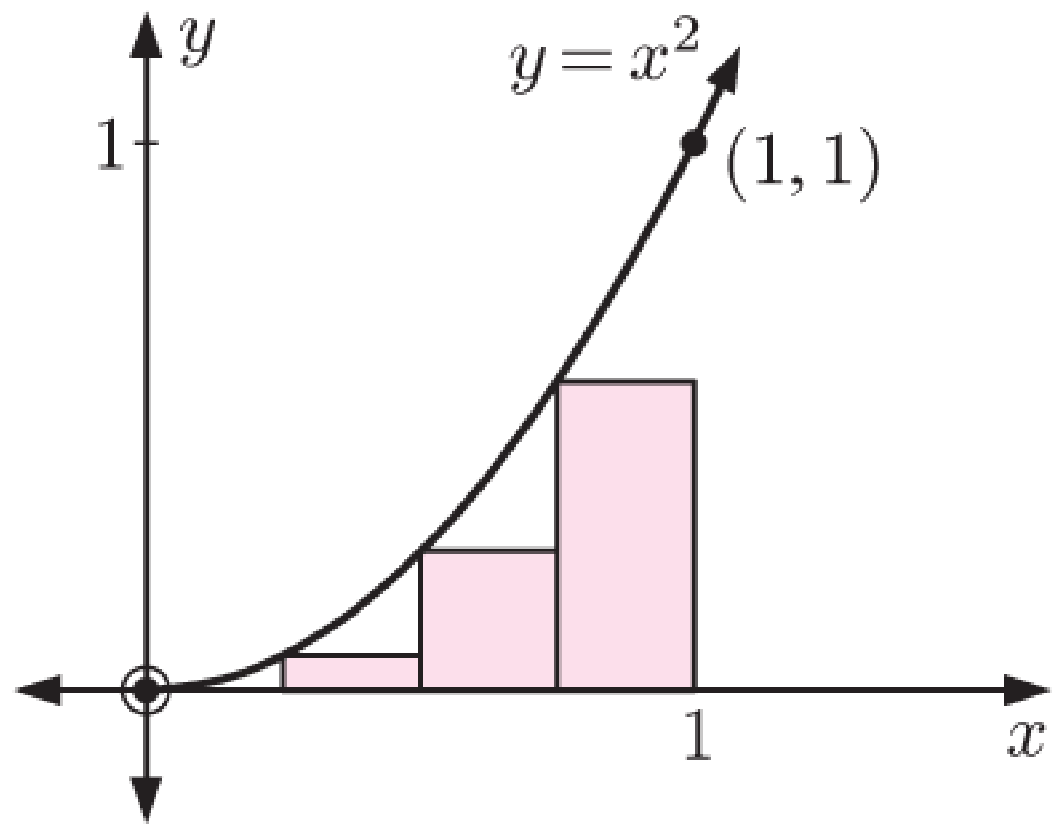
Consider the function $f(x) = x^2$ in the **Opening Problem**.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical line $x = 1$.

Suppose we divide the interval $0 \leqslant x \leqslant 1$ into 4 strips of width $\frac{1}{4}$ unit as shown. We obtain 4 subintervals of equal width.

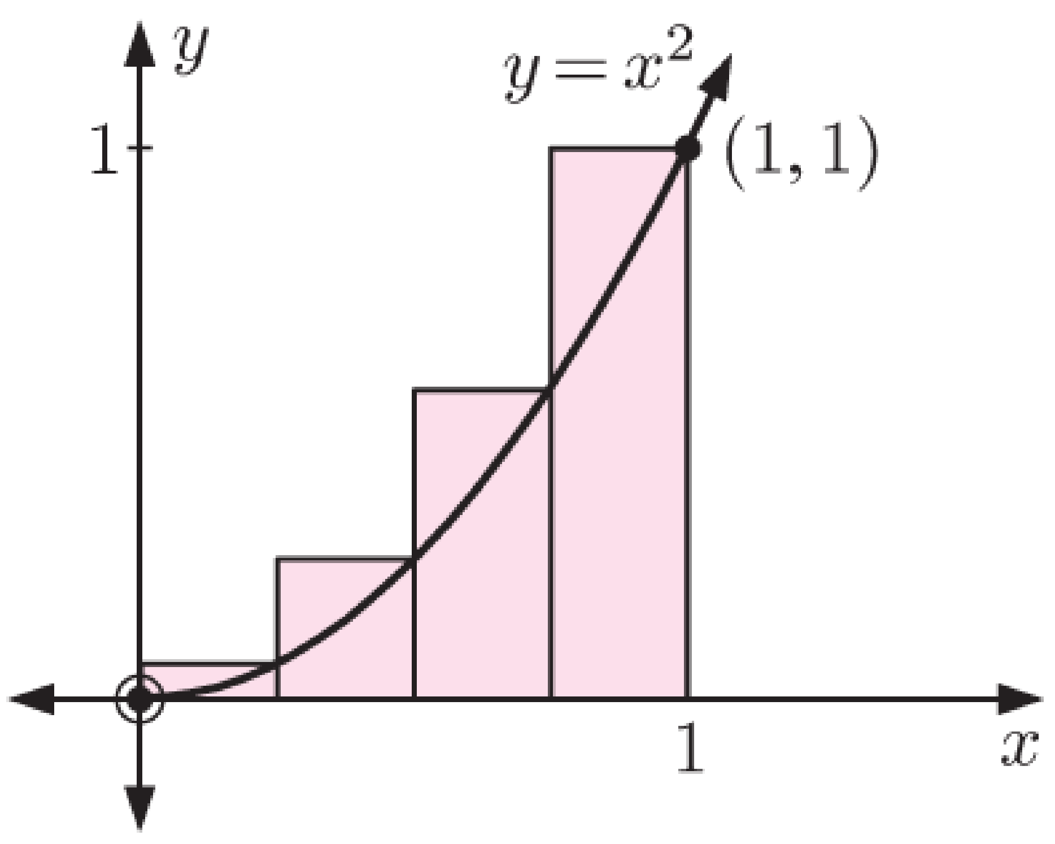


The diagram alongside shows **lower rectangles**, which are rectangles with height equal to the *lower* value of the function at the endpoints of the subinterval.



The total area of the lower rectangles is

$$\begin{aligned} A_L &= \tfrac{1}{4} \times f(0) + \tfrac{1}{4} \times f(\tfrac{1}{4}) + \tfrac{1}{4} \times f(\tfrac{1}{2}) + \tfrac{1}{4} \times f(\tfrac{3}{4}) \\ &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &= 0.218\,75 \end{aligned}$$



The next diagram shows **upper rectangles**, which are rectangles with height equal to the *upper* value of the function at the endpoints of the subinterval.

The total area of the upper rectangles is

$$\begin{aligned}A_U &= \tfrac{1}{4}\times f(\tfrac{1}{4}) + \tfrac{1}{4}\times f(\tfrac{1}{2}) + \tfrac{1}{4}\times f(\tfrac{3}{4}) + \tfrac{1}{4}\times f(1)\\ &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2\\ &= 0.468\,75\end{aligned}$$

Clearly, for increasing functions such as $f(x) = x^2$, $A_L < A < A_U$, so the area A lies between $0.218\,75$ units^2 and $0.468\,75$ units^2.

If the interval $0 \leqslant x \leqslant 1$ was divided into 8 subintervals instead, each of width $\frac{1}{8}$, then

$$\begin{aligned}A_L &= \tfrac{1}{8}\left[f(0) + f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8})\right]\\ &= \tfrac{1}{8}\left[0 + \tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64}\right]\\ &\approx 0.273\,44\end{aligned}$$

$$\begin{aligned}A_U &= \tfrac{1}{8}\left[f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8}) + f(1)\right]\\ &= \tfrac{1}{8}\left[\tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64} + 1\right]\\ &\approx 0.398\,44\end{aligned}$$

From this refinement we conclude that the area A lies between $0.273\,44$ units^2 and $0.398\,44$ units^2.

As we create more subintervals, the estimates A_L and A_U will become more and more accurate. In fact, as the subinterval width is reduced further and further, both A_L and A_U will **converge** to A.

Now suppose there are n subintervals between $x = 0$ and $x = 1$, each of width $\dfrac{1}{n}$.

You can use the **area finder** software or your **graphics calculator** to help calculate A_L and A_U for large values of n.

AREA FINDER

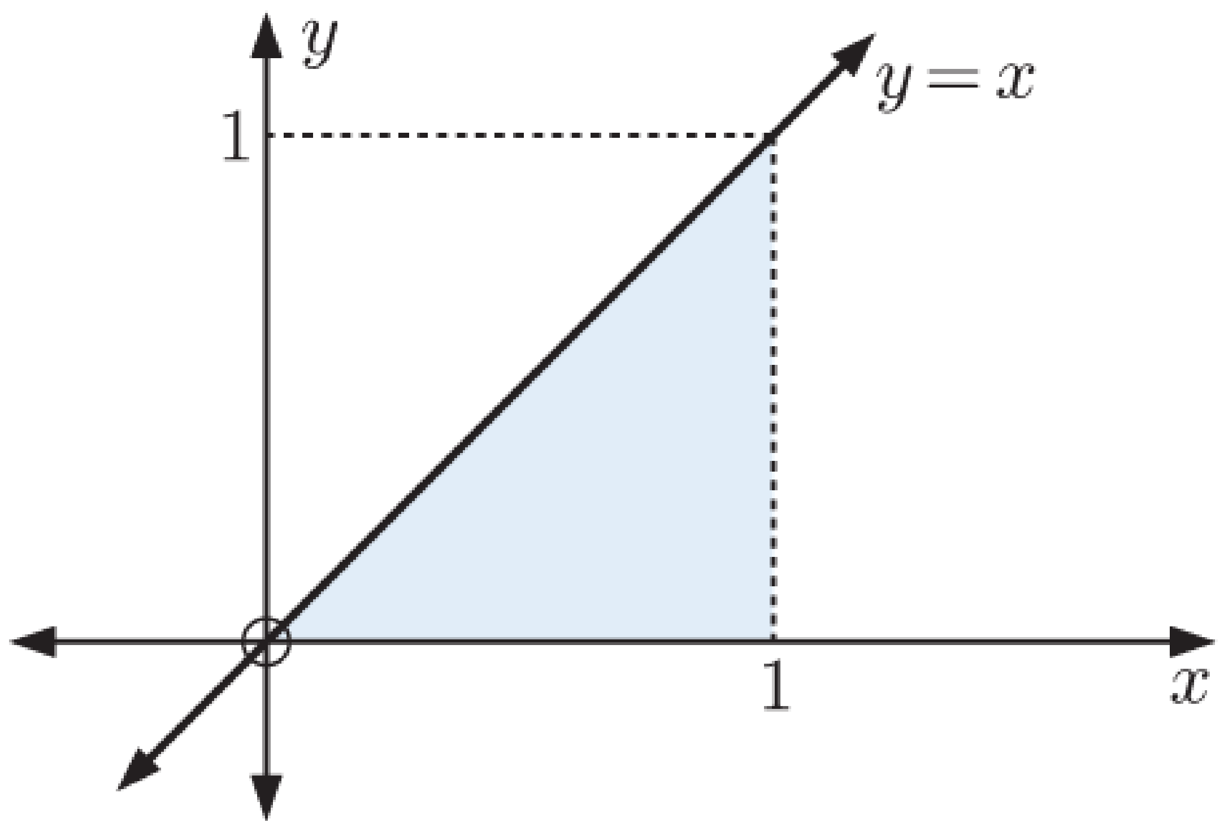
GRAPHICS CALCULATOR INSTRUCTIONS

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
8	0.273 44	0.398 44	0.335 50
16	0.302 73	0.365 23	0.333 98
50	0.323 40	0.343 40	0.333 40
200	0.330 84	0.335 84	0.333 38
1000	0.332 83	0.333 83	0.333 33
10 000	0.333 28	0.333 38	0.333 33

The table alongside summarises the results you should obtain for $n = 4, 8, 16, 50, 200, 1000$, and $10\,000$.

From the table, it appears that both A_L and A_U are *converging* to $\frac{1}{3}$ as n increases.

EXERCISE 13A.1



1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

- **a** Divide the interval into 5 subintervals of equal width, then estimate the area using:
  - **i** lower rectangles
  - **ii** upper rectangles.
- **b** Calculate the actual area and compare it with your answers in **a**.

2 Consider the area between $y = \frac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 subintervals of equal width, then estimate the area using:

a lower rectangles
b upper rectangles.

3 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10, 25, 50, 100$, and 500. Give your answers to 4 decimal places.
As n gets larger, both A_L and A_U converge to the same rational number. What is it?

AREA FINDER

4 **a** Use lower and upper rectangle sums to estimate the area between each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$. Use $n = 5, 10, 50, 100, 500, 1000$, and $10\,000$. Give your answer to 5 decimal places in each case.

i $y = x$
ii $y = x^3$
iii $y = x^4$
iv $y = x^5$

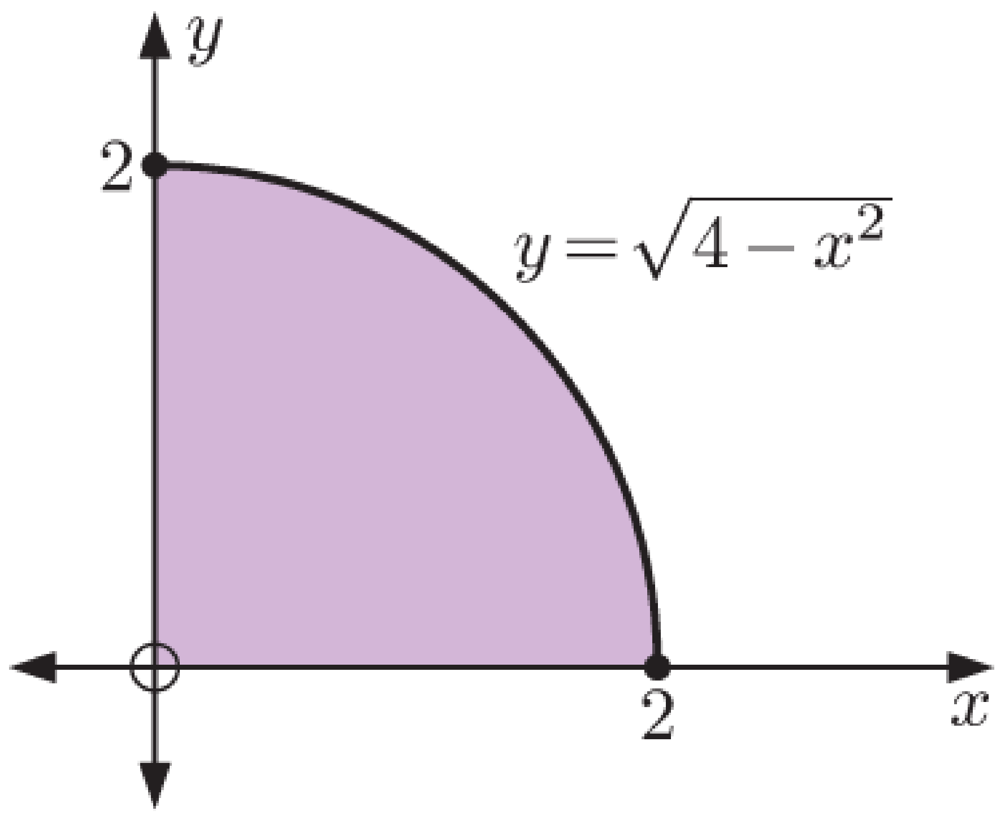
b For each case in **a**, A_L and A_U converge to a rational number. Write down the number for each case.

c Using your answer to **b**, predict the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any integer $a > 0$.

5 Consider the quarter circle with centre $(0, 0)$ and radius 2 units illustrated.

Its area is $\frac{1}{4}$(full circle with radius 2 units)

$$= \tfrac{1}{4} \times \pi \times 2^2$$
$$= \pi \text{ units}^2$$



a Estimate the area using lower and upper rectangles for $n = 10, 50, 100, 200, 1000$, and $10\,000$. Hence find rational bounds for π.

b Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
For what value of n is your estimate for π better than that of Archimedes?

THE TRAPEZOIDAL RULE

For a function which is either increasing on the whole interval, or decreasing on the whole interval, the actual area under the curve is guaranteed to lie between the lower rectangle and upper rectangle estimates: $A_L \leqslant A \leqslant A_U$.

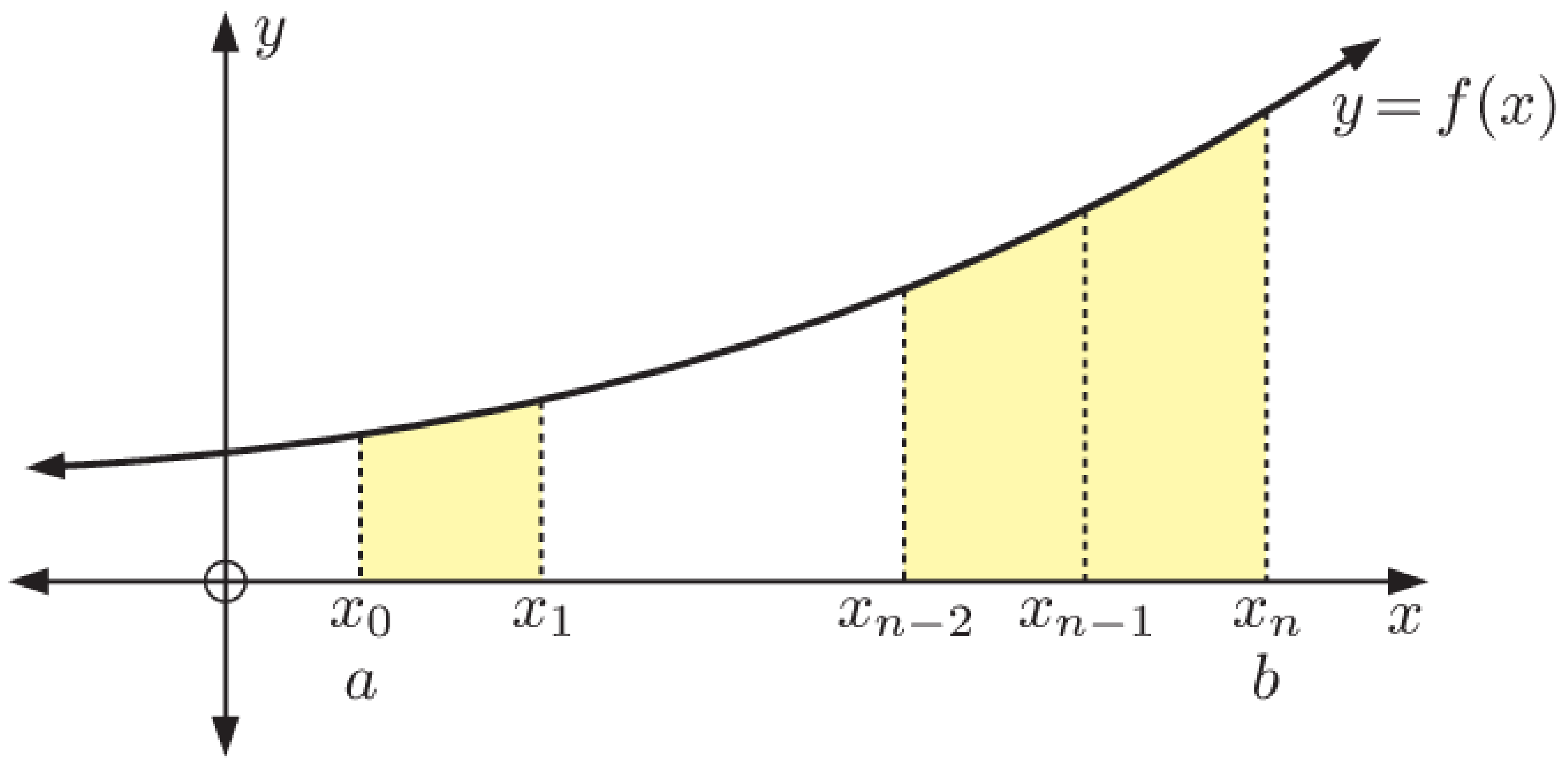
For functions with turning points on the interval, we no longer have that guarantee, and it is inefficient to ask a computer to calculate A_L and A_U. This is because the computer has to make extra decisions about which is the lower and which is the upper value for each subinterval.

So, while we have seen that increasing the number of subintervals will improve the accuracy of A_L and A_U, we really want a more efficient way to numerically approximate the area under a curve.

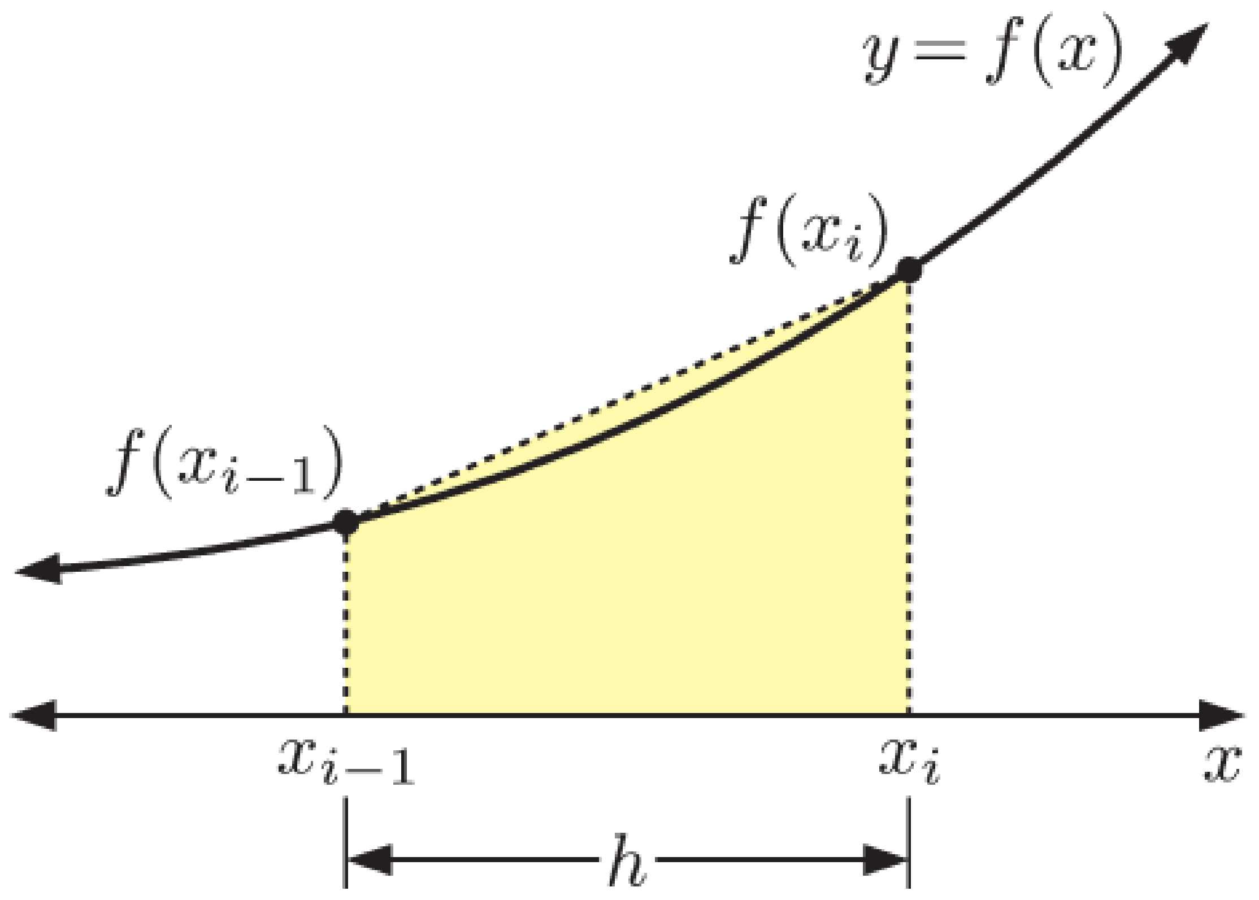
Suppose we use a trapezium to approximate the area under the curve on each subinterval. In doing this, we use the *average* of the values of the function at the left and right endpoints. The value calculated for each subinterval will always lie between the area of the lower rectangle and the area of the upper rectangle, and generally be a better approximation of the actual area.

Suppose we divide the interval $a \leqslant x \leqslant b$ into n subintervals of equal width $h = \frac{b-a}{n}$.

We let $x_i = a + ih$ for $i = 0, 1, 2, ..., n$, so the ith subinterval is $x_{i-1} \leqslant x \leqslant x_i$.



The area for the ith subinterval is the shaded trapezium below:



In calculating this sum, we only need to evaluate the function $n+1$ times.

Our approximation for the area under the curve is therefore

$$A \approx h\left(\frac{f(x_0)+f(x_1)}{2}\right) + h\left(\frac{f(x_1)+f(x_2)}{2}\right) + ... + h\left(\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n))$$

$$\approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$

Example 1

Self Tutor

Use the trapezoidal rule with 6 subintervals to estimate the area between $f(x) = \sqrt{6-x^2}$ and the x-axis from $x = 1$ to $x = 2$.

$n = 6, \quad a = 1, \quad b = 2, \quad f(x) = \sqrt{6-x^2}$

$h = \frac{b-a}{n} = \frac{1}{6}$

$x_i = 1 + \frac{1}{6}i$

i	x_i	$f(x_i)$
0	1	2.236 068
1	$1\frac{1}{6}$	2.153 808
2	$1\frac{1}{3}$	2.054 805
3	$1\frac{1}{2}$	1.936 492
4	$1\frac{2}{3}$	1.795 055
5	$1\frac{5}{6}$	1.624 466
6	2	1.414 214

GRAPHING PACKAGE

Using the trapezoidal rule, the area

$$\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_5) + f(x_6))$$

$$\approx 1.8983 \text{ units}^2$$

EXERCISE 13A.2

For questions **1** to **3**, construct a table of values manually to answer the question. You can then check your answers by constructing the table of values directly on your calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

1 Use the trapezoidal rule with 4 subintervals to approximate the area between the x-axis and:

a $f(x) = \frac{2}{\sqrt{x}}$ from $x = 2$ to $x = 4$

b $f(x) = -x^2 + 6x - 4$ from $x = 1$ to $x = 3$.

2 **a** Use the trapezoidal rule with 6 subintervals to calculate the area between the x-axis and $f(x) = 3 - x$ from $x = 0$ to $x = 3$.

b Explain why your calculation is exact for this function.

3 Use the trapezoidal rule with 8 subintervals to approximate the area between the x-axis and:

a $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$

b $f(x) = \sqrt{x}e^{-\pi x}$ from $x = 0$ to $x = 1$

c $f(x) = x^3 - 2x^2 + 1$ from $x = -0.6$ to $x = 1$.

4 Use the trapezoidal method in the software to estimate the area under $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$.

AREA FINDER

a Copy and complete this table of results:

n	*Area estimate*
8	
40	
100	
1000	

b For what value of n is your estimate of π more accurate than that of Archimedes (see **Exercise 13A.1** question **5 b**)?

HISTORICAL NOTE

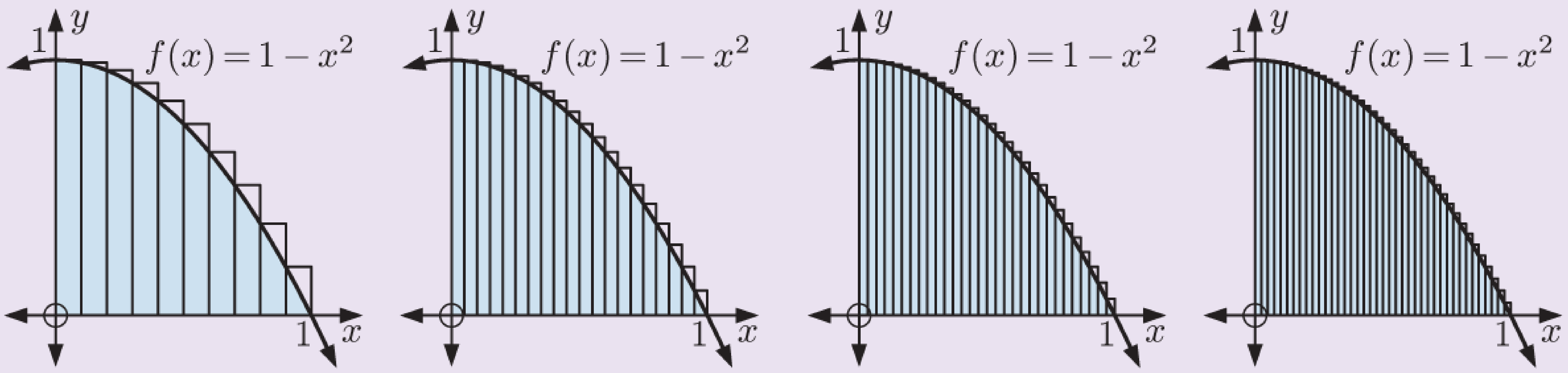
Italian Mathematician **Bonaventura Cavalieri** (1598 - 1647) became Professor of Mathematics at Bologna in 1629. He published tables for many trigonometric and logarithmic functions. However, his best known contribution to mathematics was the invention of **indivisibles**.

Bonaventura Cavalieri

In his Method of Indivisibles, Cavalieri considered that a moving point could be used to sketch a curve. The curve could therefore be considered as the set of an infinite number of points, each with no length.

In a similar way, the "indivisibles" that made up a surface were an infinite number of lines. Almost every introduction to integral calculus starts with the division of an area into a number of rectangular strips with finite width.

Cavalieri's important step was to make the strips narrower and narrower until they were infinitely thin lines. This reduces the "jagged" steps of the strips until they exactly define the curved boundary of the area.

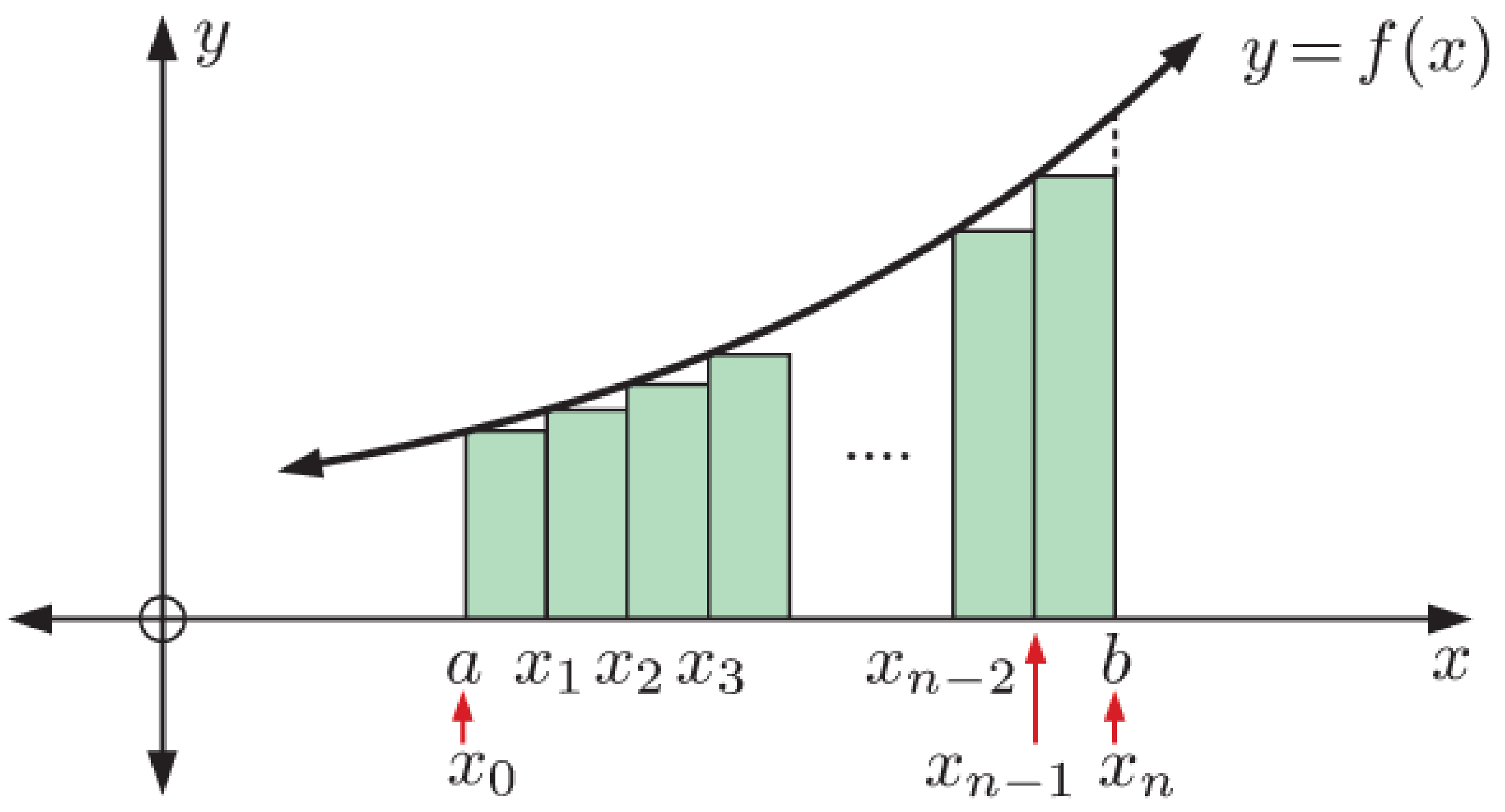


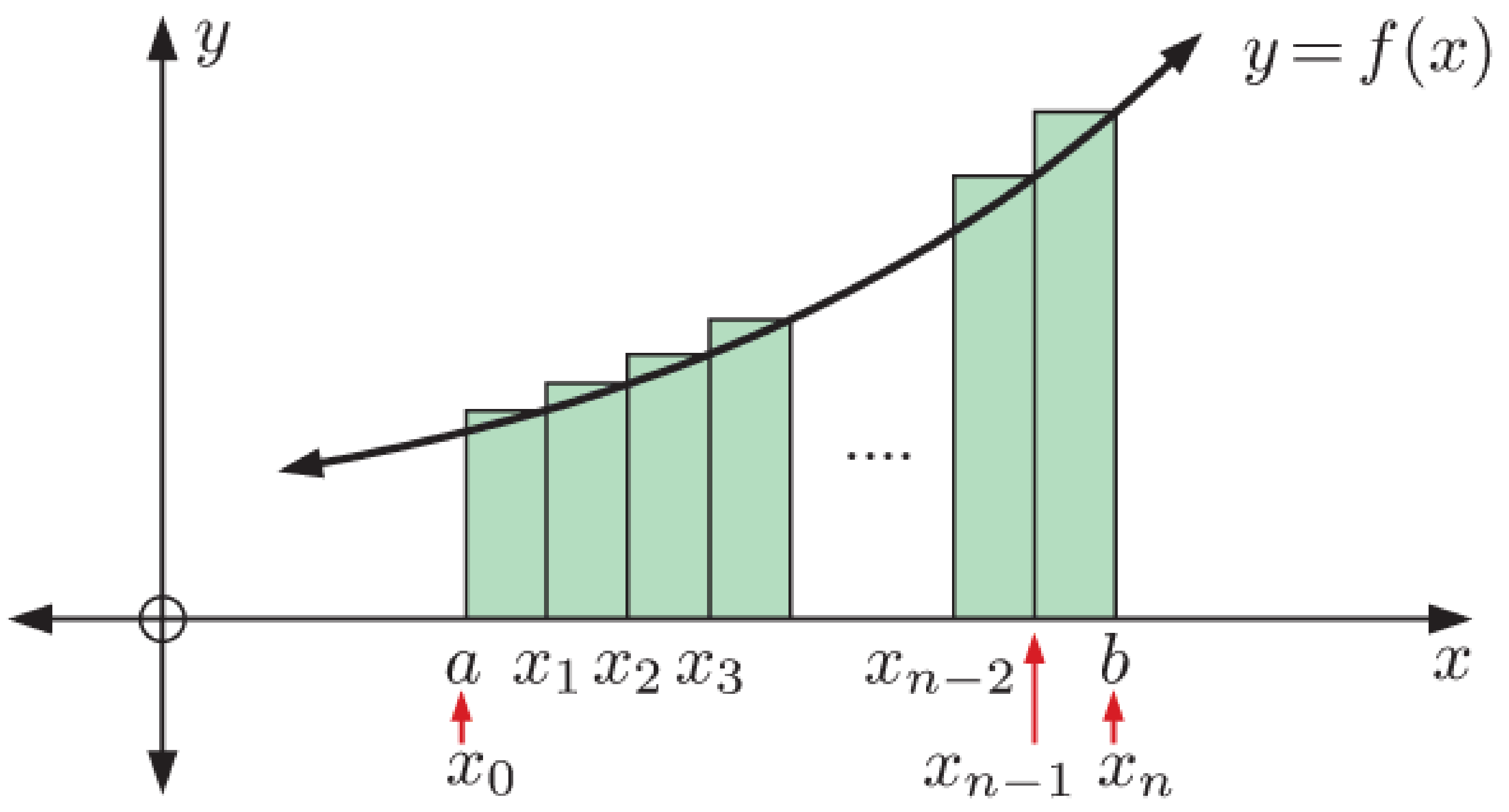
It was not until Englishman **Sir John Wallis** (1616 - 1703) formally introduced the idea of a **limit** in 1656 that Cavalieri's Method of Indivisibles progressed into the foundation for Integral Calculus.

B THE RIEMANN INTEGRAL

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subintervals, each of width $w = \dfrac{b-a}{n}$.





Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + \dots + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + \dots + w\,f(x_{n-1}) + w\,f(x_n) = w\sum_{i=1}^{n} f(x_i)$$

$$\therefore \quad A_U - A_L = w\,(f(x_n) - f(x_0))$$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

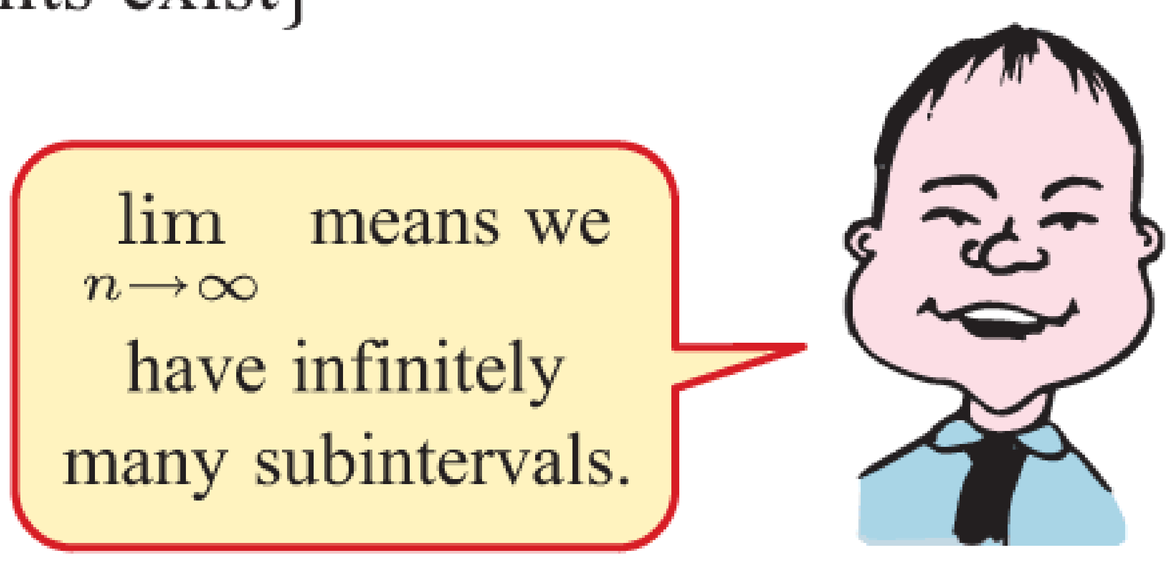
Following Cavalieri's suggestion, we allow there to be infinitely many subintervals, so $n \to \infty$.

In this case $\displaystyle\lim_{n\to\infty}(A_U - A_L) = 0$ $\quad \{\text{since } \displaystyle\lim_{n\to\infty}\frac{1}{n} = 0\}$

$\therefore \displaystyle\lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U$ $\quad$ {provided both limits exist}

$\therefore$ since $A_L < A < A_U$ for all values of n, it follows that

$$\lim_{n\to\infty} A_L = A = \lim_{n\to\infty} A_U$$

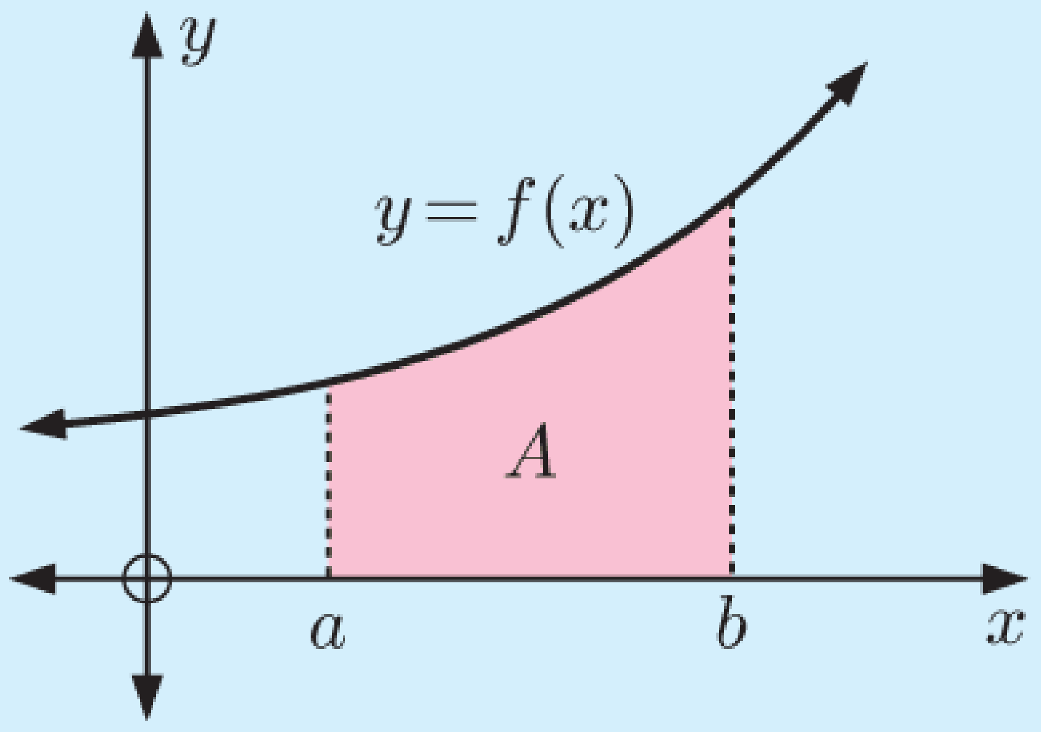

$\displaystyle\lim_{n\to\infty}$ means we have infinitely many subintervals.

We can obtain a result like this for every increasing and decreasing interval of a positive function provided the function is *continuous*. This means that the function must have a defined value $f(k)$ for all $a \leqslant k \leqslant b$, and that $\displaystyle\lim_{x\to k} f(x) = f(k)$ for all $a \leqslant k \leqslant b$.

If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx$ is equal to the shaded area A.

This is known as the **Riemann integral**.

We would say "the integral of $f(x)$ from a to b with respect to x".



The symbol $\int$ is called an **integral sign**.

HISTORICAL NOTE

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas of the thin rectangular strips are put together into one whole area.

The theory of integration was developed independently by **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**.

It was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866), whose name is given to the integral which calculates the area under a curve.

Bernhard Riemann

Example 2 **Self Tutor**

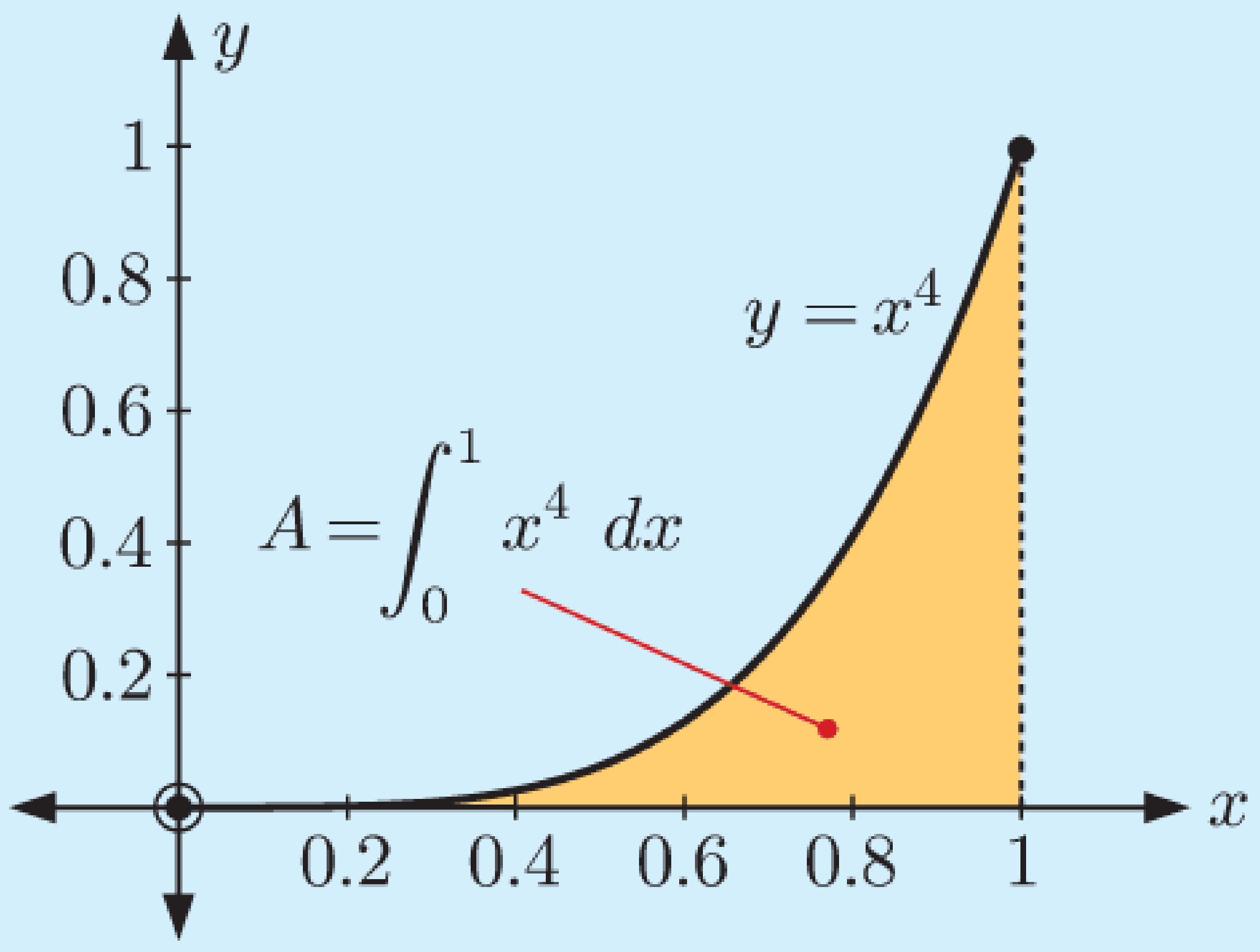
a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4\,dx$.

b Use technology to calculate the lower and upper rectangle sums for n equal subintervals where $n = 5, 10, 50, 100$, and 500.

c Hence evaluate $\int_0^1 x^4\,dx$ to 2 significant figures.

d Approximate $\int_0^1 x^4\,dx$ using the trapezoidal method with 10 subintervals. Comment on your answer.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4\, dx < A_U$, $\int_0^1 x^4\, dx \approx 0.20$

d $n = 10$, $a = 0$, $b = 1$, $f(x) = x^4$

$h = \frac{b-a}{n} = \frac{1}{10}$

$x_i = \frac{1}{10} i$

Using the trapezoidal rule, the area

$\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10}))$

$\approx 0.203\,33$

With just 10 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 100$.

i	x_i	$f(x_i)$
0	0	0
1	0.1	0.0001
2	0.2	0.0016
3	0.3	0.0081
4	0.4	0.0256
5	0.5	0.0625
6	0.6	0.1296
7	0.7	0.2401
8	0.8	0.4096
9	0.9	0.6561
10	1	1

EXERCISE 13B

AREA FINDER

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\int_0^1 \sqrt{x}\, dx$.

b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

c Hence evaluate $\int_0^1 \sqrt{x}\, dx$ to 2 significant figures.

d Approximate $\int_0^1 \sqrt{x}\, dx$ using the trapezoidal method with 8 subintervals. Comment on your answer.

GRAPHING PACKAGE

2 Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

a Write expressions for the lower and upper rectangle sums using n subintervals where $n \in \mathbb{N}$.

b Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

c Hence estimate $\int_0^2 \sqrt{1 + x^3}\, dx$.

d Approximate $\int_0^2 \sqrt{1 + x^3}\, dx$ using the trapezoidal method with 10 subintervals.

3 The integral $\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx$ is of considerable interest to statisticians.

a Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

b Calculate the lower and upper rectangle sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.

c Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to estimate $\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx$. Compare your answer with $\sqrt{2\pi}$.

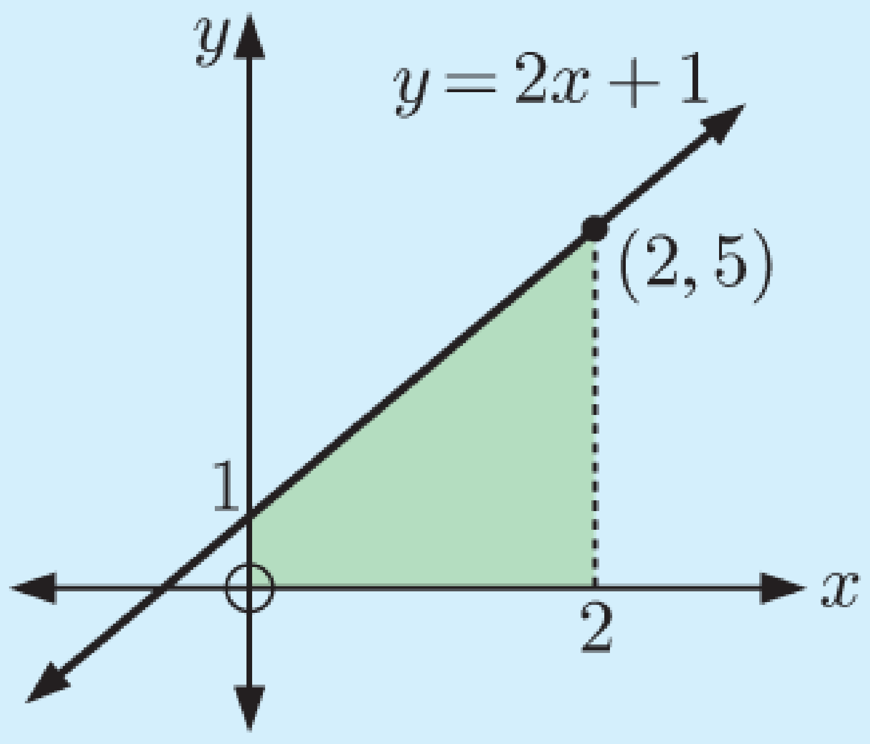
d How many subintervals are necessary with the trapezoidal rule to estimate $\int_0^3 e^{-\frac{x^2}{2}}\, dx$ more accurately than your lower and upper rectangle sums in **b**?

Example 3 **Self Tutor**

Use graphical evidence and known area facts to find:

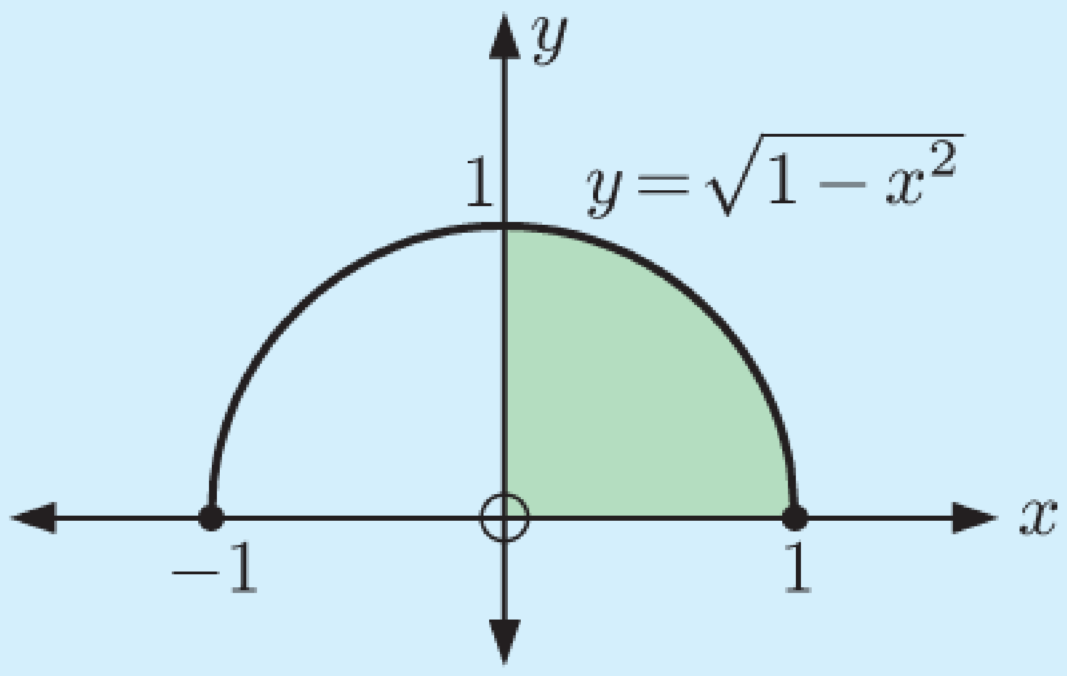
a $\displaystyle\int_0^2 (2x+1)\,dx$ **b** $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

a



$$\int_0^2 (2x+1)\,dx$$
$$= \text{shaded area}$$
$$= \left(\frac{1+5}{2}\right) \times 2$$
$$= 6$$

b If $y=\sqrt{1-x^2}$ then $y^2=1-x^2$ and so $x^2+y^2=1$. This is the equation of a circle with radius 1 unit, and $y=\sqrt{1-x^2}$ is the upper half.



$$\int_0^1 \sqrt{1-x^2}\,dx$$
$$= \text{shaded area}$$
$$= \tfrac{1}{4} \times \pi \times 1^2$$
$$= \tfrac{\pi}{4}$$

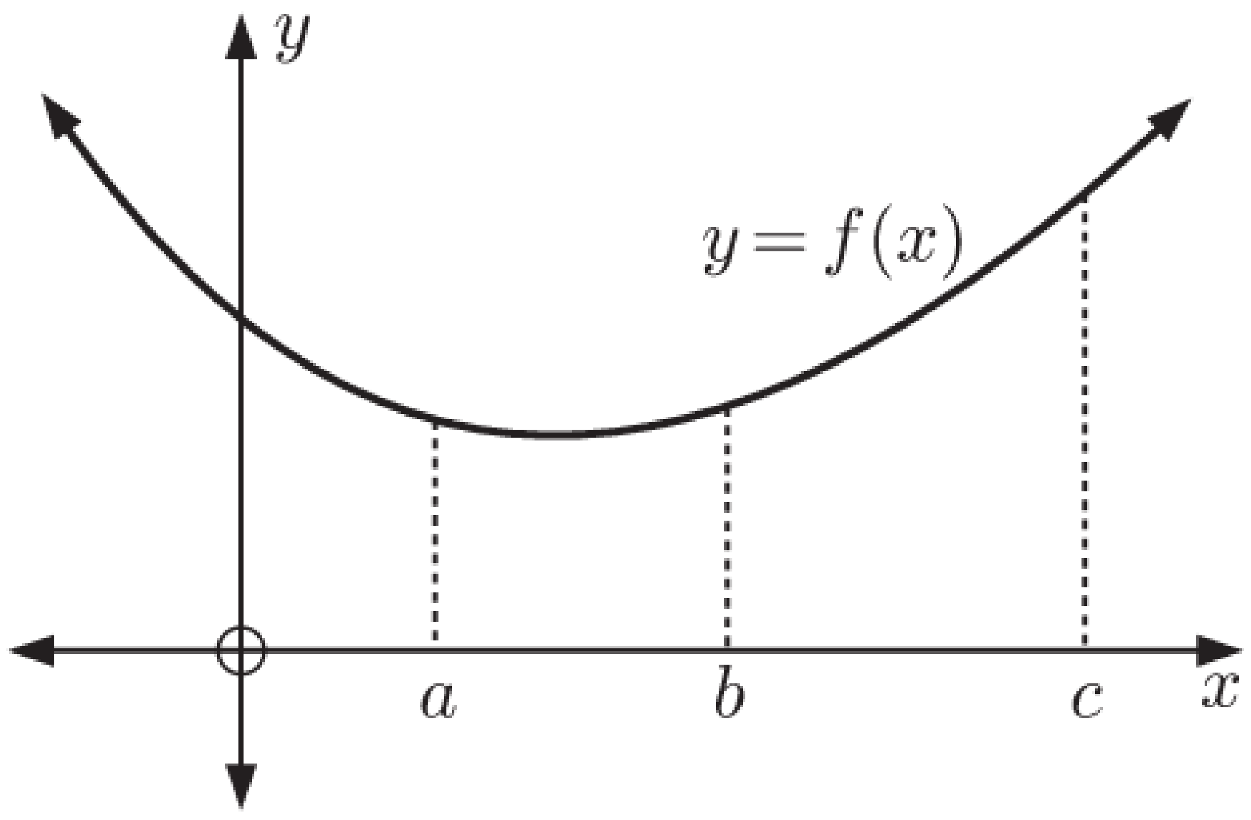
4 Use graphical evidence and known area facts to find:

a $\displaystyle\int_1^3 (1+4x)\,dx$ **b** $\displaystyle\int_{-1}^2 (2-x)\,dx$ **c** $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

5 **a** Use the diagram alongside to show that for any positive function $f(x)$:

i $\displaystyle\int_a^a f(x)\,dx = 0$

ii $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



b For a positive function $f(x)$, $\displaystyle\int_2^5 f(x)\,dx = 10$, and $\displaystyle\int_5^9 f(x)\,dx = 12$. Find:

i $\displaystyle\int_5^5 f(x)\,dx$ **ii** $\displaystyle\int_2^9 f(x)\,dx$

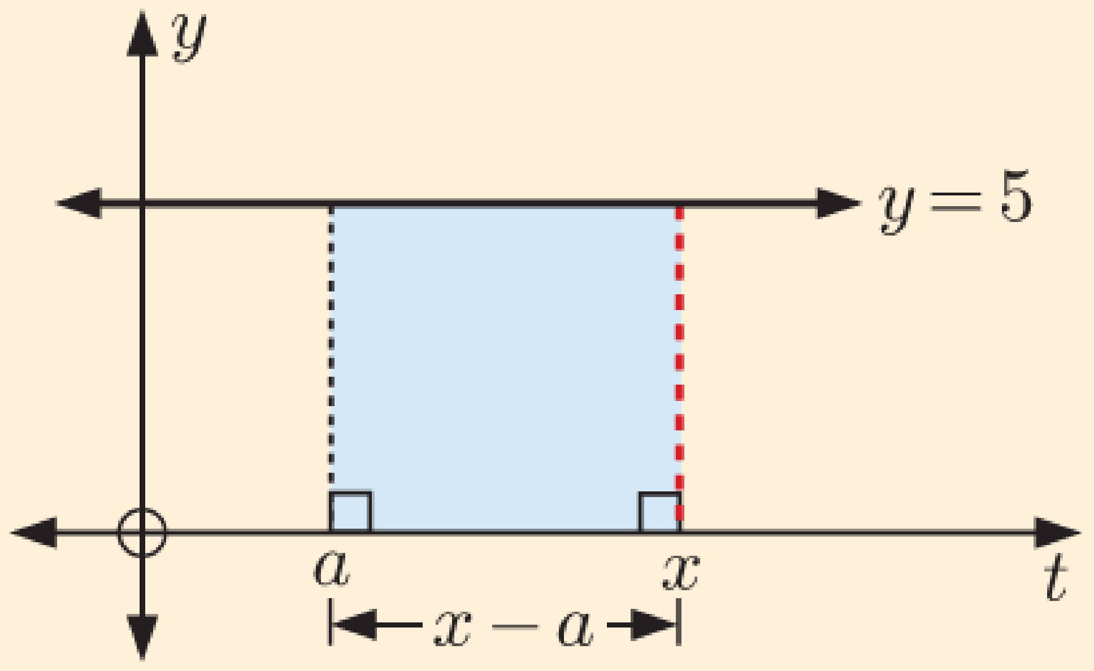
C THE FUNDAMENTAL THEOREM OF CALCULUS

In this Section we explore the link between the Riemann integral we just saw for the area under a curve, and the differential calculus we saw in the previous Chapters. This link is called the **Fundamental Theorem of Calculus**.

INVESTIGATION 1 THE AREA FUNCTION

Consider the constant function $f(t) = 5$.

We wish to find an **area function** which will give the area under the function between $t = a$ and some other value of t which we will call x.



The area function is $A(x) = \int_a^x 5\,dt$ = shaded area

$= (x-a)5$

$= 5x - 5a$

Notice that $f(t) = 5$ is the derivative of the function $F(t) = 5t$. We say that $F(t)$ is the **antiderivative** of $f(t)$. We can therefore write $A(x) = 5x - 5a$ in the form $F(x) - F(a)$.

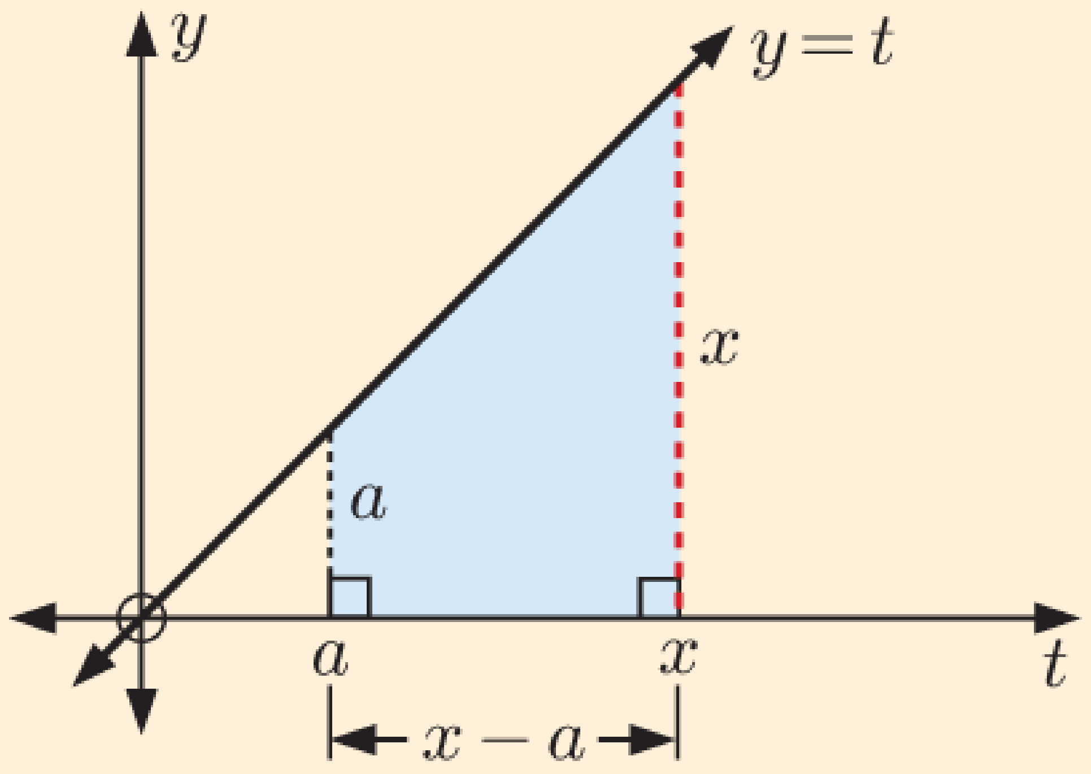
What to do:

1 Consider the simplest linear function $f(t) = t$.
The corresponding area function is

$$A(x) = \int_a^x t\,dt$$

= shaded area

$$= \left(\frac{x+a}{2}\right)(x-a) \qquad \{\text{area of a trapezium}\}$$



a Show that $A(x)$ can be written in the form $F(x) - F(a)$ where $F(x) = \frac{1}{2}x^2$.

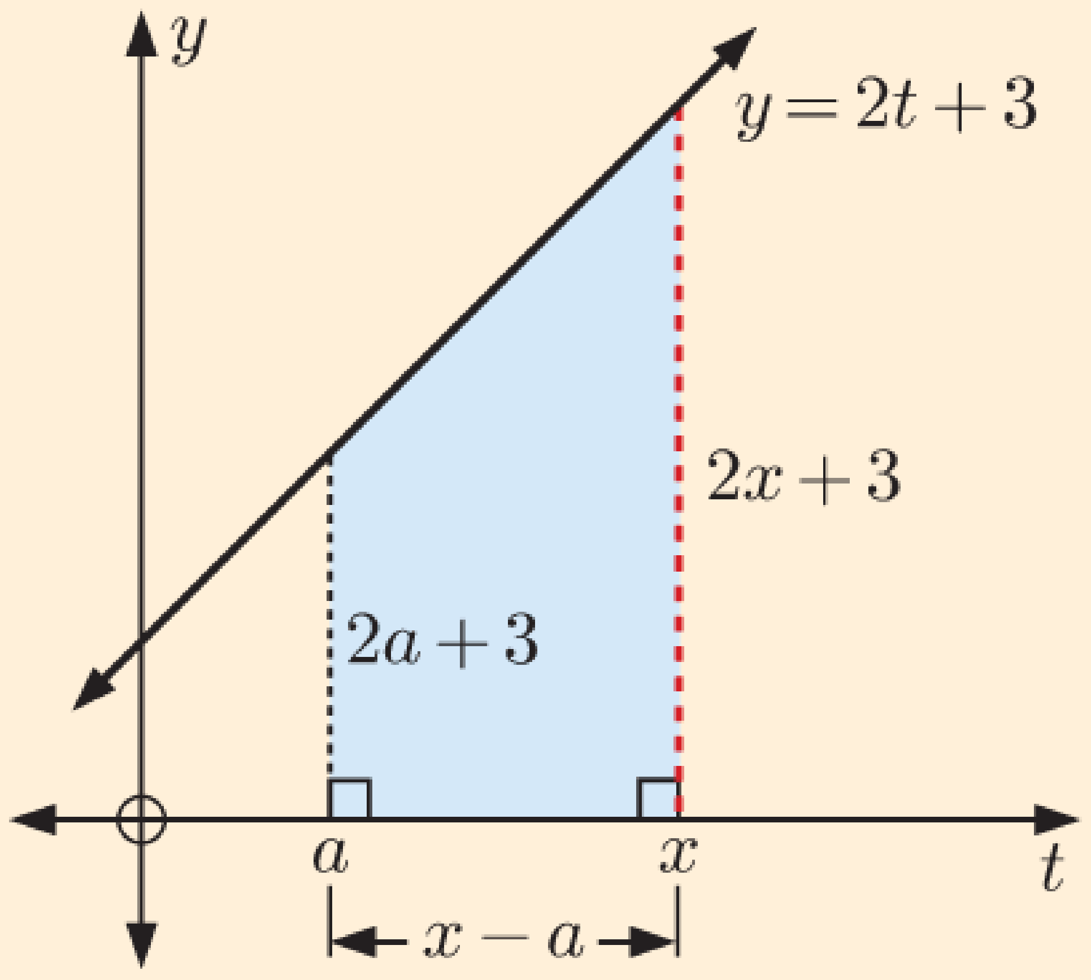
b What is the derivative $F'(t)$? How does it relate to the function $f(t)$?

2 Consider $f(t) = 2t + 3$. The corresponding area function is

$$A(x) = \int_a^x (2t+3)\,dt$$

= shaded area

$$= \left(\frac{2x+3+2a+3}{2}\right)(x-a) \qquad \{\text{area of a trapezium}\}$$



a Show that $A(x)$ can be written in the form $F(x) - F(a)$ where $F(x) = x^2 + 3x$.

b What is the derivative $F'(t)$?
How does it relate to the function $f(t)$?

3 Repeat the procedure in **1** and **2** to find area functions for:

a $f(t) = \frac{1}{2}t + 3$ **b** $f(t) = 5 - 2t$

Do your results fit with your earlier observations?

4 If $f(t) = 3t^2 + 4t + 5$, predict what $F(t)$ will be without performing the algebraic procedure.

From the **Investigation** you should have found that, for $f(t) \geqslant 0$,

$$\int_a^x f(t)\,dt = F(x) - F(a) \quad \text{where} \quad F'(t) = f(t).$$ $F(t)$ is the **antiderivative** of $f(t)$.

Letting $x = b$, $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$

This result is in fact true for *all* continuous functions $f(t)$, even if they are negative.

However, in situations where a function is negative, the area between the curve and the x-axis would be counted as negative.

THE FUNDAMENTAL THEOREM OF CALCULUS

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

In general, $\displaystyle\int_a^b f(x)\,dx$ is called a **definite integral**.

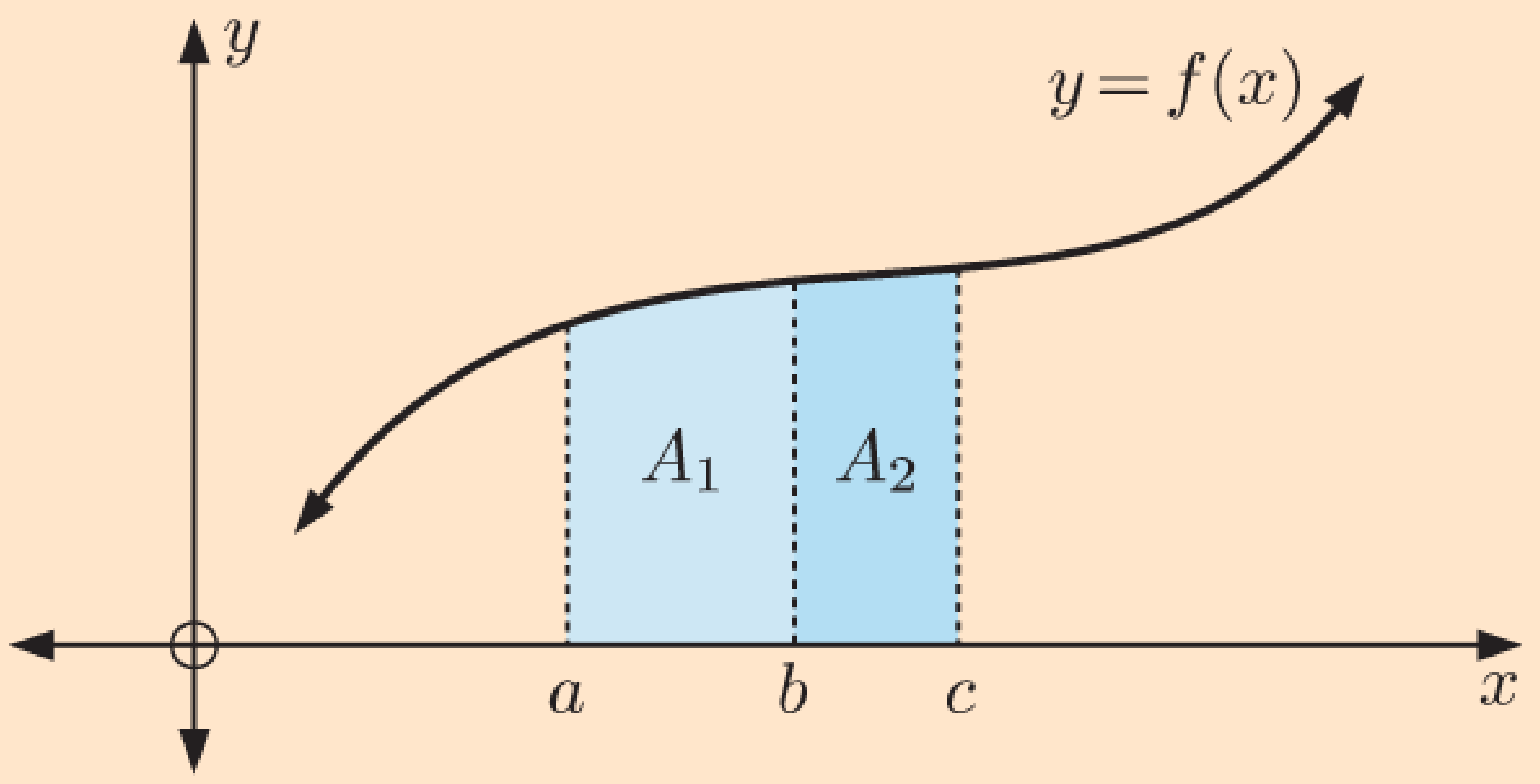
PROPERTIES OF DEFINITE INTEGRALS

The following properties of definite integrals can be deduced from the Fundamental Theorem of Calculus:

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b k\,dx = k(b-a)$ $\{k$ is a constant$\}$
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example proof:

$$\begin{aligned} &\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \\ &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \int_a^c f(x)\,dx \end{aligned}$$



In particular, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$$

The Fundamental Theorem of Calculus allows us to calculate areas under curves that we could previously only estimate.

Example 4 **Self Tutor**

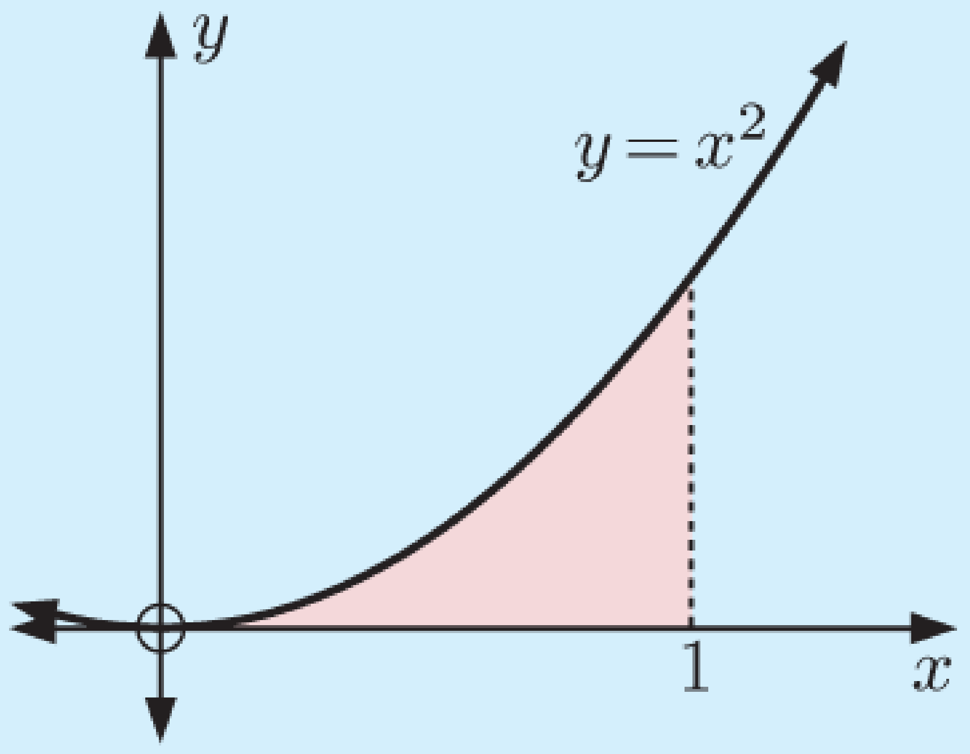
a Differentiate $F(x) = \frac{x^3}{3}$, and hence find the antiderivative of x^2.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$.

a If $F(x) = \frac{x^3}{3}$ then $F'(x) = \frac{1}{3}(3x^2) = x^2$

$\therefore$ $F(x)$ is the antiderivative of x^2.

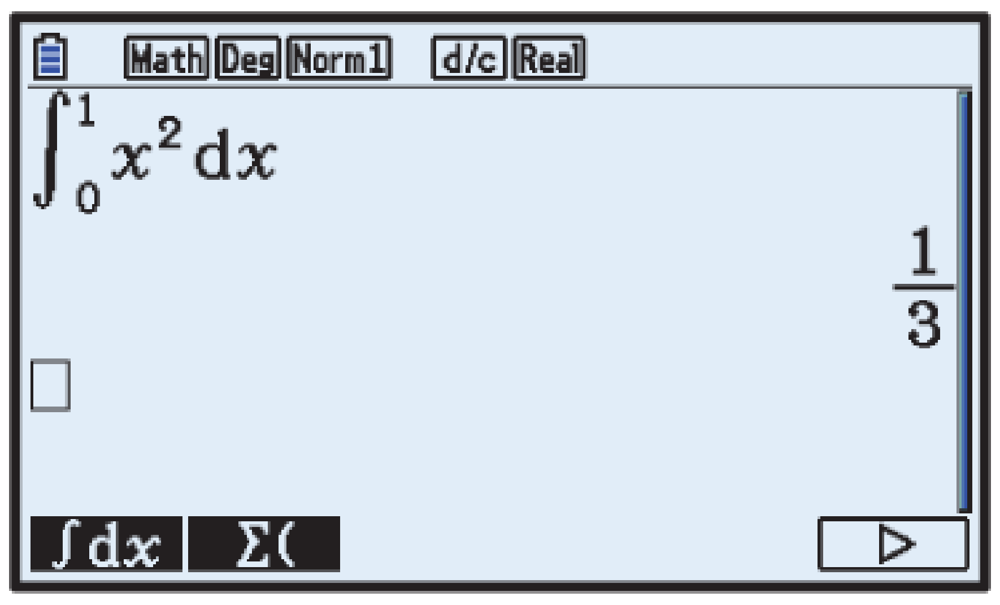
b



$$\begin{aligned}\text{Shaded area} &= \int_0^1 x^2\,dx \\ &= F(1) - F(0) \\ &= \tfrac{1}{3} - 0 \\ &= \tfrac{1}{3}\text{ units}^2\end{aligned}$$

Instructions for evaluating definite integrals on your calculator can be found by clicking on the icon.

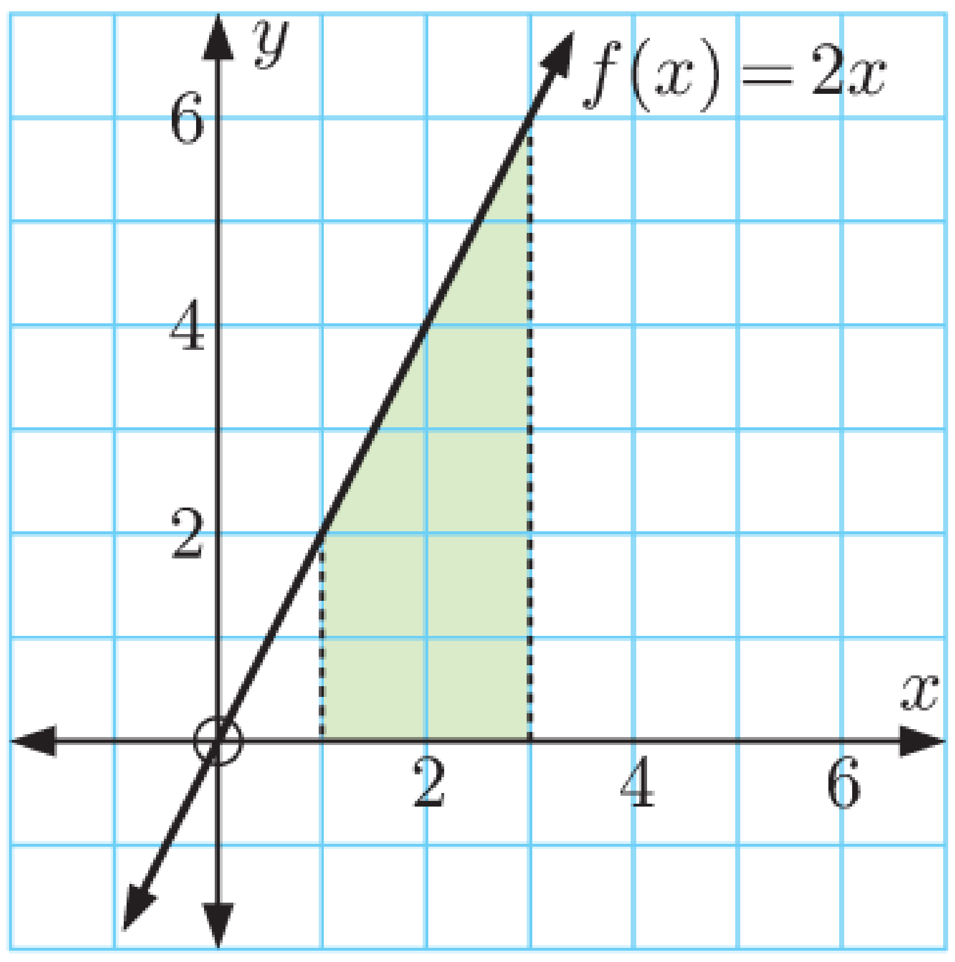
GRAPHICS CALCULATOR INSTRUCTIONS



$\int_0^1 x^2\,dx$

$\frac{1}{3}$

EXERCISE 13C



1 **a** Differentiate $F(x) = x^2$, and hence find the antiderivative of $2x$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $f(x) = 2x$ from $x = 1$ to $x = 3$.

c Use graphical methods to check your answer.

2 **a** Differentiate $F(x) = x^3$, and hence find the antiderivative of $3x^2$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = 3x^2$ from $x = 0$ to $x = 1$.

3 **a** Differentiate $F(x) = \frac{1}{4}x^4$, and hence find the antiderivative of x^3.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^3$ from:

i $x = 0$ to $x = 2$ **ii** $x = 2$ to $x = 3$ **iii** $x = 0$ to $x = 3$.

c Check your answers using technology.

d Comment on your answers in **b**.

4 Use the Fundamental Theorem of Calculus to show that:

- **a** $\displaystyle\int_a^a f(x)\,dx = 0$
- **b** $\displaystyle\int_a^b k\,dx = k(b-a)$, k a constant
- **c** $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- **d** $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, k a constant
- **e** $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- **f** $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

D ANTIDIFFERENTIATION AND INDEFINITE INTEGRALS

The Fundamental Theorem of Calculus gives us motivation to find *antiderivatives.* In other words, given a derivative $\dfrac{dy}{dx}$, we need to know y in terms of x.

The process of finding y from $\dfrac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.

$$\begin{tikzcd} y \text{ or } f(x) \arrow[r, bend left, "\text{differentiation}"] & \dfrac{dy}{dx} \text{ or } f'(x) \arrow[l, bend left, "\text{antidifferentiation}"] \end{tikzcd}$$

Consider $\dfrac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\dfrac{dy}{dx} = x^2$.

However, we might notice that for *all* of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\dfrac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant, which will give $\dfrac{dy}{dx} = x^2$.

We say that:

- The **antiderivative** of x^2 is $\frac{1}{3}x^3$. $\frac{1}{3}x^3$ is the simplest function with derivative x^2.
- The **indefinite integral** of x^2 with respect to x is $\frac{1}{3}x^3 + c$. We write this as $\int x^2\,dx = \frac{1}{3}x^3 + c$.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$
- the **antiderivative** of $f(x)$ is $F(x)$
- the **indefinite integral** of $f(x)$ with respect to x is $\int f(x)\,dx = F(x) + c$.

The constant c is called the **constant of integration**.

Example 5 **Self Tutor**

Differentiate x^4 and hence find:

a the antiderivative of x^3 **b** $\int x^3\,dx$

$$\frac{d}{dx}\left(x^4\right) = 4x^3$$

$$\therefore \quad \frac{d}{dx}\left(\tfrac{1}{4}x^4\right) = x^3$$

a The antiderivative of x^3 is $\frac{1}{4}x^4$. **b** $\int x^3\,dx = \frac{1}{4}x^4 + c$.

EXERCISE 13D

1 Differentiate x^2 and hence find: **a** the antiderivative of x **b** $\int x\,dx$

2 Differentiate x^3 and hence find: **a** the antiderivative of x^2 **b** $\int x^2\,dx$

3 Differentiate x^{-1} and hence find: **a** the antiderivative of x^{-2} **b** $\int x^{-2}\,dx$

4 Differentiate x^{-2} and hence find: **a** the antiderivative of x^{-3} **b** $\int x^{-3}\,dx$

5 **a** Use your observations from the questions above to predict the antiderivative of x^n, $n \in \mathbb{Z}$, $n \neq -1$.

b Hence predict the antiderivative of x^5. Check your answer by differentiation.

6 Find the derivative of $x^3 + x^2$, and hence find $\int (3x^2 + 2x)\,dx$.

7 Find the derivative of $3x^4 - 2x^2$, and hence find $\int (3x^3 - x)\,dx$.

8 Differentiate $\frac{1}{x} + 2x$ and hence find $\int \left(\frac{1}{x^2} - 2\right) dx$.

E RULES FOR INTEGRATION

In **Chapter 10** we developed some rules to help us differentiate functions more efficiently. These rules or combinations of them can be used to differentiate all of the functions we consider in this course. There are also many other rules for differentiation we have not covered, and these allow mathematicians to differentiate almost any function.

Integration is not so simple. For the functions we consider in this course, we will apply the rules of differentiation in reverse to give us integrals. However, there are many types of function for which no simple formula for the integral exists. This is one reason why numerical methods for integration such as the trapezoidal rule are important.

HISTORICAL NOTE

Robert Henry Risch (1939 -) is an American mathematician. He studied at the University of California, Berkeley.

In his doctorate studies in 1968, Risch devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

After completing his doctorate, he worked at the IBM Thomas Watson Research Centre.

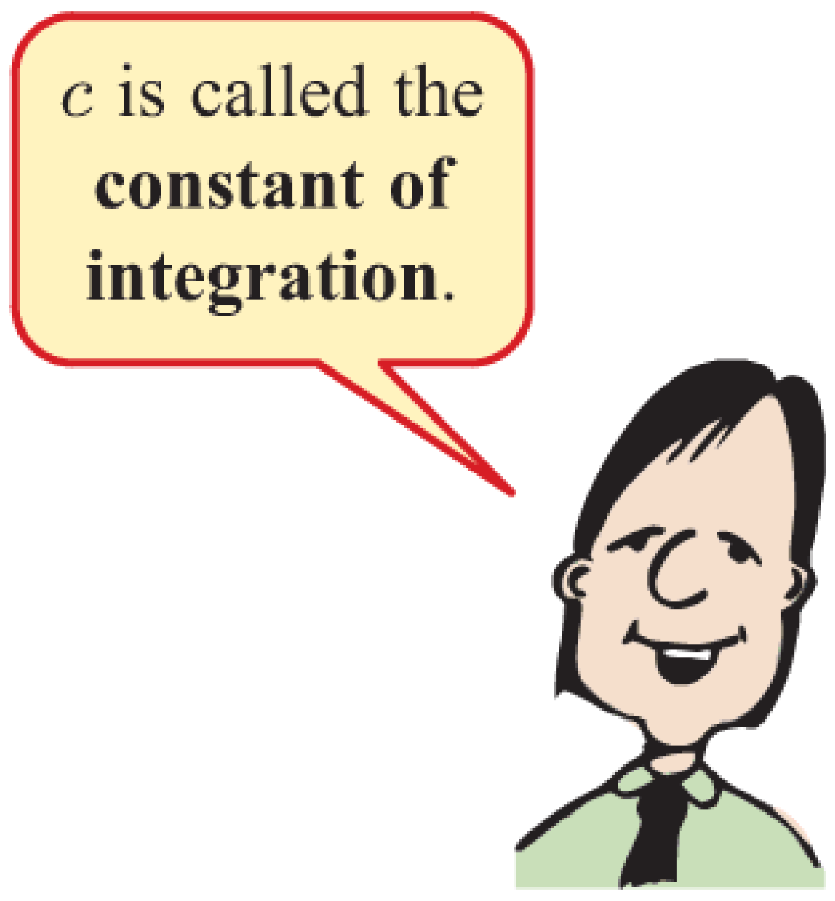
INTEGRATING BASIC FUNCTION TYPES

- For k a constant, $\frac{d}{dx}(kx+c) = k$

$$\therefore \quad \boxed{\int k\,dx = kx + c}$$

- If $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{(n+1)x^n}{n+1} = x^n$

$$\therefore \quad \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1}$$


c is called the **constant of integration**.

DISCUSSION

In the rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$, why do we exclude the value $n = -1$?

GENERAL RULES FOR INTEGRATION

- Any constant factor within the integral may be written in front of the integral sign.

$$\boxed{\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}}$$

Proof: Consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\begin{aligned}\frac{d}{dx}(k\,F(x)) &= k\,F'(x) = k\,f(x)\\ \therefore \int k\,f(x)\,dx &= k\,F(x) + c\\ &= k\int f(x)\,dx\end{aligned}$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

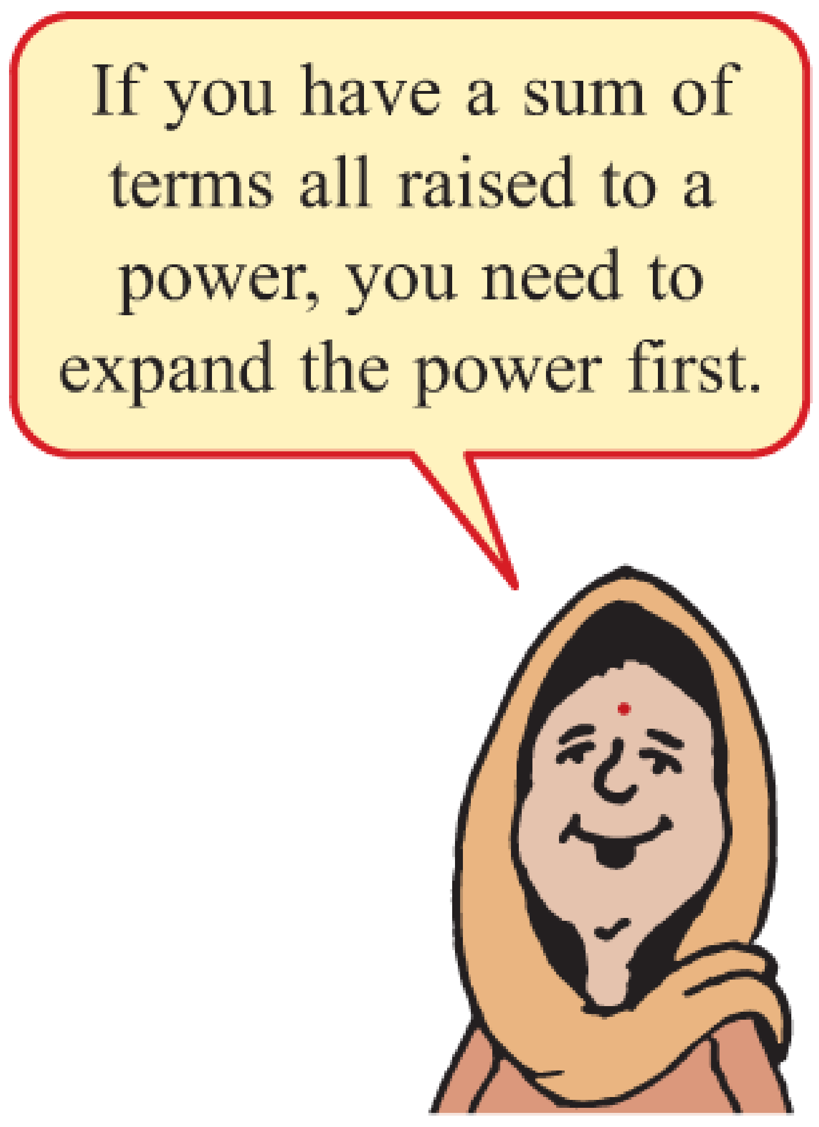
Example 6 **Self Tutor**

Find:

a $\int (-2x^3 + 5x - 2)\,dx$ **b** $\int \left(x - \frac{1}{x}\right)^2 dx$

a
$$\begin{aligned}&\int (-2x^3 + 5x - 2)\,dx\\ &= \frac{-2x^4}{4} + \frac{5x^2}{2} - 2x + c\\ &= -\tfrac{1}{2}x^4 + \tfrac{5}{2}x^2 - 2x + c\end{aligned}$$

b
$$\begin{aligned}&\int \left(x - \frac{1}{x}\right)^2 dx\\ &= \int \left(x^2 - 2x\left(\frac{1}{x}\right) + \frac{1}{x^2}\right) dx\\ &= \int (x^2 - 2 + x^{-2})\,dx\\ &= \frac{x^3}{3} - 2x + \frac{x^{-1}}{(-1)} + c\\ &= \tfrac{1}{3}x^3 - 2x - \frac{1}{x} + c\end{aligned}$$


If you have a sum of terms all raised to a power, you need to expand the power first.

EXERCISE 13E

1 Find:

a $\int 3\,dx$ **b** $\int (-2)\,dx$ **c** $\int 2x\,dx$

d $\int 3x^2\,dx$ **e** $\int 5x^4\,dx$ **f** $\int \left(-\tfrac{1}{2}x^3\right) dx$

g $\int \tfrac{2}{3}x^4\,dx$ **h** $\int \frac{2}{x^2}\,dx$ **i** $\int \frac{1}{2x^3}\,dx$

2 Find:

a $\int (2x - 1)\,dx$ **b** $\int (x + 3)\,dx$ **c** $\int (4 - x)\,dx$

d $\int \frac{3x + 1}{2}\,dx$ **e** $\int (x^2 - 2)\,dx$ **f** $\int (5 - x^2)\,dx$

g $\int \left(\tfrac{2}{3}x + 3x^2\right) dx$ **h** $\int \frac{2x^3 - 4}{3}\,dx$ **i** $\int \frac{x^3 + 1}{x^2}\,dx$

3 Find y if:

a $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = \dfrac{1}{x^2}$

d $\dfrac{dy}{dx} = 2x^3 - 4$ **e** $\dfrac{dy}{dx} = 4x^3 + 3x^2$ **f** $\dfrac{dy}{dx} = 2 - \dfrac{1}{x^2}$

4 Find:

a $\int (x^2 + 3x - 2)\, dx$ **b** $\int (2x^2 - 3x + 1)\, dx$

c $\int (-x^3 + 4x^2 - 3)\, dx$ **d** $\int \left(\frac{1}{2}x + x^2 + x^3\right) dx$

e $\int (x^4 - x^2 - x + 2)\, dx$ **f** $\int \left(\dfrac{1}{3x^3} - \dfrac{2}{x^2}\right) dx$

g $\int \left(\frac{1}{2}x^3 - x^4 + x\right) dx$ **h** $\int \left(\dfrac{4}{x^2} + x^2 - \frac{1}{4}x^3\right) dx$

You can check your integration by differentiating the resulting function.

5 Find:

a $\int (2x+1)^2\, dx$ **b** $\int \left(x + \dfrac{1}{x}\right)^2 dx$ **c** $\int \left(3 - x^2\right)^2 dx$

d $\int \left(\dfrac{2}{x^2} + 1\right)^2 dx$ **e** $\int (x+1)^3\, dx$ **f** $\int (x-1)^4\, dx$

6 Suppose $F(x)$ and $G(x)$ have the derivative functions $f(x)$ and $g(x)$ respectively.

a Find the derivative of $F(x) + G(x)$.

b Show that $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$.

F PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 7 **Self Tutor**

Find $f(x)$ given that $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$.

$$f'(x) = x^3 - 2x^2 + 3$$

$$\therefore\ f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$.

EXERCISE 13F

1 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = 3 - 2x^2$ and $f(1) = 1$

d $f'(x) = x - \dfrac{2}{x^2}$ and $f(1) = 2$

e $f'(x) = x^3 - 2$ and $f(4) = 0$

f $f'(x) = 3x^2 - 4x + 1$ and $f(0) = 12$.

2 A curve has gradient function $\dfrac{dy}{dx} = x - 2x^2$ and passes through $(2, 4)$. Find the equation of the curve.

3 A curve has gradient function $\dfrac{dy}{dx} = 1 - \dfrac{3}{x^2}$ and x-intercept 3. Find the equation of the curve.

4 A curve has gradient function $f'(x) = ax + 1$ where a is a constant. Find $f(x)$ given that $f(0) = 3$ and $f(3) = -3$.

5 A curve has gradient function $f'(x) = ax^2 + bx$ where a, b are constants. Find $f(x)$ given that $f(-1) = -2$, $f(0) = 1$, and $f(1) = 4$.

G DEFINITE INTEGRALS

We have already seen how the Fundamental Theorem of Calculus gives meaning to a **definite integral** over a particular domain:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

$\displaystyle\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x"
or "the integral from a to b of $f(x)$ with respect to x".

It is common to write $F(b) - F(a)$ as $[F(x)]_a^b$, so $\displaystyle\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

For continuous functions, we can list the following properties of definite integrals:

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, k is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 8 **Self Tutor**

Find:

a $\displaystyle\int_1^2 (x^3 - 4x + 5)\,dx$ **b** $\displaystyle\int_1^4 \left(2x + \frac{3}{x^2}\right)dx$

a
$$\begin{aligned}&\int_1^2 (x^3 - 4x + 5)\,dx\\ &= \left[\tfrac{1}{4}x^4 - 2x^2 + 5x\right]_1^2\\ &= \left(\tfrac{2^4}{4} - 2(2)^2 + 5(2)\right) - \left(\tfrac{1^4}{4} - 2(1)^2 + 5(1)\right)\\ &= 6 - 3\tfrac{1}{4}\\ &= \tfrac{11}{4}\end{aligned}$$

We omit c for definite integrals.

b
$$\begin{aligned}&\int_1^4 \left(2x + \frac{3}{x^2}\right)dx\\ &= \int_1^4 (2x + 3x^{-2})\,dx\\ &= \left[x^2 - \frac{3}{x}\right]_1^4\\ &= \left(16 - \tfrac{3}{4}\right) - (1 - 3)\\ &= 17\tfrac{1}{4}\end{aligned}$$

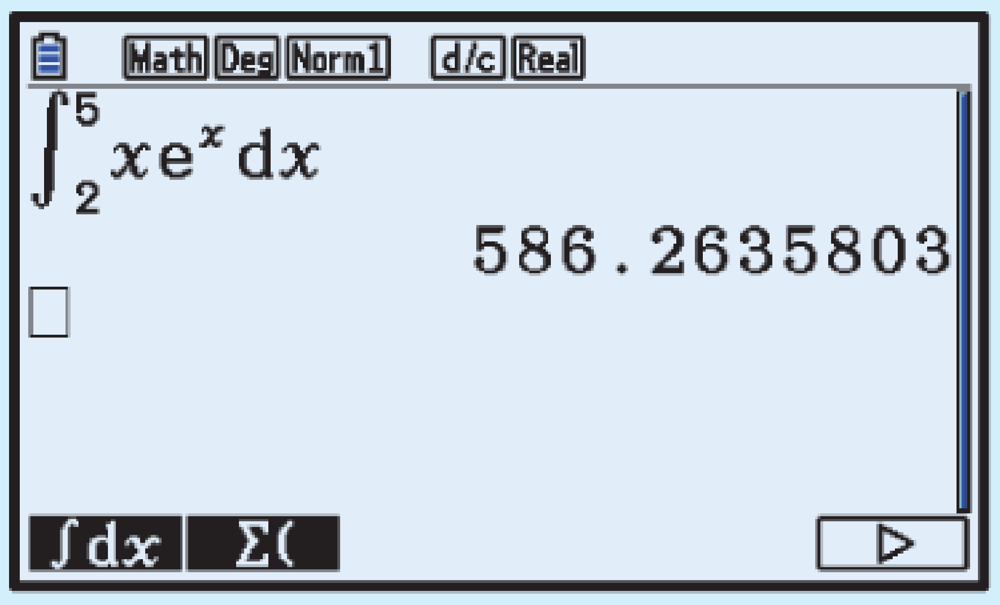
Some definite integrals are difficult or even impossible to evaluate analytically. In these cases you can use a graphics calculator to evaluate the integral.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 9 **Self Tutor**

Evaluate $\displaystyle\int_2^5 xe^x\,dx$ correct to 4 significant figures.

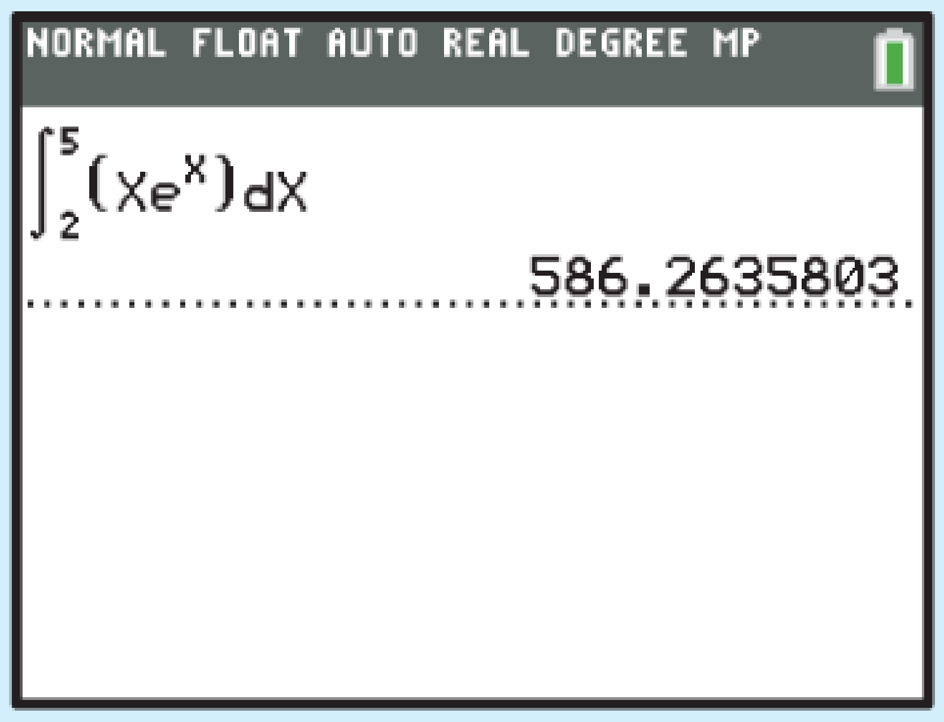
Casio fx-CG50



$\int_2^5 xe^x\,dx$

586.2635803

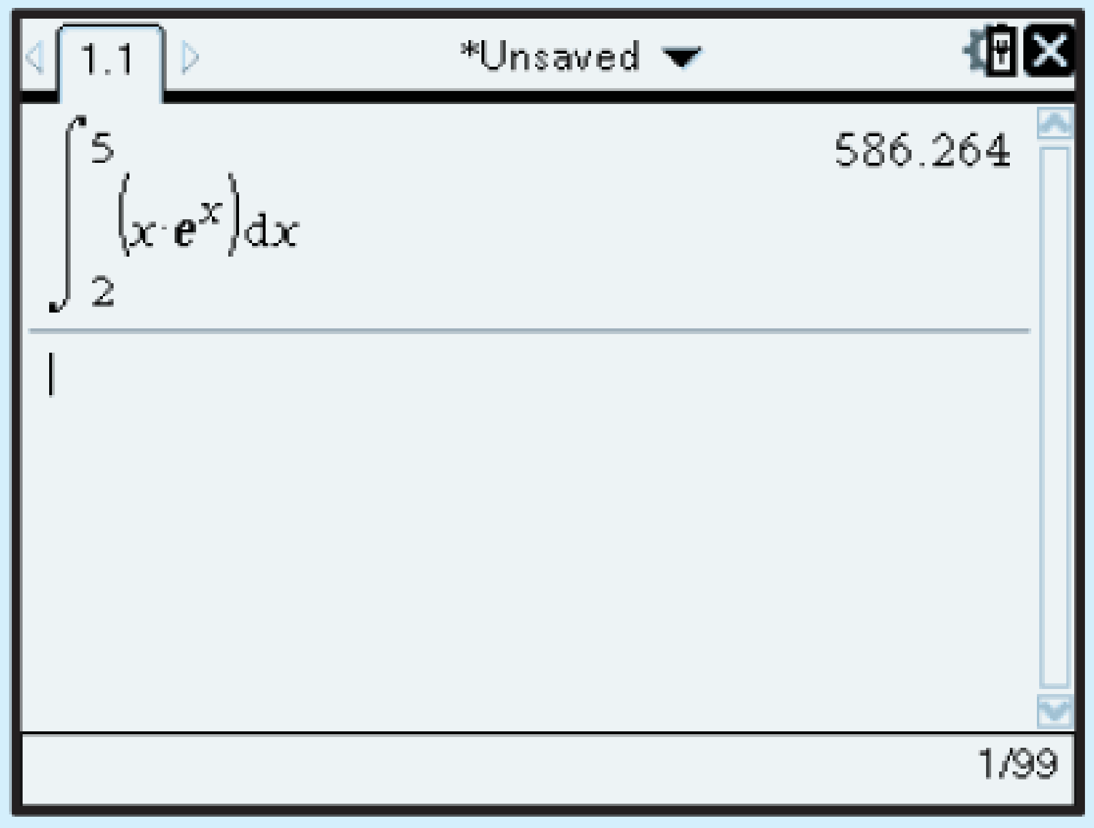
TI-84 Plus CE



$\int_2^5 (Xe^X)\,dX$

586.2635803

TI-*n*spire



$\int_2^5 (x\cdot e^x)\,dx$ 586.264

$$\int_2^5 xe^x\,dx \approx 586.3$$

EXERCISE 13G

1 Find:

a $\displaystyle\int_1^2 x^3\,dx$ and $\displaystyle\int_1^2 (-x^3)\,dx$ **b** $\displaystyle\int_0^1 x^7\,dx$ and $\displaystyle\int_0^1 (-x^7)\,dx$

2 Find:

a $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_1^2 x^2\,dx$

c $\displaystyle\int_0^2 x^2\,dx$ **d** $\displaystyle\int_0^1 3x^2\,dx$

Use questions **1** to **4** to check the properties of definite integrals.

3 Find:

a $\displaystyle\int_0^2 (x^3-4x)\,dx$ **b** $\displaystyle\int_2^3 (x^3-4x)\,dx$ **c** $\displaystyle\int_0^3 (x^3-4x)\,dx$

4 Find:

a $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_0^1 (-3x)\,dx$ **c** $\displaystyle\int_0^1 (x^2-3x)\,dx$

5 Evaluate:

a $\displaystyle\int_0^1 x^3\,dx$ **b** $\displaystyle\int_0^2 (x^2-x)\,dx$ **c** $\displaystyle\int_0^2 (3x^2-x+6)\,dx$

d $\displaystyle\int_1^4 (3+2x-x^2)\,dx$ **e** $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$ **f** $\displaystyle\int_1^2 (x+3)^2\,dx$

g $\displaystyle\int_1^2 \left(x^2+\frac{1}{x^2}\right)dx$ **h** $\displaystyle\int_1^4 \frac{x^2+5}{x^2}\,dx$ **i** $\displaystyle\int_2^3 \left(\frac{4}{x^3}-\frac{6}{x^4}\right)dx$

6 Find m such that:

a $\displaystyle\int_0^m (3x^2+2)\,dx = 72$ **b** $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$

7 Use technology to evaluate, correct to 4 significant figures:

a $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **b** $\displaystyle\int_{-3}^0 \sqrt{1-x}\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

d $\displaystyle\int_0^1 xe^{1-x}\,dx$ **e** $\displaystyle\int_1^3 \ln x\,dx$ **f** $\displaystyle\int_{-1}^1 e^{-x^2}\,dx$

8 Write as a single integral:

a $\displaystyle\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$ **b** $\displaystyle\int_4^5 f(x)\,dx - \int_6^5 f(x)\,dx$

c $\displaystyle\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$

9 **a** If $\displaystyle\int_1^3 f(x)\,dx = 2$ and $\displaystyle\int_1^6 f(x)\,dx = -3$, find $\displaystyle\int_3^6 f(x)\,dx$.

b If $\displaystyle\int_0^2 f(x)\,dx = 5$, $\displaystyle\int_4^6 f(x)\,dx = -2$, and $\displaystyle\int_0^6 f(x)\,dx = 7$, find $\displaystyle\int_2^4 f(x)\,dx$.

10 Suppose $\int_{-1}^{1} f(x)\,dx = -4$. Determine the value of:

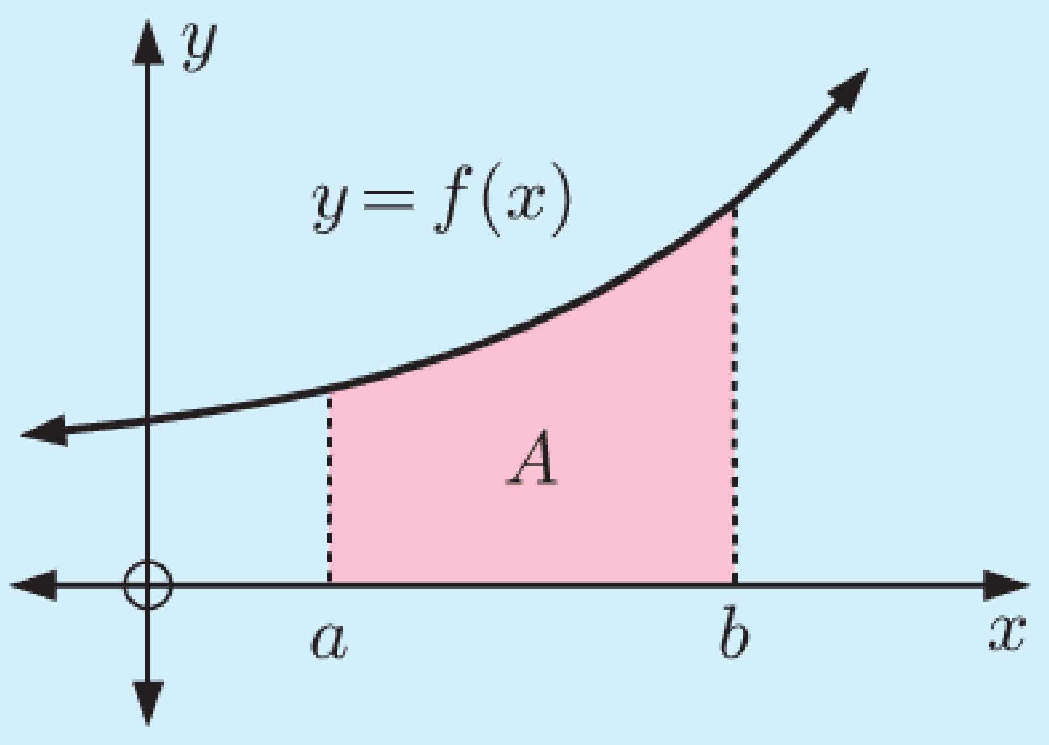
a $\int_{1}^{-1} f(x)\,dx$ **b** $\int_{-1}^{1} (2 + f(x))\,dx$ **c** $\int_{-1}^{1} 2f(x)\,dx$

d k such that $\int_{-1}^{1} kf(x)\,dx = 7$

H THE AREA UNDER A CURVE

Having now studied techniques for integration, we return to the **Riemann integral**, or definite integral for a positive function, which gives the area under a curve.

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

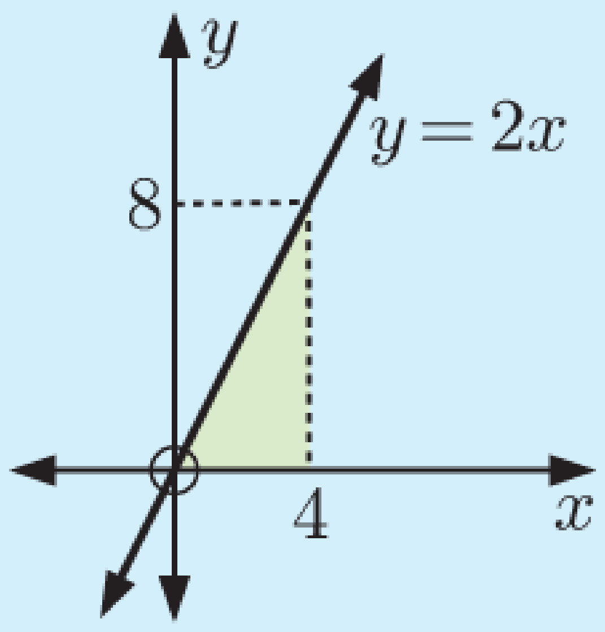


Example 10 **Self Tutor**

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$, and $x = 4$ using:

a a geometric argument **b** integration.

a

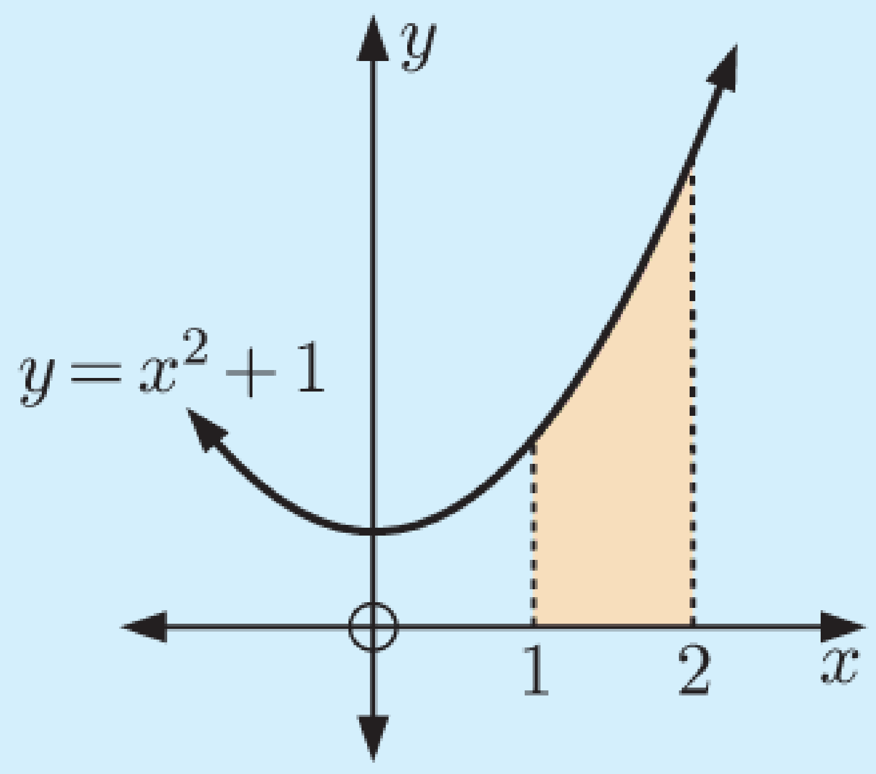


$$\text{Area} = \tfrac{1}{2} \times 4 \times 8$$
$$= 16 \text{ units}^2$$

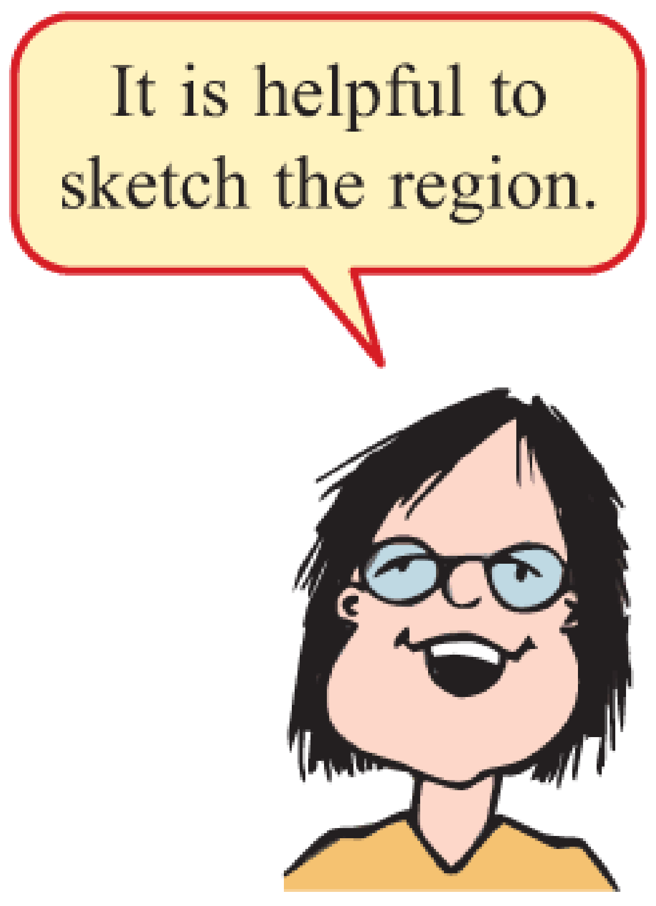
b
$$\text{Area} = \int_0^4 2x\,dx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0^2$$
$$= 16 \text{ units}^2$$

Example 11 **Self Tutor**

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$, and $x = 2$.



$$\text{Area} = \int_1^2 (x^2 + 1)\,dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$


It is helpful to sketch the region.

We can check this result using technology.

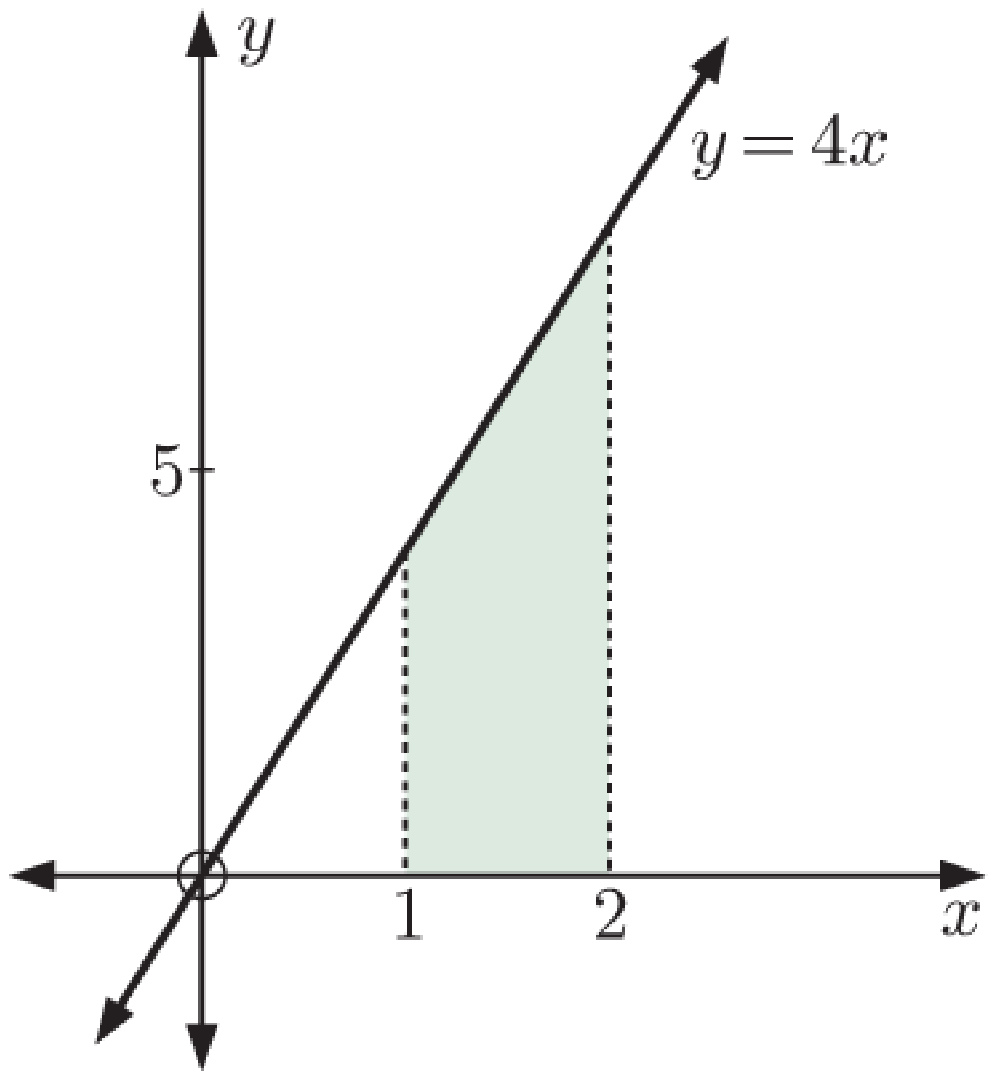
GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 13H

1 Find the shaded area using:

- **a** a geometric argument
- **b** integration.



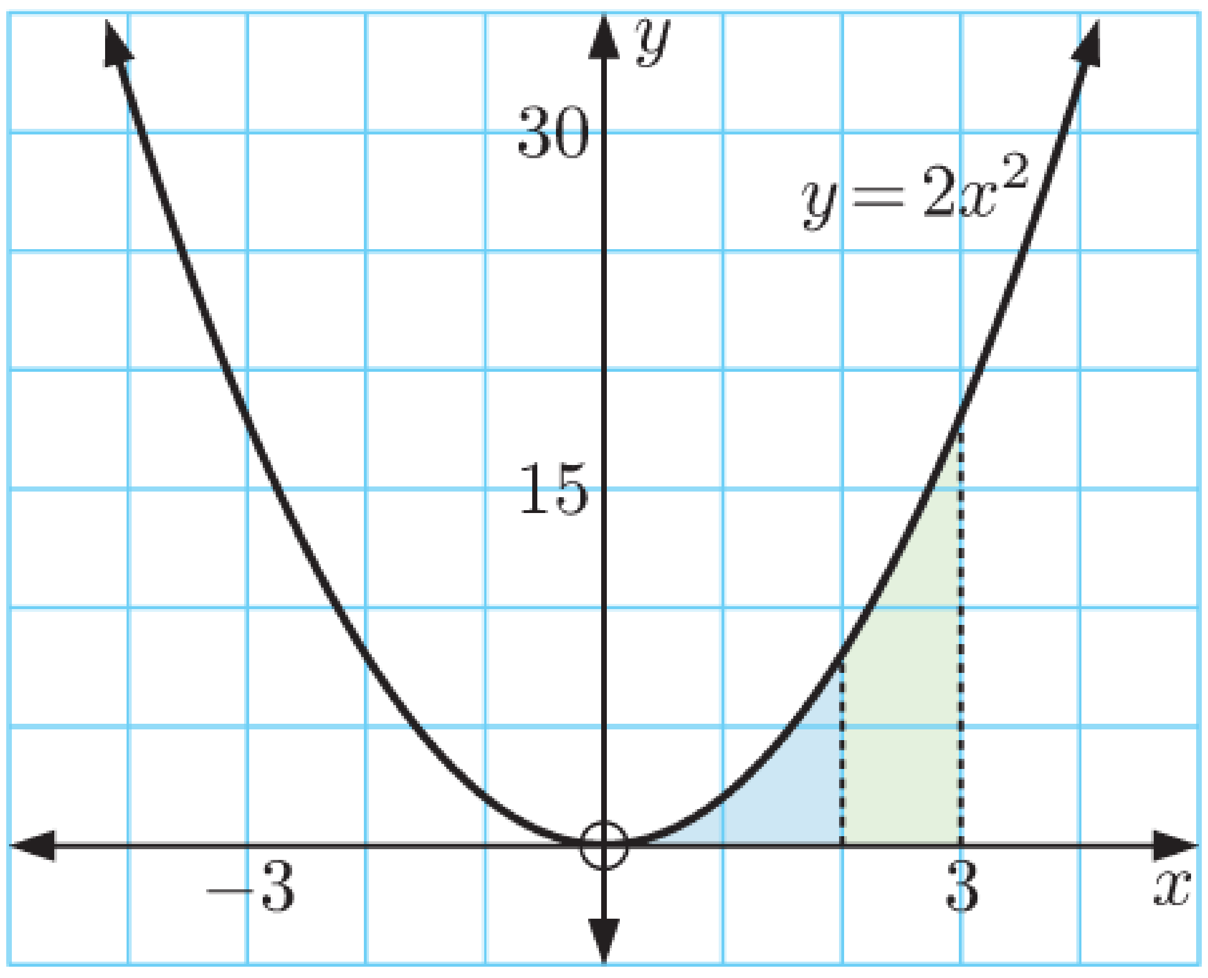
2 Find the area of each region described below using:

i a geometric argument **ii** integration

- **a** the region enclosed by $y = 5$, the x-axis, $x = -6$, and $x = 0$
- **b** the region enclosed by $y = x$, the x-axis, $x = 4$, and $x = 5$
- **c** the region enclosed by $y = -3x$, the x-axis, $x = -3$, and $x = 0$.

3 Find the exact area of:

- **a** the blue shaded region
- **b** the green shaded region.

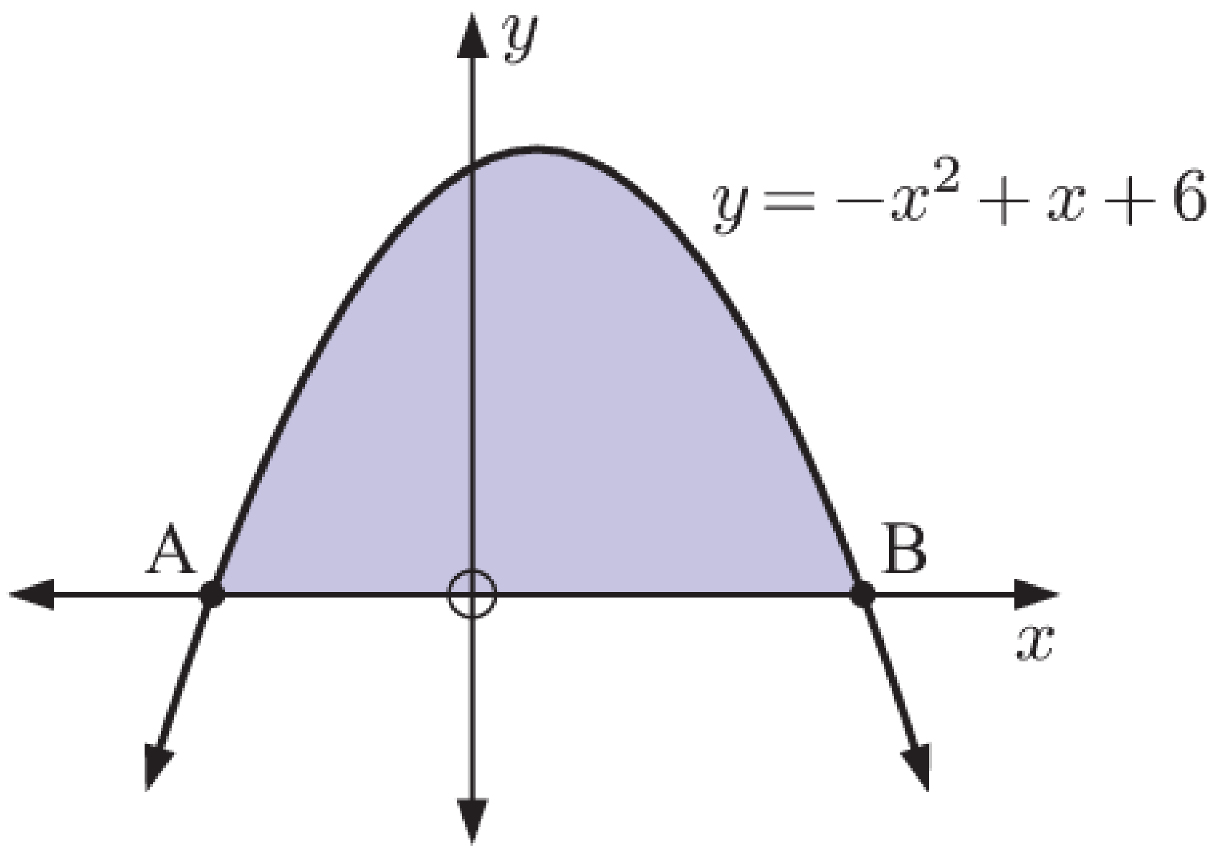


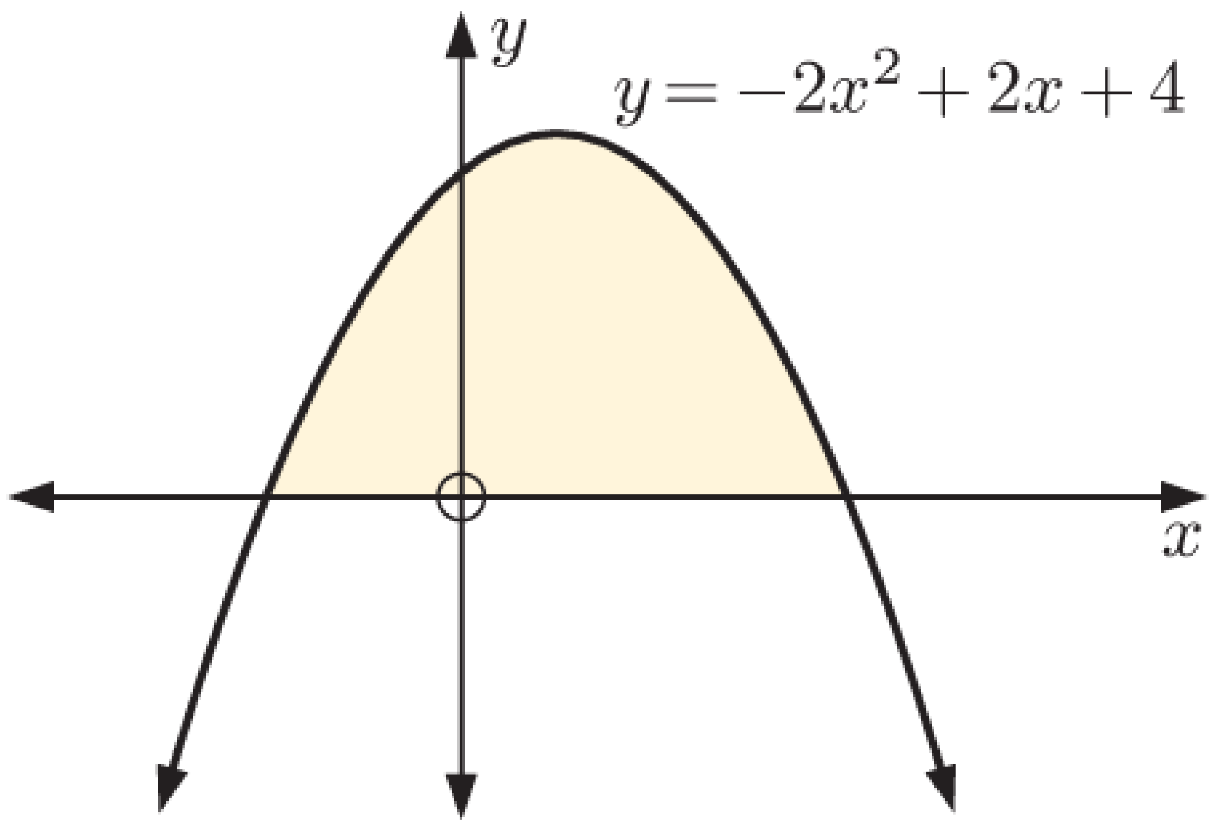
4 Find the area of the region bounded by:

- **a** $y = x^2$, the x-axis, and $x = 1$
- **b** $y = x^3$, the x-axis, $x = 1$, and $x = 4$
- **c** $y = \frac{1}{2}x^2 - 1$, the x-axis, $x = 2$, and $x = 3$.

5 The graph of $y = -x^2 + x + 6$ is shown alongside.

- **a** Find the coordinates of A and B.
- **b** Hence find the shaded area.



6 

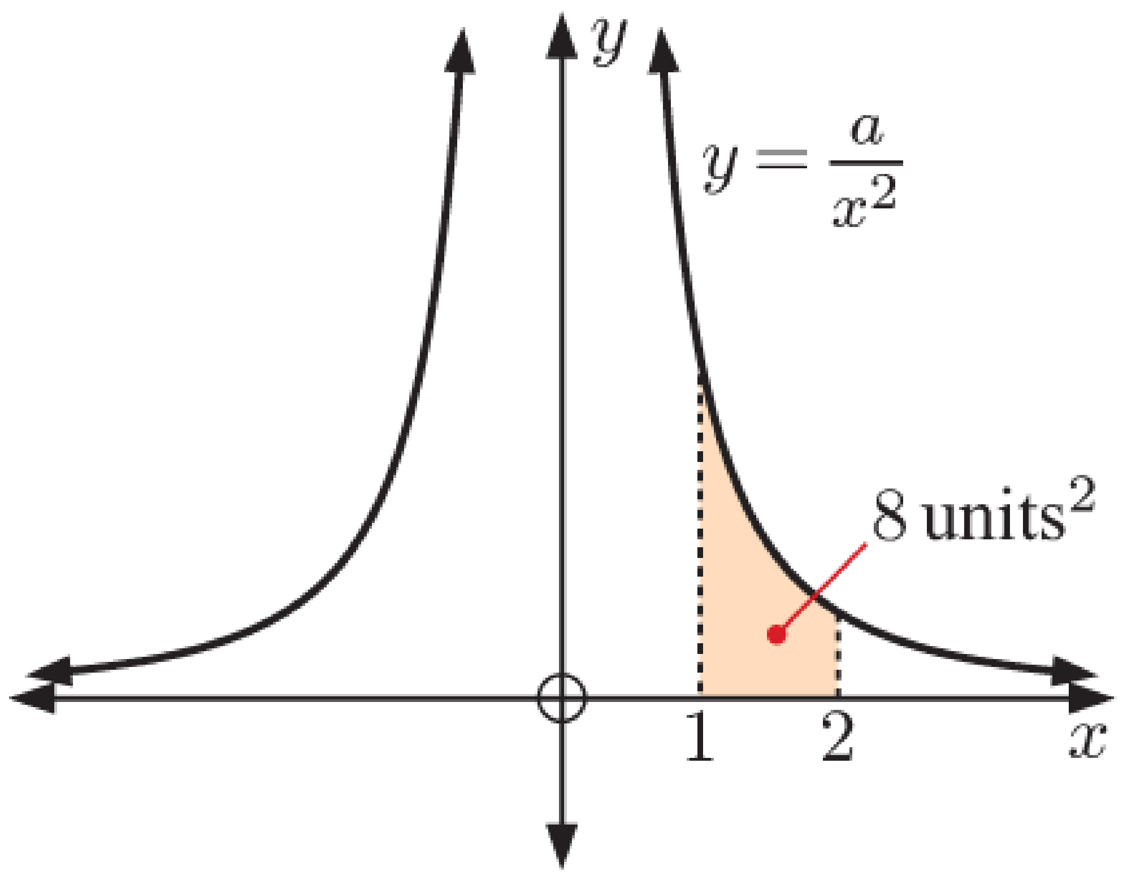
Find the area enclosed by $y=-2x^2+2x+4$ and the x-axis.

7 Find the area enclosed by each curve and the x-axis:

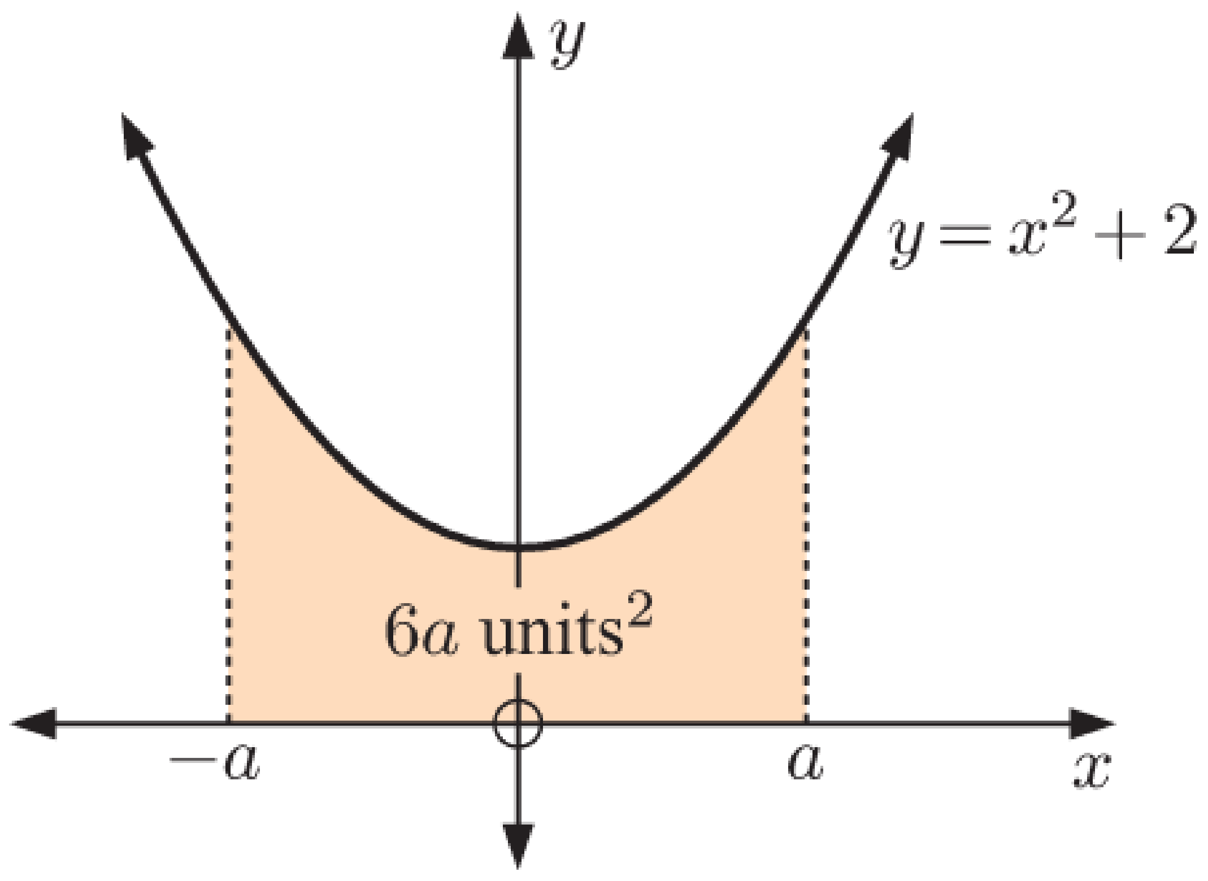
a $y=-(x-2)(x-5)$ **b** $y=3-x^2$

8 Find the area of the region bounded by $y=\frac{1}{x^2}$, the x-axis, $x=1$, and $x=2$.

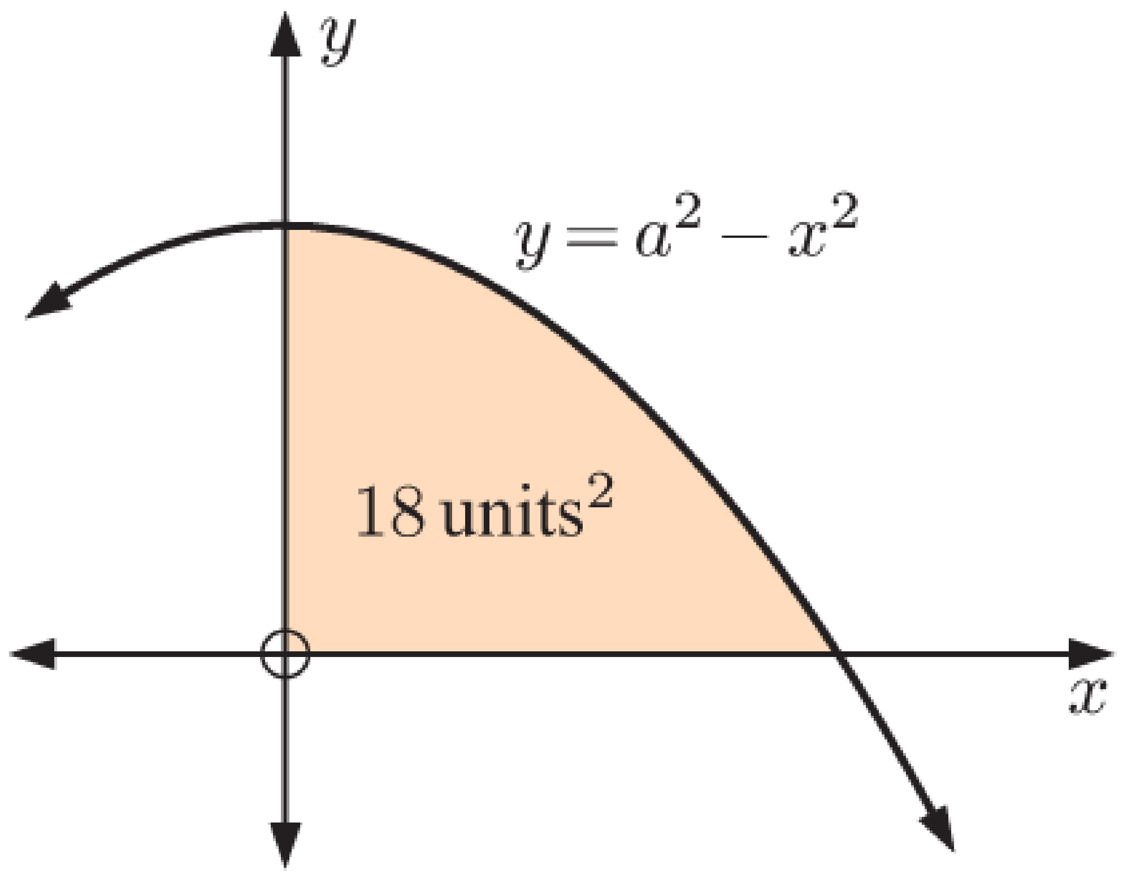
9 Find the exact value of a:

a 

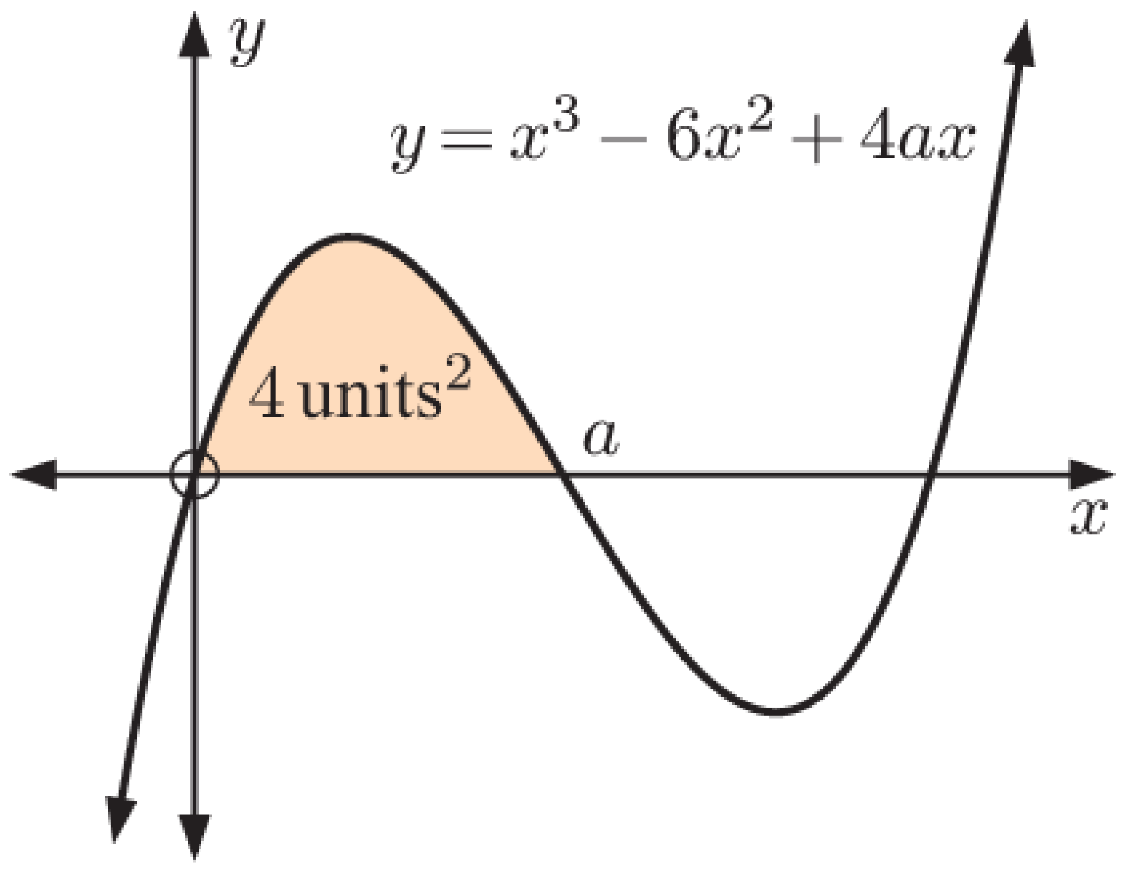
y
$y=\frac{a}{x^2}$
8 units^2
1
2
x

b 

y
$y=x^2+2$
$6a$ units^2
$-a$
a
x

c 

y
$y=a^2-x^2$
18 units^2
x

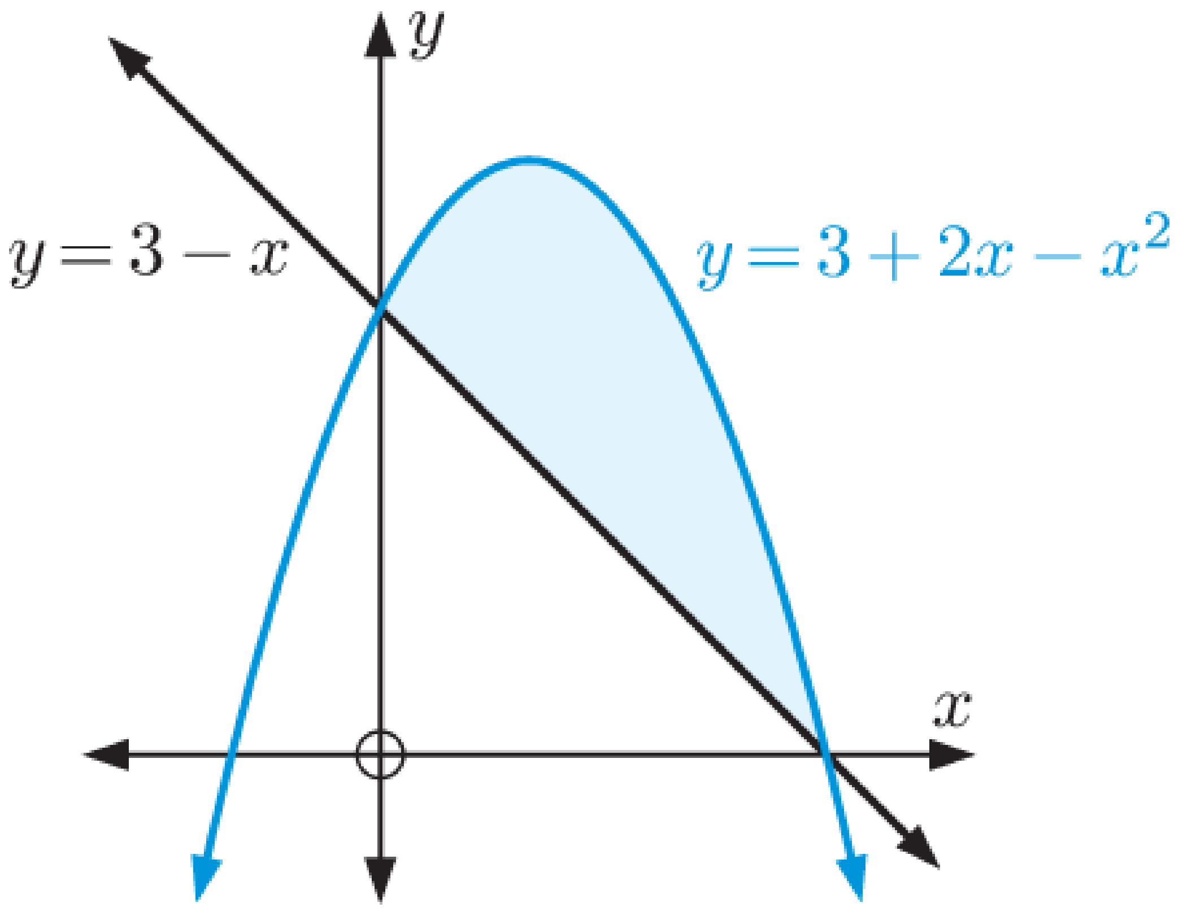
d 

y
$y=x^3-6x^2+4ax$
4 units^2
a
x

10 **a** Find the area of the region bounded by the x-axis, $x=0$, $x=3$, and:

i $y=3-x$ **ii** $y=3+2x-x^2$.

b Hence find the shaded area:



y
$y=3-x$
$y=3+2x-x^2$
x

INVESTIGATION 2 $\int_a^b f(x)\,dx$ AND AREAS

Can $\int_a^b f(x)\,dx$ always be interpreted as an area?

What to do:

1 **a** Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.

b Use a graph to explain why the first integral in **a** gives an area, whereas the second integral does not.

c Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.

d Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

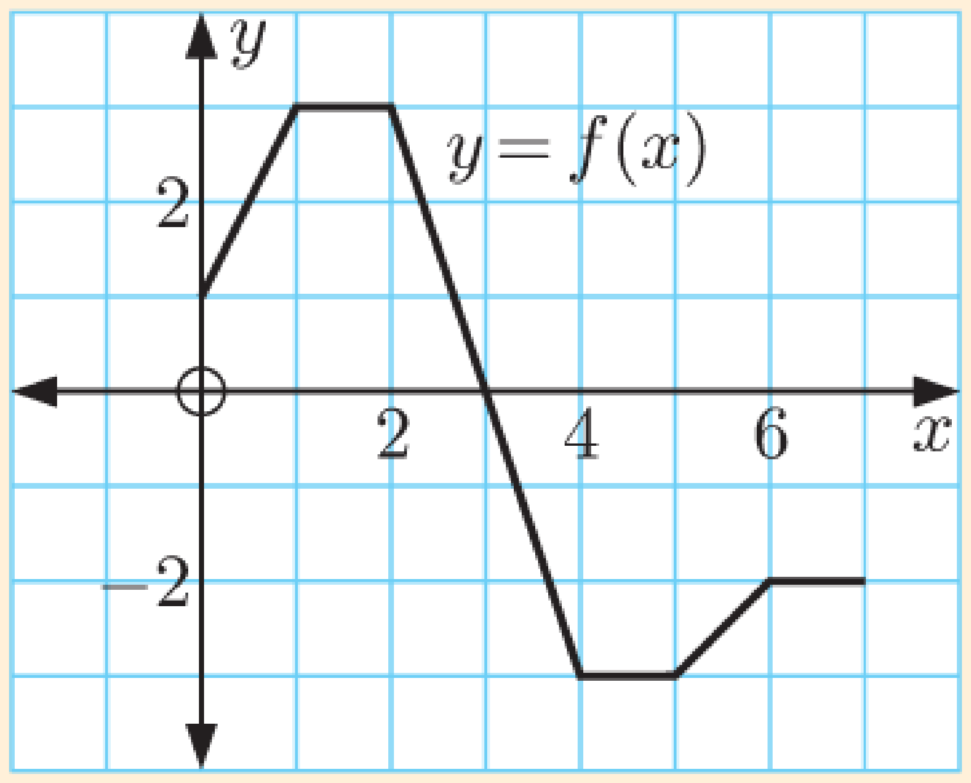
e Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.

2 Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Write an expression for the area between the function and the x-axis for $a \leqslant x \leqslant b$.

3 Evaluate using area interpretation:

a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

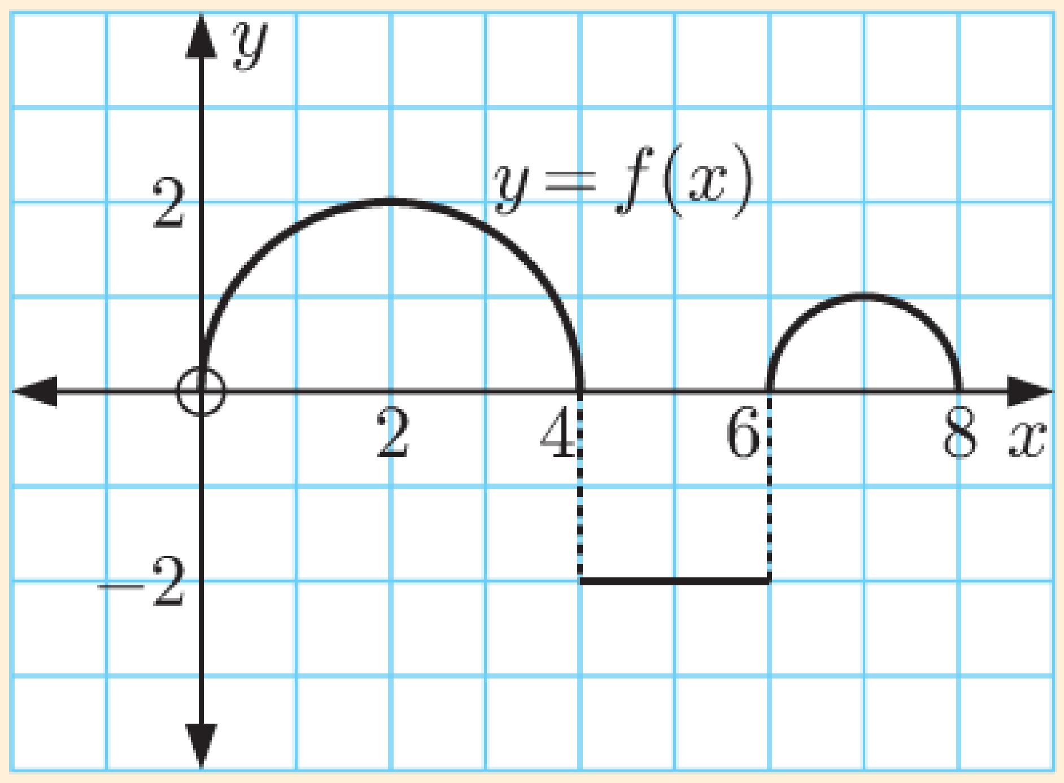
c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$



4 Evaluate using area interpretation:

a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

c $\int_6^8 f(x)\,dx$ **d** $\int_0^8 f(x)\,dx$



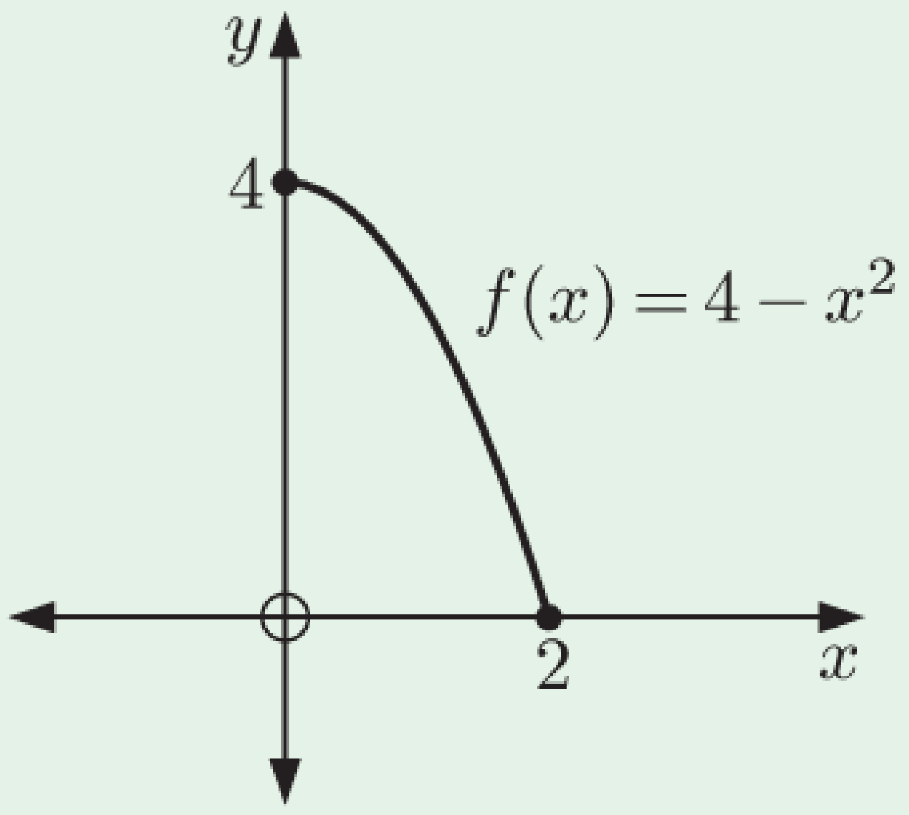
REVIEW SET 13A

1 **a** Use *four* lower and upper rectangles to find rational numbers A and B such that:

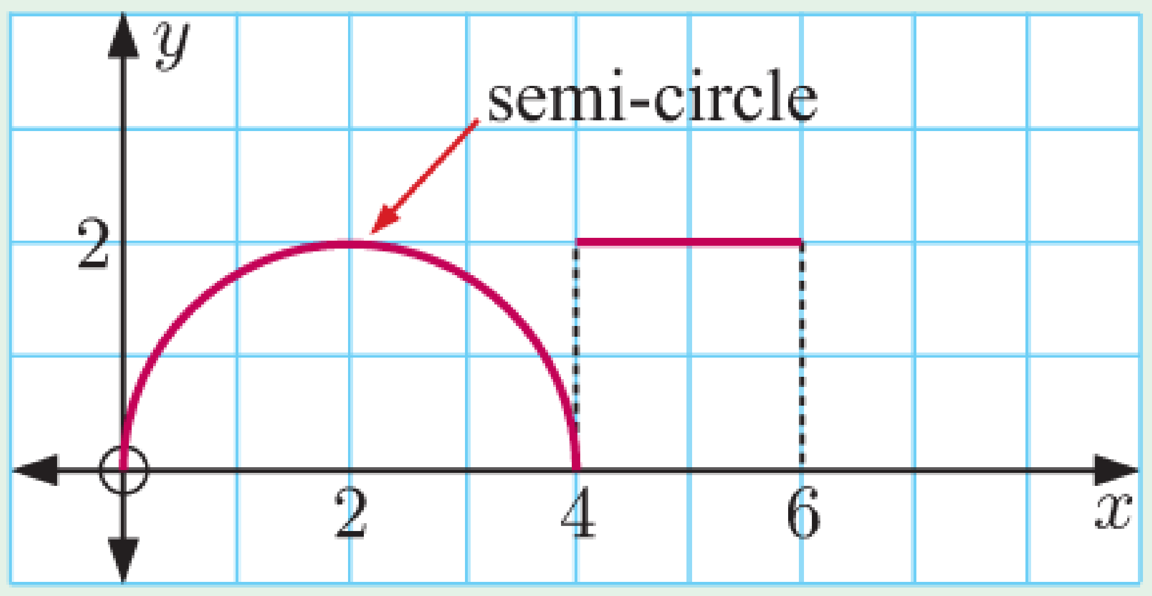
$$A < \int_0^2 (4 - x^2)\,dx < B.$$

b Estimate $\int_0^2 (4 - x^2)\,dx$ using the trapezoidal method with 8 subintervals.

c Compare your answers with the exact value of $\int_0^2 (4 - x^2)\,dx$.



2 The graph of $y = f(x)$ is illustrated:
Evaluate the following using area interpretation:



a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

3 Use graphical evidence and known area facts to find:

a $\int_2^4 (2x-1)\,dx$ **b** $\int_{-1}^1 \sqrt{1-x^2}\,dx$

4 Find the derivative of $x^4 - x^2$, and hence find $\int (2x^3 - x)\,dx$.

5 Find:

a $\int 5\,dx$ **b** $\int 6x^2\,dx$ **c** $\int (3-2x)\,dx$

6 Integrate with respect to x:

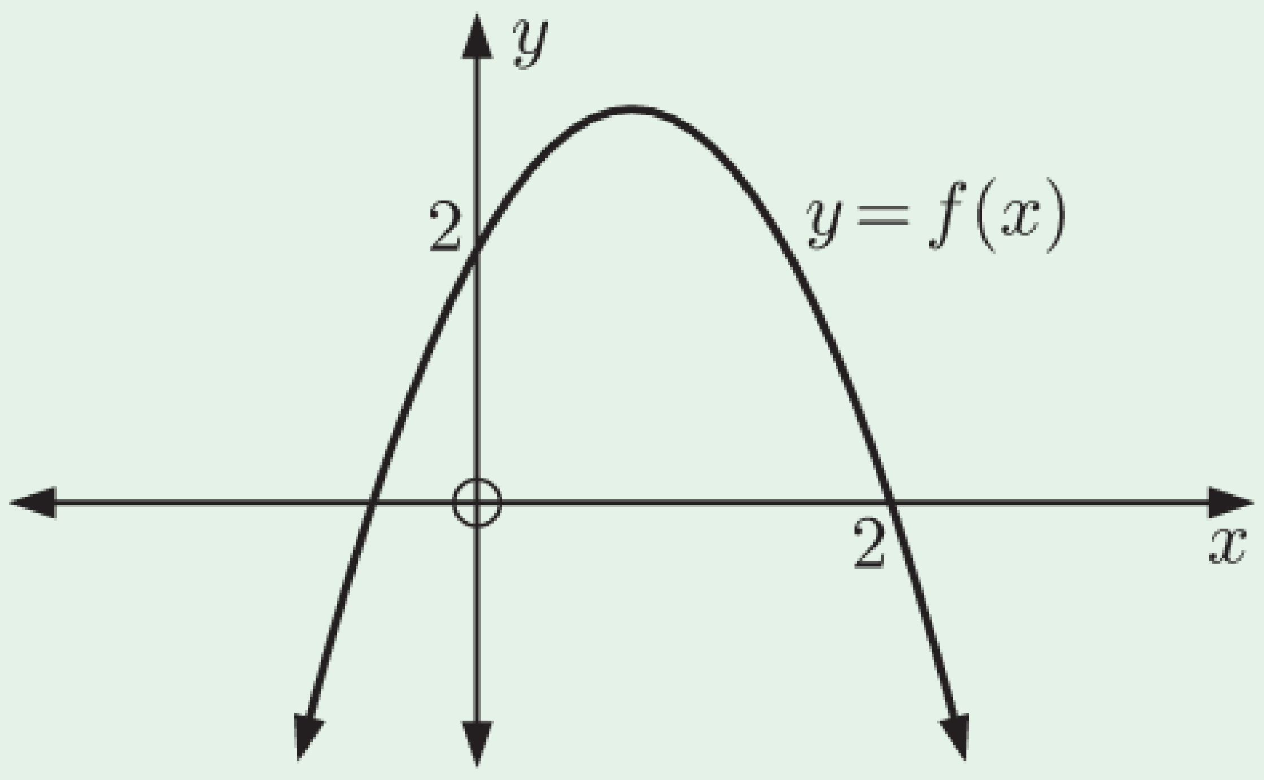
a $4x - \frac{2}{x^2}$ **b** $\frac{1}{3}x^3 + 2x$ **c** $\frac{1-2x}{x^3}$

7 Find:

a $\int (-3x^4 + 6x^2)\,dx$ **b** $\int \frac{3x^3 - x^2 - 1}{x^2}\,dx$ **c** $\int (2x-3)^2\,dx$

8 Given that $f'(x) = 3x^2 - 4x + 1$ and $f(0) = 2$, find $f(x)$.

9 The curve $y = f(x)$ shown alongside has gradient function $f'(x) = ax + 3$.
Find the equation of the curve.



10 Find the exact value of:

a $\int_{-2}^0 (1-3x)\,dx$ **b** $\int_0^{\frac{1}{2}} (x^2 - x)\,dx$ **c** $\int_1^2 (x^2+1)^2\,dx$

11 Find b such that:

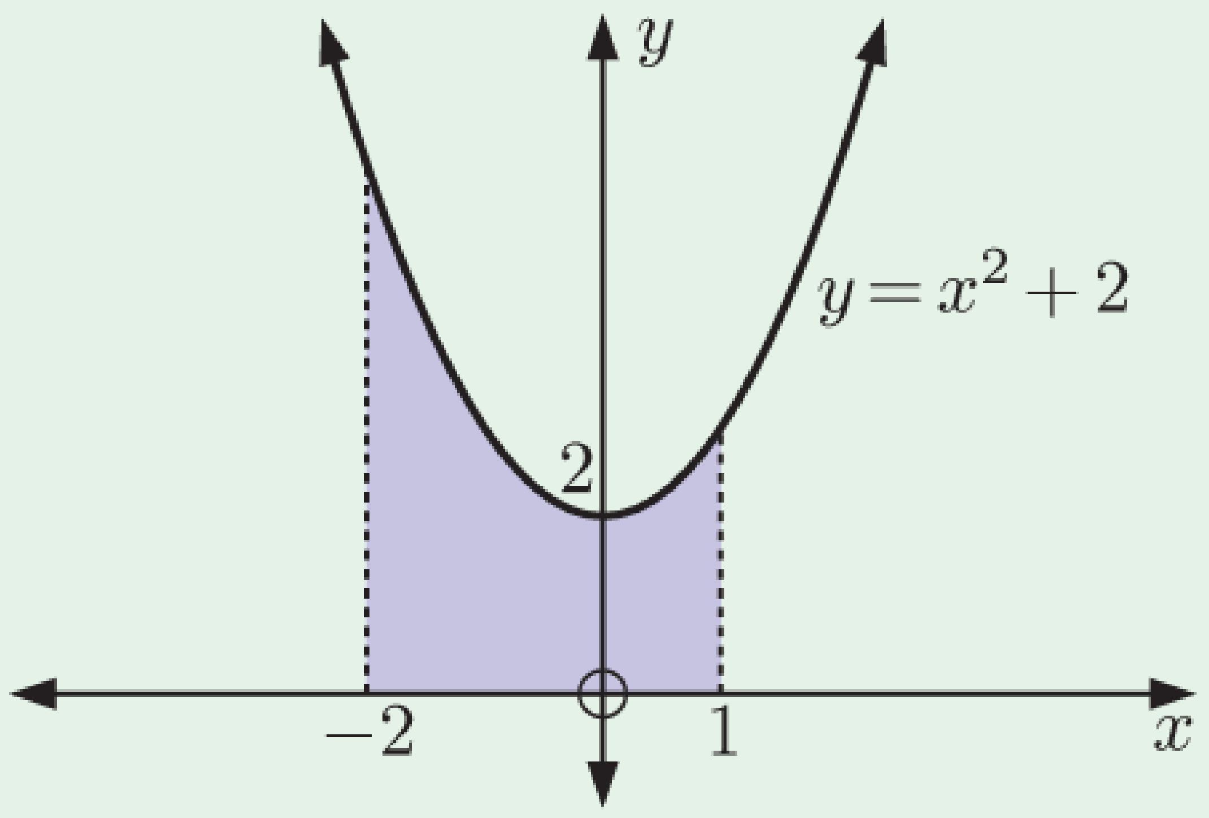
a $\int_0^b (x-b)^2\,dx = 9$ **b** $\int_0^b \left(x^2 + \frac{1}{2}x\right)dx = 3$

12 Evaluate correct to 6 significant figures:

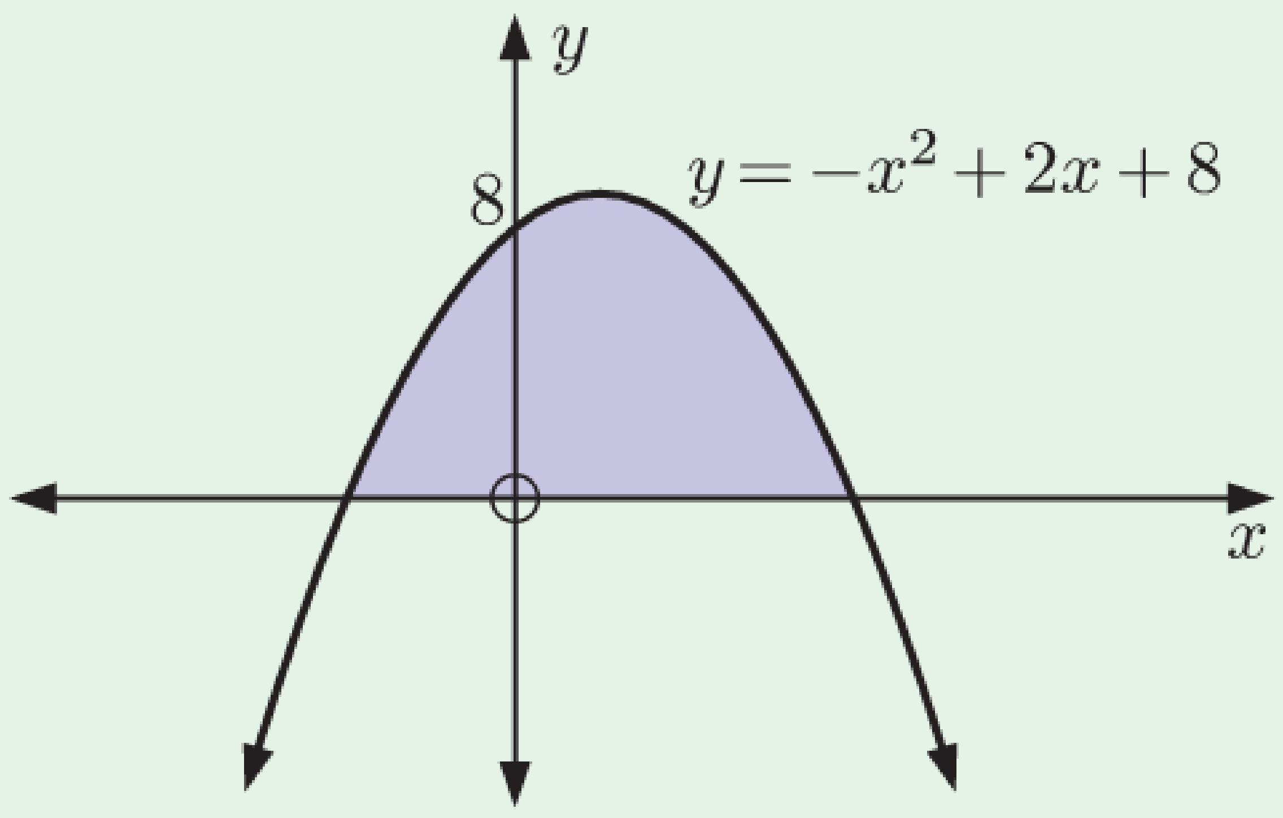
a $\int_3^4 \frac{x}{\sqrt{2x+1}}\,dx$ **b** $\int_0^1 x^2 e^{x+1}\,dx$

13 Find the shaded area:

a



b



14 Find the area of the region bounded by:

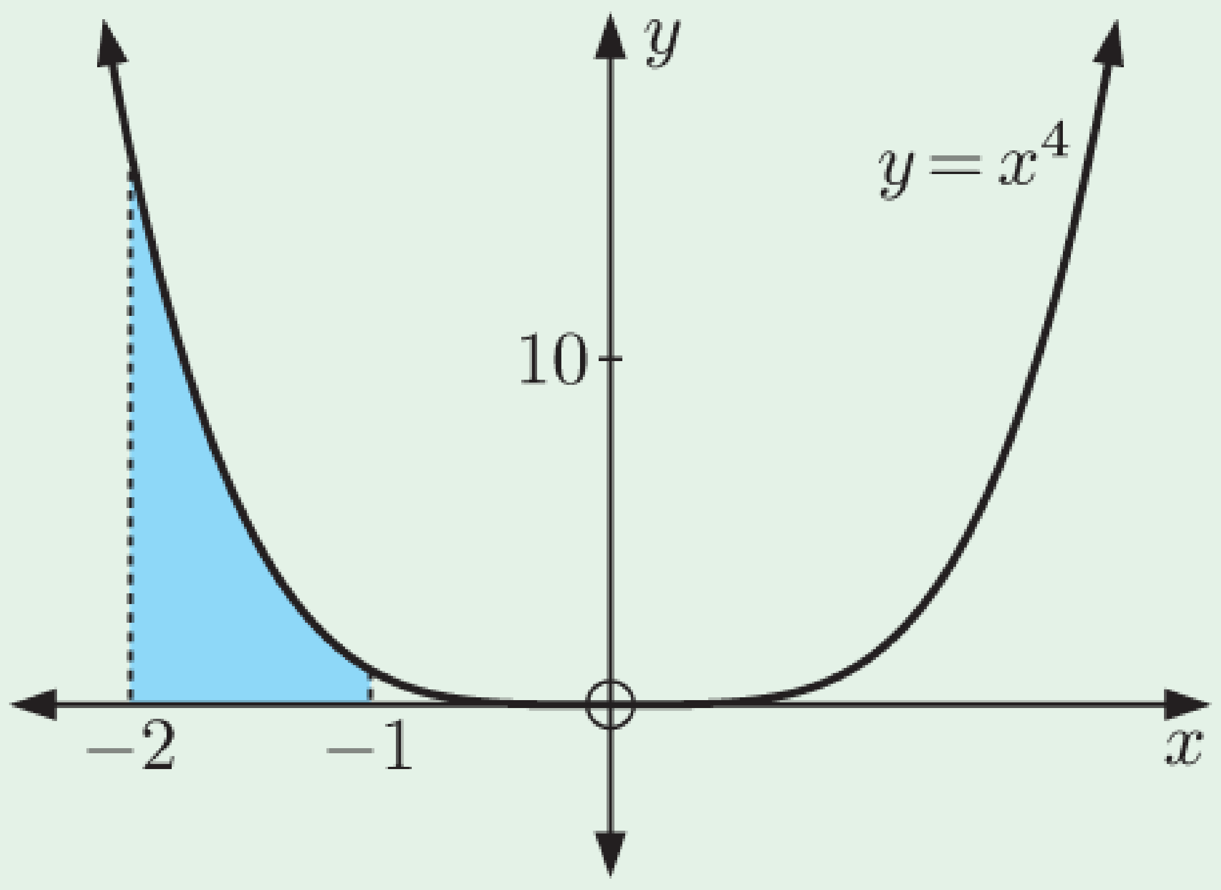
a $y = x^2$, the x-axis, $x = 2$, and $x = 5$ **b** $y = 8 - x^3$ and the axes.

15 The graph of $y = x^4$ is shown alongside.

a Use the trapezoidal rule with 8 subintervals to estimate $\int_{-2}^{-1} x^4\,dx$.

b Use the graph to explain whether your estimate is an over estimate or an under estimate.

c Check your answer to **b** by performing the integration.



REVIEW SET 13B

1 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$.

Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Use the area finder software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.

AREA FINDER

c Give your best estimate for $\int_0^1 \dfrac{4}{1+x^2}\,dx$.

d Estimate $\int_0^1 \dfrac{4}{1+x^2}\,dx$ using the trapezoidal method with 10 subintervals.

e Compare your results with the exact answer π.

2 **a** Sketch the region between $y = e^x - 1$ and the x-axis for $0 \leqslant x \leqslant 1$.

b With 10 subintervals, estimate $\int_0^1 (e^x - 1)\,dx$ using:

i lower rectangles **ii** upper rectangles **iii** the trapezoidal rule.

c Comment on your results.

3 Find the derivative of $2x - \frac{3}{x}$, and hence find $\int \left(-\frac{3}{x^2} - 2\right) dx$.

4 Find the antiderivative of:

a $-5x^2$ **b** $6x^{-2}$ **c** $1 + \dfrac{x^2}{6}$

5 Integrate with respect to x:

a $\dfrac{x^2 - 2}{x^2}$ **b** $(3x - 4)^2$ **c** $4 - 2x^2$

6 Find:

a $\int \left(\frac{x+3}{3}\right) dx$ **b** $\int (3x^2 - 2)\, dx$ **c** $\int (3+2x)^2\, dx$

7 Given that $f'(x) = x^2 - 3x + 2$ and $f(1) = 3$, find $f(x)$.

8 Explain why we include a constant of integration when we find an indefinite integral.

9 Find the exact value of:

a $\int_2^4 \frac{4}{x^2}\, dx$ **b** $\int_1^4 (x - \frac{1}{2}x^2)\, dx$ **c** $\int_0^1 (x^2 + \frac{1}{3})^2\, dx$

10 If $\int_0^a (x^2 - \frac{1}{2}x)\, dx = \frac{9}{16}$, find the exact value of a.

11 Evaluate correct to 4 significant figures:

a $\int_{-2}^0 4e^{-x^2}\, dx$ **b** $\int_0^1 \frac{10x}{\sqrt{3x+1}}\, dx$

12 If $\int_1^4 f(x)\, dx = 3$, determine:

a $\int_1^4 (f(x) + 1)\, dx$ **b** $\int_1^2 f(x)\, dx - \int_4^2 f(x)\, dx$

c k given that $\int_4^1 k\, f(x)\, dx = 5$.

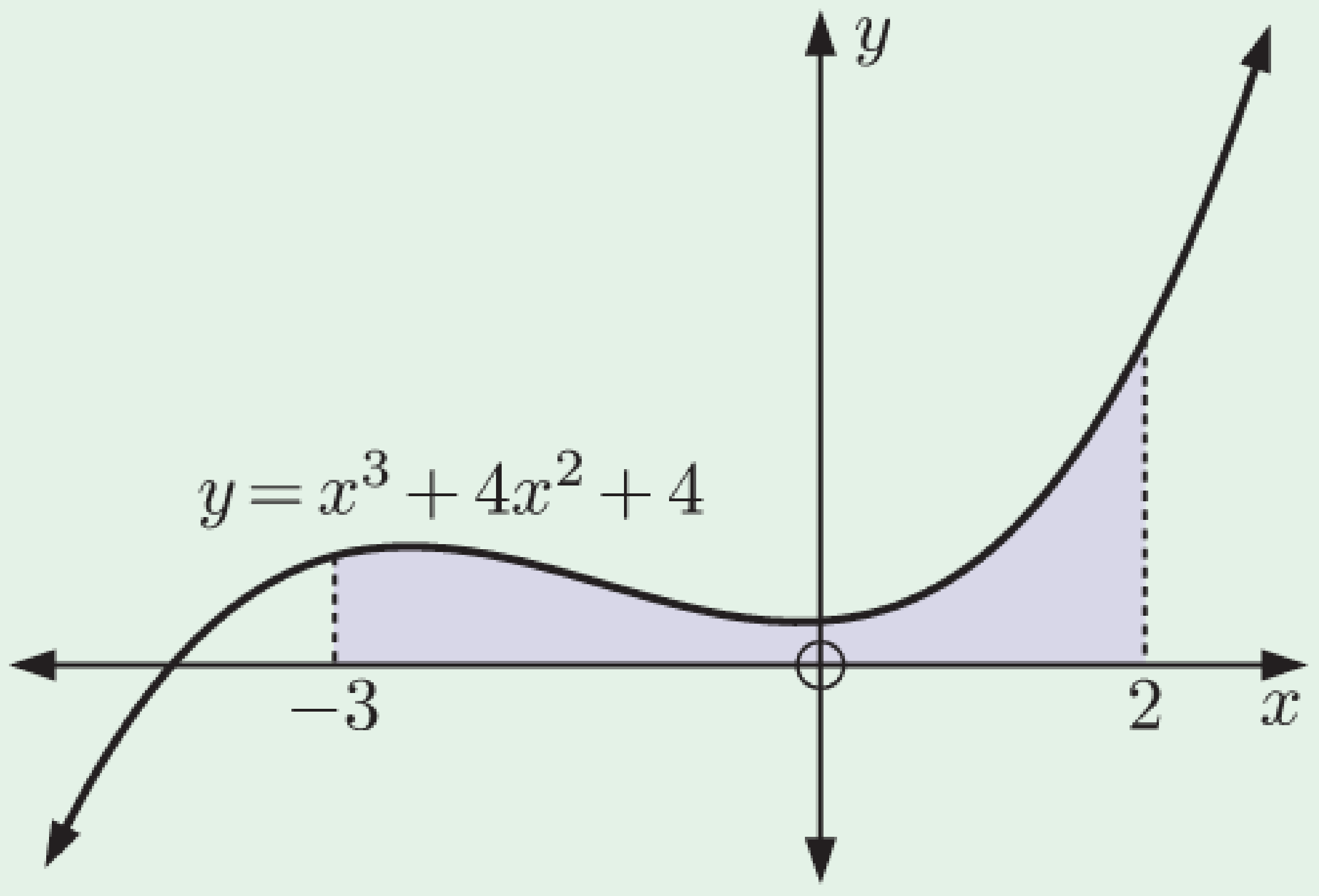
13 Find the area of the region bounded by:

a $y = x^3 + 1$, the x-axis, $x = 1$, and $x = 3$

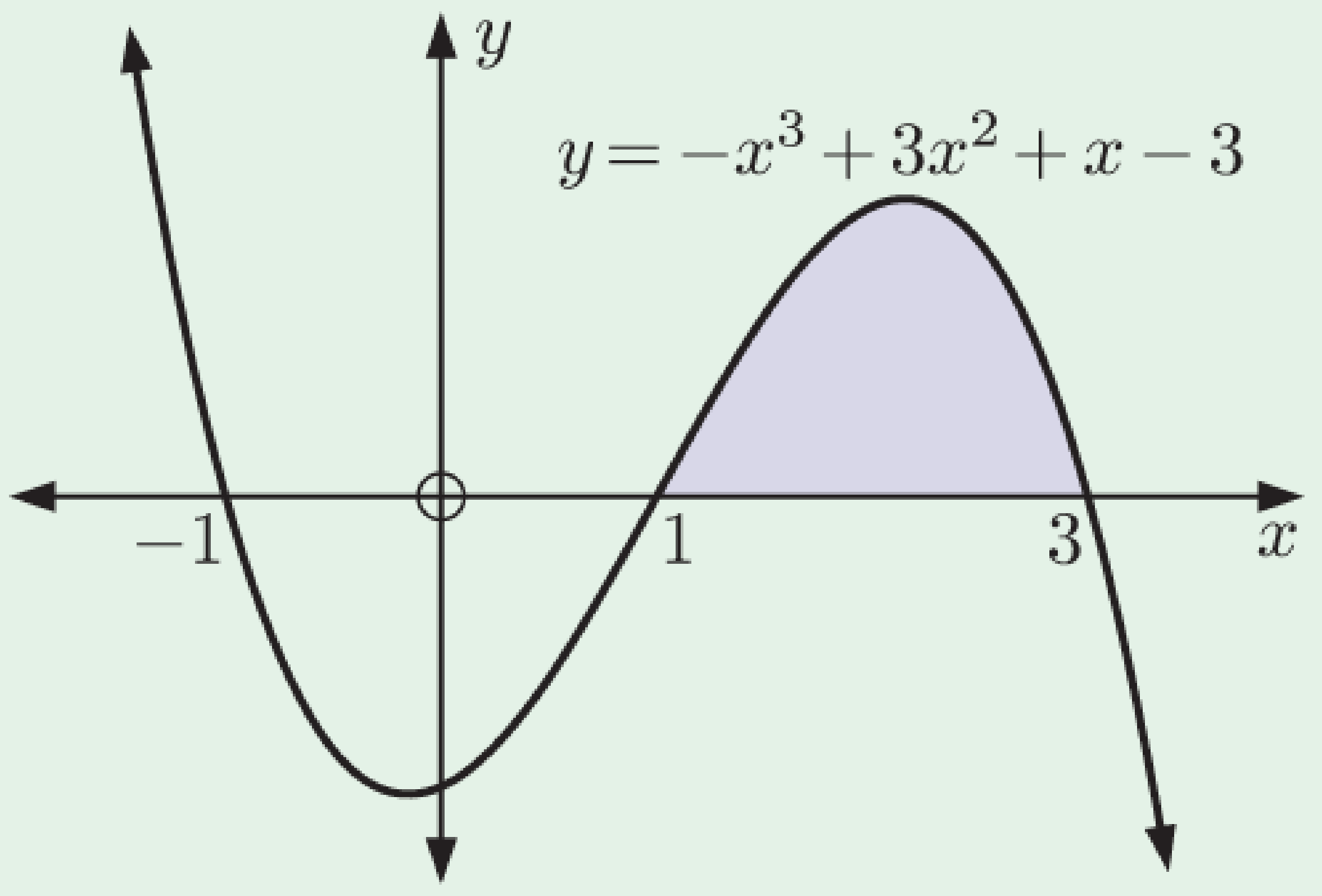
b $y = \frac{1}{x^2}$, the x-axis, $x = 3$, and $x = 9$.

14 Find the shaded area:

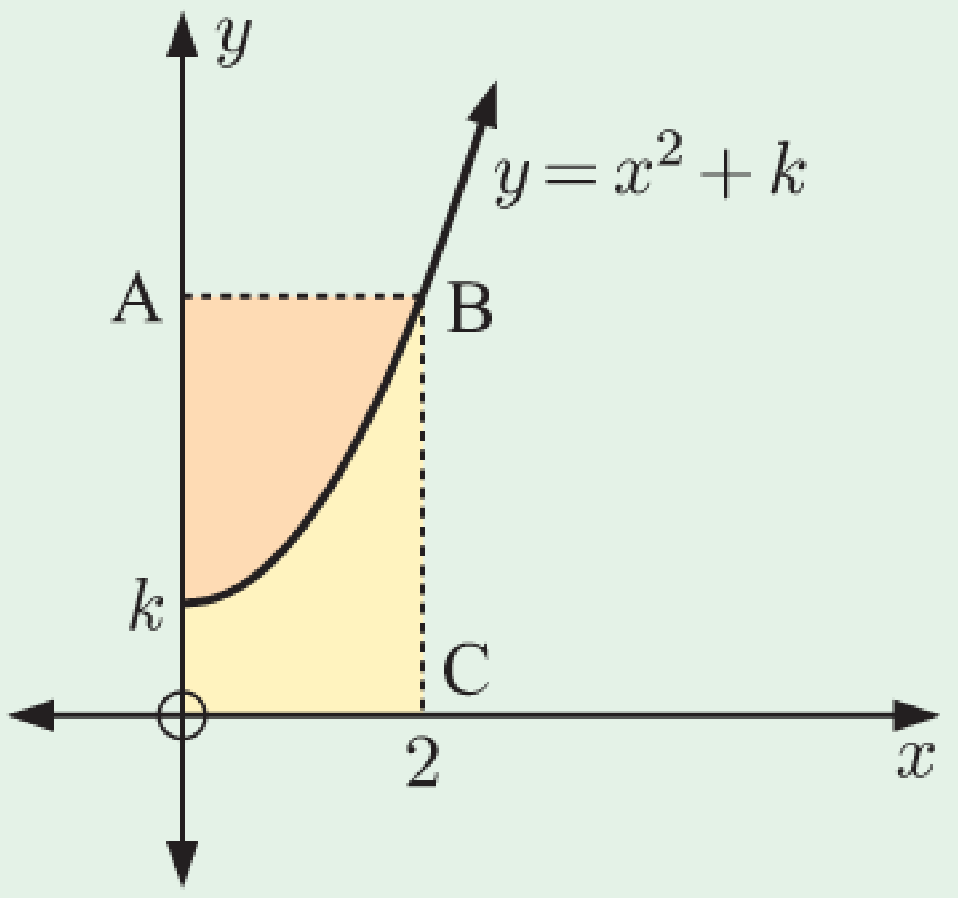
a



b



15 OABC is a rectangle and the two shaded regions are equal in area. Find k.



Chapter 14

Discrete random variables

Contents:

A Random variables
B Discrete probability distributions
C Expectation
D The binomial distribution
E Using technology to find binomial probabilities
F The mean and standard deviation of a binomial distribution

OPENING PROBLEM

In a sideshow game, players have a 50% chance of winning a prize worth \$2, \$5, \$10, or \$20. The probabilities of winning these prizes are given in the table below.

Prize value	\$0	\$2	\$5	\$10	\$20
Probability	0.5	0.35	0.1	0.04	0.01

Things to think about:

- **a** What is the sum of the probabilities in the table? Why must this be the answer?
- **b** What is the most likely outcome from playing the game?
- **c** What is the *average* result from playing the game?
- **d** What is a *fair* price for playing the game?

Many variables in the world around us depend on chance events. Examples of such variables are:

- the number of players in your football team who will score a goal in the next match
- the time it will take you to travel to school tomorrow
- the sum of the values when three dice are rolled.

Because of the element of chance in these variables, we cannot predict the exact value they will take when next measured. However, we can often determine the *possible values* the variable can take, and we can assign to each possible value the **probability** of it occurring.

In this Chapter we will extend the ideas of probability we have studied to model the random variation or **distribution** of numerical variables.

A RANDOM VARIABLES

A **random variable** uses numbers to describe the possible outcomes which could result from a random experiment.

A random variable is often represented by a capital letter such as X.

Random variables can be either **discrete** or **continuous**.

A **discrete random variable** X has a set of distinct possible values.

For example, X could be:

- the number of wickets a bowler takes in an innings of cricket, so X could take the values 0, 1, 2, ..., 10
- the number of defective light bulbs in a purchase order of 50, so X could take the values 0, 1, 2, ..., 50.

To determine the value of a discrete random variable, we need to **count**.

A **continuous random variable** X can take any value within some interval on the number line.

For example, X could be:

- the heights of men, which lie in the interval $50 \text{ cm} < X < 250 \text{ cm}$
- the volume of water in a tank, which could lie in the interval $0 \text{ m}^3 < X < 100 \text{ m}^3$.

To determine the value of a continuous random variable, we need to **measure**.

DISCRETE RANDOM VARIABLES

In this course we will only consider discrete random variables with a finite number of outcomes, so we label them $x_1, x_2, x_3, ..., x_n$.

Example 1 **Self Tutor**

A supermarket has three checkouts A, B, and C. A government inspector checks the weighing scales for accuracy at each checkout. The random variable X is the number of accurate weighing scales at the supermarket.

a List the possible outcomes and the corresponding values of X.

b What value(s) of X correspond to there being:

i one accurate scale **ii** at least one accurate scale?

a Possible outcomes:

A	B	C	X
✗	✗	✗	0
✓	✗	✗	1
✗	✓	✗	1
✗	✗	✓	1
✗	✓	✓	2
✓	✗	✓	2
✓	✓	✗	2
✓	✓	✓	3

b **i** $X = 1$

ii $X = 1, 2, \text{ or } 3$

EXERCISE 14A

1 Classify each random variable as continuous or discrete:

a the quantity of fat in a sausage

b the mark out of 50 for a geography test

c the weight of a Year 12 student

d the volume of water in a cup of coffee

e the number of trout in a lake

f the number of hairs on a cat

g the length of a horse's mane

h the height of a skyscraper.

2 For each scenario:

i Identify the random variable being considered.

ii State whether the variable is continuous or discrete.

iii Give possible values for the random variable.

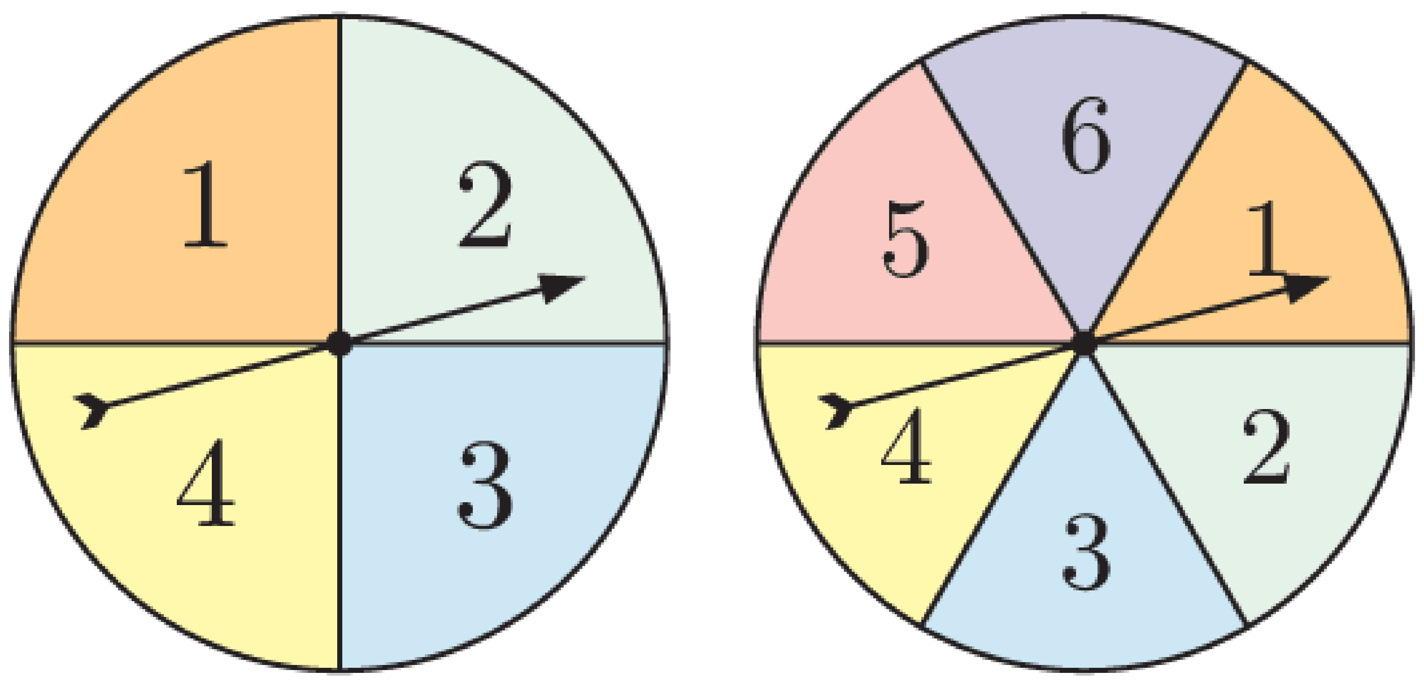
a To measure the rainfall over a 24-hour period in Singapore, water is collected in a rain gauge.

b To investigate the stopping distance for a tyre with a new tread pattern, a braking experiment is carried out.

c To check the reliability of a new type of light switch, switches are repeatedly turned off and on until they fail.

3 Suppose the spinners alongside are spun, and X is the sum of the numbers.

- **a** Explain why X is a discrete random variable.
- **b** State the possible values of X.



4 In the finals series of a baseball championship, the first team to win 4 games wins the championship. Let X represent the number of games played in the finals series.

- **a** State the possible values of X.
- **b** What value(s) of X correspond to the series lasting:
  - **i** exactly 5 games
  - **ii** at least 6 games?

5 A supermarket has four checkouts A, B, C, and D. Management checks the weighing devices at each checkout. The random variable X is the number of weighing devices which are accurate.

- **a** What values can X have?
- **b** List the possible outcomes and the corresponding values of X.
- **c** What value(s) of X correspond to:
  - **i** exactly two devices being accurate
  - **ii** at least two devices being accurate?

6 Suppose three coins are tossed simultaneously. Let X be the number of heads that result.

- **a** State the possible values of X.
- **b** List the possible outcomes and the corresponding values of X.
- **c** Are the possible values of X equally likely to occur? Explain your answer.

B DISCRETE PROBABILITY DISTRIBUTIONS

For any random variable, there is a corresponding **probability distribution** which describes the probability that the variable will take a particular value.

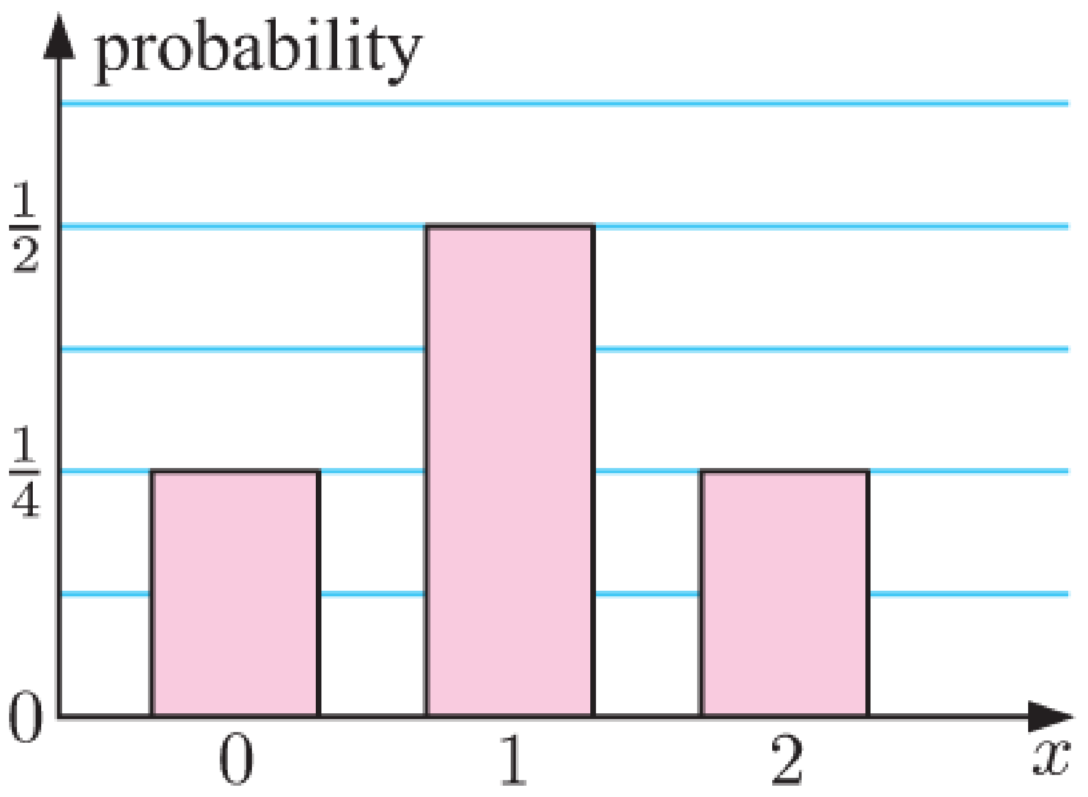
The probability that the variable X takes value x is denoted $\mathrm{P}(X = x)$.

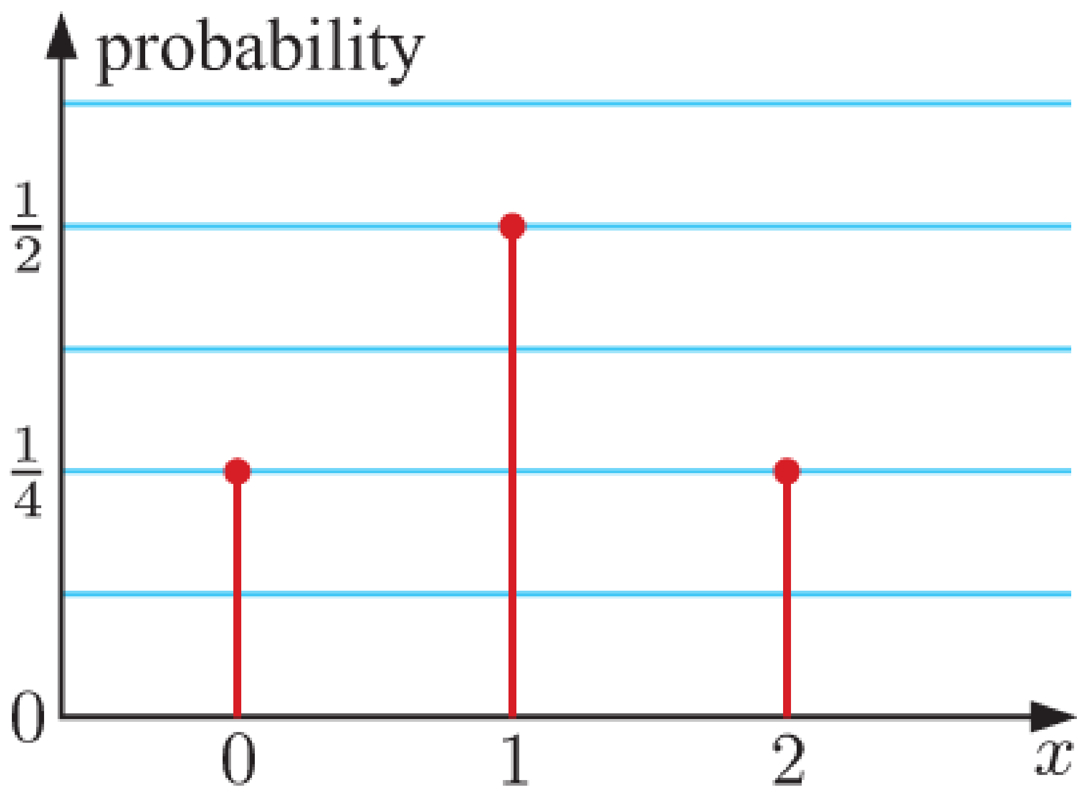
> If X is a random variable with possible values $\{x_1, x_2, x_3, ..., x_n\}$ and corresponding probabilities $\{p_1, p_2, p_3, ..., p_n\}$ such that $\mathrm{P}(X = x_i) = p_i$, $i = 1, ..., n$, then:
>
> - $0 \leqslant p_i \leqslant 1$ for all $i = 1, ..., n$
> - $\sum_{i=1}^{n} p_i = p_1 + p_2 + p_3 + ... + p_n = 1$
> - $\{p_1, ..., p_n\}$ describes the **probability distribution** of X.

For example, suppose X is the number of heads obtained when 2 coins are tossed. The possible values for X are $\{0, 1, 2\}$ with corresponding probabilities $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$. We see that $0 \leqslant p_i \leqslant 1$ for each value of i, and that the probabilities add up to 1.

We can display this probability distribution in a **table** or a **graph**.

x	0	1	2
$\text{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

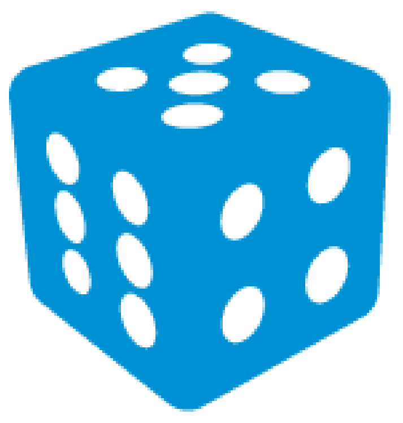




UNIFORM DISCRETE RANDOM VARIABLES

If the possible values $x_1, x_2, ..., x_n$ of a discrete random variable X all have the same probability $\frac{1}{n}$ of occurring, then X is a **uniform discrete random variable**.

An example of a uniform discrete random variable is the result X when a die is rolled. The possible values of X are 1, 2, 3, 4, 5, and 6, and each value has probability $\frac{1}{6}$ of occurring.



By contrast, if two dice are rolled, the sum of the resulting numbers Y is *not* a uniform discrete random variable.

THE MODE AND MEDIAN

The **mode** of a discrete probability distribution is the most frequently occurring value of the variable. This is the data value x_i whose probability p_i is the highest.

The **median** of the distribution corresponds to the 50th percentile. If the possible values $\{x_1, x_2, ..., x_n\}$ are listed in ascending order, the median is the value x_j when the cumulative sum $p_1 + p_2 + ... + p_j$ reaches 0.5.

Example 2 **Self Tutor**

A magazine store recorded the number of magazines purchased by its customers in one week. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four, and 5% purchased five. Let X be the number of magazines sold to a randomly selected customer.

a State the possible values of X.

b Construct a probability table for X.

c Graph the probability distribution.

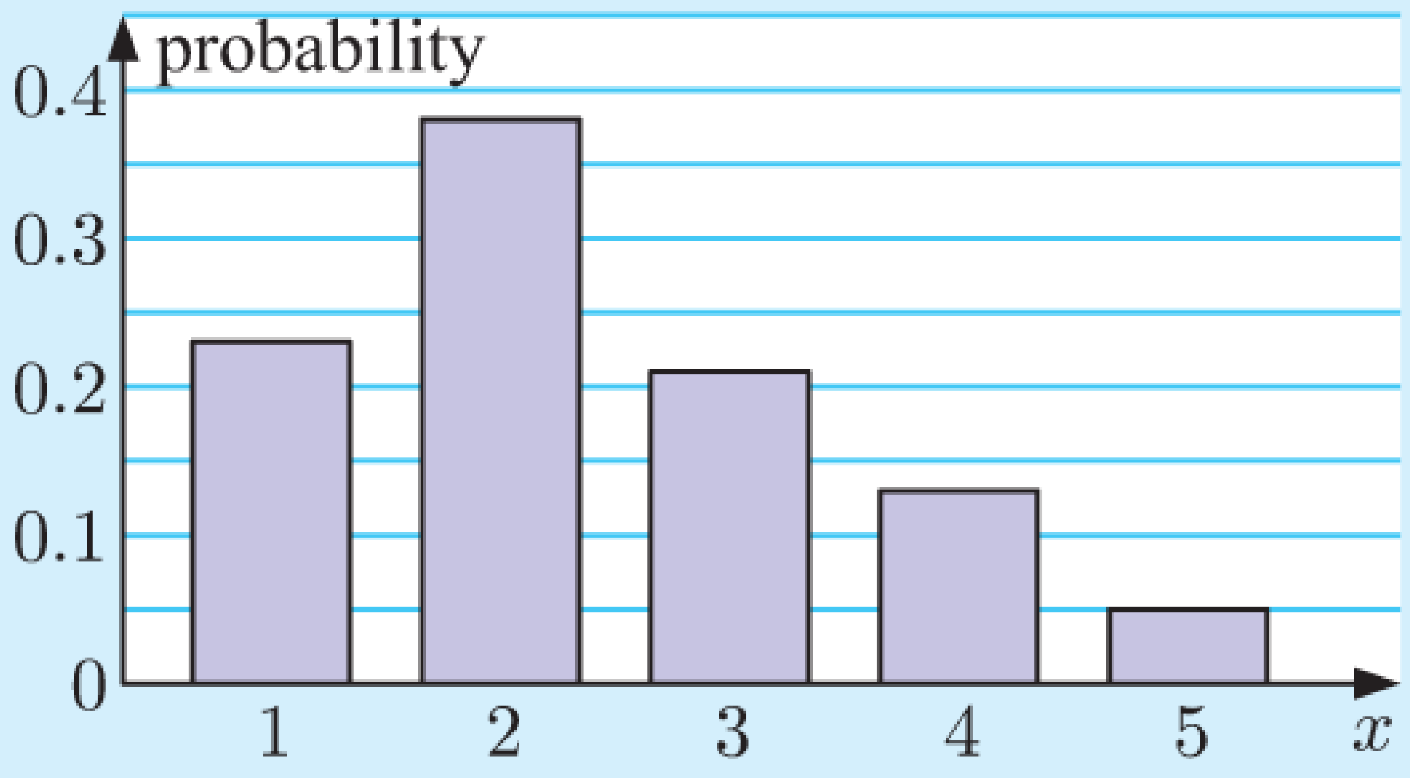
d Find the mode and median of X.

a $X = 1, 2, 3, 4, \text{ or } 5$

b

x	1	2	3	4	5
$\text{P}(X = x)$	0.23	0.38	0.21	0.13	0.05

c



d Customers are most likely to buy 2 magazines, so this is the mode of X.
We now find the median:

$$p_1 = 0.23$$
$$p_1 + p_2 = 0.23 + 0.38 = 0.61$$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2 magazines.

We can also describe the probability distribution of a discrete random variable using a **probability mass function** $P(x) = \mathrm{P}(X = x)$. The domain of the probability mass function is the set of possible values of the variable, and the range is the set of values in the probability distribution.

Example 3 **Self Tutor**

Show that $P(x) = \dfrac{x^2+1}{34}$, $x = 1, 2, 3, 4$ is a valid probability mass function.

$P(1) = \frac{2}{34}$, $P(2) = \frac{5}{34}$, $P(3) = \frac{10}{34}$, $P(4) = \frac{17}{34}$

All of these values obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum_{i=1}^{n} P(x_i) = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

EXERCISE 14B

1 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3	4
$\mathrm{P}(X = x)$	0.2	0.4	0.15	0.25

ii

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.3	0.4	0.2

iii

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

iv

x	2	3	4	5
$\mathrm{P}(X = x)$	0.3	0.4	0.5	-0.2

b For which of the probability distributions in **a** is X a uniform random variable?

2 Find k in each of these probability distributions:

a

x	0	1	2
$\mathrm{P}(X = x)$	0.3	k	0.5

b

x	0	1	2	3
$\mathrm{P}(X = x)$	k	$2k$	$3k$	k

3 Consider the probability distribution alongside.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.1	0.25	0.45	a

a Find the value of a.

b Is X a uniform discrete random variable? Explain your answer.

c State the mode of the distribution.

d Find $\mathrm{P}(X \geqslant 2)$.

4 The probability distribution for Jason scoring X home runs in each game during his baseball career is given in the following table:

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a State the value of $P(2)$.

b Find the value of a. Explain what this number means.

c Find the value of $P(1)+P(2)+P(3)+P(4)+P(5)$. Explain what this means.

d Draw a graph of $P(x)$ against x.

e Find the mode and median of the distribution.

5 A policeman inspected the safety of tyres on cars passing through a checkpoint. The number of tyres X which needed replacing on each car followed the probability distribution below.

x	0	1	2	3	4
$\mathrm{P}(X=x)$	0.68	0.2	0.06	k	0.02

a Find the value of k.

b Find the mode of the distribution.

c Find $\mathrm{P}(X>1)$, and interpret this value.

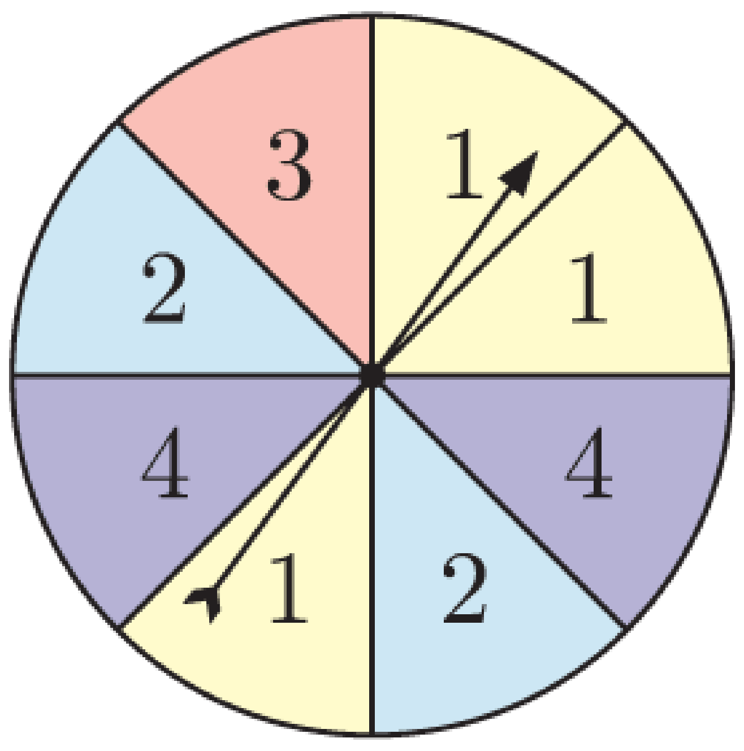
6 Let X be the result when the spinner alongside is spun.

a Display the probability distribution of X in a table.

b Graph the probability distribution.

c Find the mode and median of the distribution.

d Find $\mathrm{P}(X \leqslant 3)$.



7 100 people were surveyed about the number of bedrooms in their house. 24 people had one bedroom, 35 people had two bedrooms, 27 people had three bedrooms, and 14 people had four bedrooms. Let X be the number of bedrooms a randomly selected person has in their house.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

8 A group of 25 basketballers took shots from the free throw line until they scored a goal. 12 of the players only needed one shot, 7 players took two shots, 2 players took three shots, and the rest took four shots. Let X be the number of shots a randomly selected player needs to score a goal.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

9 Show that the following are valid probability mass functions:

a $P(x)=\dfrac{x+1}{10}$ for $x=0, 1, 2, 3$

b $P(x)=\dfrac{6}{11x}$ for $x=1, 2, 3$.

10 Find k for the following probability mass functions:

a $P(x) = k(x+2)$ for $x = 1, 2, 3$ **b** $P(x) = \dfrac{k}{x+1}$ for $x = 0, 1, 2, 3$.

11 A discrete random variable X has the probability mass function $P(x) = \dfrac{4x - x^2}{a}$ for $x = 0, 1, 2, 3$.

a Find the value of a.

b Find $\mathrm{P}(X = 1)$.

c Find the mode of the distribution.

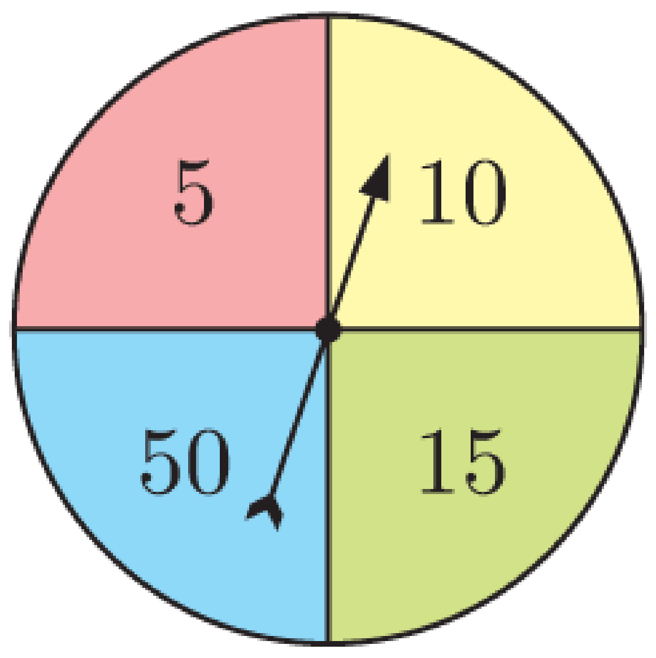
C EXPECTATION

We have already seen how probabilities can be used to predict the number of times we *expect* an event to occur when an experiment is repeated many times.

We can consider the **expected value** or **expectation** of a random variable in a similar way.

EXPECTED VALUE

When the spinner alongside is spun, players are awarded the resulting number of points. On average, how many points can we *expect* to be awarded per spin?

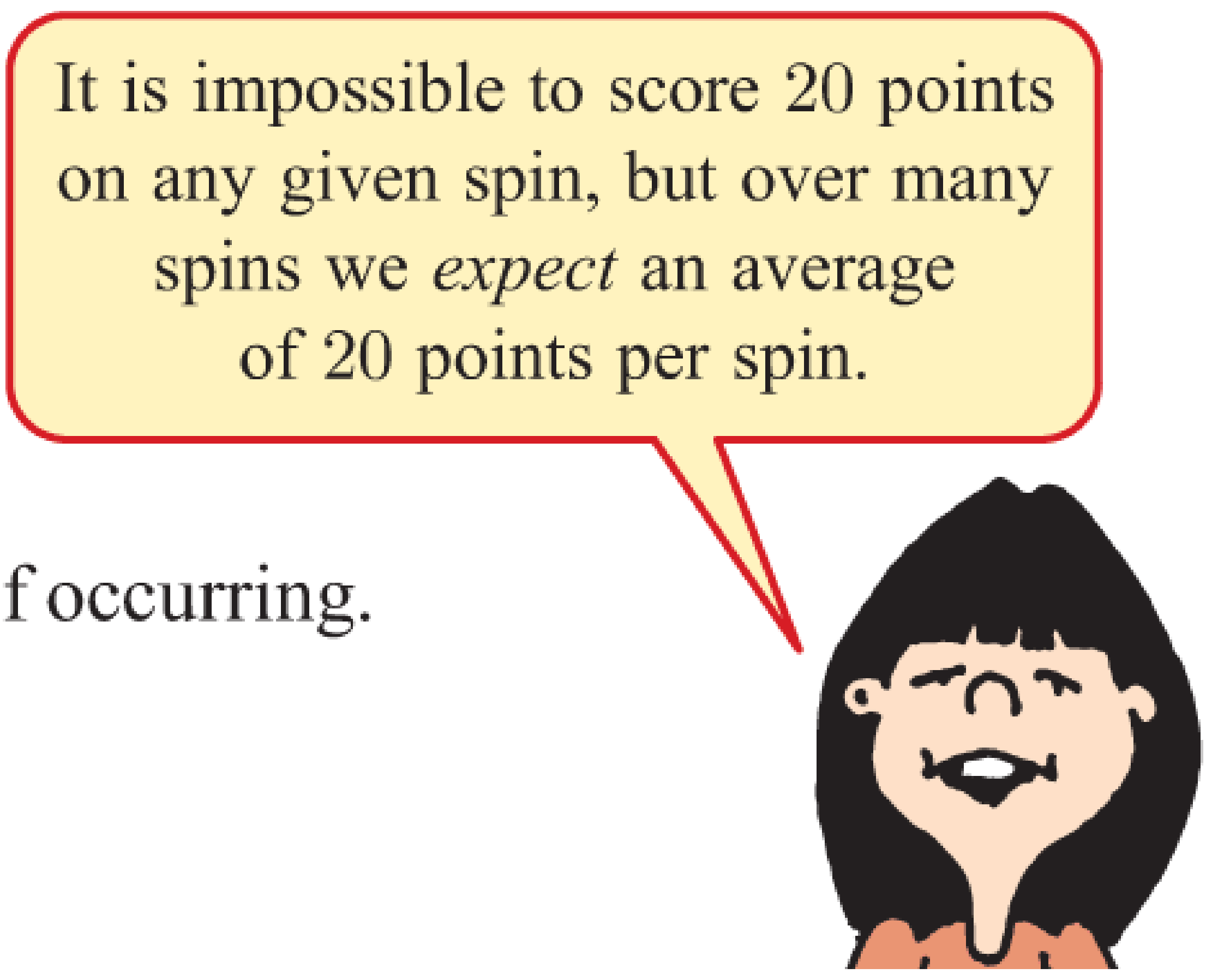


For every 4 spins, we would expect that on average, each score will be spun once. The total score in this case would be $50 + 15 + 10 + 5 = 80$, which is an average of $\frac{80}{4} = 20$ points per spin.

Alternatively, we can write the average score as

$$\begin{aligned} &\tfrac{1}{4}(50 + 15 + 10 + 5) \\ &= \tfrac{1}{4} \times 50 + \tfrac{1}{4} \times 15 + \tfrac{1}{4} \times 10 + \tfrac{1}{4} \times 5 \\ &= 20 \text{ points.} \end{aligned}$$

Notice that each score is multiplied by its probability of occurring.


It is impossible to score 20 points on any given spin, but over many spins we *expect* an average of 20 points per spin.

For a random variable X with possible values $x_1, x_2, x_3, ..., x_n$ and associated probabilities $p_1, p_2, ..., p_n$, the **expected value** of X is

$$\begin{aligned} \mathbf{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= x_1 p_1 + x_2 p_2 + ... + x_n p_n \end{aligned}$$

$\mathrm{E}(X)$ is the **mean** of the probability distribution of X. It is sometimes denoted μ.

Example 4 **Self Tutor**

Consider the magazine store from **Example 2**.

Find the expected number of magazines bought by each customer. Explain what this represents.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

In **Example 2** we found the mode and median for this distribution.

$$\begin{aligned} \text{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05) \\ &= 2.39 \end{aligned}$$

In the long term, the average number of magazines purchased per customer is 2.39.

EXERCISE 14C.1

1 Find $\text{E}(X)$ for the following probability distributions:

a

x_i	1	2	3
p_i	0.4	0.5	0.1

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

d

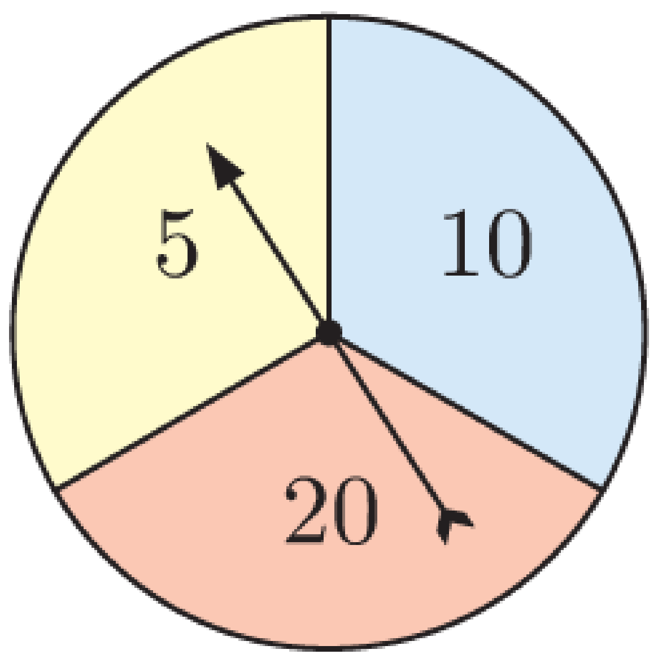
x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

2 Consider the probability distribution alongside.

x	1	3	5
$\text{P}(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

- **a** Find the value of a.
- **b** Find the mode of the distribution.
- **c** Find the mean μ of the distribution.

3 When the spinner alongside is spun, players are awarded the resulting number of points. In the long term, how many points can we expect to be awarded per spin?



4 When Ernie goes fishing, he catches 0, 1, 2, or 3 fish, with the probabilities shown. On average, how many fish would you expect Ernie to catch on a fishing trip?

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

5 Each time Pam visits the library, she borrows either 1, 2, 3, 4, or 5 books, with the probabilities shown.

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

- **a** Find the value of a.
- **b** Find the mode of the distribution.
- **c** On average, how many books does Pam borrow per visit?

6 Lachlan randomly selects a ball from a bag containing 5 red balls, 2 green balls, and 1 white ball. He is then allowed to take a particular number of lollies from a jar according to the colour of the ball.
Find the average number of lollies that Lachlan can expect to receive.

Colour	*Number of lollies*
Red	4
Green	6
White	10

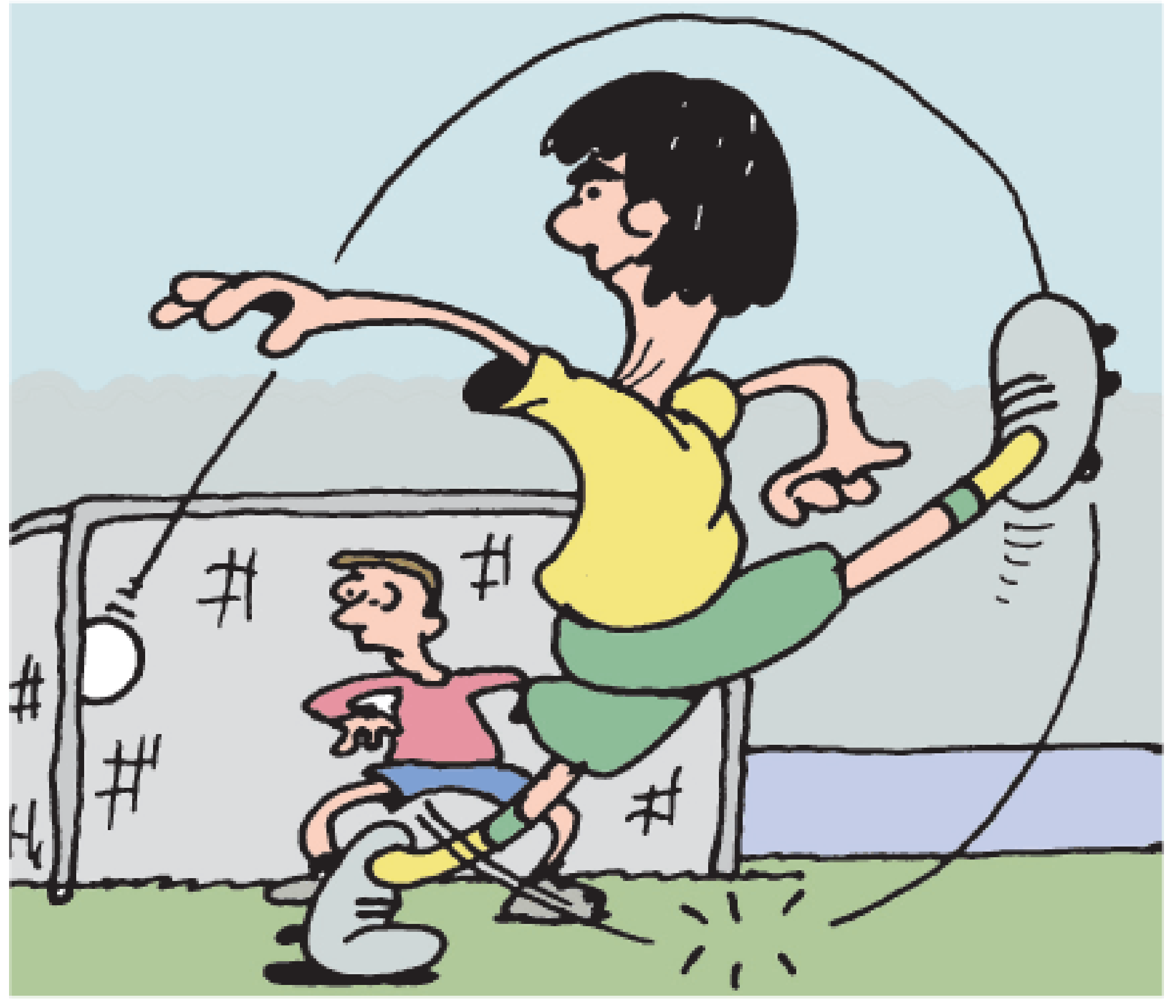
7 When ten-pin bowler Jenna bowls her first bowl of a frame, she always knocks down at least 8 pins. $\frac{1}{3}$ of the time she knocks down 8 pins, and $\frac{2}{5}$ of the time she knocks down 9 pins.

- **a** Find the probability that she knocks down all 10 pins on the first bowl.
- **b** On average, how many pins does Jenna knock down with her first bowl?

8 Given that $\text{E}(X) = 2.5$, find a and b.

x	1	2	3	4
$\text{P}(X = x)$	0.3	a	b	0.2

9 When Brad's soccer team plays an offensive strategy, they win 30% of the time and lose 55% of the time. When they play a defensive strategy, they win 20% of the time and lose 30% of the time.
On the league table, teams are awarded 3 points for a win, 1 point for a draw, and no points for a loss.



- **a** Find the probability that Brad's team will draw a match under each strategy.
- **b** Calculate the expected number of points per game under each strategy.
- **c** In the long run, is it better for the team to play an offensive or defensive strategy?
- **d** Should the strategy change if teams are awarded 4 points instead of 3 points for a win?

10 Every Thursday, Zoe meets her friends in the city for dinner. There are two car parks nearby, the costs for which are shown below:

Car park A

Time	*Cost*
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	*Cost*
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

Zoe's dinner takes 1 - 2 hours 20% of the time, 2 - 3 hours 70% of the time, and 3 - 4 hours 10% of the time.

- **a** Which car park is cheapest for Zoe if she stays:
  - **i** 1 - 2 hours
  - **ii** 2 - 3 hours
  - **iii** 3 - 4 hours?
- **b** When Zoe parks her car, she does not know how long she will stay. Which car park do you recommend for her? Explain your answer.

11 An insurance policy covers a \$20 000 sapphire ring against theft and loss. If the ring is stolen then the insurance company will pay the policy owner in full. If the ring is lost then they will pay the owner \$8000. From past experience, the insurance company knows that the probability of theft is 0.0025, and the probability of loss is 0.03. How much should the company charge to cover the ring in order that their expected return is \$100?

INSURANCE POLICY

FAIR GAMES

In gambling, the **expected gain** of the player from each game is the expected return or payout from the game, less the amount it cost them to play.

A game is said to be **fair** if the expected gain is zero.

Suppose X represents the gain of a player from each game.
The game is **fair** if $E(X) = 0$.

DISCUSSION

Would you expect a gambling game to be "fair"?

Example 5 **Self Tutor**

In a game of chance, a player spins a square spinner labelled 1, 2, 3, 4. The player wins an amount of money according to the table alongside.

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a Find the expected return for one spin of the spinner.

b Find the expected *gain* of the player if it costs \$5 to play each game.

c Discuss whether you would recommend playing this game.

a Let Y denote the return or payout from each spin.
Each outcome is equally likely, so the probability for each outcome is $\frac{1}{4}$

$\therefore$ the expected return $= E(Y) = \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4$.

b Let X denote the *gain* of the player from each game.
Since it costs \$5 to play the game, the expected gain $= E(X) = E(Y) - \$5$
$= \$4 - \$5$
$= -\$1$

c Since $E(X) \neq 0$, the game is not fair. In particular, since $E(X) = -\$1$, we expect the player to lose \$1 on average with each spin. We would not recommend that a person play the game.

EXERCISE 14C.2

1 A dice game costs \$2 to play. If an odd number is rolled, the player receives \$3. If an even number is rolled, the player receives \$1.
Determine whether the game is fair.

2 A man rolls a regular six-sided die. He wins the number of dollars shown on the uppermost face.

- **a** Find the expected return from one roll of the die.
- **b** Find the expected *gain* if it costs \$4 to play the game.
- **c** Would you advise the man to play many games?

3 A roulette wheel has 18 red numbers, 18 black numbers, and 1 green number. Each number has an equal chance of occurring. I place a bet of \$2 on red. If a red is spun, I receive my \$2 back plus another \$2. Otherwise I lose my \$2.

- **a** Calculate the expected gain from this bet.
- **b** If the same bet is made 100 times, what is the expected result?

4 A person pays \$5 to play a game with a pair of coins. If two heads appear then \$10 is won. If a head and a tail appear then \$3 is won. If two tails appear then \$1 is won.
Let X be the gain of the person from each game. Find the expected value of X.

5 In a carnival game, a player randomly selects a ticket from a box of tickets numbered 1 to 20. If the selected number is a multiple of 3, the player wins 5 tokens. If the selected number is a multiple of 10 the player wins 10 tokens.

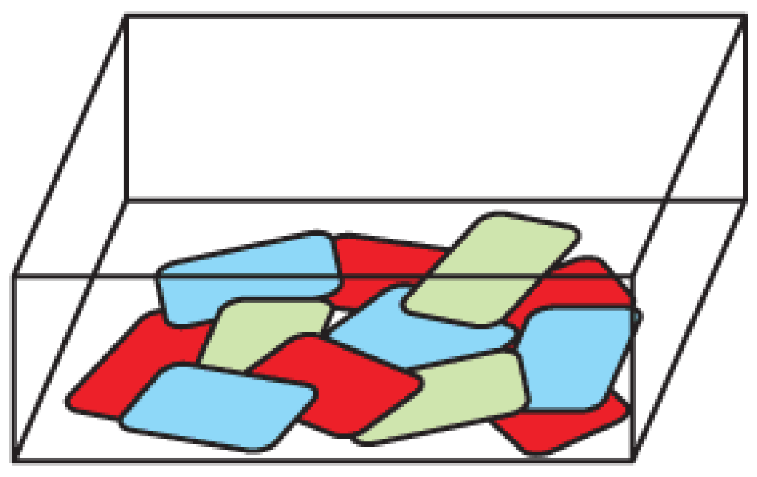
- **a** Calculate the probability of a player winning:
  - **i** 5 tokens
  - **ii** 10 tokens.
- **b** Let X be the number of tokens won from playing this game. Find the expected value of X.
- **c** If it costs 3 tokens to play the game, would you recommend playing the game many times? Explain your answer.

MULTIPLE 3
5 TOKENS
MULTIPLE 10
10 TOKENS

6 A person selects a disc from a bag containing 10 black discs, 4 blue discs, and 1 gold disc. They win \$1 for a black disc, \$5 for a blue disc, and \$20 for the gold disc. The game costs \$4 to play.

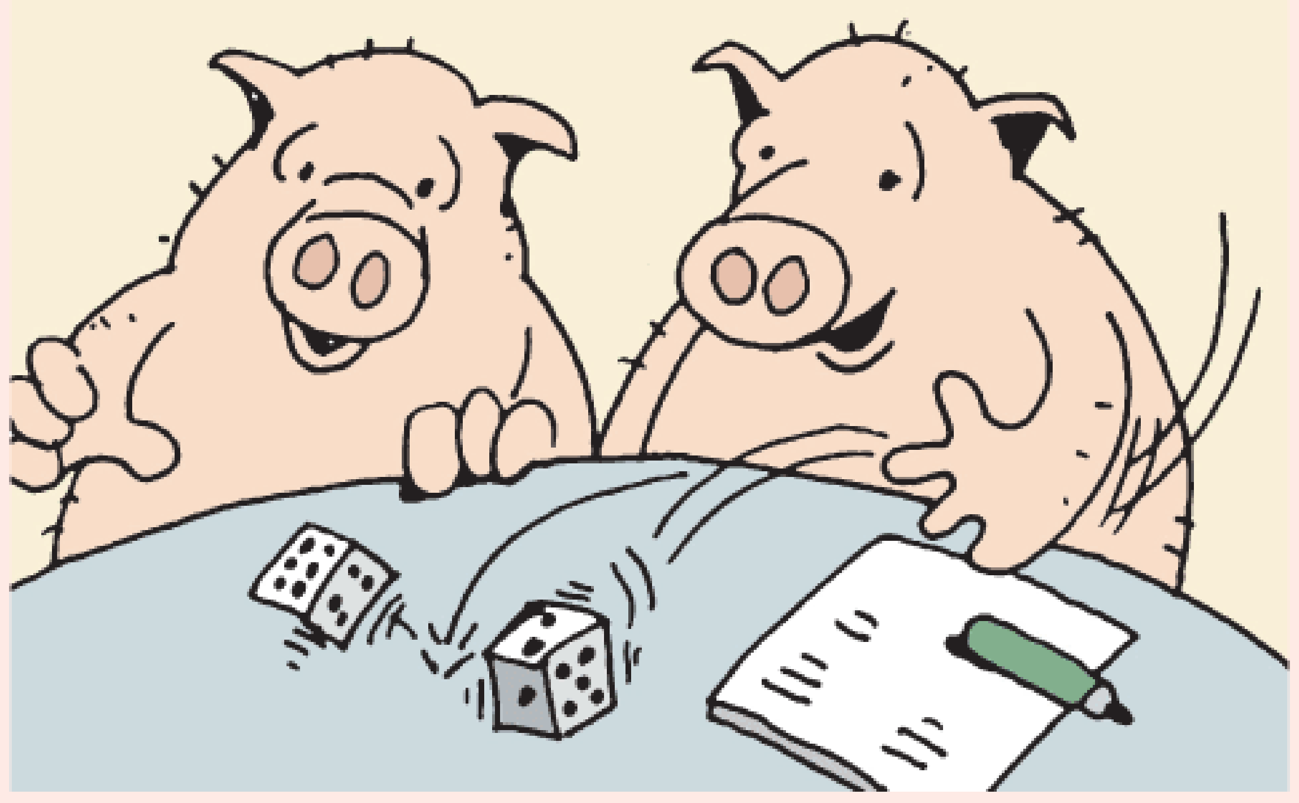
- **a** Calculate the expected gain for this game, and hence show that the game is not fair.
- **b** To make the game fair, the prize money for selecting the gold disc is increased. Find the new prize money for selecting the gold disc.

7 In a fundraising game "Lucky 11", a player selects 3 cards without replacement from a box containing 5 red, 4 blue, and 3 green cards.
The player wins \$11 if the cards drawn are all the same colour *or* are one of each colour.
If the organiser of the game wants to make an average of \$1 per game, how much should they charge to play it?



ACTIVITY GREEDY PIGS

In this Activity, we will play a variant of the dice game **Greedy Pigs**. We will use expected value to find a strategy for playing the game.



In each turn of the game, a player rolls one die a number of times, accumulating points according to the numbers rolled. After each roll, the player can either end their turn and "bank" the points accumulated so far, or continue rolling in an attempt to score more points. However, if the player rolls a 1, the player loses all of the points accumulated on that turn, and their turn is over.

What to do:

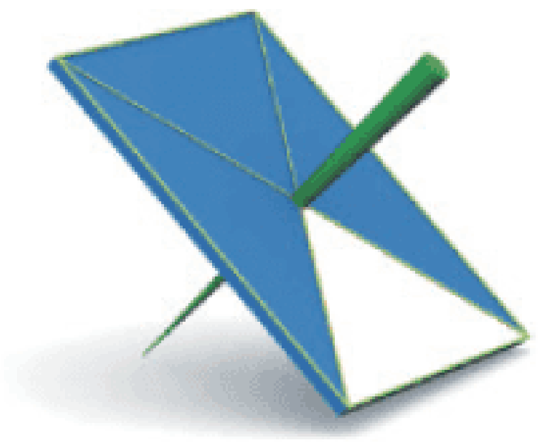
1 Play the game in pairs, so that each player has 20 turns.
Which player scored the most points in total? Discuss the strategies you used during the game.
Did your strategy change during the game?

2 Expected value can be used to find a strategy that, *on average*, will maximise a player's score.

- **a** Explain why it would not be sensible to:
  - **i** stop while you have scored less than 5 points
  - **ii** keep going if you have scored over 50 points.
- **b** Suppose you have scored 10 points so far in your turn. Let X be the gain or loss from rolling again.
  - **i** Construct a probability distribution for X.
  - **ii** Show that $\mathrm{E}(X)$ is positive.
  - **iii** What does $\mathrm{E}(X)$ tell us about whether we should roll again at 10 points?
- **c** Find the lowest score at which, on average, it is not beneficial to continue rolling. Hence describe a strategy that will maximise your score in the long term.
- **d** Can you think of situations where this strategy may not be the best strategy for winning the game?

3 How would the strategy for maximising your score change if your turn ended when a 6 was rolled, rather than a 1?

D THE BINOMIAL DISTRIBUTION



Suppose $X =$ the number of blues which result from spinning this spinner *once*.

The probability distribution of X is:

x	0	1
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{3}{4}$

Now suppose we spin the spinner n times and count the number of blues that result. The probability that we get a blue is the same for each spin, and each spin is independent of every other spin. This is an example of a **binomial experiment**.

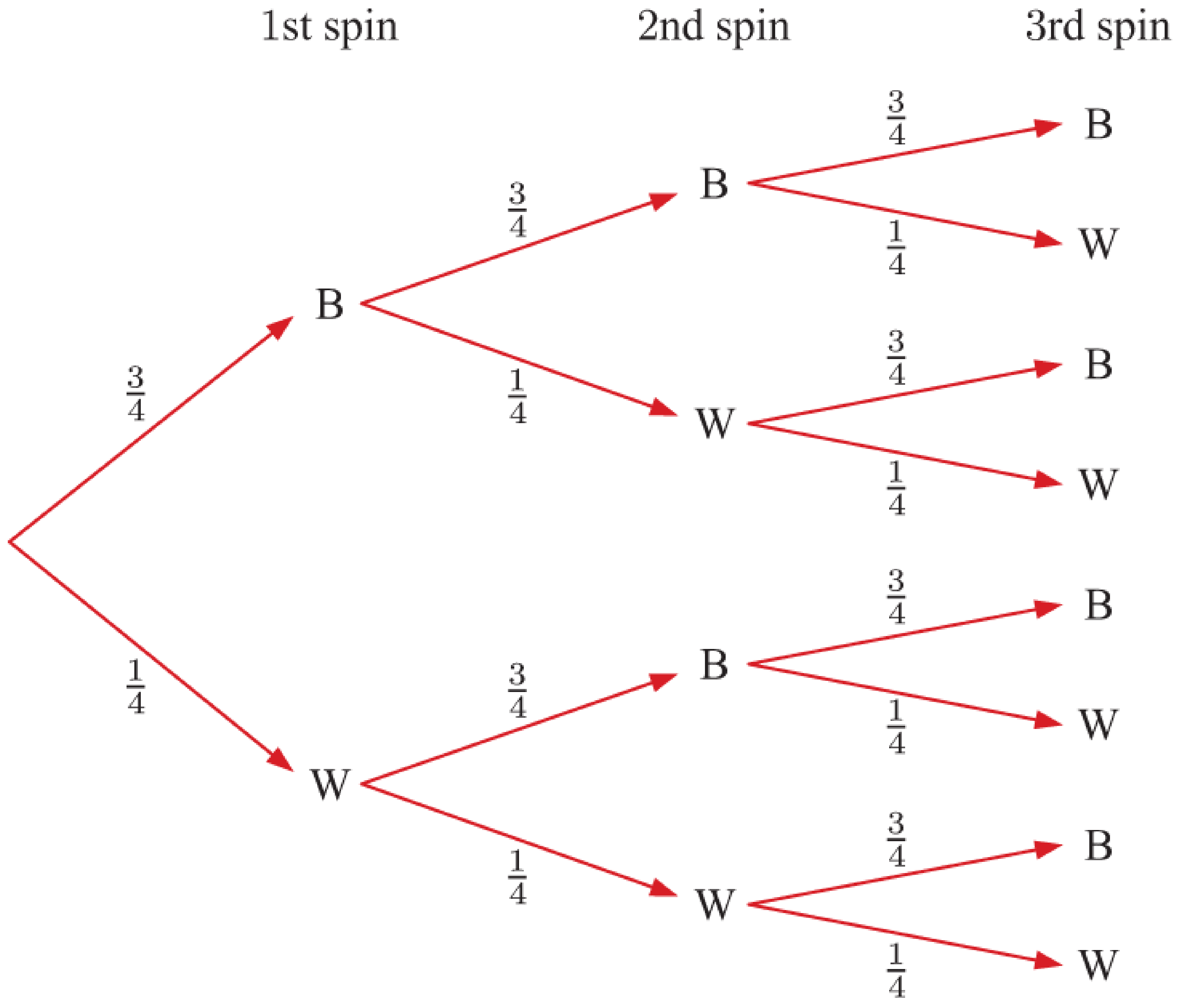
In a **binomial experiment**:

- there are a fixed number of **independent trials**
- there are only *two* possible results for each trial:
  success if some event occurs, or **failure** if the event does not occur
- the probability of success is the same for each trial.

If X is the number of successes in a binomial experiment with n trials, each with probability of success p, then X is a **binomial random variable**.

Consider the spinner on the previous page. Suppose a "success" is a blue result and let X be the number of "successes" in 3 spins of the spinner. X is a binomial random variable with $n = 3$ and $p = \frac{3}{4}$, and can take the values 0, 1, 2, or 3.

To help determine the probability distribution of X, we first draw a tree diagram and find the probabilities associated with each possible outcome. We let B represent blue and W represent white.



outcome	probability
BBB	$\left(\frac{3}{4}\right)^3$
BBW	$\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$
BWB	$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)$
BWW	$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2$
WBB	$\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2$
WBW	$\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$
WWB	$\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)$
WWW	$\left(\frac{1}{4}\right)^3$

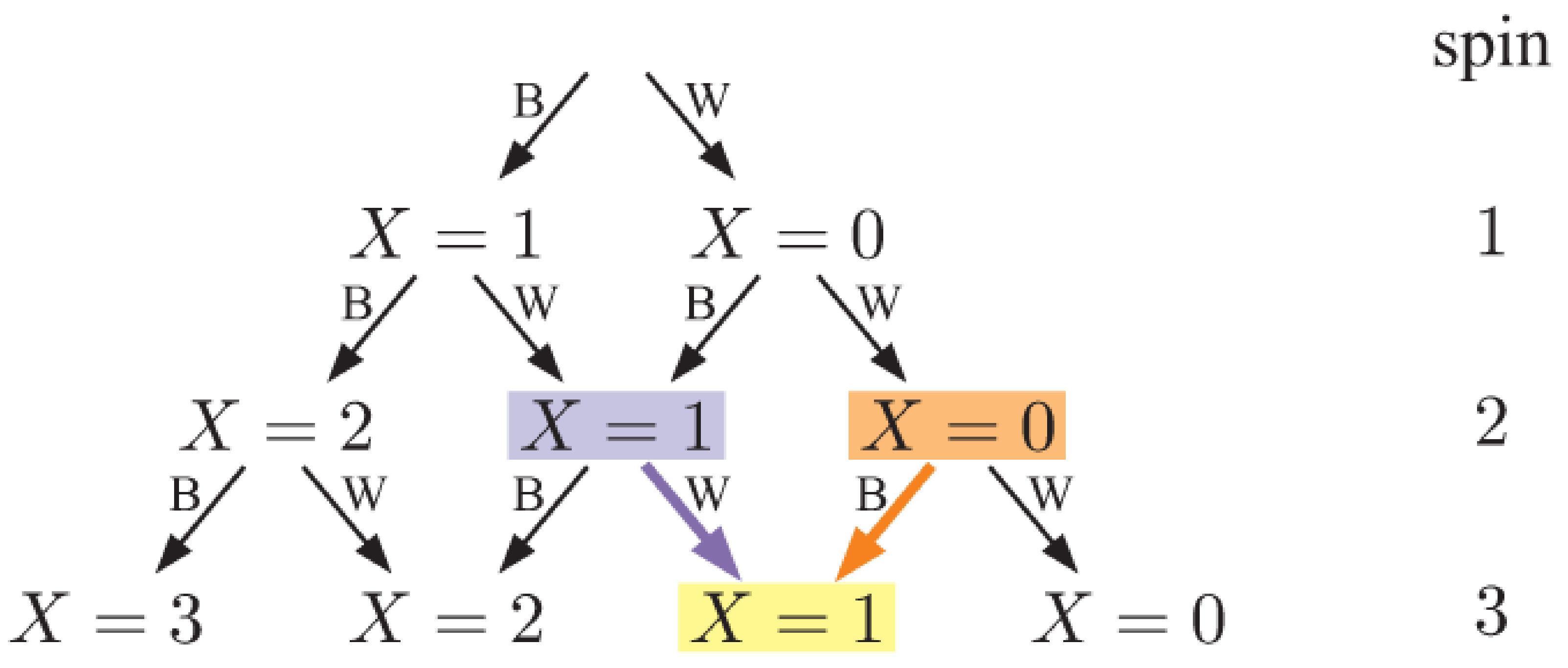
The outcomes have been shaded according to the value of X.
The probabilities associated with each value of X are:

$$\begin{aligned}
\mathrm{P}(X=0) &= \mathrm{P}(\mathrm{WWW}) \\
&= \left(\tfrac{1}{4}\right)^3 \qquad \{\text{outcome shaded green}\} \\
\mathrm{P}(X=1) &= \mathrm{P}(\mathrm{BWW\ or\ WBW\ or\ WWB}) \\
&= \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right) \\
&= 3 \times \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 \qquad \{\text{outcomes shaded yellow}\} \\
\mathrm{P}(X=2) &= \mathrm{P}(\mathrm{BBW\ or\ BWB\ or\ WBB}) \\
&= \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 \\
&= 3 \times \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) \qquad \{\text{outcomes shaded red}\} \\
\mathrm{P}(X=3) &= \mathrm{P}(\mathrm{BBB}) \\
&= \left(\tfrac{3}{4}\right)^3 \qquad \{\text{outcome shaded blue}\}
\end{aligned}$$

Notice that for any particular value of X, each outcome with this property will have the same probability of occurring. The *order* in which blues and whites appear does not matter.

PASCAL'S TRIANGLE

We can simplify our tree diagram on the previous page to only consider the value of X after each spin:



We can use this diagram to help find the number of ways of obtaining each value of X after each spin.

Suppose that after 3 spins we have had 1 blue in total, so $X = 1$. This outcome is highlighted in yellow.

To reach this position, either:

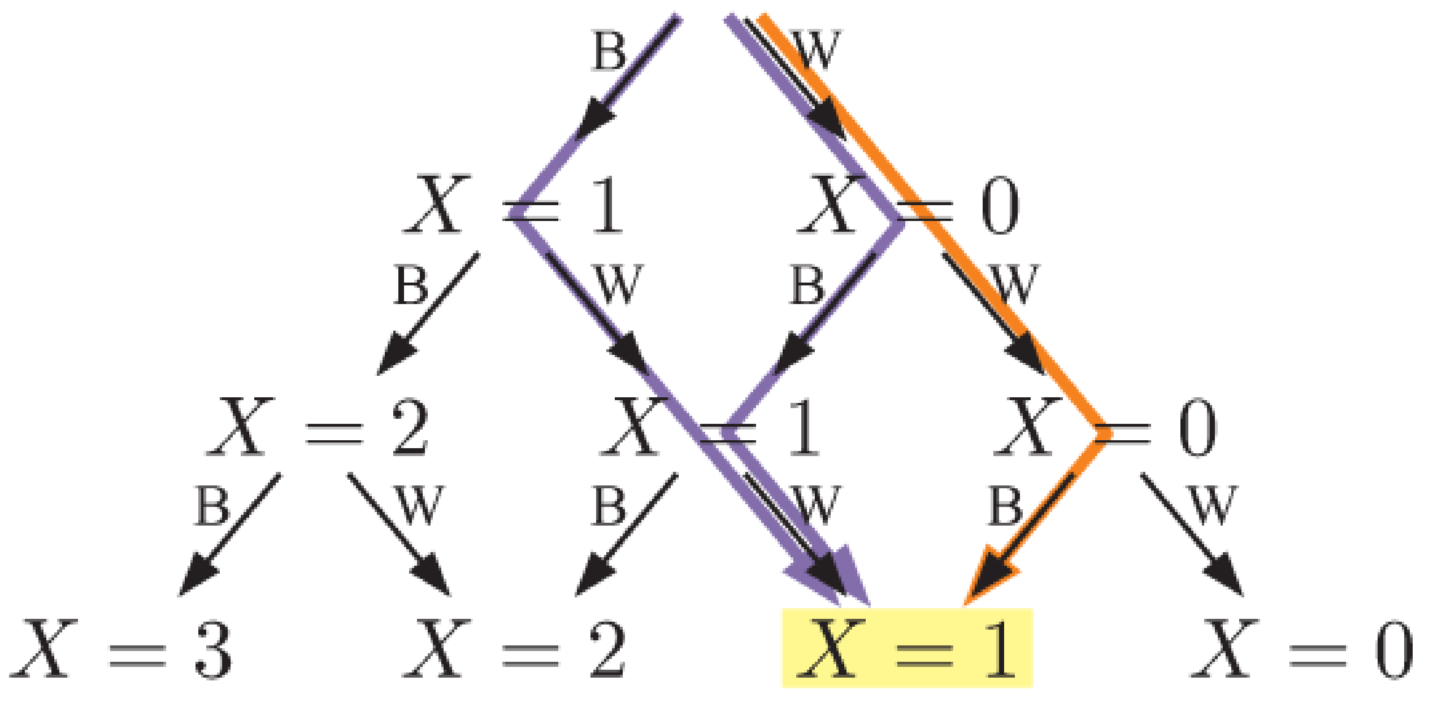
- we had one blue after 2 spins, and we spun a white on the third spin (purple path), or
- we had no blues after 2 spins, and we spun a blue on the third spin (orange path).

$\therefore$ the total number of ways we can get 1 blue in 3 spins

$=$ number of ways of getting 1 blue in 2 spins $+$ number of ways of getting 0 blues in 2 spins

$= 2 + 1 = 3$

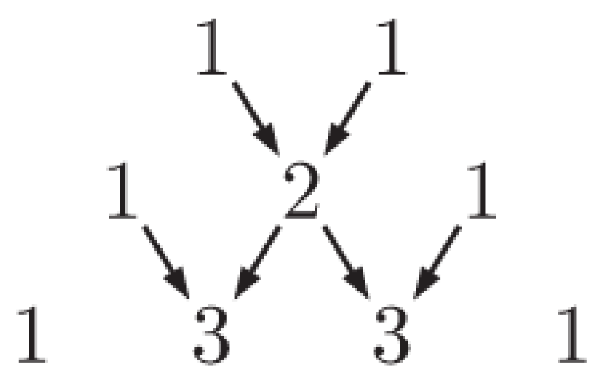
The 3 paths are highlighted alongside.



Click on the icon to observe this process for the rest of the tree diagram.

DEMO

We can simplify our diagram even further by only writing the number of ways of reaching each point on the tree.



This diagram is the first three lines of a triangle of numbers called **Pascal's triangle**.

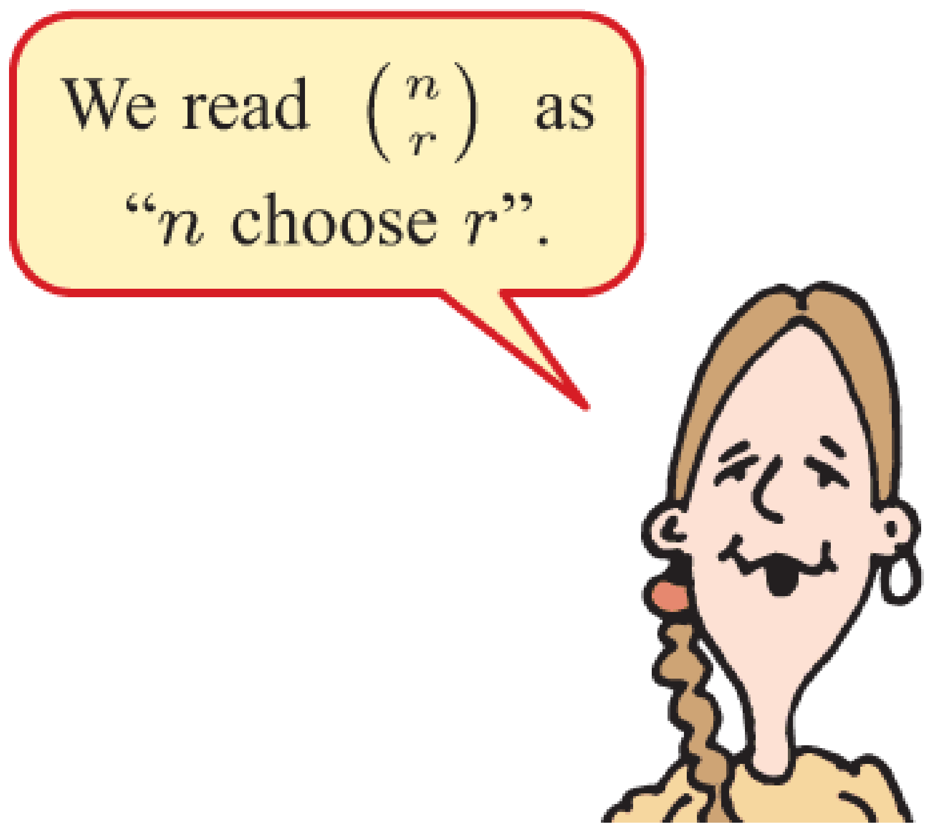
Notice that:

- the values on the end of each row are always 1
- each of the remaining values is found by adding the two values diagonally above it
- the triangle can be continued indefinitely.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
⋮ ⋮

The number of ways of getting r successes from n trials is the $(r+1)$th value in the nth row of Pascal's triangle.

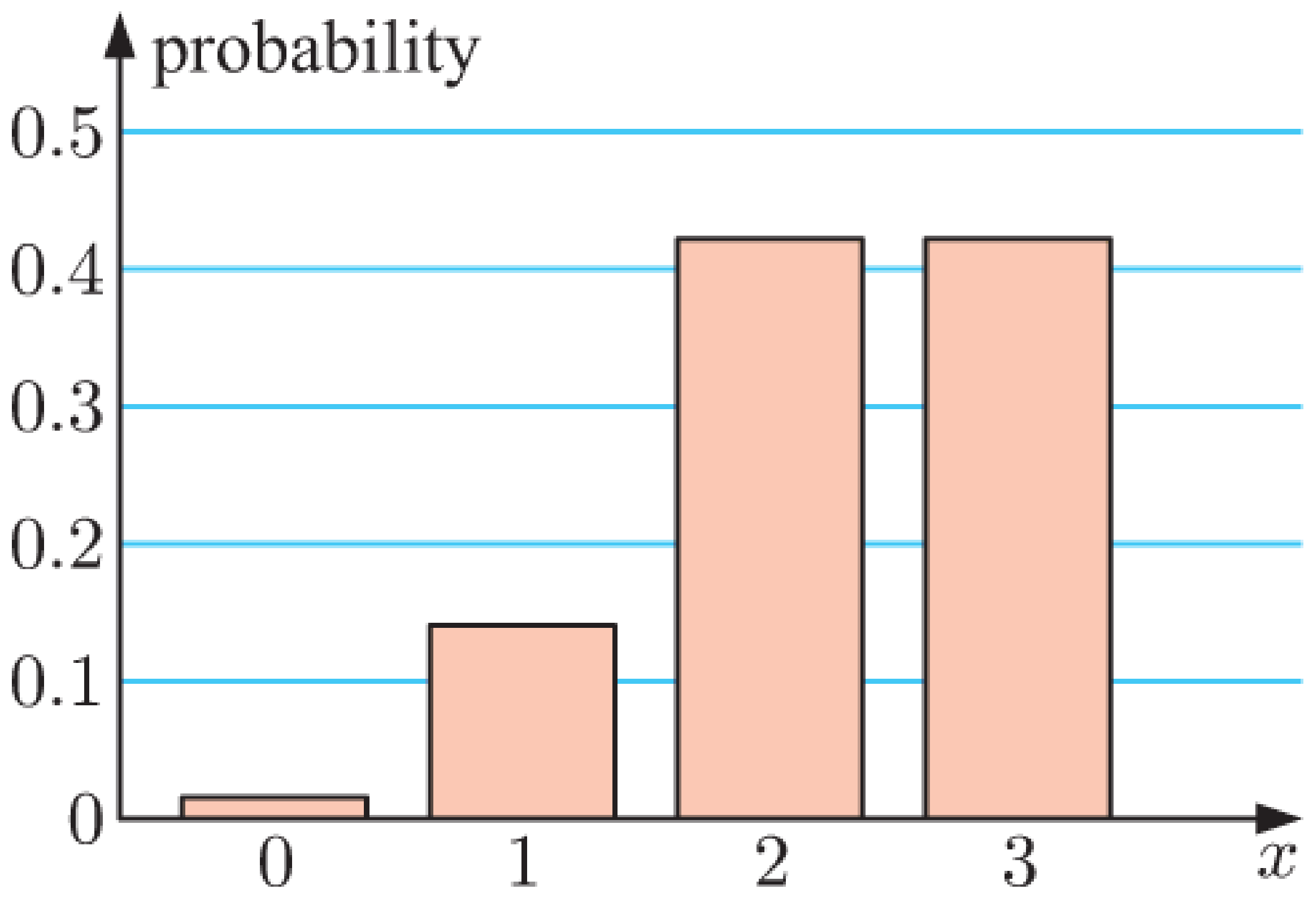
It is written as $\binom{n}{r}$ and is called the **binomial coefficient**.


We read $\binom{n}{r}$ as "n choose r".

Returning to the spinner with 3 blue sectors and 1 white sector, where X is the number of blues after 3 spins, we can now write the probabilities using the binomial coefficient:

$$\begin{aligned}
\mathrm{P}(X=0) &= \left(\tfrac{1}{4}\right)^3 &&= \tbinom{3}{0}\left(\tfrac{3}{4}\right)^0\left(\tfrac{1}{4}\right)^3 \approx 0.0156\\
\mathrm{P}(X=1) &= 3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1 &&= \tbinom{3}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \approx 0.1406\\
\mathrm{P}(X=2) &= 3\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 &&= \tbinom{3}{2}\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)^1 \approx 0.4219\\
\mathrm{P}(X=3) &= \left(\tfrac{3}{4}\right)^3 &&= \tbinom{3}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^0 \approx 0.4219
\end{aligned}$$

So, $\mathrm{P}(X=x) = \binom{3}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$.

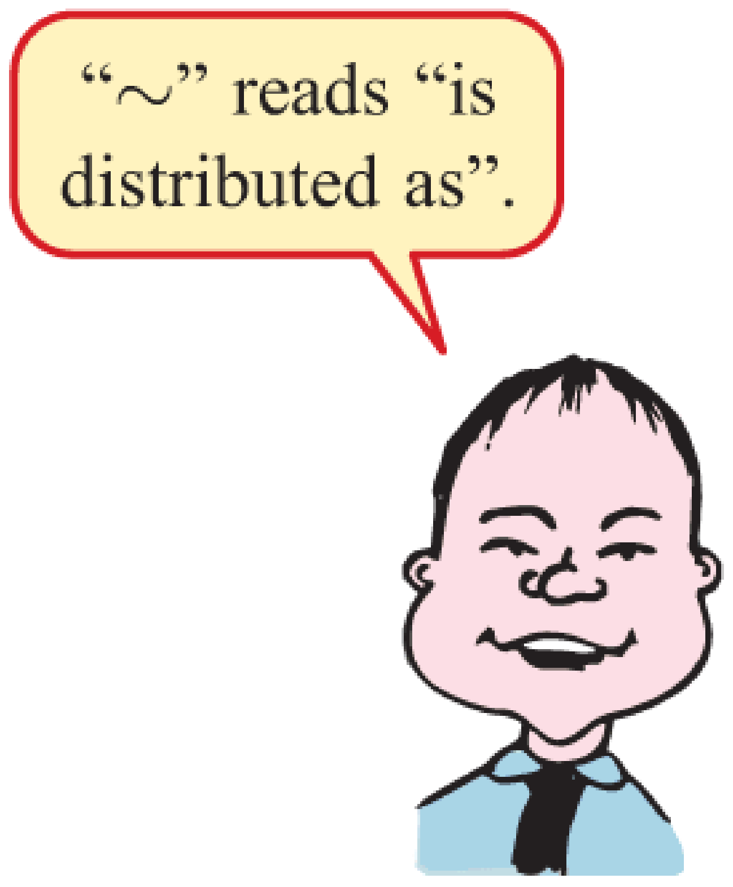


THE PROBABILITY DISTRIBUTION OF A BINOMIAL RANDOM VARIABLE

Suppose X is a binomial random variable with n independent trials and probability of success p. The probability mass function of X is:

$$P(x) = \mathrm{P}(X=x) = \underbrace{\binom{n}{x}}_{\text{number of ways of obtaining } x \text{ successes from } n \text{ trials}} \underbrace{p^x(1-p)^{n-x}}_{\text{probability of obtaining } x \text{ successes and } n-x \text{ failures in a particular order}} \quad \text{where } x = 0, 1, 2, \ldots, n.$$

The probability distribution of X is called the **binomial distribution**, and we write $\boldsymbol{X \sim \mathrm{B}(n, p)}$.

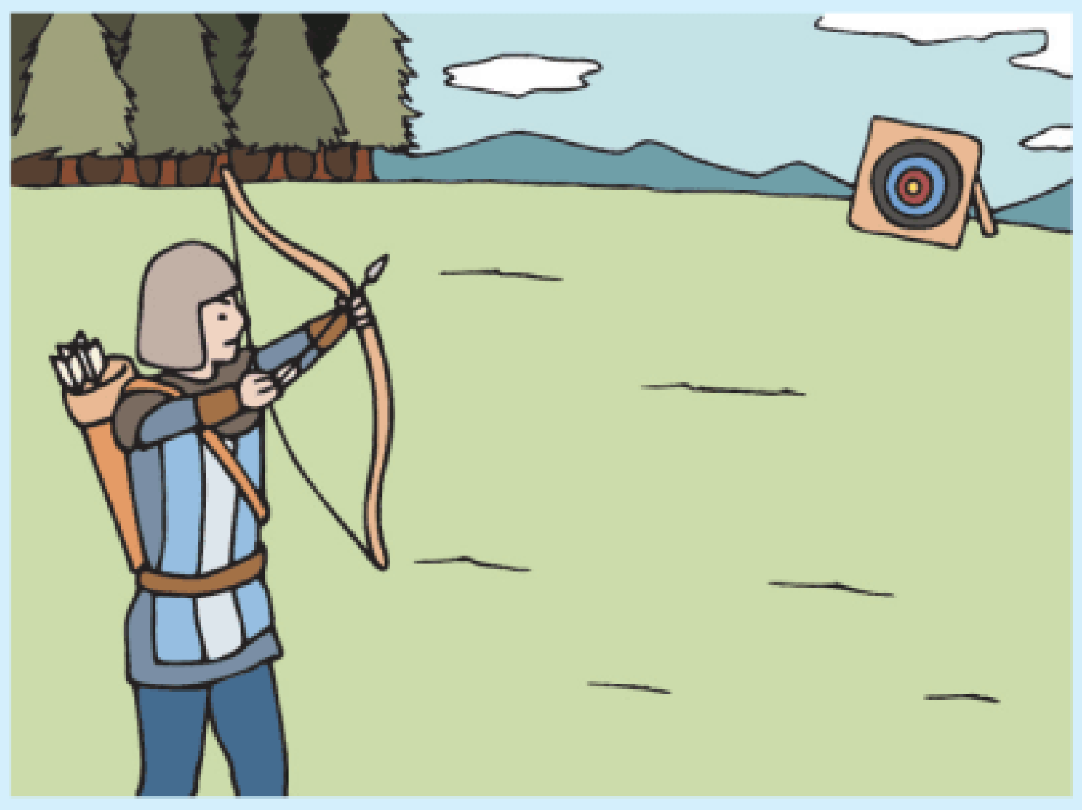

"$\sim$" reads "is distributed as".

Example 6 **Self Tutor**

a Write down the first 5 rows of Pascal's triangle.

b An archer has a 90% chance of hitting a target with each arrow. If 5 arrows are fired, determine the chance of hitting the target:

  i twice only **ii** at most 3 times.

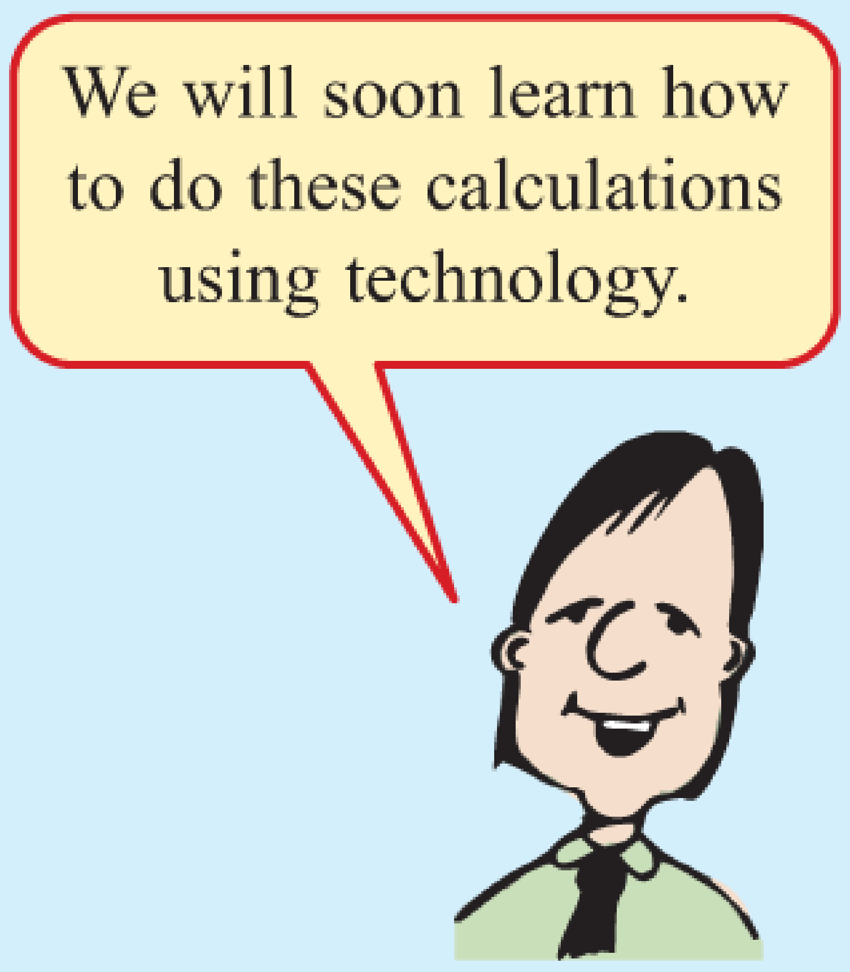


a

$$\begin{array}{ccccccccccccl}
 & & & & 1 & & 1 & & & & & & n=1 \\
 & & & 1 & & 2 & & 1 & & & & & n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & n=5
\end{array}$$

b The number of trials is $n = 5$.
The probability of success with each arrow is $p = \frac{9}{10}$.
Let X be the number of arrows that hit the target.
$\therefore \quad X \sim \mathrm{B}\left(5, \frac{9}{10}\right)$ and the probability mass function of X is:

$$\begin{aligned}
\mathrm{P}(X=x) &= \binom{5}{x}\left(\tfrac{9}{10}\right)^x\left(1-\tfrac{9}{10}\right)^{5-x} \\
&= \binom{5}{x}\left(\tfrac{9}{10}\right)^x\left(\tfrac{1}{10}\right)^{5-x}
\end{aligned}$$


We will soon learn how to do these calculations using technology.

i P(hits twice only) $= \mathrm{P}(X = 2)$

$$\begin{aligned}
&= \binom{5}{2}\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^{5-2} \\
&= 10\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^3 \\
&= 0.0081
\end{aligned}$$

ii P(hits at most 3 times)

$$\begin{aligned}
&= \mathrm{P}(X \leqslant 3) \\
&= \mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3) \\
&= \binom{5}{0}\left(\tfrac{9}{10}\right)^0\left(\tfrac{1}{10}\right)^{5-0}+\binom{5}{1}\left(\tfrac{9}{10}\right)^1\left(\tfrac{1}{10}\right)^{5-1}+\binom{5}{2}\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^{5-2}+\binom{5}{3}\left(\tfrac{9}{10}\right)^3\left(\tfrac{1}{10}\right)^{5-3} \\
&= \left(\tfrac{1}{10}\right)^5+5\left(\tfrac{9}{10}\right)\left(\tfrac{1}{10}\right)^4+10\left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right)^3+10\left(\tfrac{9}{10}\right)^3\left(\tfrac{1}{10}\right)^2 \\
&\approx 0.0815
\end{aligned}$$

EXERCISE 14D

1 For which of these probability experiments does the binomial distribution apply? Explain your answers.

- **a** A coin is thrown 100 times. The variable is the number of heads.
- **b** One hundred coins are each thrown once. The variable is the number of heads.
- **c** A box contains 5 blue and 3 red marbles. I draw out 5 marbles one at a time, replacing the marble before the next is drawn. The variable is the number of red marbles drawn.
- **d** A box contains 5 blue and 3 red marbles. I draw out 5 marbles without replacement. The variable is the number of red marbles drawn.
- **e** A large bin contains ten thousand bolts, 1% of which are faulty. I draw a sample of 10 bolts from the bin. The variable is the number of faulty bolts.

2 Write down the first 6 rows of Pascal's triangle.

3 If a coin is tossed *four* times, what is the probability of getting:

a 4 heads **b** 3 heads **c** 2 heads?

4 If *five* coins are tossed simultaneously, what is the probability of getting:

- **a** 4 heads and 1 tail in any order
- **b** 2 heads and 3 tails in any order
- **c** 4 heads and then 1 tail?

5 A box of chocolates contains strawberry creams and almond centres in the ratio 2 : 1. Four chocolates are selected at random, with replacement. Find the probability of getting:

- **a** all strawberry creams
- **b** two of each type
- **c** at least 2 strawberry creams.

6 In New Zealand in 1946 there were two different coins of value one florin. These were "normal" kiwis and "flat back" kiwis, in the ratio 3 : 1. From a very large batch of 1946 florins, six were selected at random with replacement. Find the probability that:

- **a** two were "flat backs"
- **b** at least 3 were "flat backs"
- **c** at most 3 were "normal" kiwis.

NEW·ZEALAND
ONE·FLORIN
NEW·ZEALAND
ONE·FLORIN·1946

INVESTIGATION 1 THE GRAPH OF A BINOMIAL DISTRIBUTION

In this Investigation we will explore the graph of a binomial distribution and how its shape varies with changes to n and p.

What to do:

1 Click on the icon to access the demonstration. It shows the graph of the binomial distribution for $X \sim \text{B}(n, p)$. Set $n = 20$ and $p = 0.1$.

DEMO

- **a** What is the mode of X?
- **b** Describe the shape of the distribution.

2 Use the slider to change the value of p. Describe how the *shape* of the distribution changes as p changes.

3 Reset p to 0.1. Use the slider to change the value of n.
How does this affect the shape of the distribution? What happens to the shape of the binomial distribution as the number of trials n increases?

E USING TECHNOLOGY TO FIND BINOMIAL PROBABILITIES

We can quickly calculate binomial probabilities using a graphics calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

For example:

- To find the probability $\text{P}(X = k)$ that the variable takes the value k, we use the **binomial probability function**.
- To find the probability that the variable takes a *range* of values, such as $\text{P}(X \leqslant k)$ or $\text{P}(X \geqslant k)$, we use the **binomial cumulative probability function**.

Some calculator models, such as the **TI-84 Plus CE**, only allow you to calculate $P(X \leqslant k)$. To find the probability $P(X \geqslant k)$ for these models, it is often easiest to find the complement $P(X \leqslant k-1)$ and use $P(X \geqslant k) = 1 - P(X \leqslant k-1)$.

Example 7

Self Tutor

72% of union members are in favour of a certain change to their conditions of employment. A random sample of five members is taken. Find the probability that:

- **a** three members are in favour of the change in conditions
- **b** at least three members are in favour of the changed conditions.

Let X denote the number of members in the sample in favour of the change.

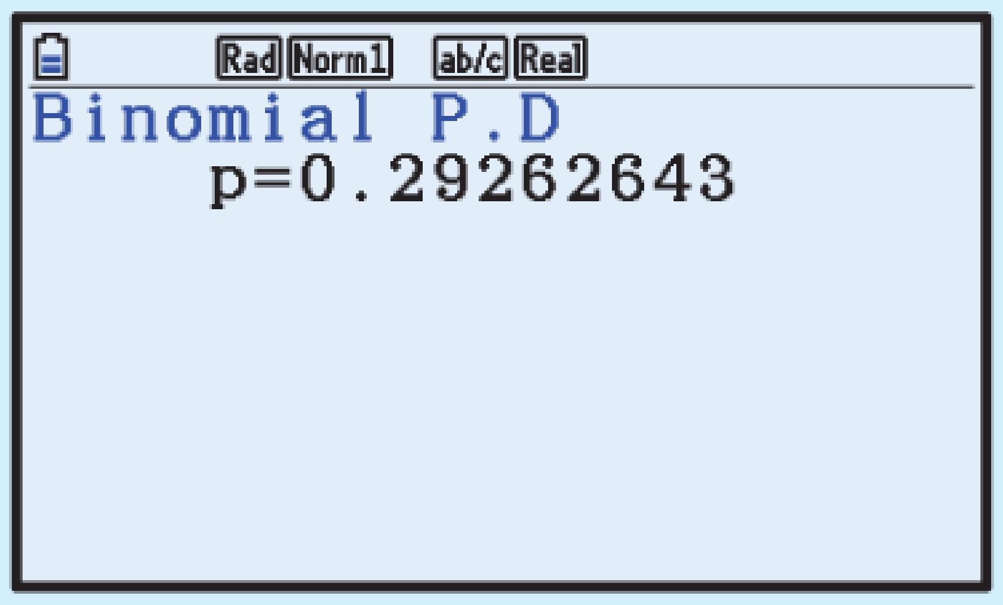
$n = 5$, so $X = 0, 1, 2, 3, 4$, or 5, and $p = 72\% = 0.72$

$\therefore\ X \sim B(5, 0.72)$.

a $P(X = 3) = \binom{5}{3}(0.72)^3(0.28)^2$

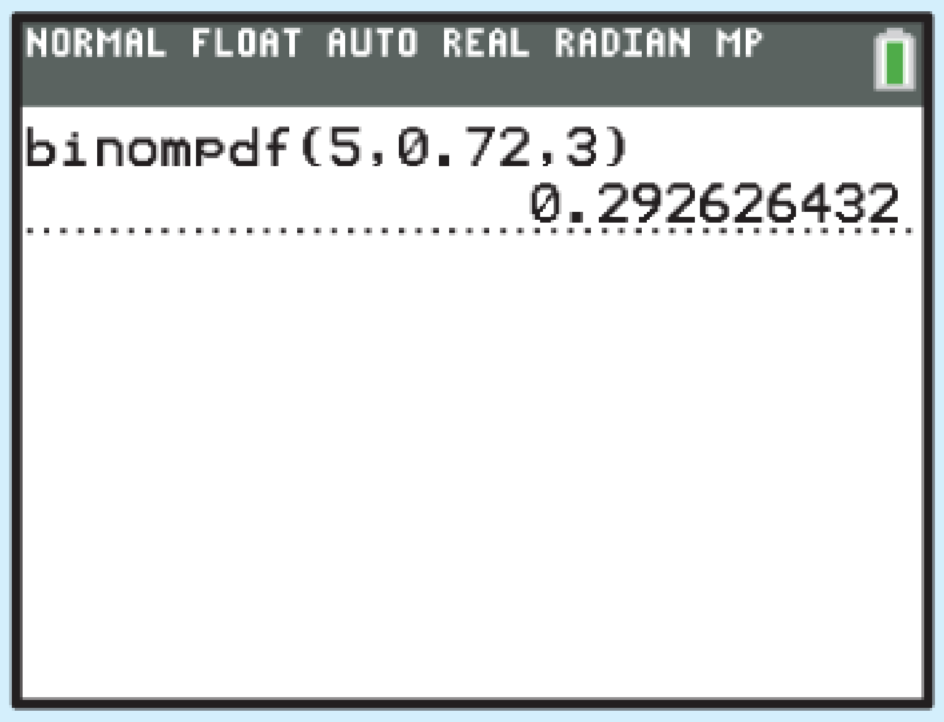
≈ 0.293

Casio fx-CG50



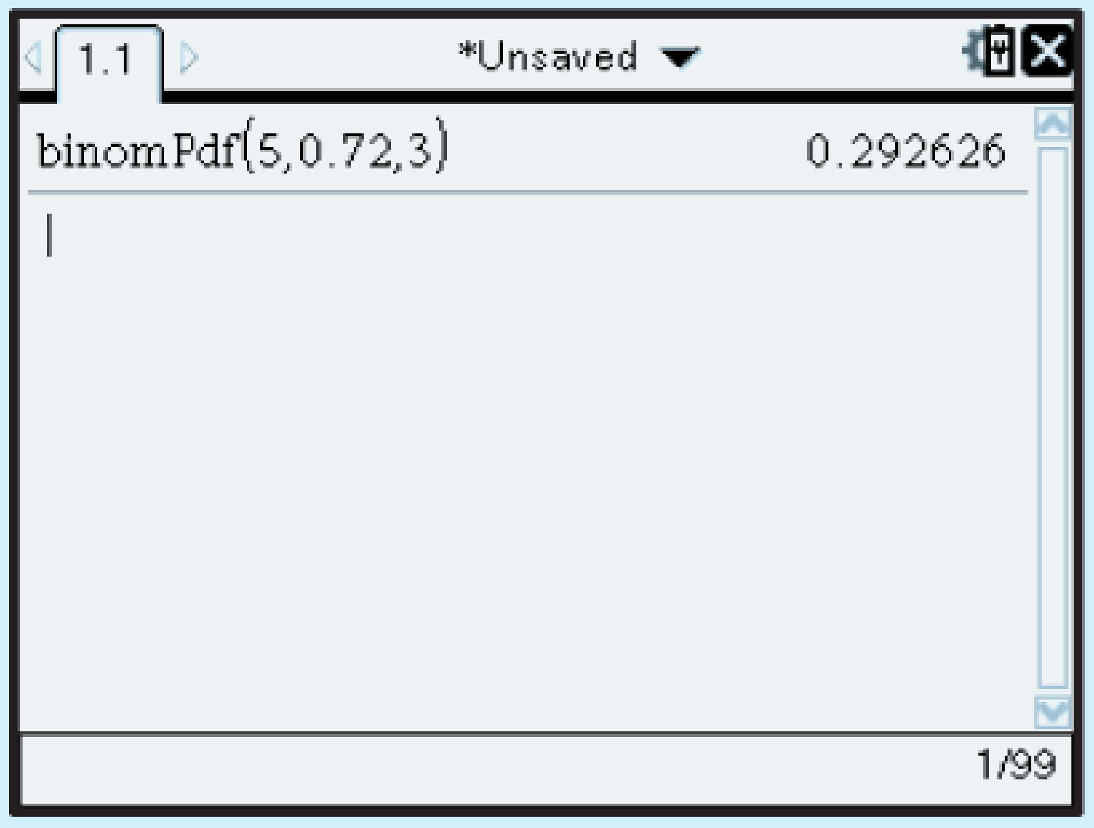
```
Binomial P.D
    p=0.29262643
```

TI-84 Plus CE



```
binompdf(5,0.72,3)
                  0.292626432
```

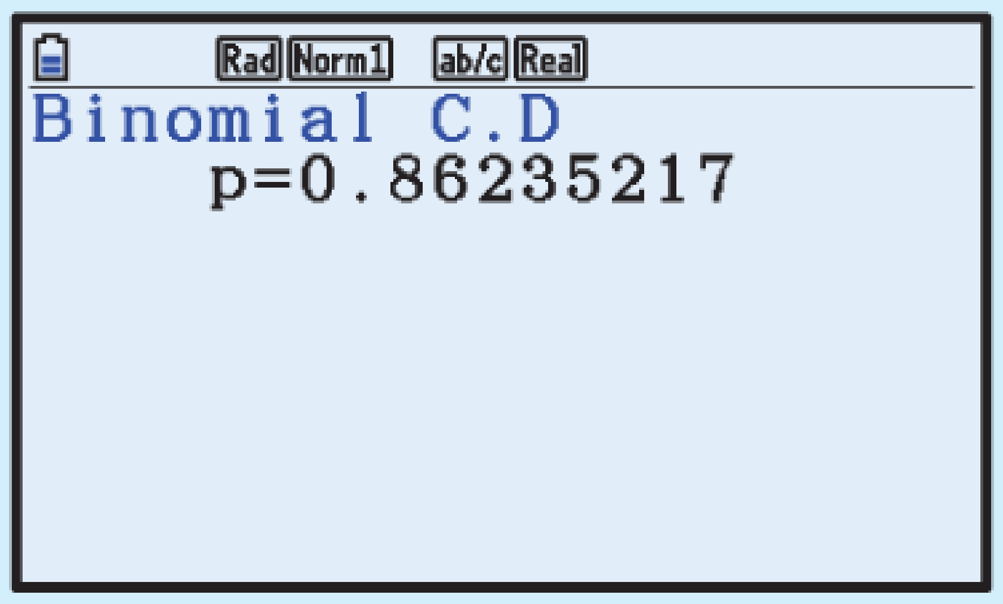
TI-*n*spire



```
binomPdf(5,0.72,3)              0.292626
```

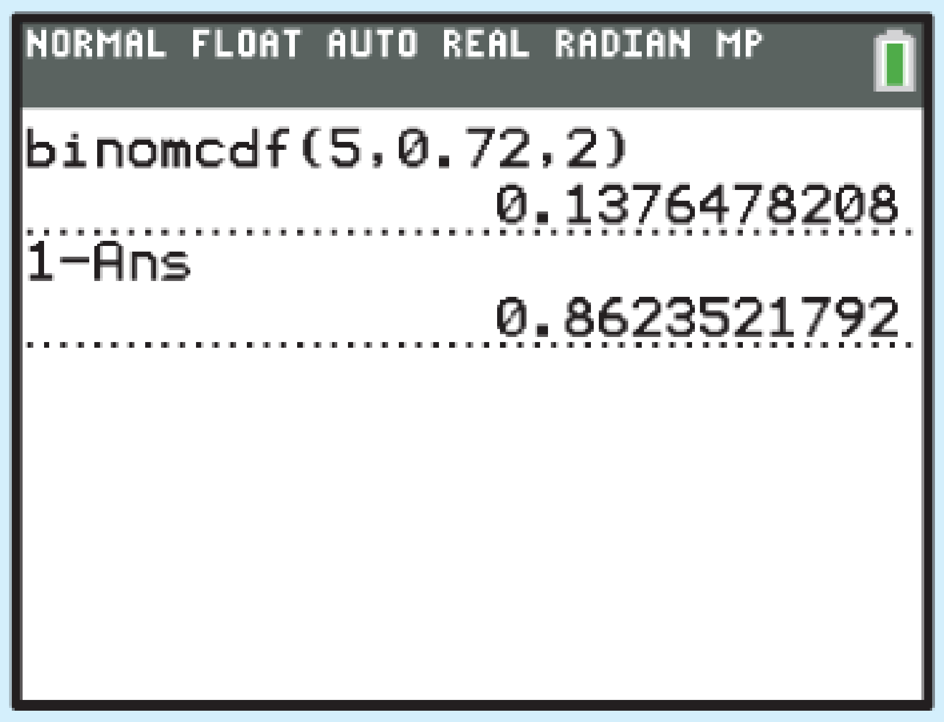
b $P(X \geqslant 3) \approx 0.862$

Casio fx-CG50



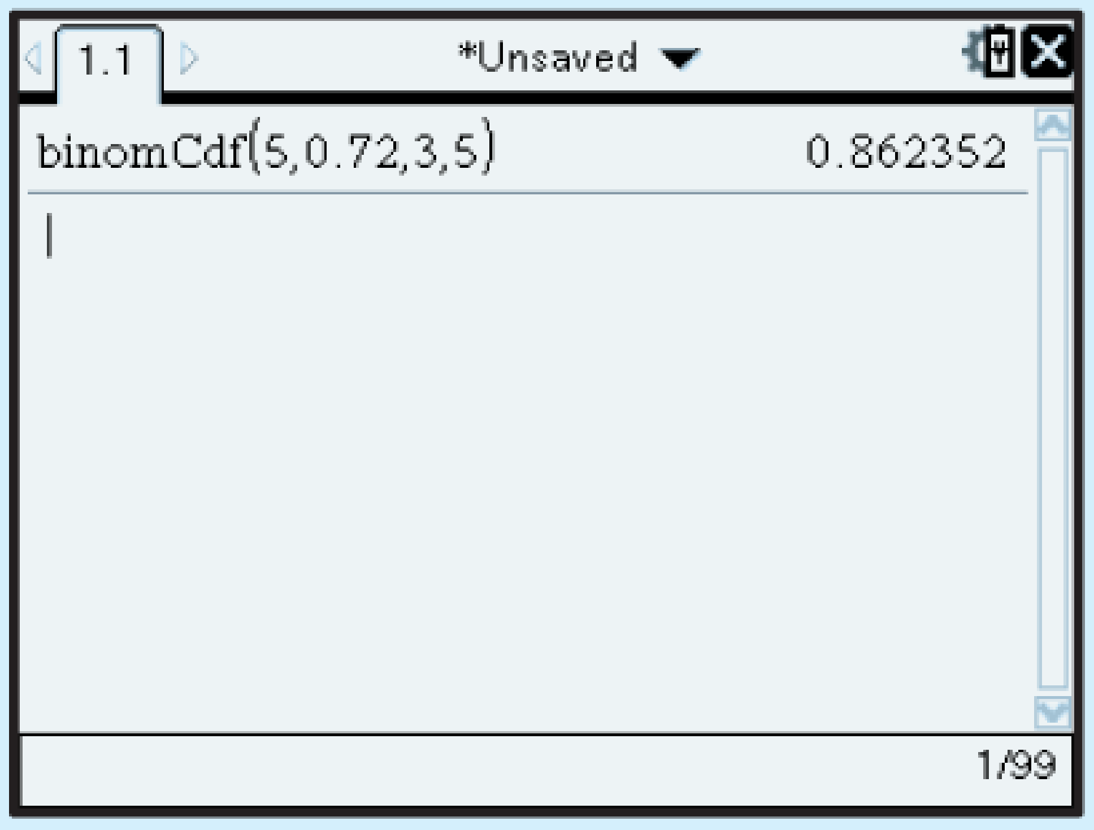
```
Binomial C.D
    p=0.86235217
```

TI-84 Plus CE



```
binomcdf(5,0.72,2)
                 0.1376478208
1-Ans
                 0.8623521792
```

TI-*n*spire



```
binomCdf(5,0.72,3,5)            0.862352
```

EXERCISE 14E

1 5% of electric light bulbs are defective at manufacture. 6 bulbs are randomly tested, with each one being replaced before the next is chosen. Determine the probability that:

- **a** two are defective
- **b** at least one is defective.

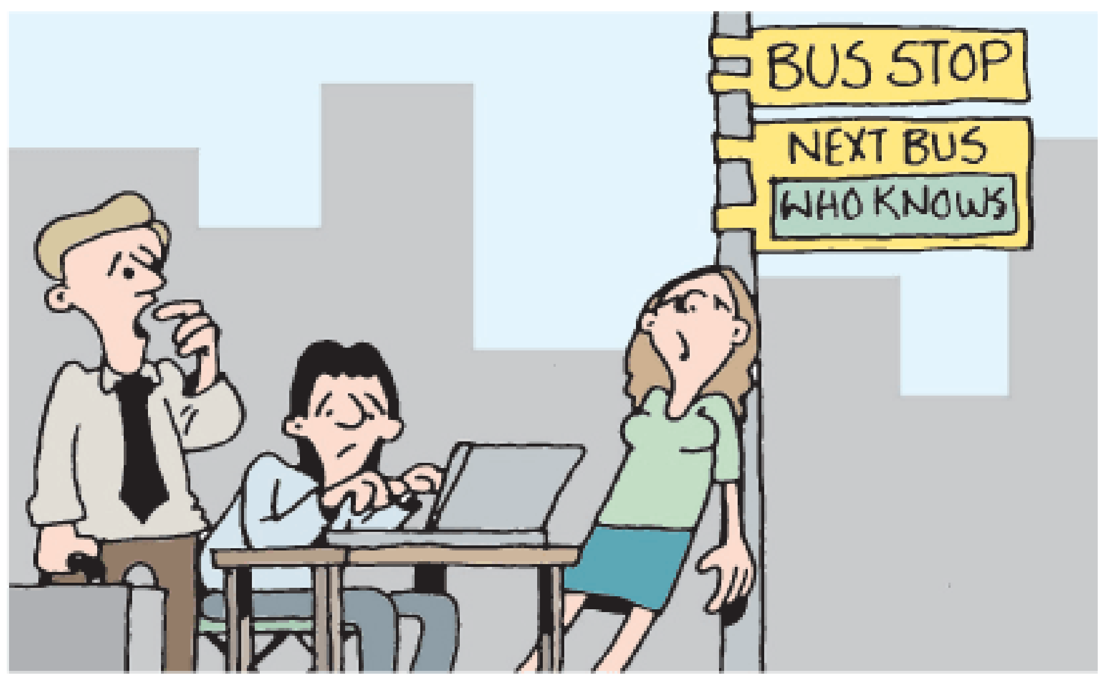
2 Records show that 6% of the items assembled on a production line are faulty. A random sample of 12 items is selected with replacement. Find the probability that:

- **a** none will be faulty
- **b** at most one will be faulty
- **c** at least two will be faulty
- **d** less than four will be faulty.

3 The local bus service does not have a good reputation. The 8 am bus will run late on average two days out of every five. For any week of the year taken at random, find the probability of the 8 am bus being on time:

a all 7 days **b** only on Monday

c on any 6 days **d** on at least 4 days.



4 In a multiple choice test there are 10 questions. Each question has 5 choices, one of which is correct. Raj knows absolutely nothing about the subject, and guesses each answer at random. Given that the pass mark is 70%, determine the probability that he will pass.

5 An infectious flu virus is spreading through a school. The probability of a randomly selected student having the flu next week is 0.3. Mr C has a class of 25 students.

a Calculate the probability that 2 or more students from Mr C's class will have the flu next week.

b If more than 20% of the students have the flu next week, a class test will have to be cancelled. What is the probability that the test will be cancelled?

6 During a season, a basketball player has an 85% success rate in shooting from the free throw line. In one match the basketballer has 20 shots from the free throw line.
Find the probability that the basketballer is successful with:

a all 20 throws **b** at least 18 throws **c** between 14 and 17 (inclusive) throws.

7 Martina beats Jelena in 2 games out of 3 at tennis. What is the probability that Jelena wins a set of tennis 6 games to 4?
Hint: What does the score after 9 games need to be?

8 A fair coin is tossed 200 times. Find the probability of obtaining:

a between 90 and 110 (inclusive) heads **b** more than 95 but less than 105 heads.

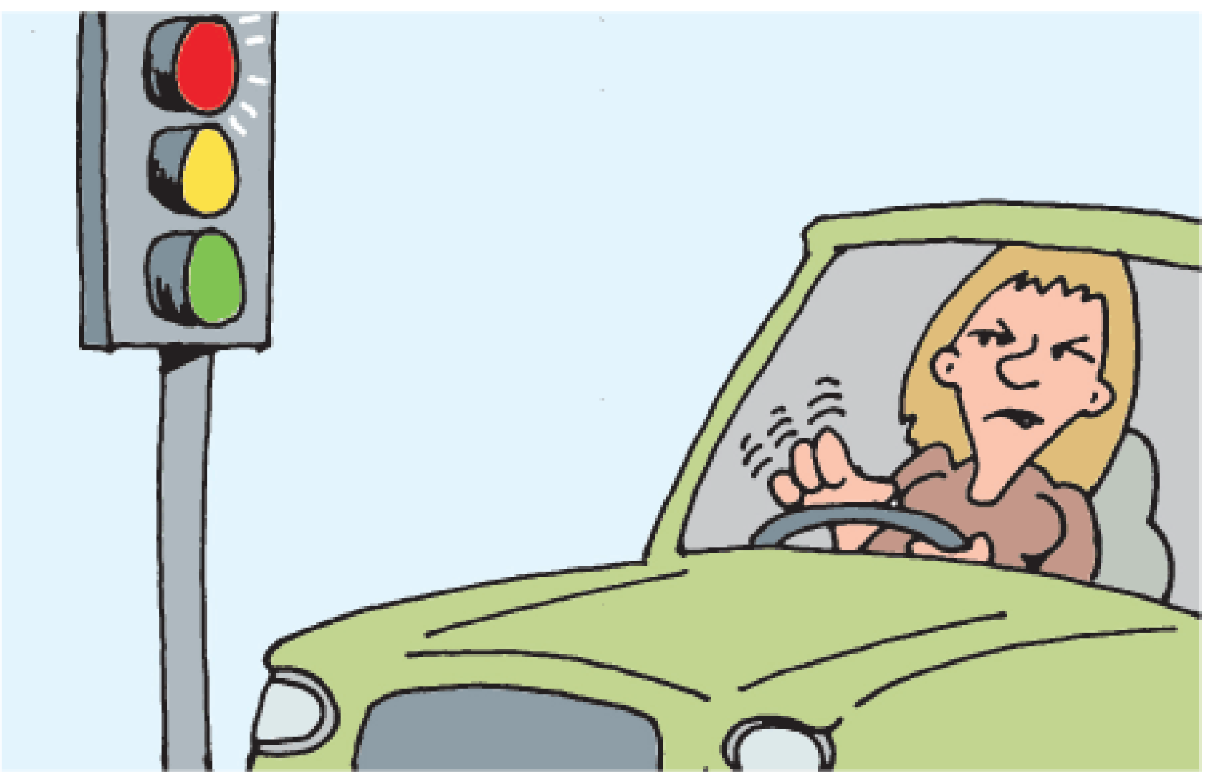
9 **a** Find the probability of rolling double sixes with a pair of dice.

b Suppose a pair of dice is rolled 500 times. Find the probability of rolling between 10 and 20 (inclusive) double sixes.

10 Shelley must pass through 15 traffic lights on her way to work. She has probability 0.6 of being stopped at any given traffic light. If she is stopped at more than 11 traffic lights, she will be late for work.

a Find the probability that Shelley will be late for work on a given day.

b Find the probability that Shelley is on time for work each day of a 5 day week.



c Shelley wants to increase the probability in **b** to at least 80%. She decides to leave home a little earlier, so she must now be stopped at more than 12 traffic lights in order to be late. Has Shelley achieved her goal? Justify your answer.

11 A hot water unit relies on 20 solar components for its power, and will operate provided at least one of its 20 components is working. The probability that an individual solar component will fail in a year is 0.85, and the failure of each individual component is independent of the others.

a Find the probability that the hot water unit will fail within one year.

b Find the smallest number of solar components required to ensure that a hot water service like this one is operating at the end of one year with a probability of at least 0.98.

F THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

In the **Core Topics SL** book we saw that:

- the **mean** is a measure of the **centre** of a distribution
- the **standard deviation** is a measure of the **spread** of the distribution.

INVESTIGATION 2 THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

In this Investigation we will use a calculator to calculate the mean and standard deviation of several binomial distributions. A spreadsheet can also be used to speed up the process.

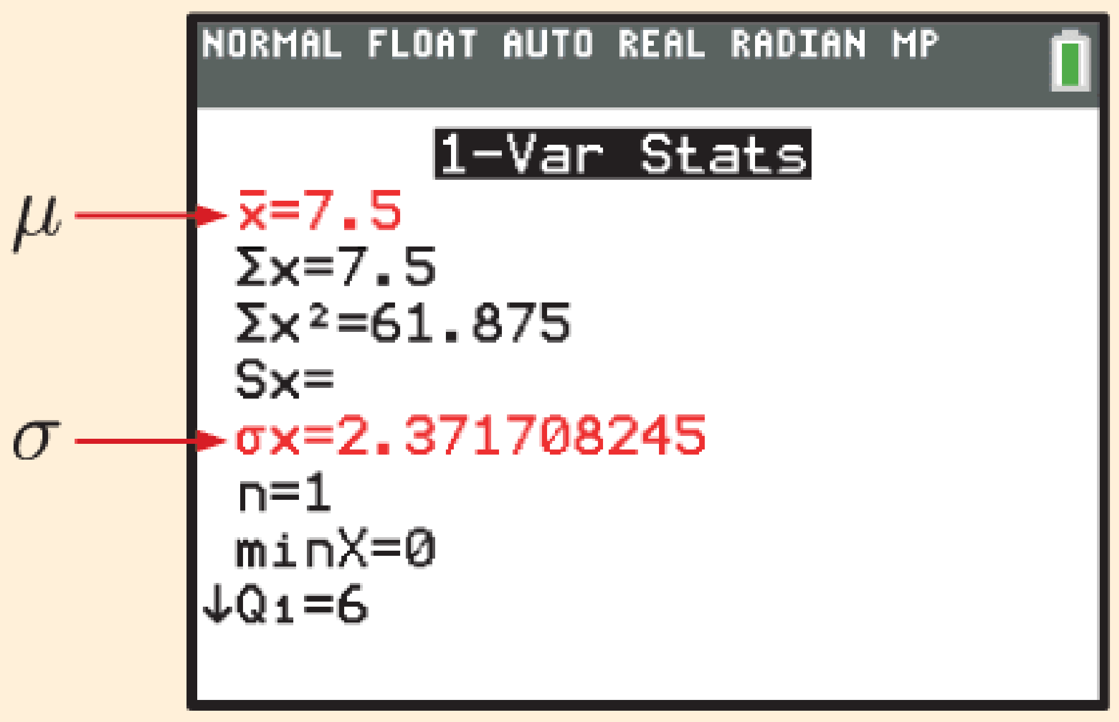
GRAPHICS CALCULATOR INSTRUCTIONS

What to do:

1 We will first calculate the mean and standard deviation for the variable $X \sim \text{B}(30,\, 0.25)$.

a Enter the possible values for X from $x = 0$ to $x = 30$ into **List 1**, and their corresponding binomial probabilities $\text{P}(X = x) = \binom{30}{x}(0.25)^x(0.75)^{30-x}$ into **List 2**.

b Calculate the descriptive statistics for the distribution. You should obtain the results in the screenshot.



```
NORMAL FLOAT AUTO REAL RADIAN MP
1-Var Stats
x̄=7.5
Σx=7.5
Σx²=61.875
Sx=
σx=2.371708245
n=1
minX=0
↓Q₁=6
```

2 Copy and complete the following table for distributions with other values of n and p.

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$				
$n = 30$		$\mu = 7.5$ $\sigma \approx 2.3717$		
$n = 50$				

3 Compare your values with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

From the **Investigation** you should have observed the following results:

Suppose X is a binomial random variable with parameters n and p, so $X \sim \text{B}(n,\, p)$.

- The **mean** of X is $\boldsymbol{\mu = np}$.
- The **variance** of X is $\boldsymbol{\sigma^2 = np(1-p)}$.
- The **standard deviation** of X is $\boldsymbol{\sigma = \sqrt{np(1-p)}}$.

Example 8 **Self Tutor**

A fair die is rolled twelve times, and X is the number of sixes that result.
Find the mean, variance, and standard deviation of X.

This is a binomial distribution with $n = 12$ and $p = \frac{1}{6}$, so $X \sim \mathrm{B}(12, \frac{1}{6})$.

$$\begin{aligned} \mu &= np \\ &= 12 \times \tfrac{1}{6} \\ &= 2 \end{aligned} \qquad \begin{aligned} \sigma^2 &= np(1-p) \\ &= 12 \times \tfrac{1}{6} \times \tfrac{5}{6} \\ &= \tfrac{5}{3} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \sigma &= \sqrt{\sigma^2} \\ &= \sqrt{\tfrac{5}{3}} \\ &\approx 1.291 \end{aligned}$$

We expect a six to be rolled 2 times, with variance $\frac{5}{3}$ and standard deviation 1.291 .

EXERCISE 14F

1 Suppose $X \sim \mathrm{B}(6, p)$. For each of the following cases:

- **i** Find the mean and standard deviation of X.
- **ii** Graph the distribution using a column graph.
- **iii** Comment on the shape of the distribution.

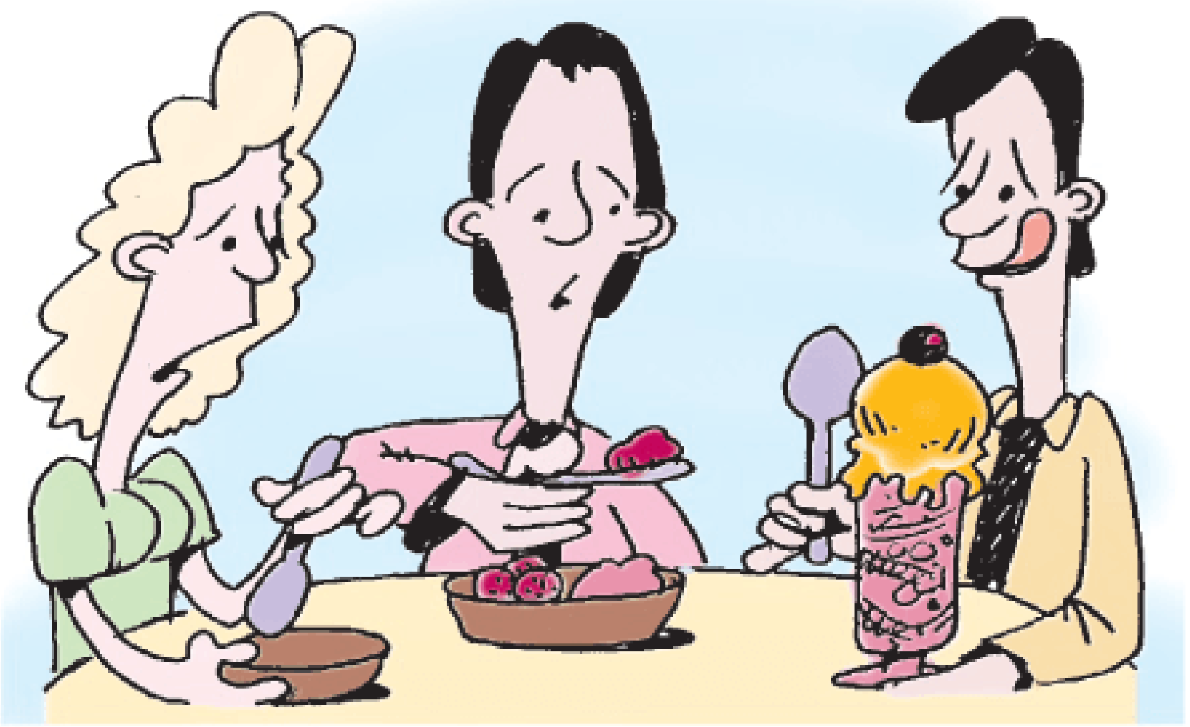
a $p = 0.5$ **b** $p = 0.2$ **c** $p = 0.8$

2 A coin is tossed 10 times and X is the number of heads which occur. Find the mean and variance of X.

3 Bolts produced by a machine vary in quality. The probability that a given bolt is defective is 0.04 . Random samples of 30 bolts are taken from the week's production.

- **a** If X is the number of defective bolts in a sample, find the mean and standard deviation of X.
- **b** If Y is the number of non-defective bolts in a sample, find the mean and standard deviation of Y.

4 A city restaurant knows that 13% of reservations are not honoured, which means the group does not arrive. Suppose the restaurant receives 30 reservations. Let the random variable X be the number of groups that do not arrive. Find the mean and standard deviation of X.



5 A new drug has a 75% probability of curing a patient within one week. Suppose 38 patients are treated using this drug. Let X be the number of patients who are cured within a week.

- **a** Find the mean μ and standard deviation σ of X.
- **b** Find $\mathrm{P}(\mu - \sigma < X < \mu + \sigma)$.

6 Let X be the number of heads which occur when a coin is tossed 100 times, and Y be the number of ones which occur when a die is rolled 300 times.

- **a** Show that the mean of both distributions is 50.
- **b** Calculate the standard deviation of each distribution.
- **c** Which variable do you think is more likely to lie between 45 and 55 (inclusive)? Explain your answer.
- **d** Find: **i** $\mathrm{P}(45 \leqslant X \leqslant 55)$ **ii** $\mathrm{P}(45 \leqslant Y \leqslant 55)$

REVIEW SET 14A

1 Determine whether the following variables are discrete or continuous:

- **a** the number of attempts to pass a driving test
- **b** the length of time before a phone loses its battery charge
- **c** the number of phone calls made before a salesperson has sold 3 products.

2 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3
$\mathrm{P}(X=x)$	0.6	0.25	0.15

ii

x	0	2	5	10
$\mathrm{P}(X=x)$	0.3	0.5	0.1	0.2

iii

x	0	1	2	3
$\mathrm{P}(X=x)$	0.4	-0.2	0.35	0.45

iv

x	2	3	4	5
$\mathrm{P}(X=x)$	0.25	0.25	0.25	0.25

v

x	2	3
$\mathrm{P}(X=x)$	0.7	0.3

vi

x	0	1
$\mathrm{P}(X=x)$	0.28	0.72

b For which of the probability distributions in **a** is X a uniform discrete random variable?

3 $\mathrm{P}(X=x)=\dfrac{a}{x^2+1}$, $x=0, 1, 2, 3$ is a probability mass function.

- **a** Find the value of a.
- **b** Find $\mathrm{P}(X \geqslant 1)$.

4 A random variable X has the probability mass function $P(x)$ described in the table.

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

- **a** Find k.
- **b** Find $\mathrm{P}(X \geqslant 3)$.
- **c** Find the mode of the distribution.
- **d** Find the expected value $\mathrm{E}(X)$ for the distribution.

5 Three green balls and two yellow balls are placed in a hat. Two balls are randomly drawn without replacement, and X is the number of green balls drawn.

- **a** Explain why X is a discrete random variable.
- **b** State the possible values of X.
- **c** Construct a probability table for X.
- **d** Find the expected number of green balls drawn.

6 The faces of a die are labelled 1, 3, 3, 4, 6, 6. Let X be the result when the die is rolled. Find the expected value of X.

7 Lakshmi rolls a regular six-sided die. She wins twice the number of dollars as the number rolled.

- **a** How much does Lakshmi expect to win from one roll of the die?
- **b** If it costs $8 to play the game, would you advise Lakshmi to play many games? Explain your answer.

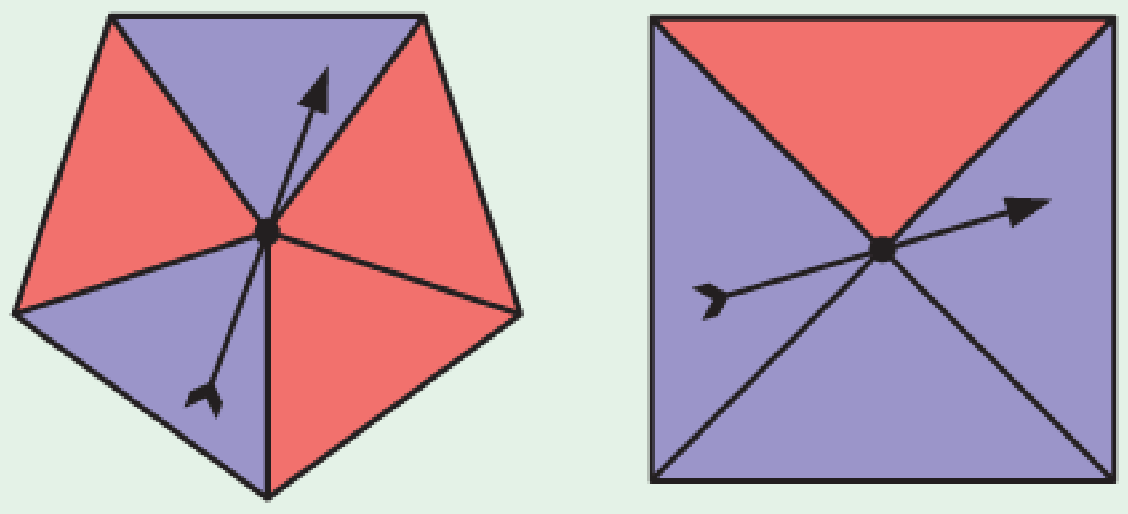
8 With every attempt, Jack has an 80% chance of kicking a goal. In one quarter of a match he has 5 kicks for goal. Use Pascal's triangle to help determine the probability that he scores:

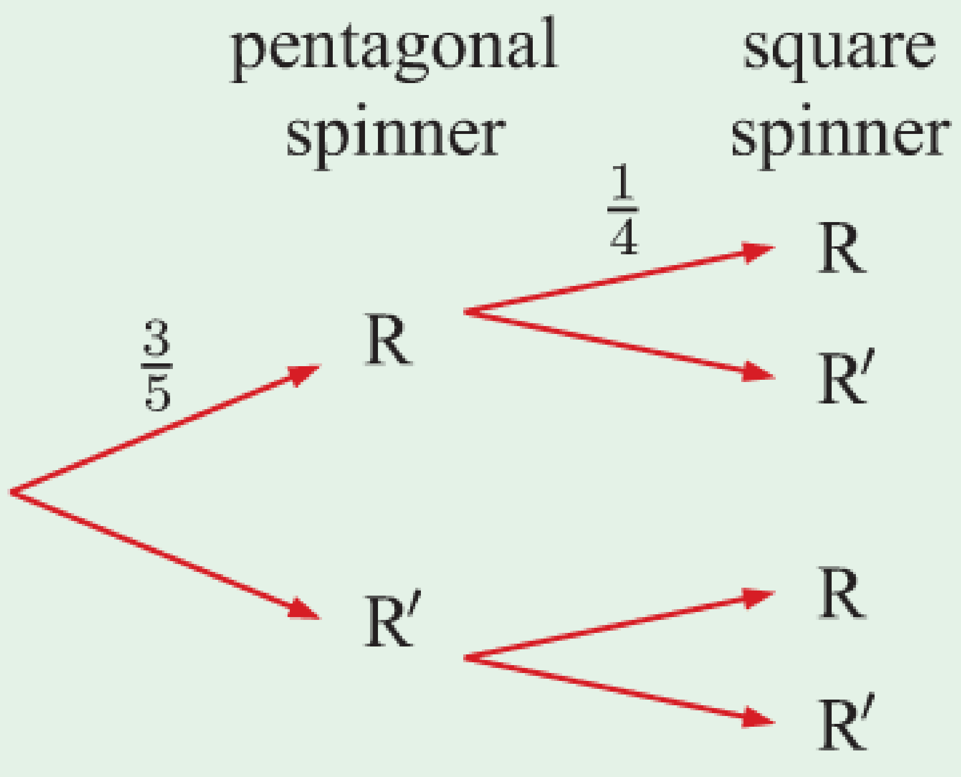
- **a** 3 goals then misses twice
- **b** 3 goals and misses twice.

9 Suppose X is the number of marsupials entering a park at night. It is suspected that X has a probability mass function $P(X) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, \ldots., 8$.

- **a** Find the constant a.
- **b** Find the expected number of marsupials entering the park on a given night.

10 Consider the two spinners illustrated:





- **a** Copy and complete the tree diagram which shows all possible results when the two spinners are spun together.
- **b** Calculate the probability that exactly one red will occur.
- **c** The pair of spinners is now spun 10 times. Let X be the number of times that exactly one red occurs.
  - **i** State the distribution of X.
  - **ii** Write down expressions for $P(X = 1)$ and $P(X = 9)$. Hence determine which of these outcomes is more likely.

11 A school volleyball team has 9 players, each of whom has a 75% chance of coming to any given game. The team needs at least 6 players to avoid forfeiting the game.

- **a** Find the probability that for a randomly chosen game, the team will:
  - **i** have all of its players
  - **ii** have to forfeit the game.
- **b** The team plays 30 games for the season. How many games would you expect the team to forfeit?

12 It is observed that 3% of all batteries produced by a company are defective. For a random sample of 20 batteries, calculate the probability that:

- **a** none are defective
- **b** at least one is defective.

REVIEW SET 14B

1 Sally's number of hits in each softball match has the probability distribution shown.

x	0	1	2	3	4	5
$P(X = x)$	0.07	0.14	k	0.46	0.08	0.02

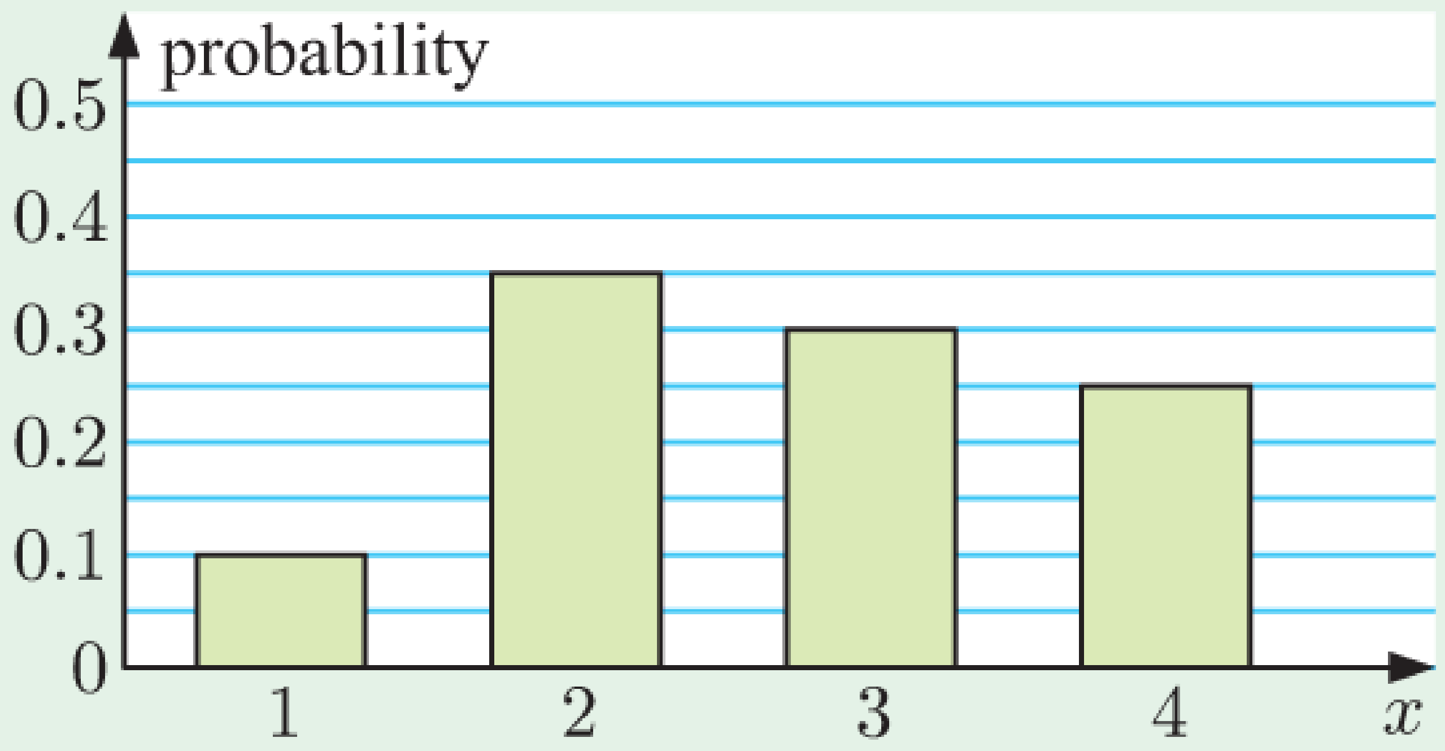
- **a** State clearly what the random variable represents.
- **b** Find: **i** k **ii** $P(X \geqslant 2)$ **iii** $P(1 \leqslant X \leqslant 3)$
- **c** Find the mode and median number of hits.

2 Show that the following are valid probability mass functions:

- **a** $P(x) = \dfrac{e^x}{1+e}$, $x = 0, 1$
- **b** $P(x) = \dfrac{x^2 + x}{40}$, $x = 1, 2, 3, 4$

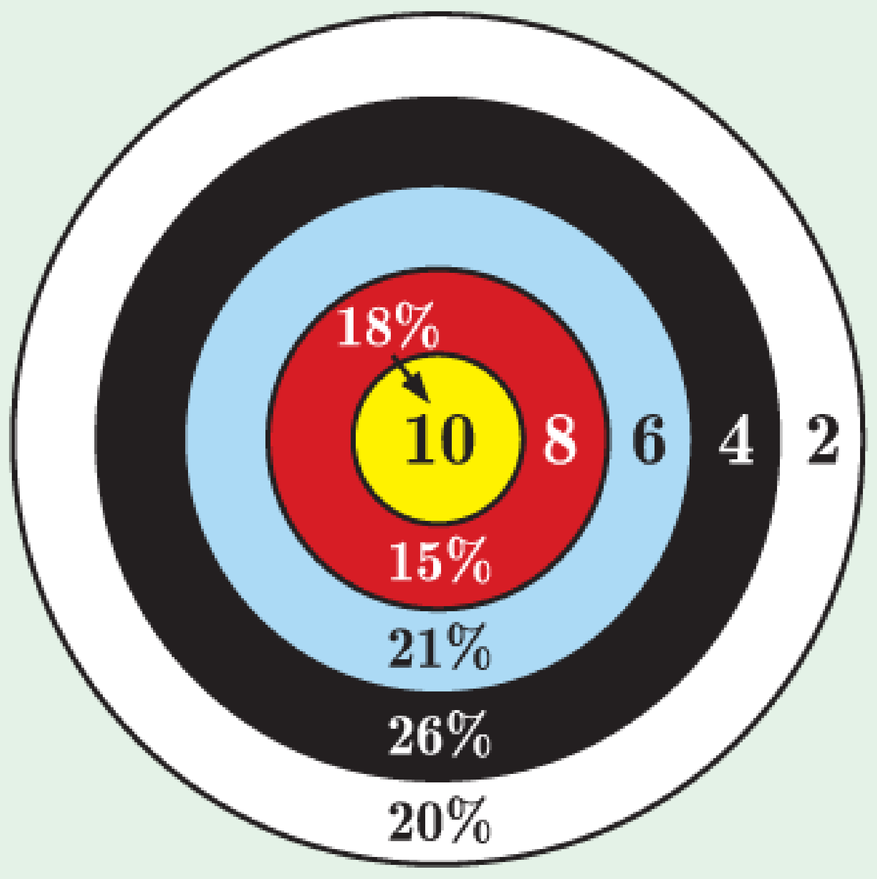
3 The probability distribution of a random variable X is graphed alongside. Find:

- **a** the mode of X
- **b** the median of X
- **c** the expected value of X.

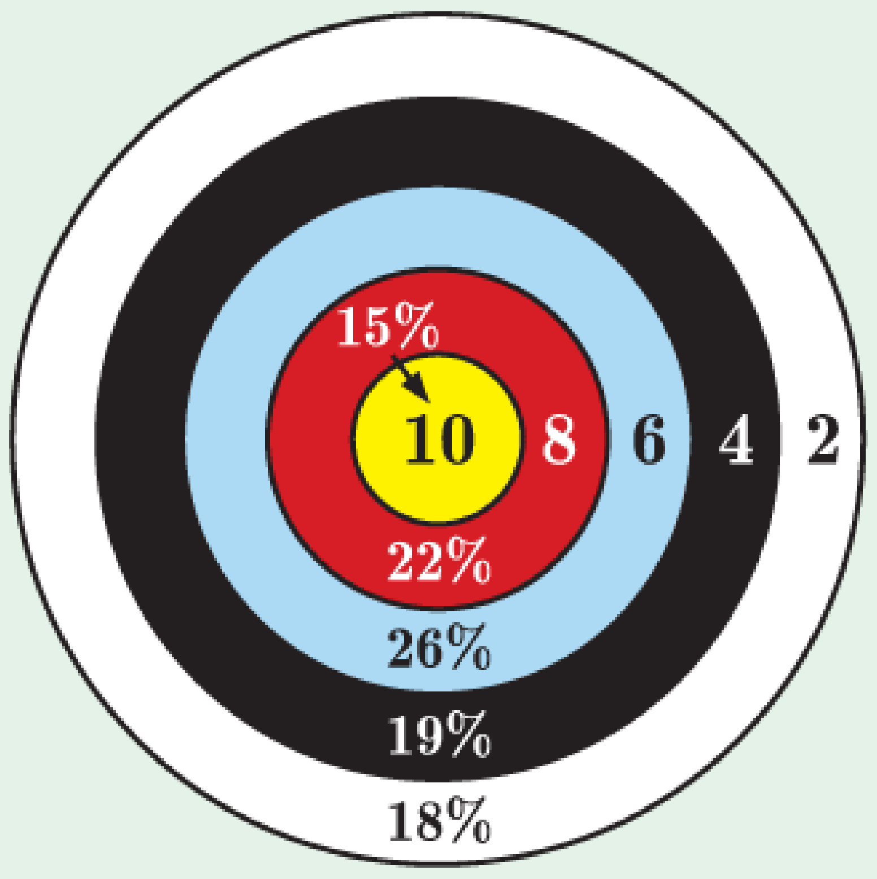


4 The probabilities of Naomi and Rosslyn hitting each section of an archery target is shown alongside.

- **a** On a single shot, who is more likely to score:
  - **i** 10 points
  - **ii** at least 6 points?
- **b** In the long run, who would you expect to score more points per shot?



Naomi



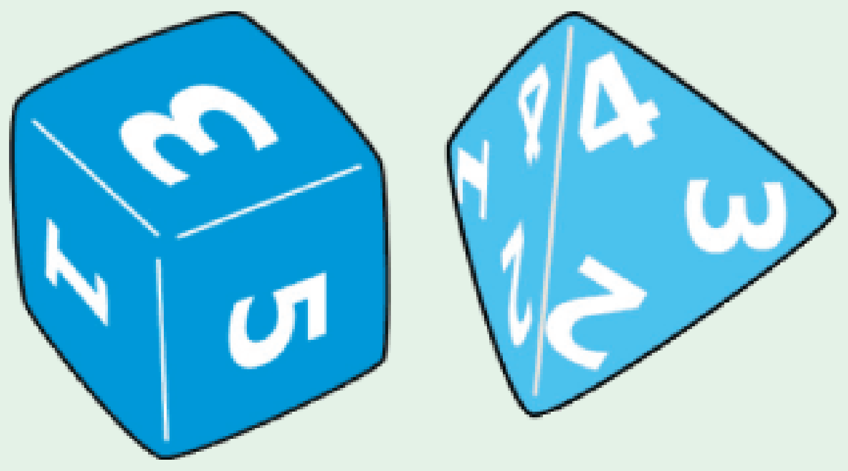
Rosslyn

5 The numbers from 1 to 20 are written on tickets and placed in a bag. A person draws out a number at random. The person wins \$3 if the number is even, \$6 if the number is a square number, and \$9 if the number is both even and square.

- **a** Calculate the probability that the player wins:
  - **i** \$3
  - **ii** \$6
  - **iii** \$9
- **b** How much should be charged to play the game so that it is fair?

6 A 6-sided and 4-sided die are rolled simultaneously. Let X be the number of twos rolled.

- **a** Explain why X is not a binomial random variable.
- **b** Find the probability distribution of X.
- **c** Find the mean of X.



7 Suppose X has the probability distribution alongside. Given that $E(X) = 2.8$, find a and b.

x_i	1	2	3	4
p_i	0.2	a	0.3	b

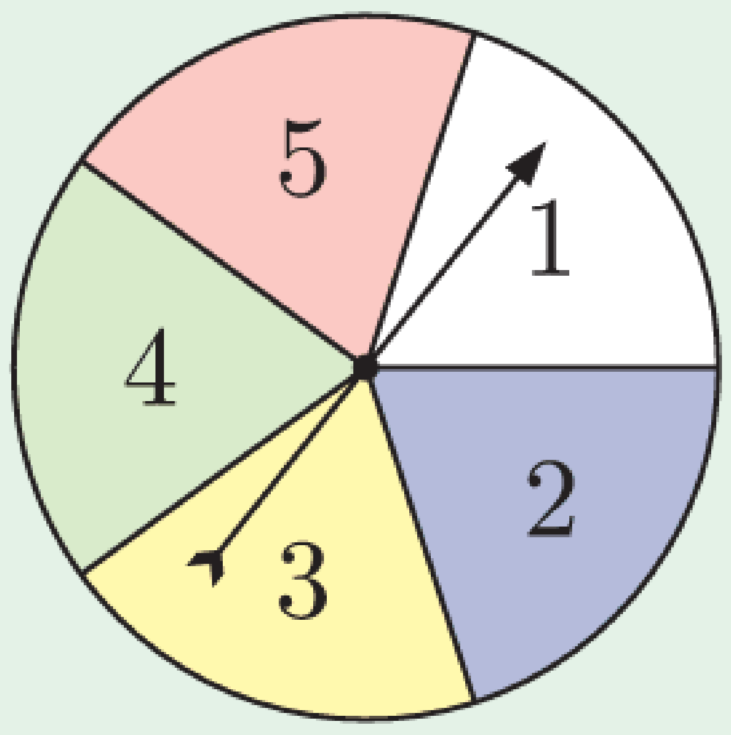
8 Caleb is thrown a baseball 4 times. Let X be the number of times Caleb catches the ball. The probability distribution of X is shown alongside.

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

- **a** Find $E(X)$.
- **b** Let Y be the number of times Caleb *drops* the ball. Find $E(Y)$.

9 The spinner alongside is spun 20 times. Let X be the number of threes spun.

- **a** Explain why X is a binomial random variable.
- **b** Find the mean and standard deviation of X.



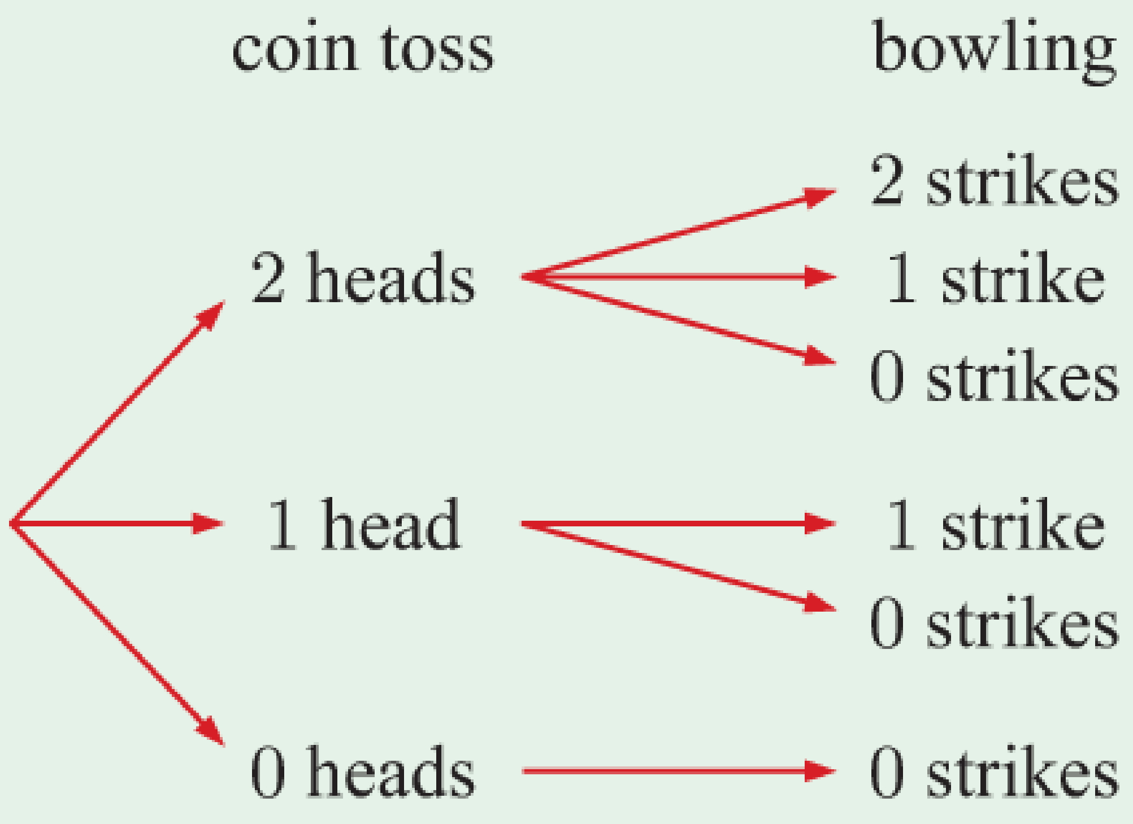
10 24% of visitors to a museum make voluntary donations. On a certain day the museum has 175 visitors.

- **a** Find the expected number of donations.
- **b** Find the probability that:
  - **i** less than 40 visitors make a donation
  - **ii** between 50 and 60 (inclusive) visitors make a donation.

11 Suvi plays a game involving 2 coins and a set of bowling pins. The coins are flipped and the number of heads that result is the number of attempts she gets to knock down the pins. If she knocks all of the pins down on a given attempt, it is called a "strike".

Suvi wins a prize worth $10 multiplied by the number of strikes she gets. On each attempt, the probability that Suvi gets a strike is $\frac{1}{3}$.

- **a** Copy and complete this tree diagram of possible outcomes:



- **b** Let X be the number of strikes that Suvi gets. Find the probability distribution of X.
- **c** Calculate Suvi's expected return per game.
- **d** Find Suvi's expected gain if the game costs $5 to play. Would you advise her to play the game many times?

Chapter 15

The normal distribution

Contents:

A Introduction to the normal distribution
B Calculating probabilities
C Quantiles

OPENING PROBLEM

A salmon breeder is interested in the distribution of the *weight of female adult salmon*, w.

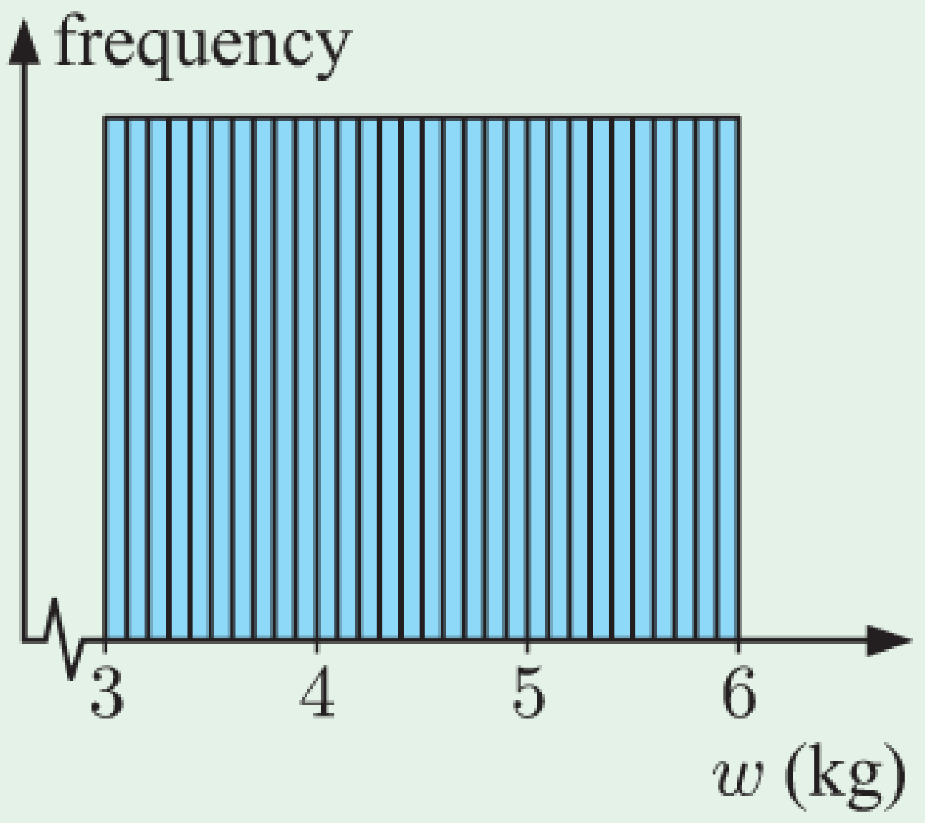
He catches hundreds of female adult fish and records their weights in a frequency table with class intervals $3 \leqslant w < 3.1$ kg, $3.1 \leqslant w < 3.2$ kg, $3.2 \leqslant w < 3.3$ kg, and so on.

The mean weight is 4.73 kg, and the standard deviation is 0.53 kg.

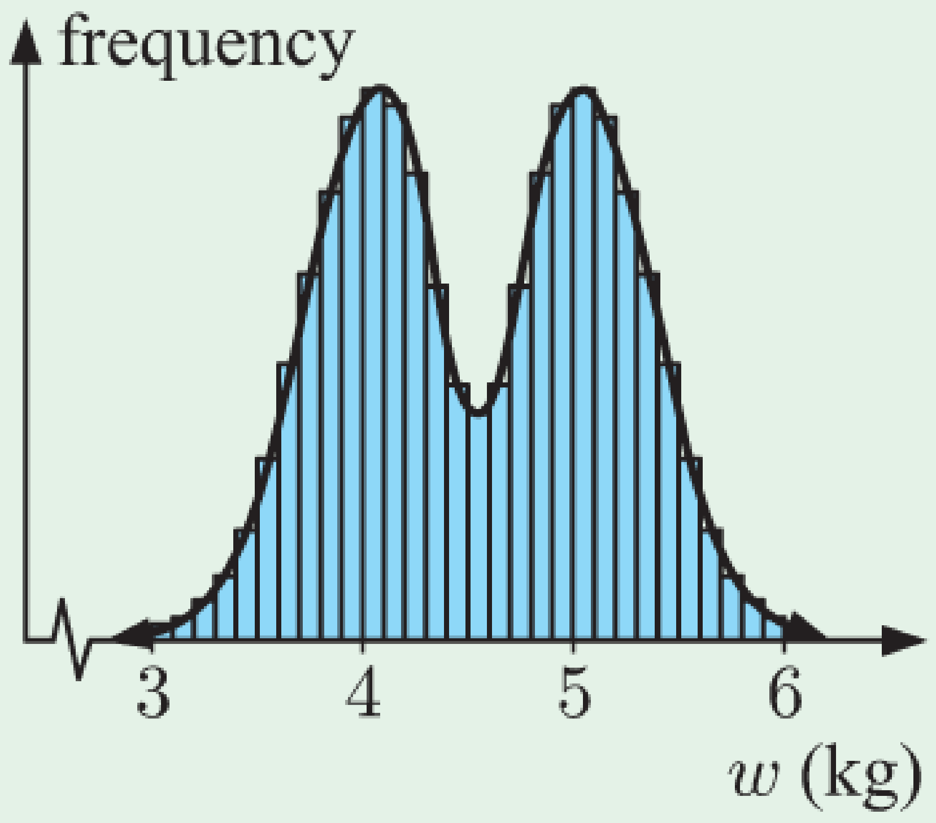
Things to think about:

a Which of these do you think is the most likely distribution for the weights of the female adult salmon?

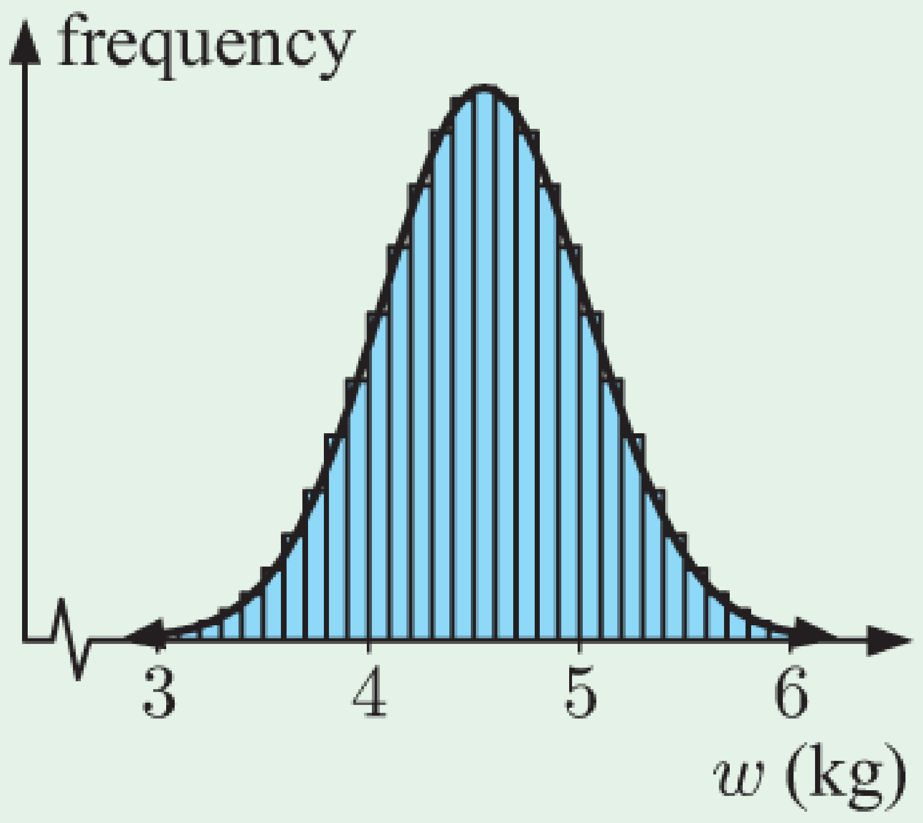
A



B



C



b How can we use the mean and standard deviation to estimate the proportion of salmon that weigh:

  i more than 6 kg
  ii between 4 kg and 6 kg?

c How can we find the weight which:

  i 90% of salmon weigh less than
  ii 25% of salmon weigh more than?

In the previous Chapter we looked at discrete random variables and examined binomial probability distributions where the random variable X could take the non-negative integer values $x = 0, 1, 2, 3, 4, ..., n$.

For a **continuous random variable** X, x can take any real value within some reasonable domain. There are infinitely many values X can take, and even if a measuring device enabled us to measure X exactly, the measurements of X from any two members of the population would never be *identical*. This means that the probability that X is exactly equal to any particular value is zero.

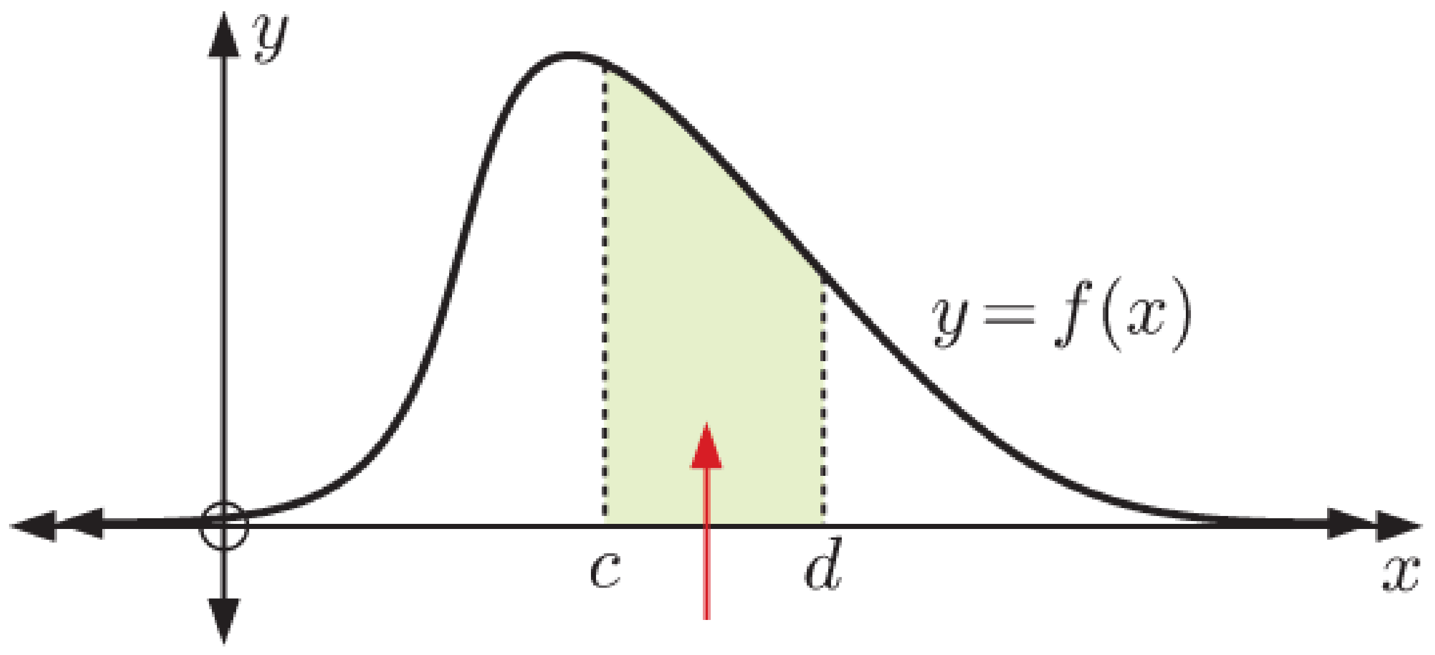
For a continuous variable X, $\text{P}(X = x) = 0$ for all x.

For example, the probability that an egg will weigh *exactly* 72.9 g is zero. If you were to weigh an egg on scales that measure to the nearest 0.1 g, a reading of 72.9 g means the weight lies somewhere between 72.85 g and 72.95 g. No matter how accurate your scales are, you can only ever know the weight of an egg within a range.

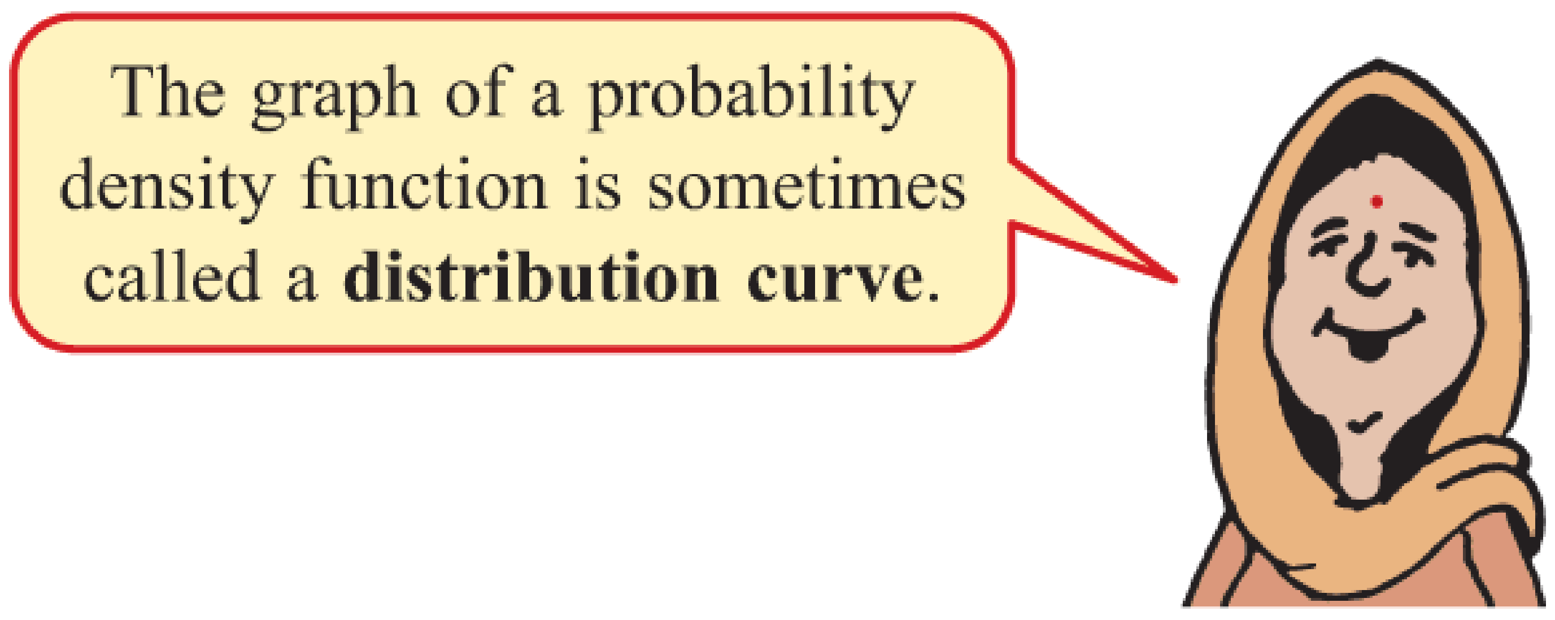
So, for a continuous variable X, we can only talk about the probability that a measured value lies in an **interval**.

Remembering that $\text{P}(X = x) = 0$ for all x,
$\text{P}(c \leqslant X \leqslant d) = \text{P}(c < X \leqslant d) = \text{P}(c \leqslant X < d) = \text{P}(c < X < d)$.

Since $P(X = x) = 0$ for all x, we cannot use a probability mass function to describe the distribution. Instead we use a function called a **probability density function**. The value of the function is not a probability. Rather, probabilities are found by calculating areas under the probability density function curve for a particular interval.



Area $= P(c \leqslant X \leqslant d)$


The graph of a probability density function is sometimes called a **distribution curve**.

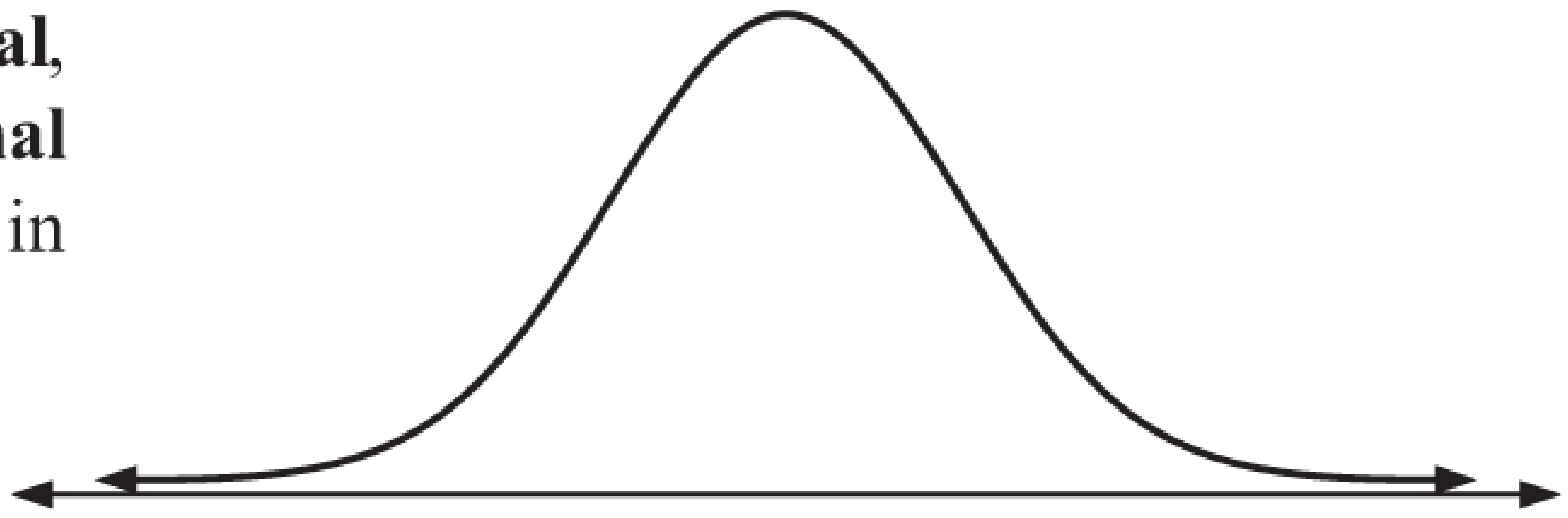
DISCUSSION

Can you explain why:

- the probability density function is always positive
- the *total* area under the curve is 1?

A INTRODUCTION TO THE NORMAL DISTRIBUTION

In this Chapter, we consider variables with **symmetrical**, **bell-shaped** distribution curves. We call this a **normal distribution**. It is the most important distribution in statistics.



The normal distribution arises in nature when many different factors affect the value of the variable.

For example, consider the apples harvested from an apple orchard. They do not all have the same weight. This variation may be due to genetic factors, the soil, the amount of sunlight reaching the leaves and fruit, weather conditions, and so on.

The result is that most of the fruit will have weights centred about the mean weight, and there will be fewer apples that are much heavier or much lighter than this mean.

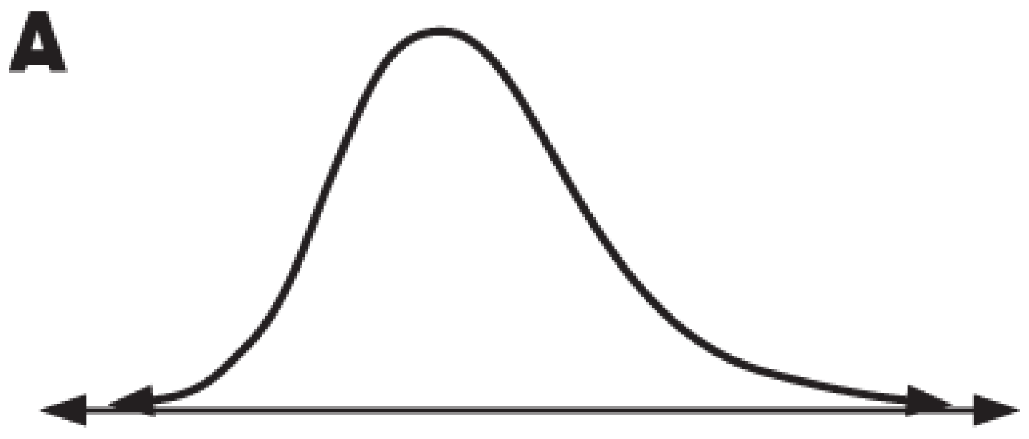
Some examples of quantities that may be normally distributed or approximately normally distributed are:

- the heights of 16 year old boys
- the lengths of adult sharks
- the yields of corn or wheat
- the volumes of liquid in soft drink cans
- the weights of peaches in a harvest
- the life times of batteries

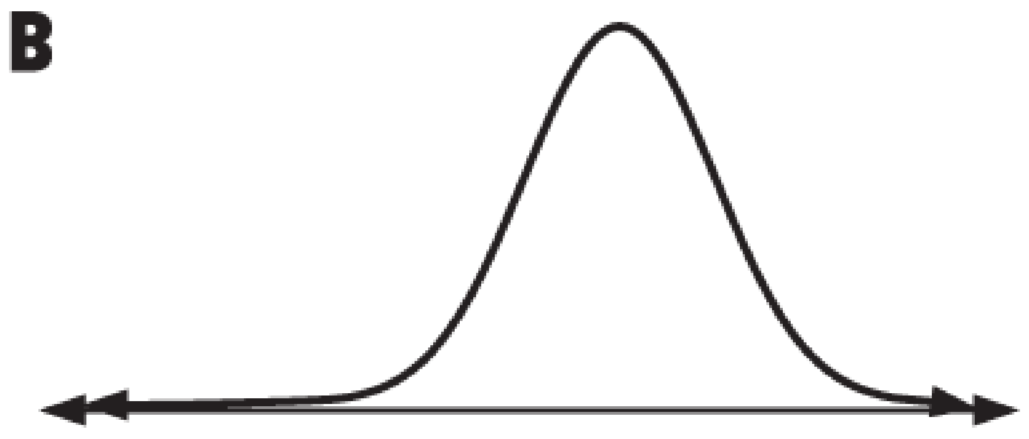
EXERCISE 15A.1

1 Which of the following appear to be normal distribution curves?

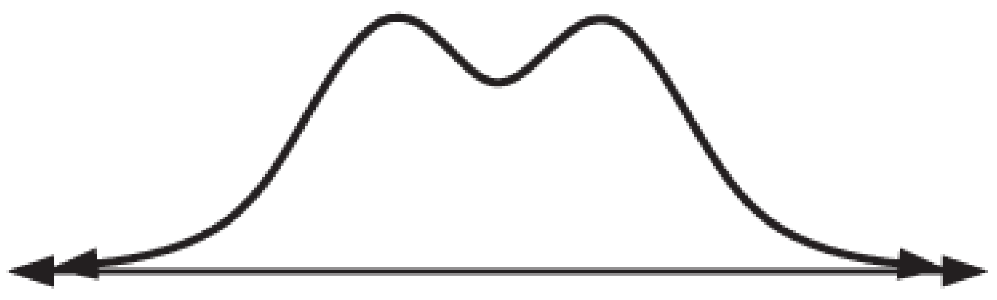
A



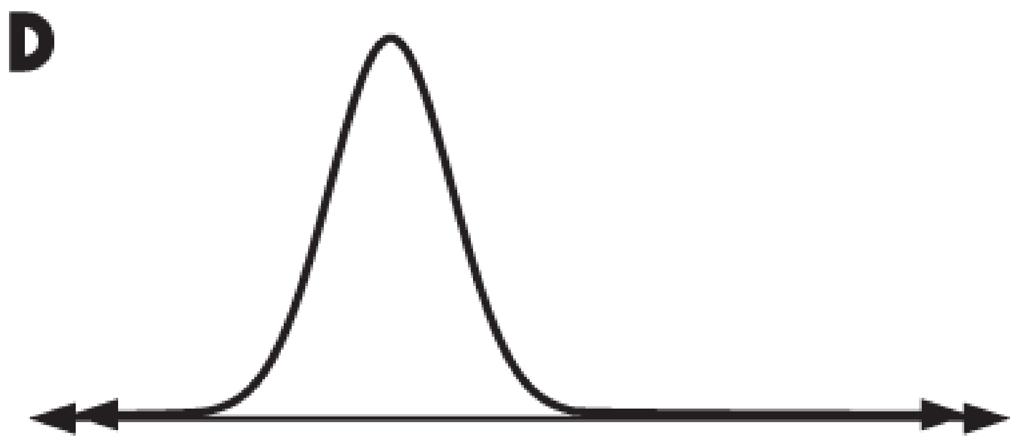
B



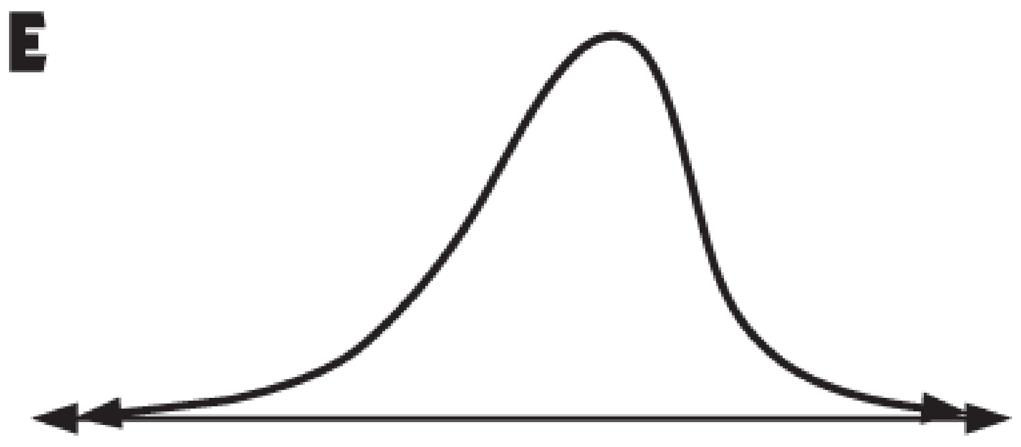
C



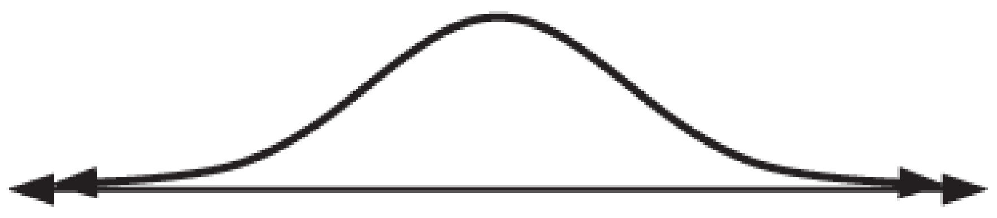
D



E



F



2 Explain why it is likely that the following variables will be normally distributed:

- **a** the diameter of wooden rods cut using a lathe
- **b** scores for tests taken by a large population
- **c** the amount of time a student takes to walk to school each day.

3 Discuss whether the following variables are likely to be normally distributed. Sketch a graph to illustrate the possible distribution of each variable.

- **a** the ages of people at a football match
- **b** the distances recorded by a long jumper
- **c** the numbers drawn in a lottery
- **d** the lengths of carrots in a supermarket
- **e** the amounts of time passengers spend waiting in a queue at an airport
- **f** the numbers of brown eggs in a sample of cartons which each contain a dozen eggs
- **g** the numbers of children in families living in Cardiff, Wales
- **h** the heights of buildings in a city.

THE NORMAL DISTRIBUTION CURVE

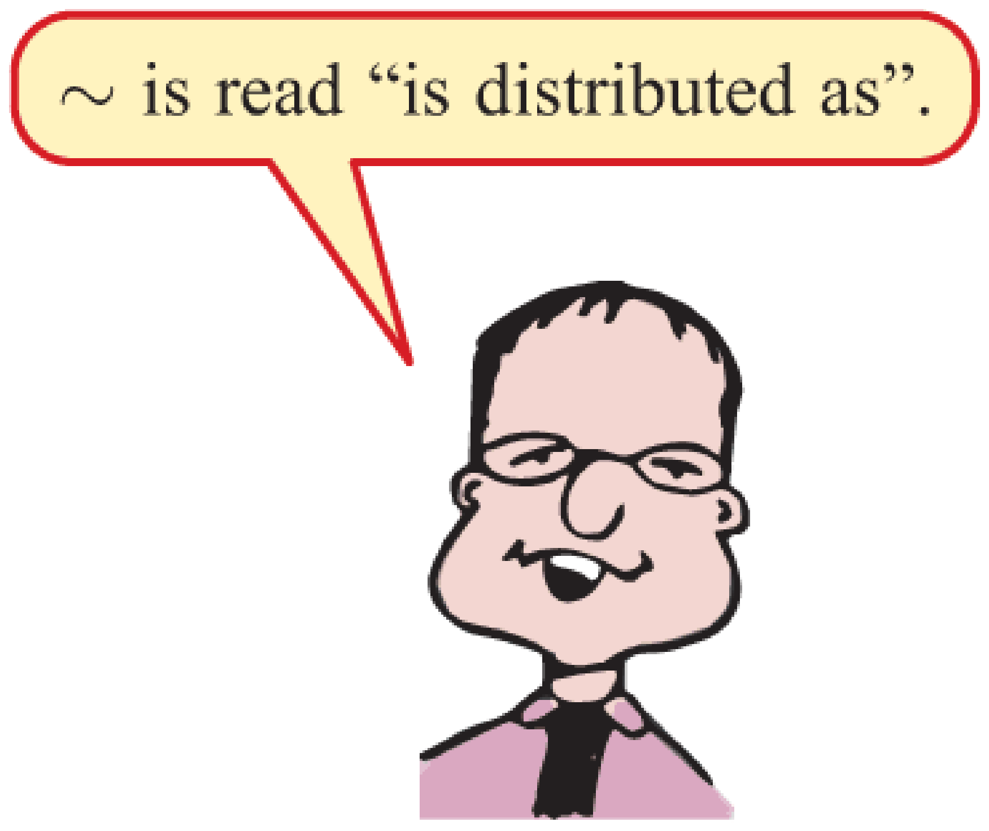
Although all normal distributions have the same general bell-shaped curve, the exact location and shape of the curve is determined by:

- the **mean** μ which measures the **centre** of the distribution
- the **standard deviation** σ which measures the **spread** of the distribution.

If X is a normally distributed random variable with mean μ and standard deviation σ, we write $X \sim \mathrm{N}(\mu, \sigma^2)$.

We say that μ and σ are the **parameters** of the distribution.

The probability density function of X is called the **normal distribution curve** or **normal curve**.



$\sim$ is read "is distributed as".

INVESTIGATION 1 PROPERTIES OF THE NORMAL CURVE

In this Investigation, we will look at some interesting properties of the normal distribution curve with the help of **graphing software**.

Click on the icon to explore the normal distribution curve and how it changes when μ and σ are altered.

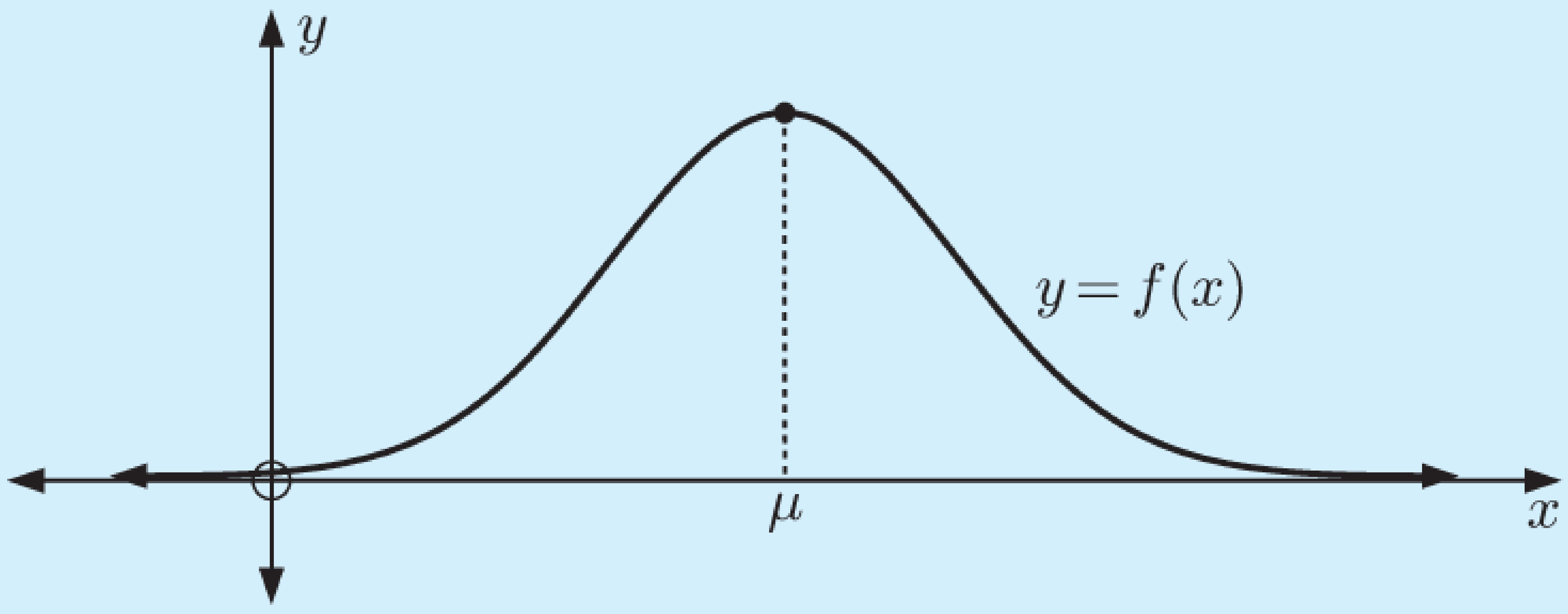
DEMO

What to do:

1. What effects do variations in μ and σ have on the curve? How do these relate to what μ and σ represent?
2. Does the curve have a line of symmetry? If so, what is it?
3. Is the function ever negative? Why is this important?
4. Discuss the behaviour of the normal curve as $x \to \pm\infty$.
5. What do you think happens to the *area* under the curve as you change μ and σ?

From the **Investigation**, you should have found that:

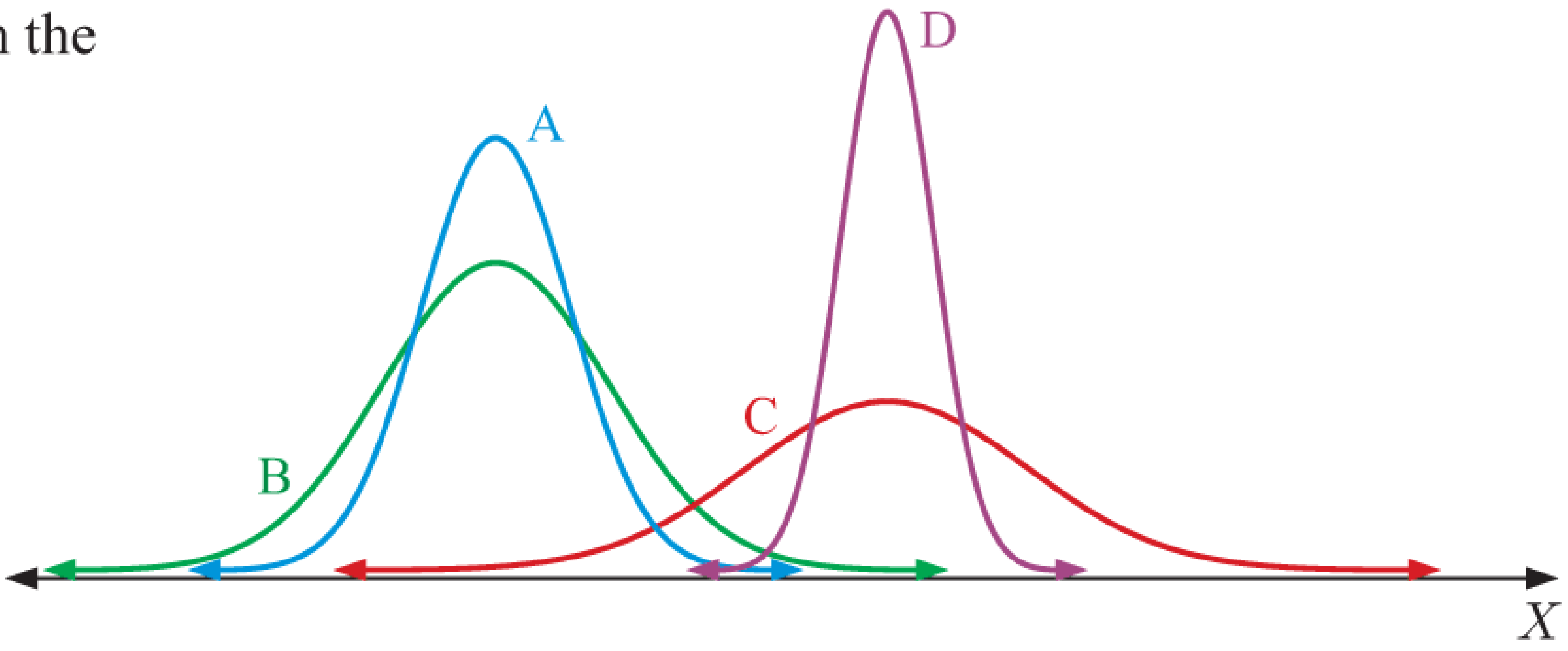
- The normal curve is symmetrical about the vertical line $x = \mu$.
- $f(x) > 0$ for all x.
- The x-axis is a horizontal asymptote.



EXERCISE 15A.2

1 Match each pair of parameters with the correct normal distribution curve:

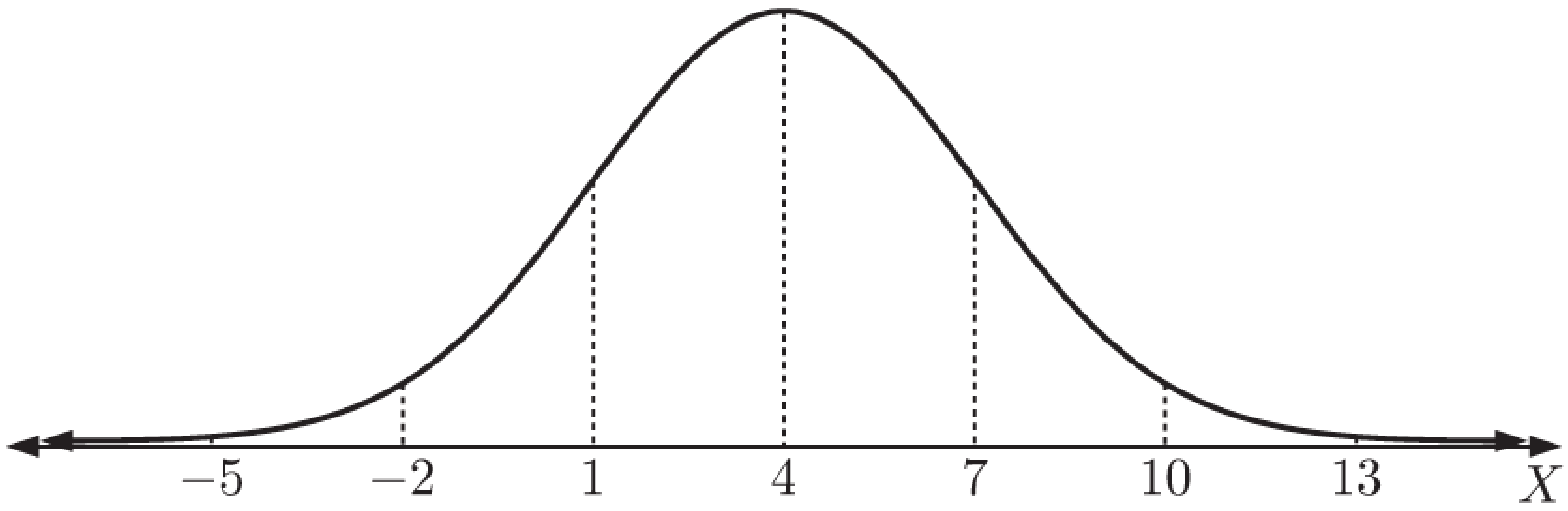
- **a** $\mu = 5,\ \sigma = 2$
- **b** $\mu = 15,\ \sigma = 0.5$
- **c** $\mu = 5,\ \sigma = 1$
- **d** $\mu = 15,\ \sigma = 3$



2 Sketch the following normal distributions on the same set of axes.

Distribution	*Mean* (mL)	*Standard deviation* (mL)
A	25	5
B	30	2
C	21	10

3 Consider the distribution curve of $X \sim \text{N}(\mu, \sigma^2)$ where $\mu = 4$ and $\sigma = 3$ shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

PRINTABLE GRAPH

a $\text{N}(\mu + 2, \sigma^2)$ **b** $\text{N}(\mu, (2\sigma)^2)$ **c** $\text{N}(\mu + 2, (2\sigma)^2)$

d $\text{N}\left(\mu - 1, \left(\frac{\sigma}{3}\right)^2\right)$ **e** $\text{N}\left(3 + \mu, \frac{\sigma^2}{4}\right)$

HISTORICAL NOTE

The normal distribution was first characterised by **Carl Friedrich Gauss** in 1809 as a way to rationalise his **method of least squares** for linear regression. In fact, the normal distribution curve is a special case of the **Gaussian function**.

Since the normal distribution has such strong ties to Gauss, it is sometimes called the **Gaussian distribution**.

Carl Friedrich Gauss

B CALCULATING PROBABILITIES

INVESTIGATION 2 PROPORTIONS FROM A NORMAL DISTRIBUTION

In this Investigation we find the proportions of normally distributed data which lie within σ, 2σ, and 3σ of the mean.

What to do:

1 Click on the icon to run a demonstration which randomly generates 1000 data values from a normal distribution with mean μ and standard deviation σ. Set $\mu = 0$ and $\sigma = 1$.

DEMO

2 Find the endpoints of the interval:

a $\mu - \sigma$ to $\mu + \sigma$ **b** $\mu - 2\sigma$ to $\mu + 2\sigma$ **c** $\mu - 3\sigma$ to $\mu + 3\sigma$

3 Use the frequency table provided to find the proportion of data values which lie between:

a $\mu - \sigma$ and $\mu + \sigma$ **b** $\mu - 2\sigma$ and $\mu + 2\sigma$ **c** $\mu - 3\sigma$ and $\mu + 3\sigma$

4 Repeat **2** and **3** for values of μ and σ of your choosing. Summarise your answers in a table like the one below.

μ	σ	$\mu-\sigma$ to $\mu+\sigma$		$\mu-2\sigma$ to $\mu+2\sigma$		$\mu-3\sigma$ to $\mu+3\sigma$	
		Interval	*Proportion*	*Interval*	*Proportion*	*Interval*	*Proportion*
0	1	$-1 \leqslant X \leqslant 1$					

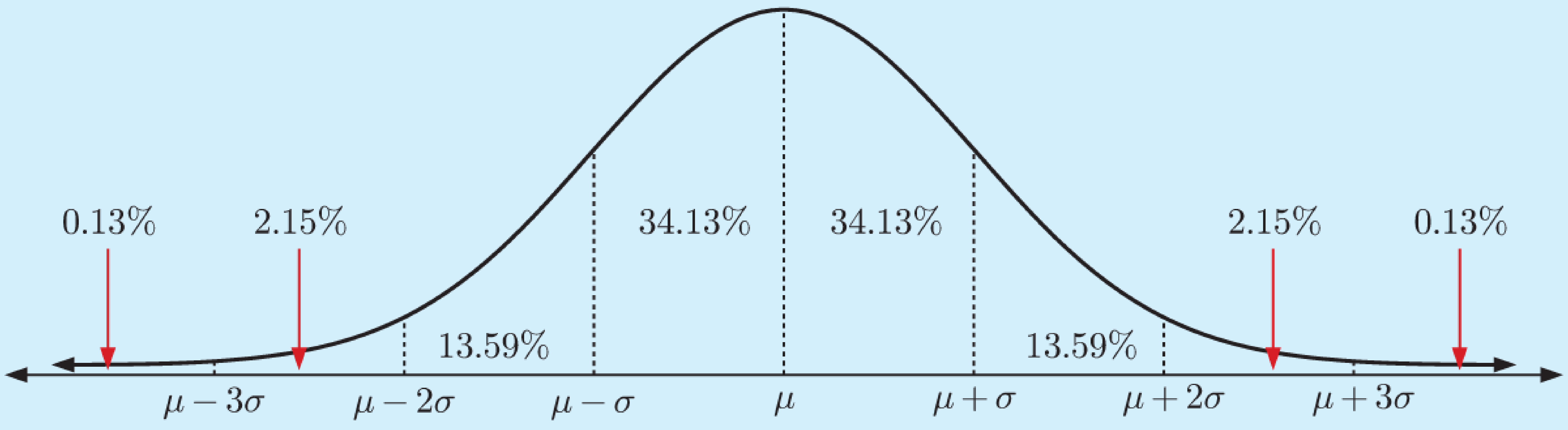
5 What do you notice about the proportion of data values in each interval?

From the **Investigation**, you should have found that:

For any population that is normally distributed with mean μ and standard deviation σ:

- approximately **0.68** or **68%** of the population will lie between $\boldsymbol{\mu-\sigma}$ and $\boldsymbol{\mu+\sigma}$
- approximately **0.95** or **95%** of the population will lie between $\boldsymbol{\mu-2\sigma}$ and $\boldsymbol{\mu+2\sigma}$
- approximately **0.997** or **99.7%** of the population will lie between $\boldsymbol{\mu-3\sigma}$ and $\boldsymbol{\mu+3\sigma}$.

The proportion of data values that lie within different ranges relative to the mean are:



For any variable that is normally distributed, we can use the mean and standard deviation to estimate the proportion of data that will lie in a given interval. This proportion tells us the probability that a randomly selected member of the population will be in that interval.

Example 1 **Self Tutor**

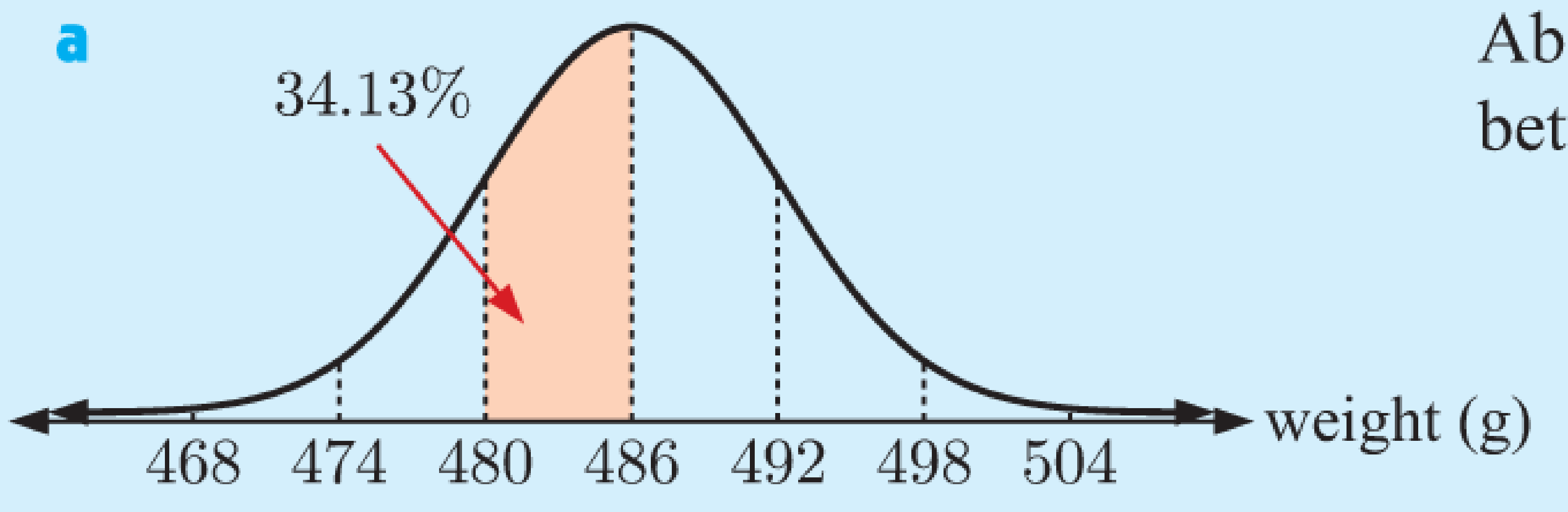
A sample of cans of peaches was taken from a warehouse, and the contents of each can was weighed. The sample mean was 486 g with standard deviation 6 g.

State the proportion of cans that weigh:

a between 480 g and 486 g **b** more than 492 g.

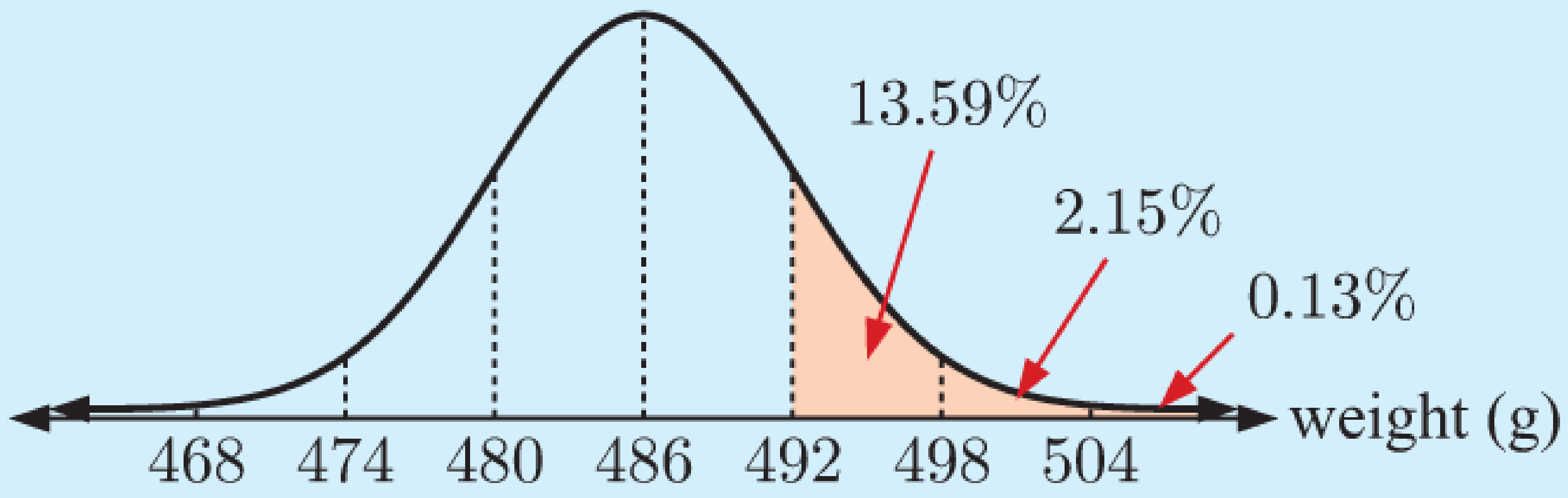
For a manufacturing process such as this, the distribution of weights is approximately normal.

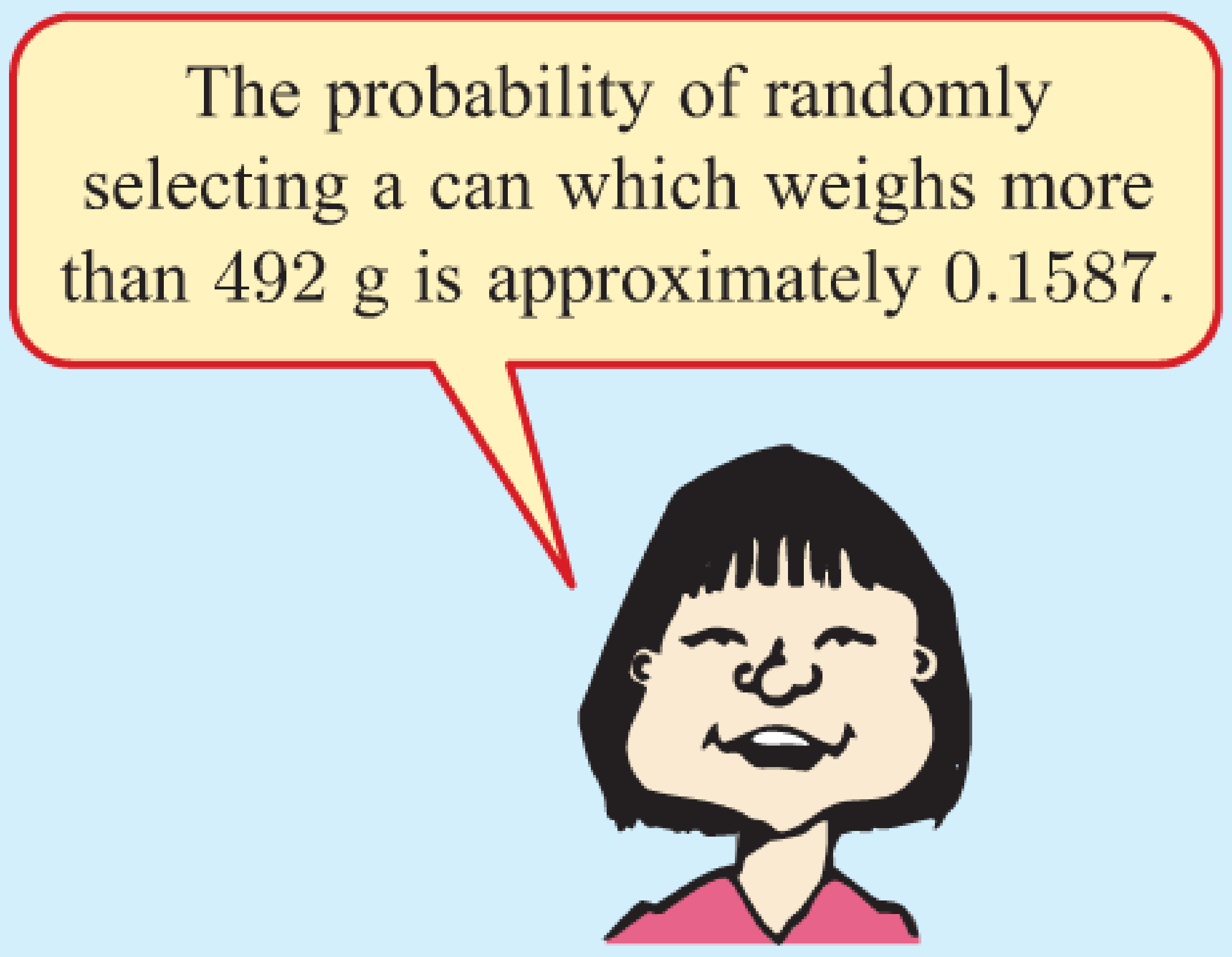
a



About 34.13% of the cans are expected to weigh between 480 g and 486 g.

b




The probability of randomly selecting a can which weighs more than 492 g is approximately 0.1587.

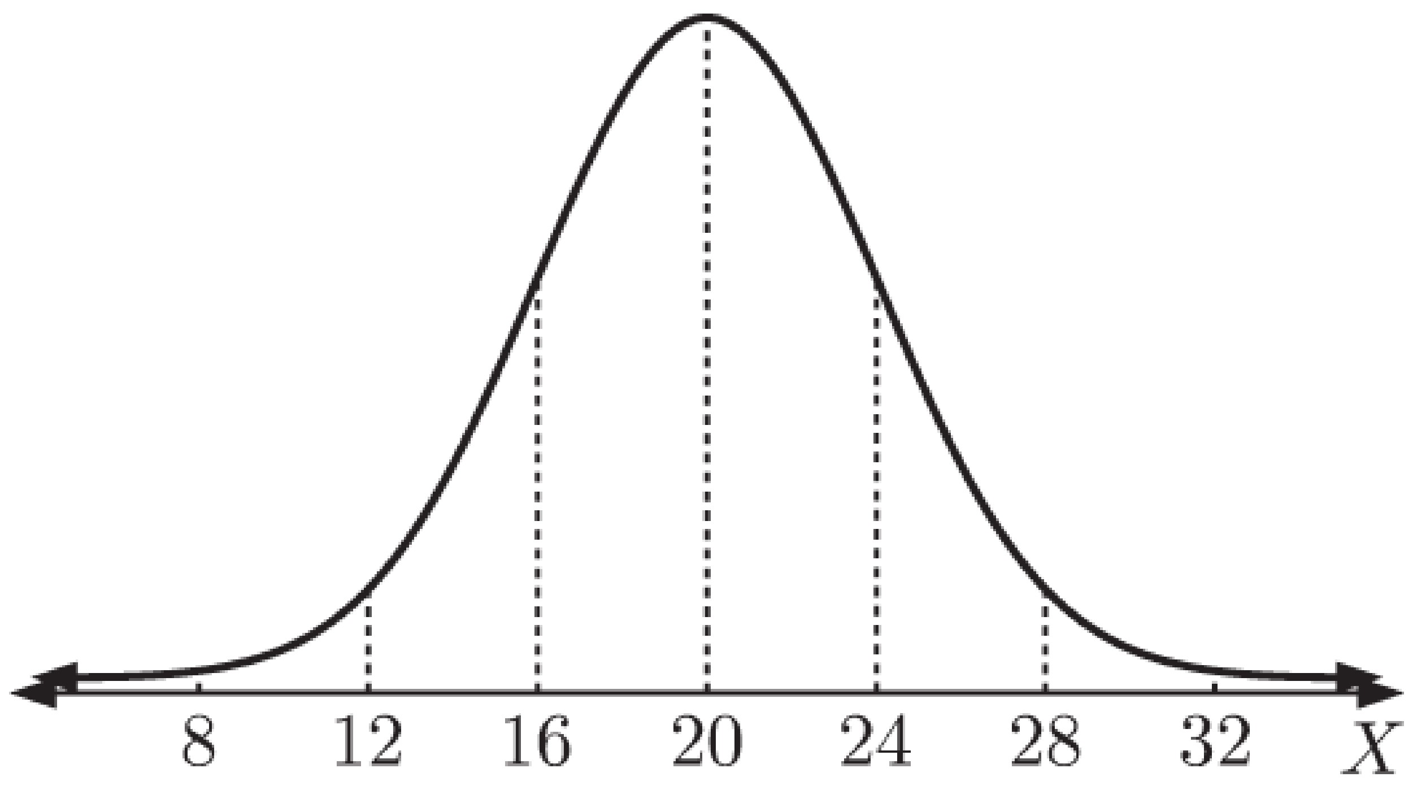
About $13.59\% + 2.15\% + 0.13\% = 15.87\%$ of the cans are expected to weigh more than 492 g.

EXERCISE 15B.1

1 Suppose X is normally distributed with mean 30 and standard deviation 5.

- **a** State the value which is:
  - **i** 2 standard deviations above the mean
  - **ii** 1 standard deviation below the mean.
- **b** Describe the following values in terms of the number of standard deviations above or below the mean:
  - **i** 35
  - **ii** 20
  - **iii** 45
- **c** Draw a curve to illustrate the distribution of X.
- **d** What proportion of values of X are between 25 and 30?
- **e** Find the probability that a randomly selected member of the population will measure between 35 and 40.

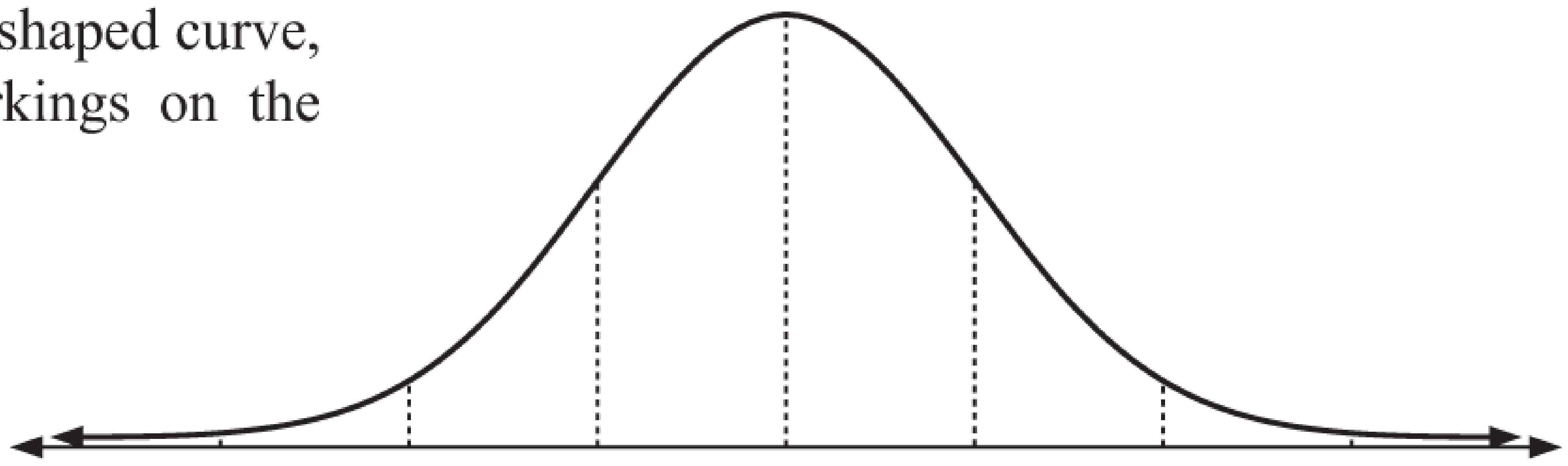
2 Suppose the variable X is normally distributed according to the curve shown.



- **a** State the mean and standard deviation of X.
- **b** Find the proportion of values of X which are:
  - **i** between 20 and 24
  - **ii** between 12 and 16
  - **iii** greater than 28.

3 A school's Grade 12 students sat for a Mathematics examination. Their marks were approximately normally distributed with mean 75 and standard deviation 8.

- **a** Copy and complete this bell-shaped curve, assigning scores to the markings on the horizontal axis.



- **b** What proportion of students would you expect to have scored:
  - **i** more than 83
  - **ii** less than 59
  - **iii** between 67 and 91?

4 State the probability that a randomly selected, normally distributed value:

- **a** lies within one standard deviation either side of the mean
- **b** is more than two standard deviations above the mean.

Example 2

Self Tutor

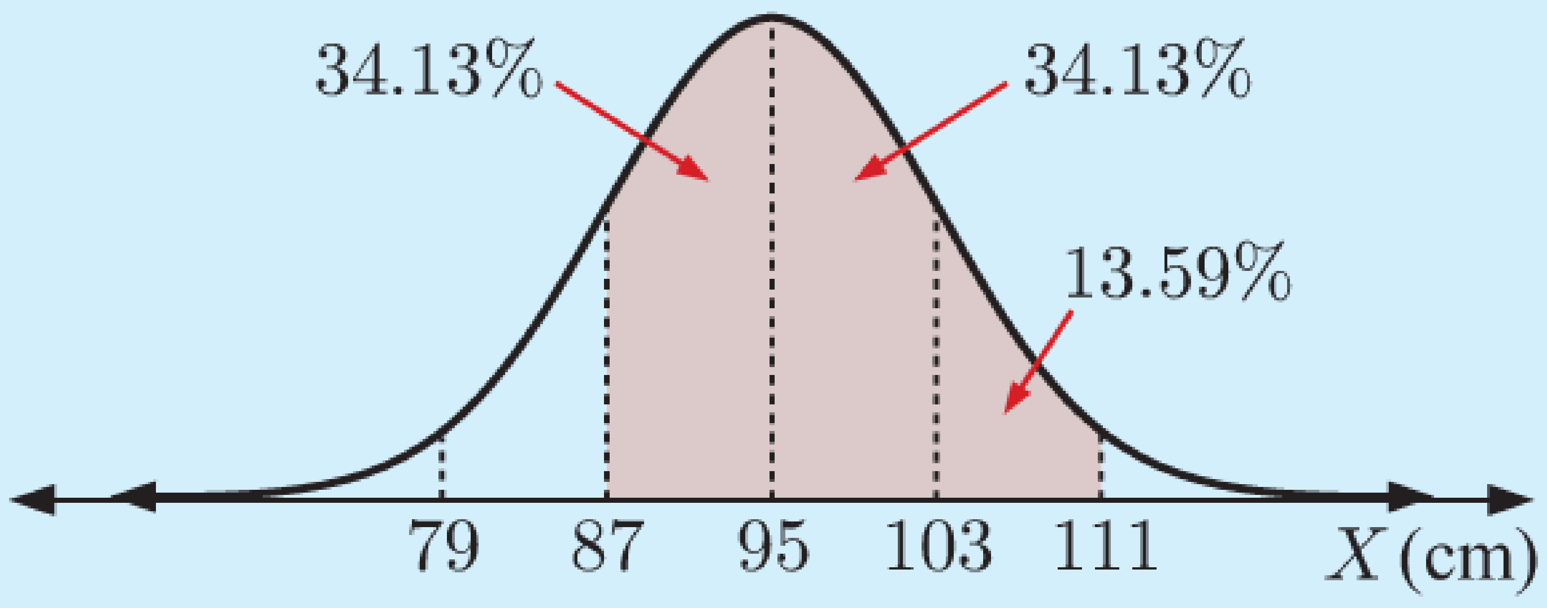
The chest measurements of 18 year old male rugby players are normally distributed with mean 95 cm and standard deviation 8 cm.

a From a group of 200 18 year old male rugby players, how many would you expect to have a chest measurement between 87 cm and 111 cm?

b Find the value of k such that approximately 16% of chest measurements are below k cm.

a About $34.13\% + 34.13\% + 13.59\% = 81.85\%$ of the rugby players have a chest measurement between 87 cm and 111 cm.

So, we would expect 81.85% of $200 \approx 164$ of the rugby players to have a chest measurement between 87 cm and 111 cm.

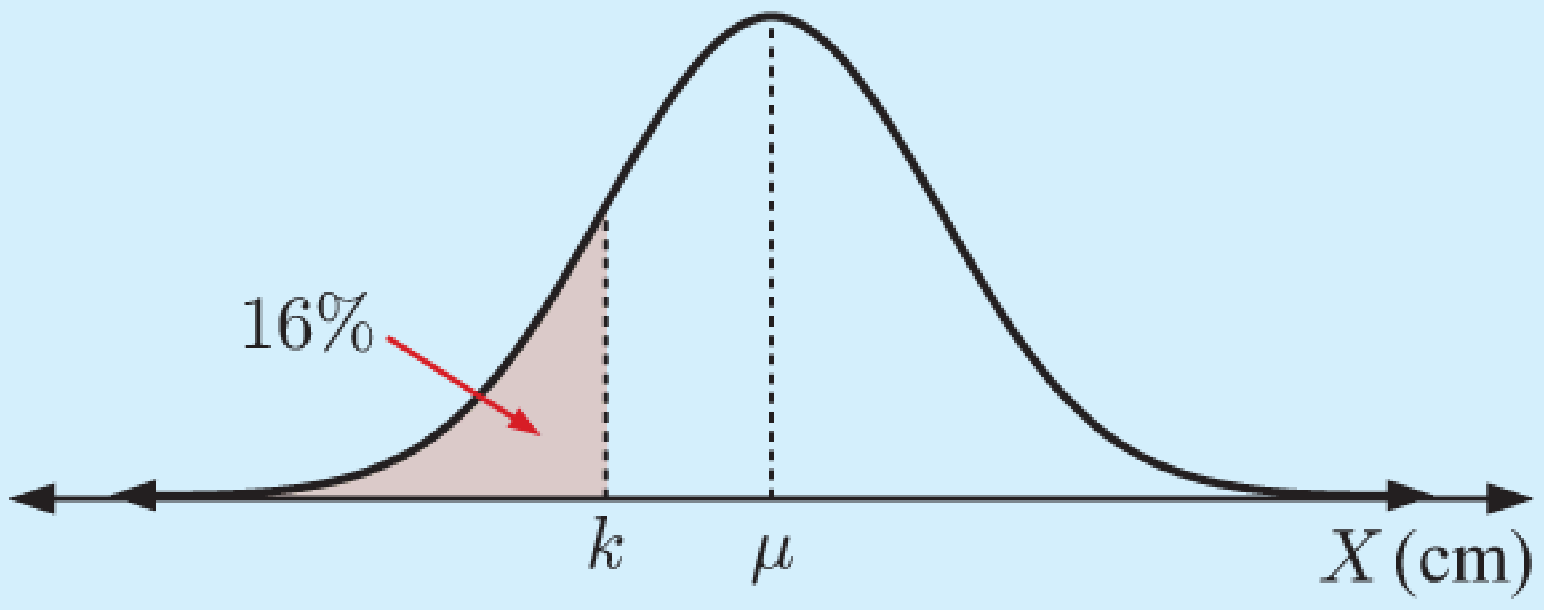


b Approximately 16% of data lies more than one standard deviation below the mean.

$\therefore$ k is σ below the mean μ

$\therefore$ $k = 95 - 8$

$= 87$



5 The height of female students at a university is normally distributed with mean 170 cm and standard deviation 8 cm.

a Find the percentage of female students whose height is:

i between 162 cm and 170 cm **ii** between 170 cm and 186 cm.

b Find the probability that a randomly chosen female student has a height:

i less than 154 cm **ii** greater than 162 cm.

c From a group of 500 female university students, how many would you expect to be between 178 cm and 186 cm tall?

d Estimate the value of k such that 16% of the female students are taller than k cm.

6 The lengths of adult female frilled sharks are normally distributed with mean 1.4 m and standard deviation 15 cm. Find the proportion of adult female frilled sharks that measure:

a more than 1.25 m **b** between 1.1 m and 1.55 m.

7 The weights of the 545 babies born at a maternity hospital last year were normally distributed with mean 3.0 kg and standard deviation 200 grams. Estimate the number that weighed:

a less than 3.2 kg **b** between 2.8 kg and 3.4 kg.

8 An industrial machine fills an average of 20 000 bottles each day with standard deviation 2000 bottles. Assuming that production is normally distributed and the year comprises 260 working days, estimate the number of working days on which:

a under 18 000 bottles are filled **b** over 16 000 bottles are filled

c between 18 000 and 24 000 bottles are filled.

9 Two hundred lifesavers competed in a swimming race. Their times were normally distributed with mean 10 minutes 30 seconds and standard deviation 15 seconds. Estimate the number of competitors who completed the race in a time:

a longer than 11 minutes **b** less than 10 minutes 15 seconds

c between 10 minutes 15 seconds and 10 minutes 45 seconds.

10 The weights of Jason's oranges are normally distributed. 84% of the crop weighs more than 152 grams and 16% weighs more than 200 grams.

a Find μ and σ for the crop.

b What percentage of the oranges weigh between 152 grams and 224 grams?

11 When a particular variety of radish is grown without fertiliser, the weights of the radishes produced are normally distributed with mean 40 g and standard deviation 10 g.
When these radishes are grown in the same conditions but with fertiliser added, their weights are also normally distributed, but with mean 140 g and standard deviation 40 g.

a Determine the proportion of radishes grown:

- **i** without fertiliser which weigh less than 50 grams
- **ii** with fertiliser which weigh less than 60 grams.

b Find the probability that a randomly selected radish weighs between 20 g and 60 g, if it is grown:

- **i** with fertiliser **ii** without fertiliser.

c One radish grown with fertiliser and one radish grown without fertiliser are selected at random. Find the probability that *both* radishes weigh more than 60 g.

USING TECHNOLOGY

When calculating normal distribution probabilities, we have so far only considered numbers that are a whole number of standard deviations from the mean.

To calculate other probabilities, we could use definite integrals of the normal probability density function. However, this function does not have an indefinite integral, so we need to use a numerical approximation. Your **graphics calculator** has built-in functions to calculate these integrals.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 3 Self Tutor

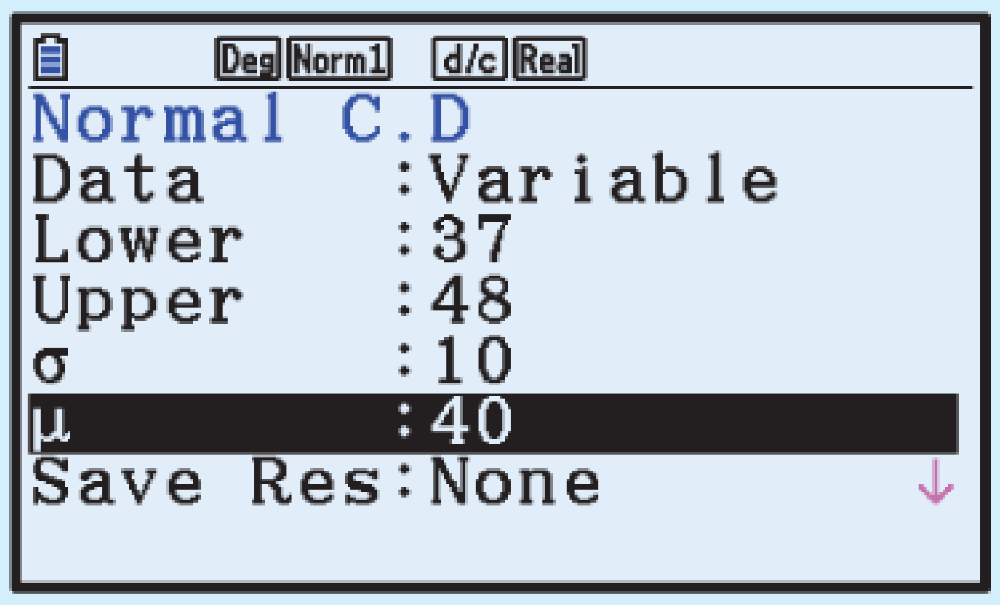
The variable X is normally distributed with mean 40 and standard deviation 10. Find:

a $P(37 < X < 48)$ **b** $P(X > 45)$ **c** $P(X < 26)$

Illustrate your answers.

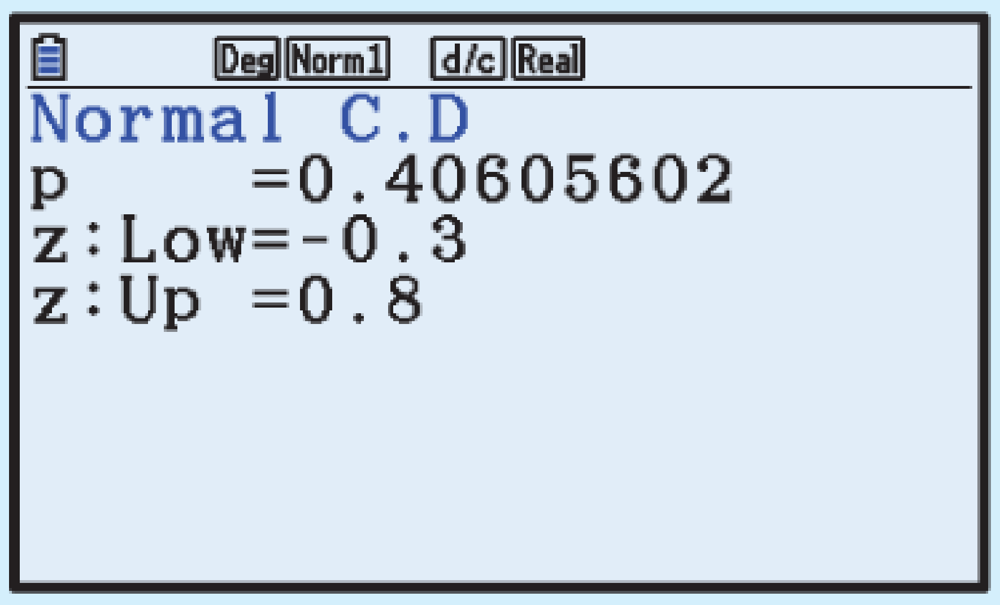
a To find $P(37 < X < 48)$, we set the lower bound to 37 and the upper bound to 48.

Casio fx-CG50

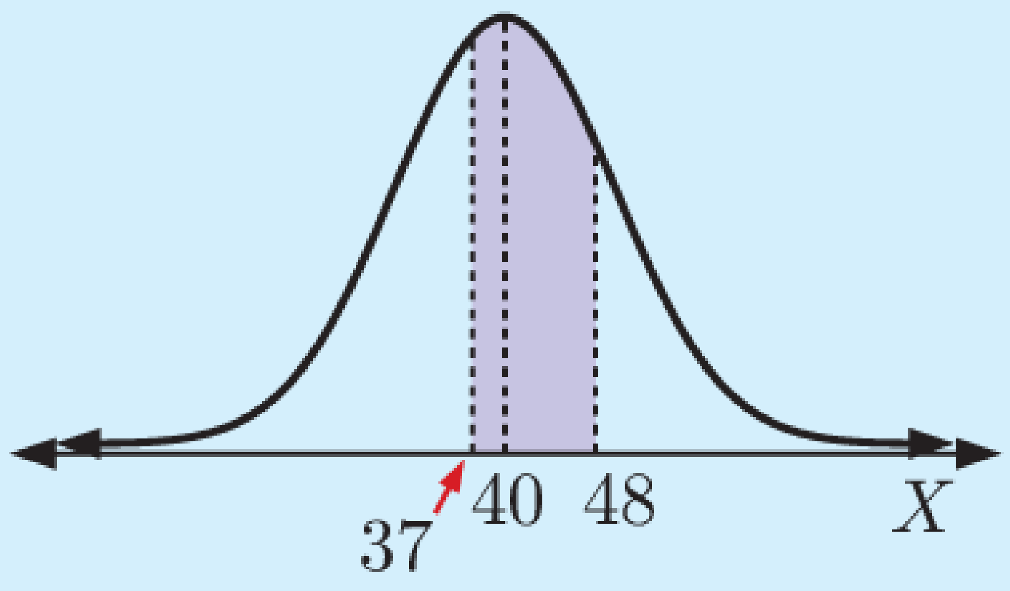


```
Normal C.D
Data    :Variable
Lower   :37
Upper   :48
σ       :10
μ       :40
Save Res:None
```

Casio fx-CG50



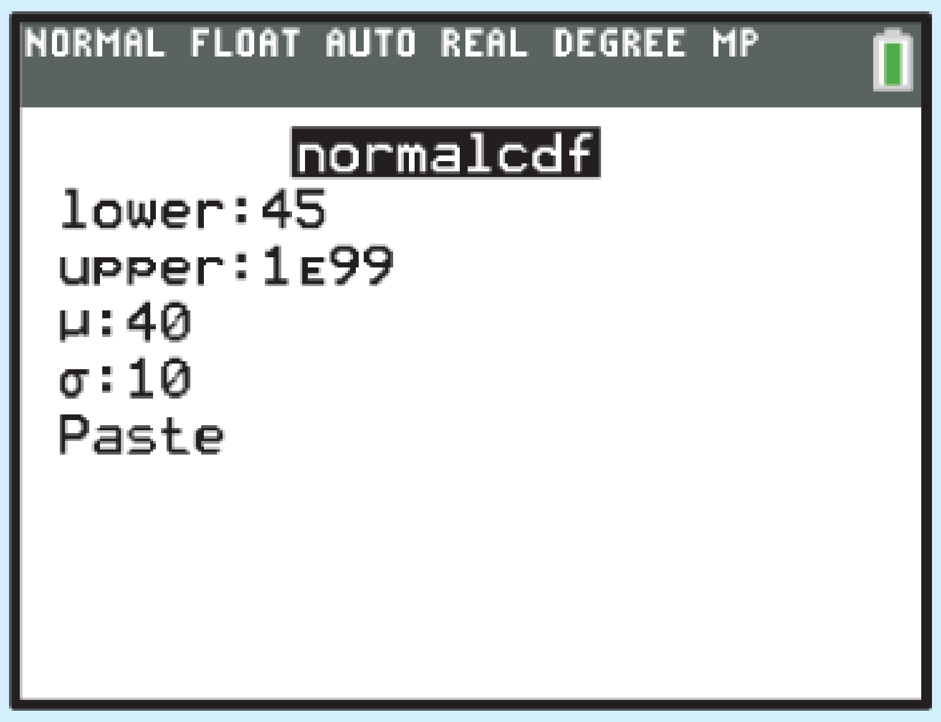
```
Normal C.D
p     =0.40605602
z:Low=-0.3
z:Up  =0.8
```



$P(37 < X < 48) \approx 0.406$

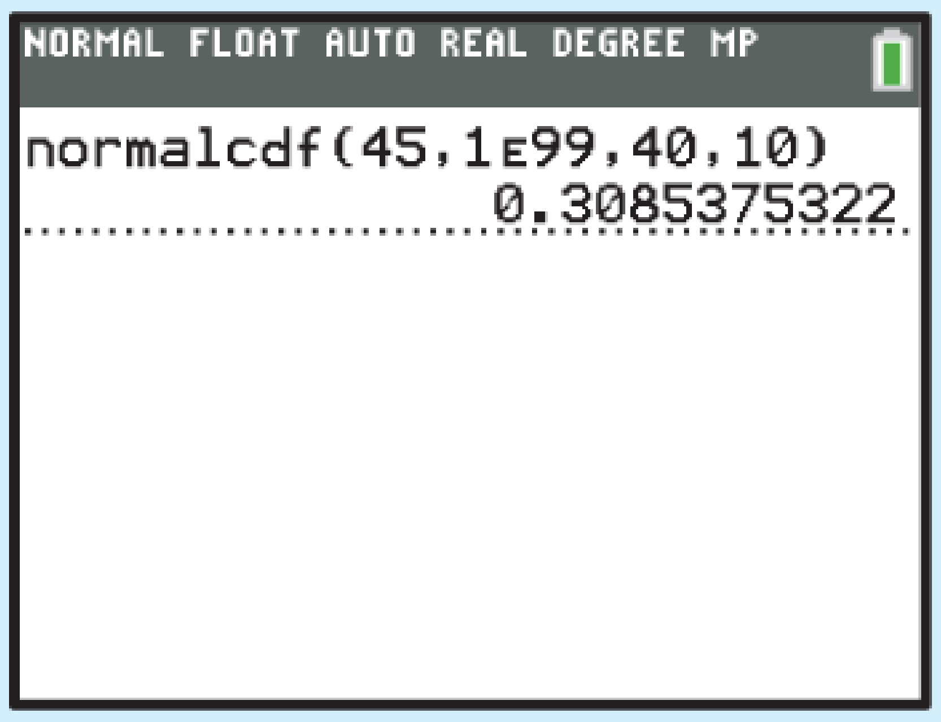
b To find $P(X > 45)$, we use a very high value such as 10^{99} to represent the upper bound.

TI-84 Plus CE

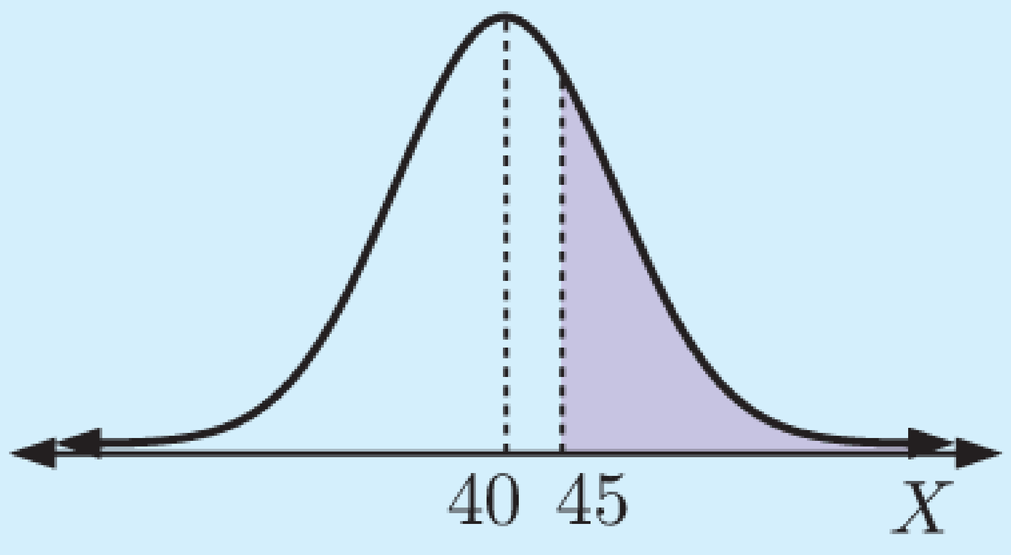


```
normalcdf
lower:45
upper:1ᴇ99
μ:40
σ:10
Paste
```

TI-84 Plus CE



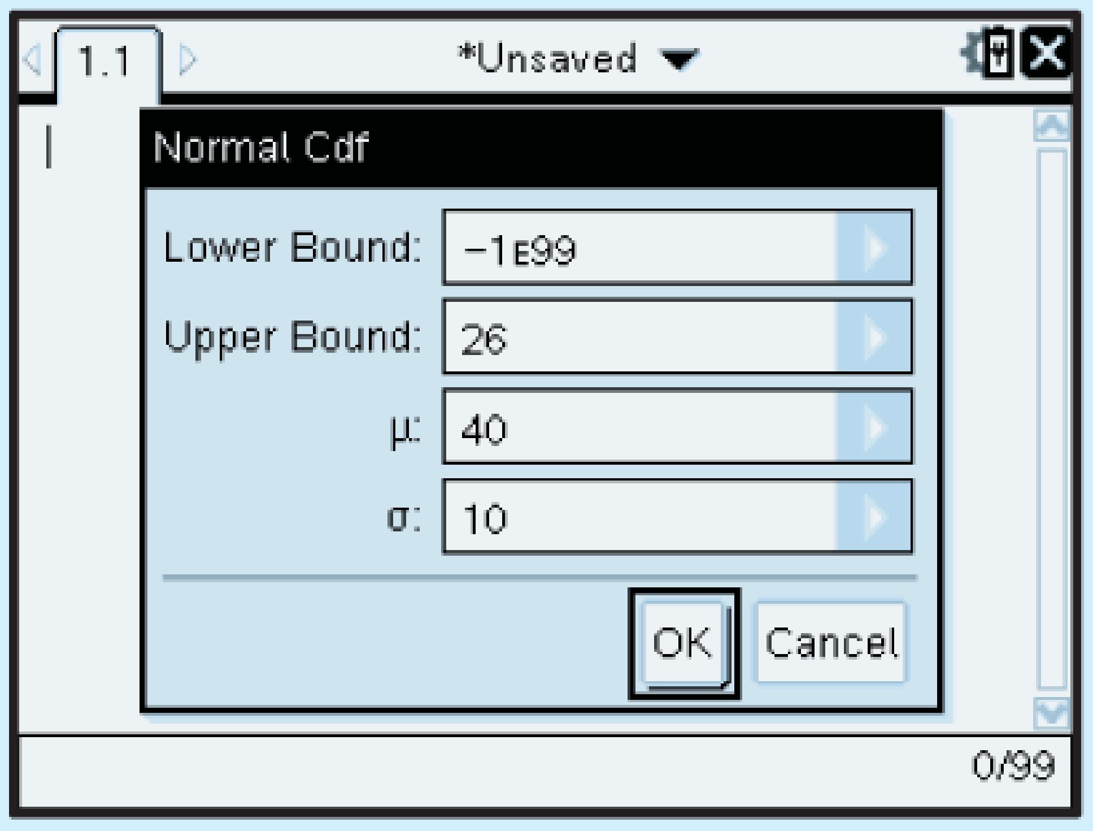
```
normalcdf(45,1ᴇ99,40,10)
                0.3085375322
```



$P(X > 45) \approx 0.309$

c To find $P(X < 26)$, we use a very low value such as -10^{99} to represent the lower bound.

TI-*n*spire



Normal Cdf

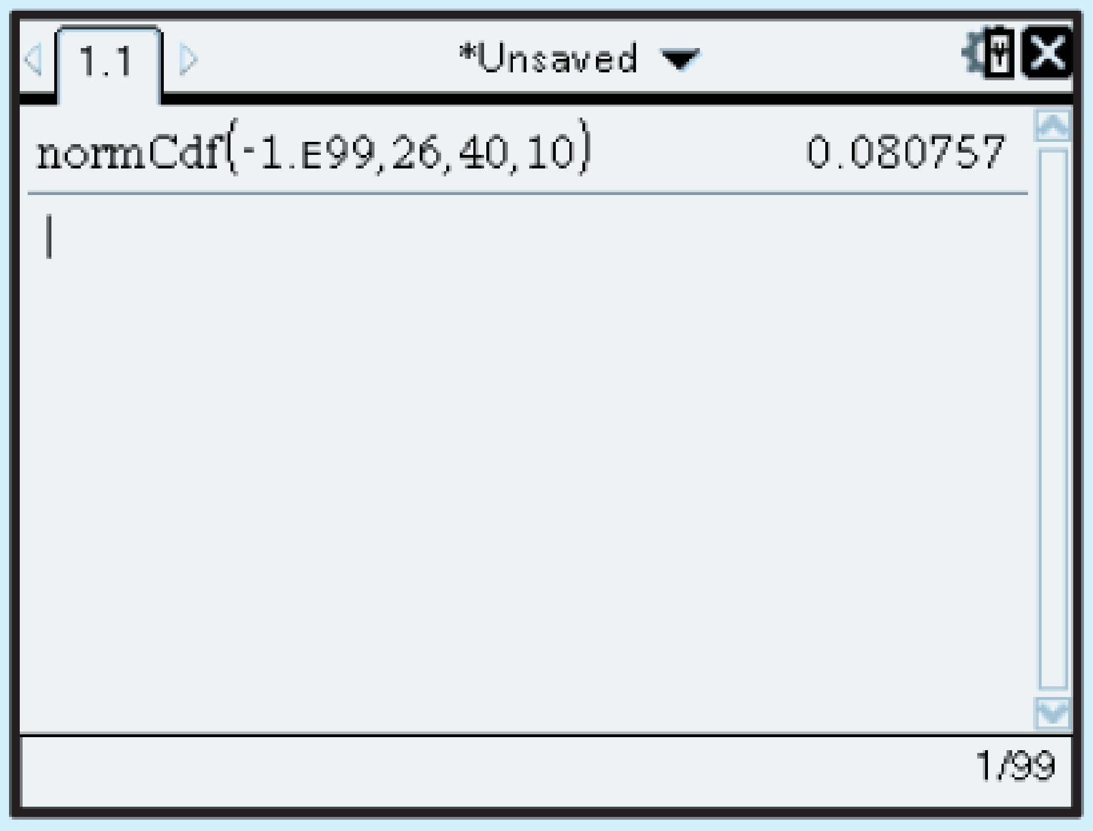
Lower Bound: -1ᴇ99

Upper Bound: 26

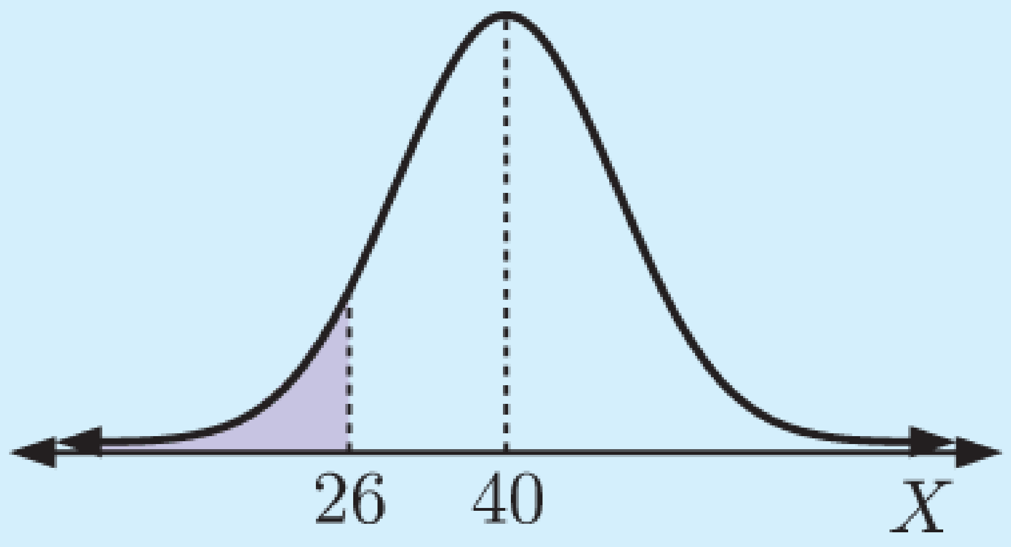
μ: 40

σ: 10

TI-*n*spire



```
normCdf(-1.ᴇ99,26,40,10)        0.080757
```



$P(X < 26) \approx 0.081$

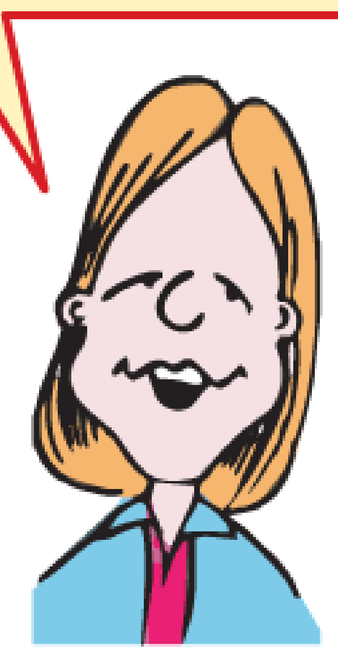
EXERCISE 15B.2

1 Suppose X is normally distributed with mean 60 and standard deviation 5. Find:

a $P(60 \leqslant X \leqslant 65)$
b $P(62 \leqslant X \leqslant 67)$
c $P(X \geqslant 64)$
d $P(X \leqslant 68)$
e $P(X \leqslant 61)$
f $P(57.5 \leqslant X \leqslant 62.5)$

Illustrate your answers.

A continuous random variable X can never be *exactly* 64, so $P(X \geqslant 64) = P(X > 64)$.



2 Suppose X is normally distributed with mean 37 and standard deviation 7.

a Use technology to find $P(X > 40)$.

b Hence find $P(37 \leqslant X \leqslant 40)$ without technology.

3 A machine produces metal bolts. The lengths of these bolts have a normal distribution with mean 19.8 cm and standard deviation 0.3 cm. If a bolt is selected at random from the machine, find the probability that it will have a length between 19.7 cm and 20 cm.

4 The speed of cars passing a supermarket is normally distributed with mean 46.3 km h^{-1} and standard deviation 7.4 km h^{-1}. Find the probability that a randomly selected car is travelling:

a between 50 and 65 km h^{-1}
b slower than 60 km h^{-1}
c faster than 50 km h^{-1}.

5 Eels are washed onto a beach after a storm. Their lengths have a normal distribution with mean 41 cm and standard deviation 5.5 cm.

- **a** If an eel is randomly selected, find the probability that it is at least 50 cm long.
- **b** Find the percentage of eels measuring between 40 cm and 50 cm long.
- **c** How many eels from a sample of 200 would you expect to measure at least 45 cm in length?

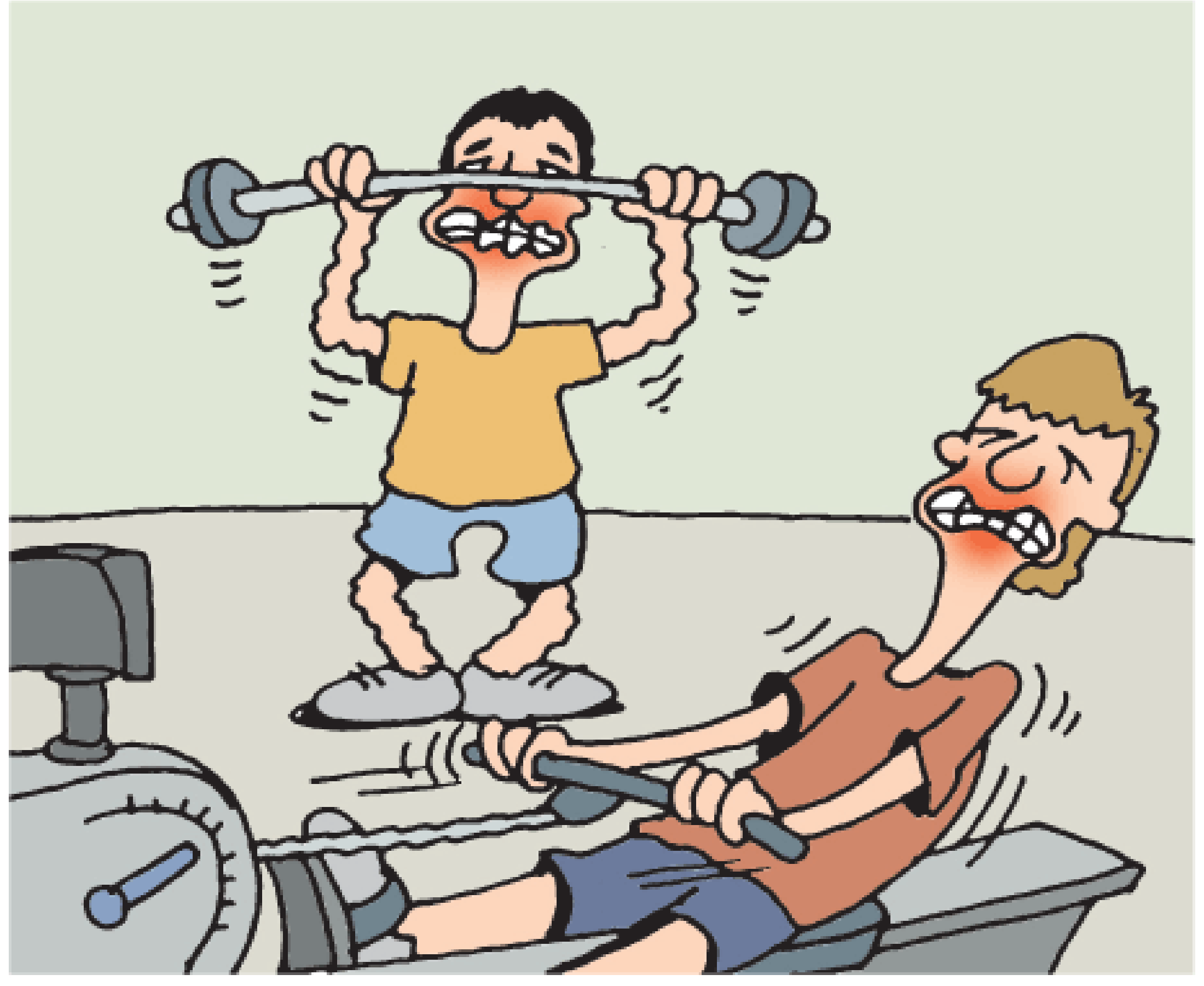
6 Max's customers put money for charity into a collection box on the front counter of his shop. The weekly collection is approximately normally distributed with mean \$40 and standard deviation \$6.

- **a** On what percentage of weeks would Max expect to collect:
  - **i** between \$30 and \$50
  - **ii** at least \$50?
- **b** How much money would you expect Max to collect in two years?

7 The amount of petrol bought by customers at a petrol station is normally distributed with mean 36 L and standard deviation 7 L.

- **a** What percentage of customers buy:
  - **i** less than 28 L of petrol
  - **ii** between 30 L and 40 L of petrol?
- **b** On a particular day, the petrol station has 600 customers.
  - **i** How much petrol would you expect the petrol station to sell on this day?
  - **ii** How many customers would you expect to buy at least 44 L of petrol?

8 The times Enrique and Damien spend working out at the gym each day are both normally distributed with mean 45 minutes. The standard deviation of Enrique's times is 9 minutes, and the standard deviation of Damien's times is 6 minutes.



- **a** On what percentage of days does:
  - **i** Enrique spend between 32 and 40 minutes at the gym
  - **ii** Damien spend less than 55 minutes at the gym?
- **b** Tomorrow, who do you think is more likely to spend:
  - **i** at least 1 hour at the gym
  - **ii** between 40 minutes and 50 minutes at the gym?

  Explain your answers.
- **c** Perform calculations to check your answers to **b**.

Example 4

Self Tutor

The times taken by students to complete a puzzle are normally distributed with mean 28.3 minutes and standard deviation 3.6 minutes. Calculate the probability that:

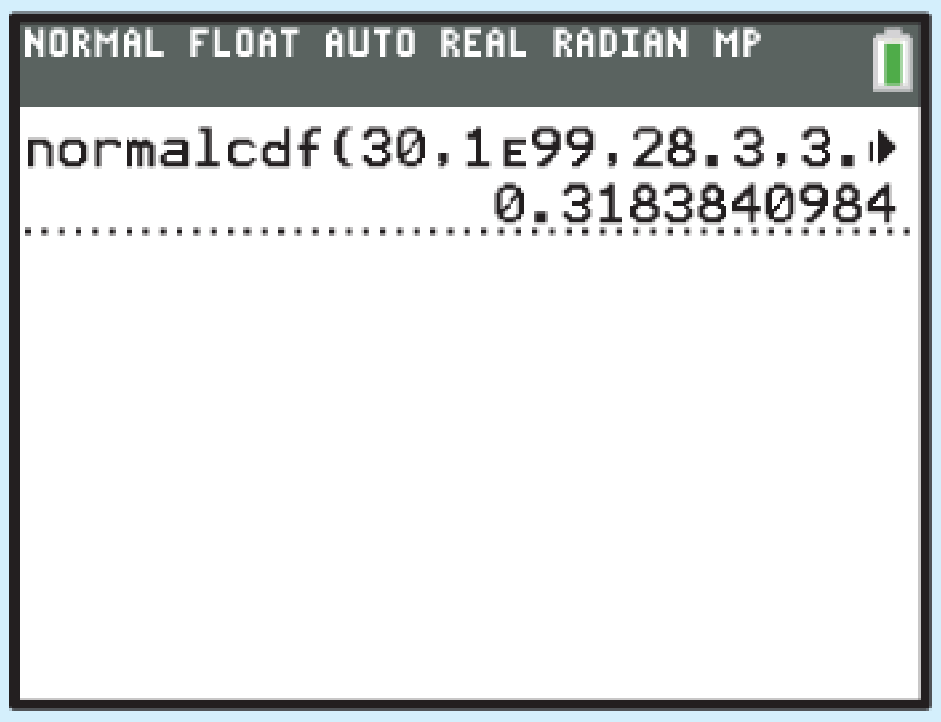
- **a** a randomly selected student took at least 30 minutes to complete the puzzle
- **b** out of 10 randomly selected students, 5 or fewer of them took at least 30 minutes to complete the puzzle.

a Let X denote the time for a student to complete the puzzle.

$X \sim \text{N}(28.3,\ 3.6^2)$

$\therefore\ \text{P}(X \geqslant 30) \approx 0.318\,38$

≈ 0.318

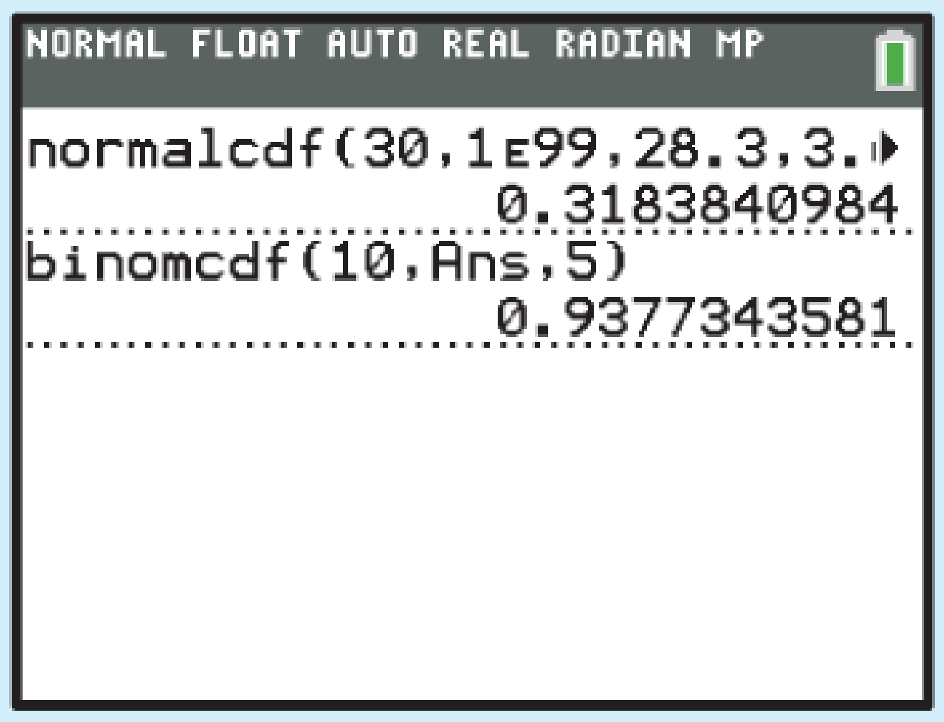


```
normalcdf(30,1E99,28.3,3.▸
                0.3183840984
```

b Let Y denote the number of students who took at least 30 minutes to complete the puzzle.

$Y \sim \text{B}(10,\ 0.318\,38)$

$\therefore\ \ \text{P}(Y \leqslant 5) \approx 0.938$



```
NORMAL FLOAT AUTO REAL RADIAN MP
normalcdf(30,1E99,28.3,3.▸
                0.3183840984
binomcdf(10,Ans,5)
                0.9377343581
```

B$(n,\ p)$ is the binomial distribution with n independent trials, each with probability of success p.

9 Apples from a grower's crop were normally distributed with mean 173 grams and standard deviation 34 grams. Apples weighing less than 130 grams were too small to sell.

a Find the percentage of apples from this crop which were too small to sell.

b Find the probability that in a picker's basket of 100 apples, more than 10 apples were too small to sell.

10 People found to have high blood pressure are prescribed a course of tablets. They have their blood pressure checked at the end of 4 weeks. The drop in blood pressure over the period is normally distributed with mean 5.9 units and standard deviation 1.9 units.

a Find the proportion of people who show a drop of more than 4 units.

b Eight people taking the course of tablets are selected at random. Find the probability that at least six of them will show a drop in blood pressure of more than 4 units.

11 The lengths of red snapper are normally distributed with mean 58 cm and standard deviation 18 cm. Fish measuring less than 38 cm long must be released back into the water.

a Find the probability that a single red snapper caught must be released back into the water.

b Rodney went fishing and caught 14 red snapper. Find the probability that Rodney can keep at least 10 fish from his catch.

C QUANTILES

Consider a population of crabs where the length of a shell, X mm, is normally distributed with mean 70 mm and standard deviation 10 mm.

A biologist wants to protect the population by allowing only the largest 5% of crabs to be harvested. He therefore wants to know what length corresponds to the 95th percentile of crabs.

To answer this question we need to find k such that $\text{P}(X \leqslant k) = 0.95$.

The number k is known as a **quantile**. In this case it is the 95% quantile.

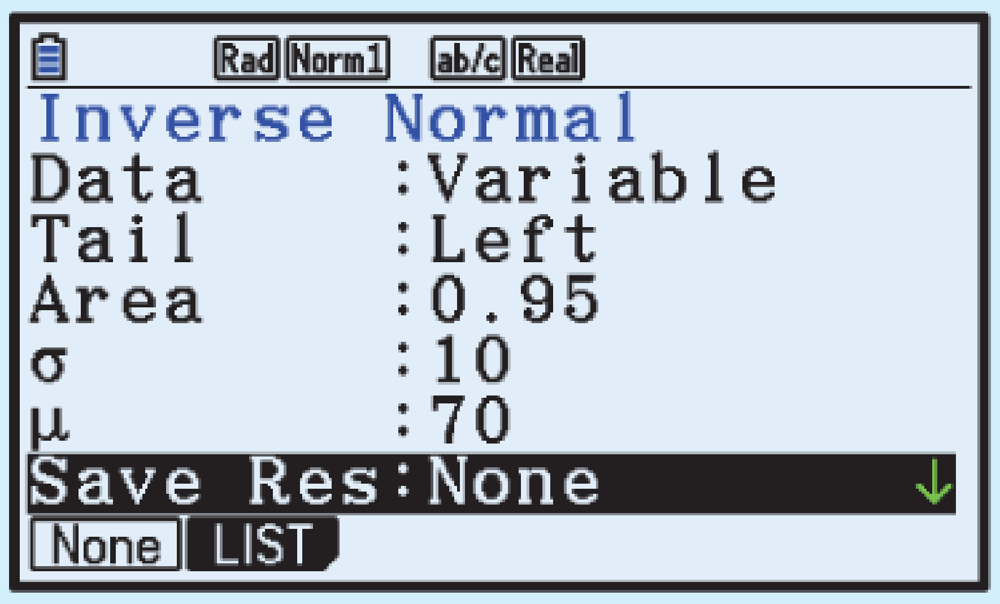
When finding quantiles, we are given a probability and are asked to calculate the corresponding measurement. This is the *inverse* of finding probabilities, so we use the **inverse normal function** on our calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

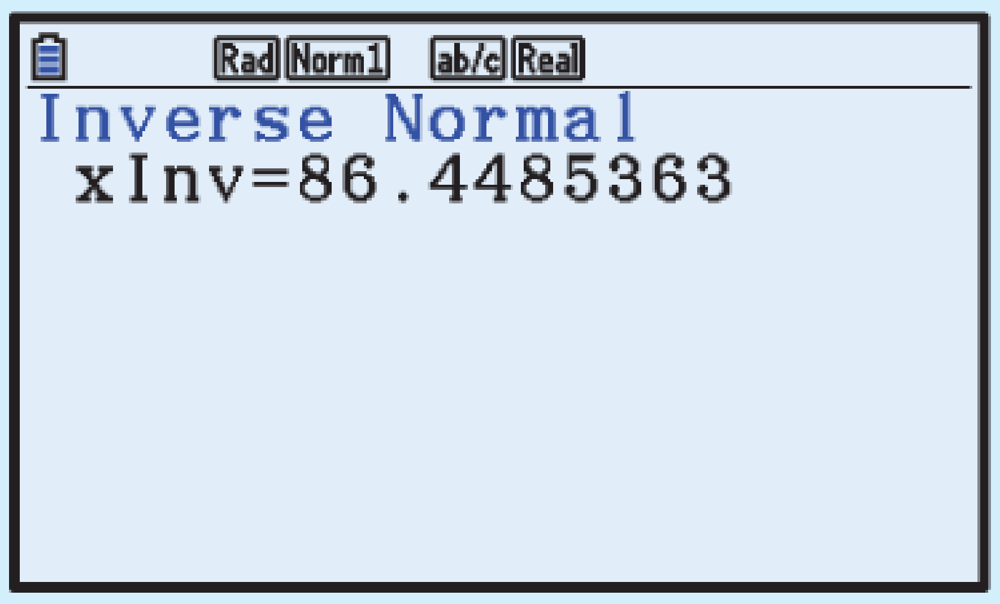
Example 5 **Self Tutor**

A population of crabs has shell length X mm. X is normally distributed with mean 70 and standard deviation 10. Find k for which $P(X < k) = 0.95$.

Casio fx-CG50

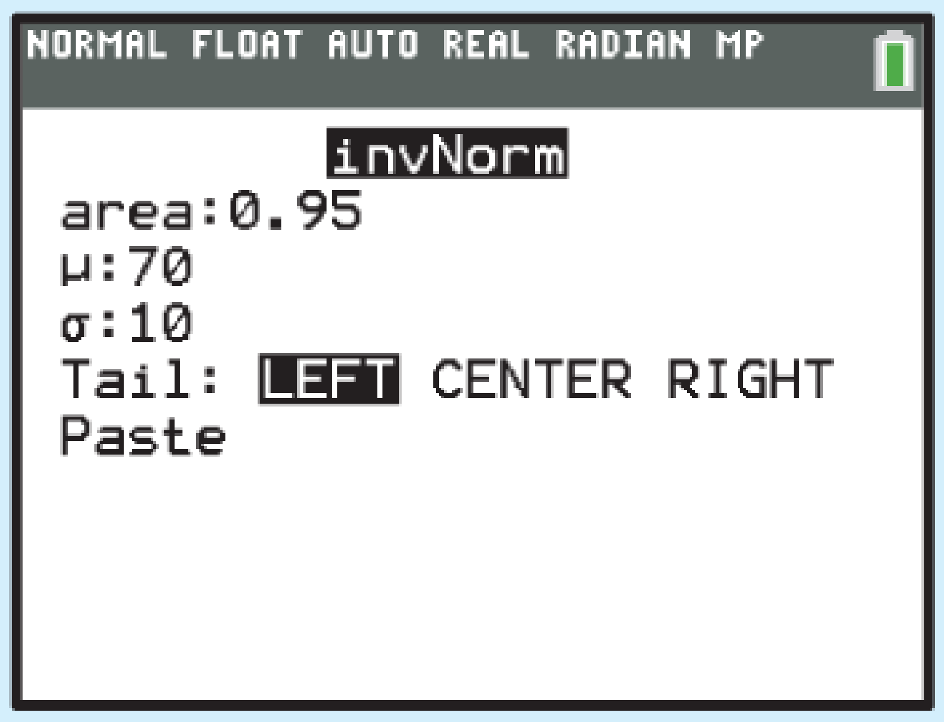


```
Inverse Normal
Data     :Variable
Tail     :Left
Area     :0.95
σ        :10
μ        :70
Save Res:None
```

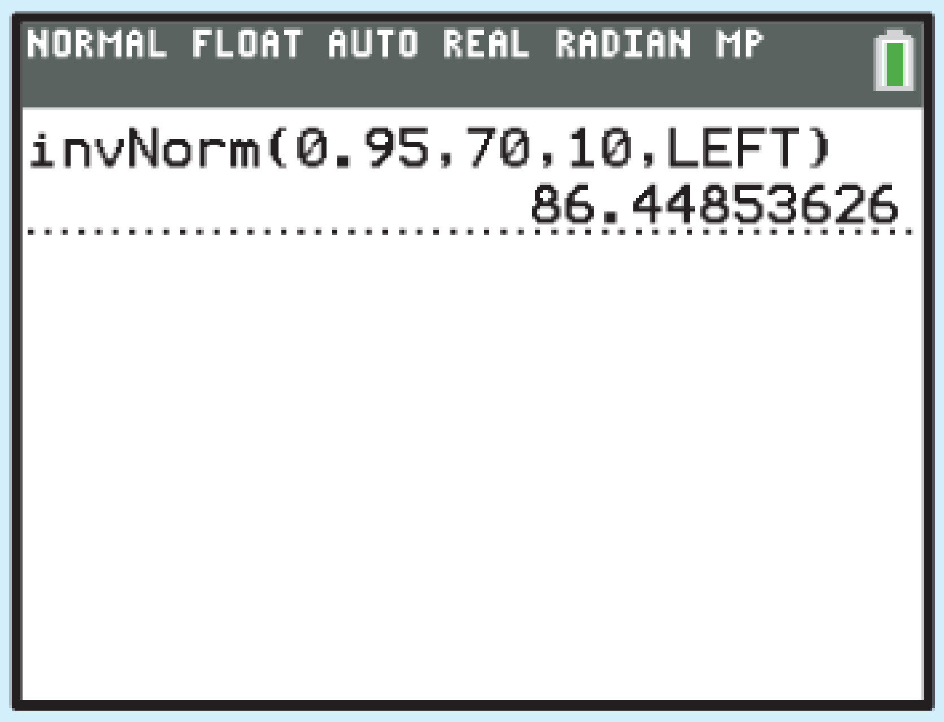


```
Inverse Normal
 xInv=86.4485363
```

TI-84 Plus CE

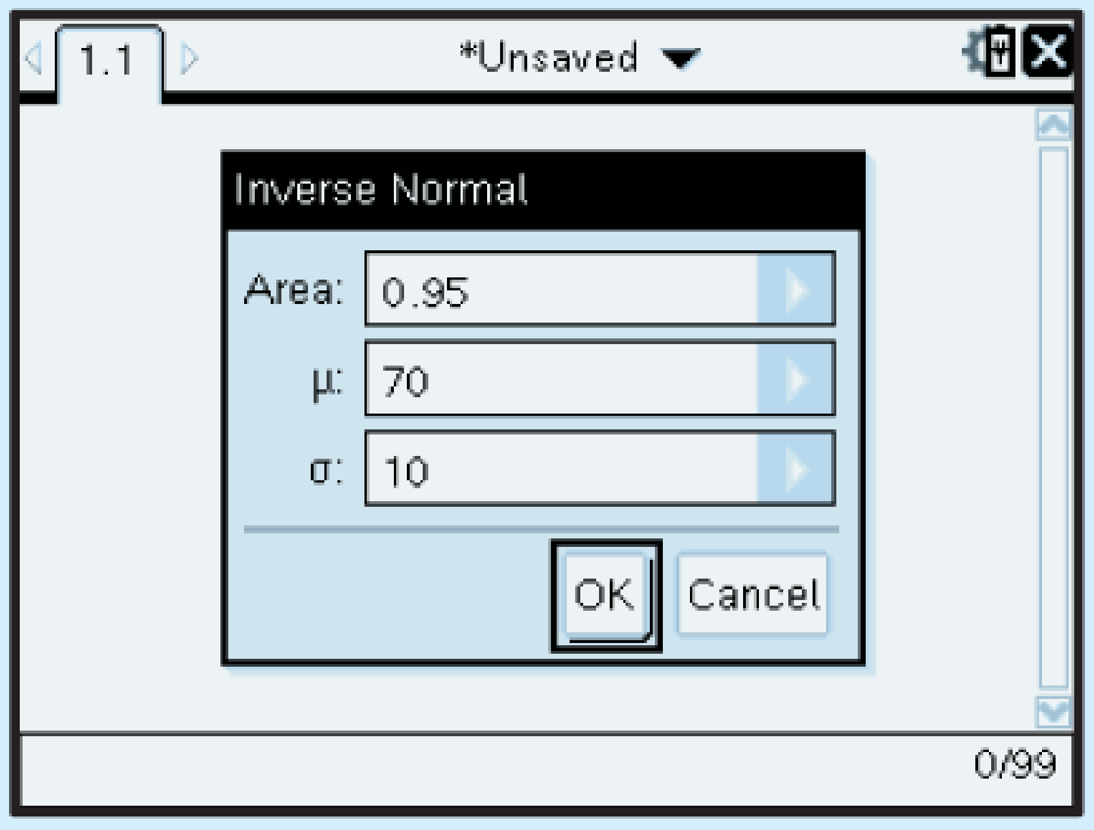


```
NORMAL FLOAT AUTO REAL RADIAN MP
        invNorm
area:0.95
μ:70
σ:10
Tail: LEFT CENTER RIGHT
Paste
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
invNorm(0.95,70,10,LEFT)
                 86.44853626
```

TI-nspire

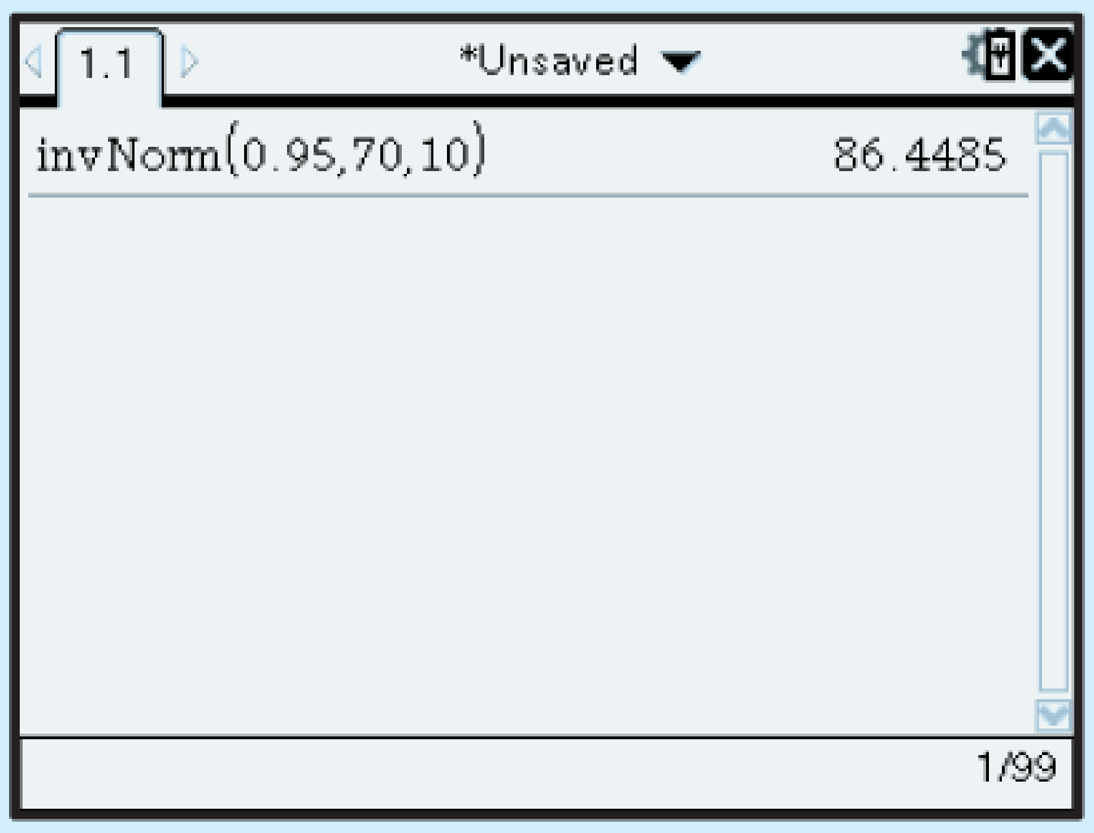


Inverse Normal

Area: 0.95

μ: 70

σ: 10

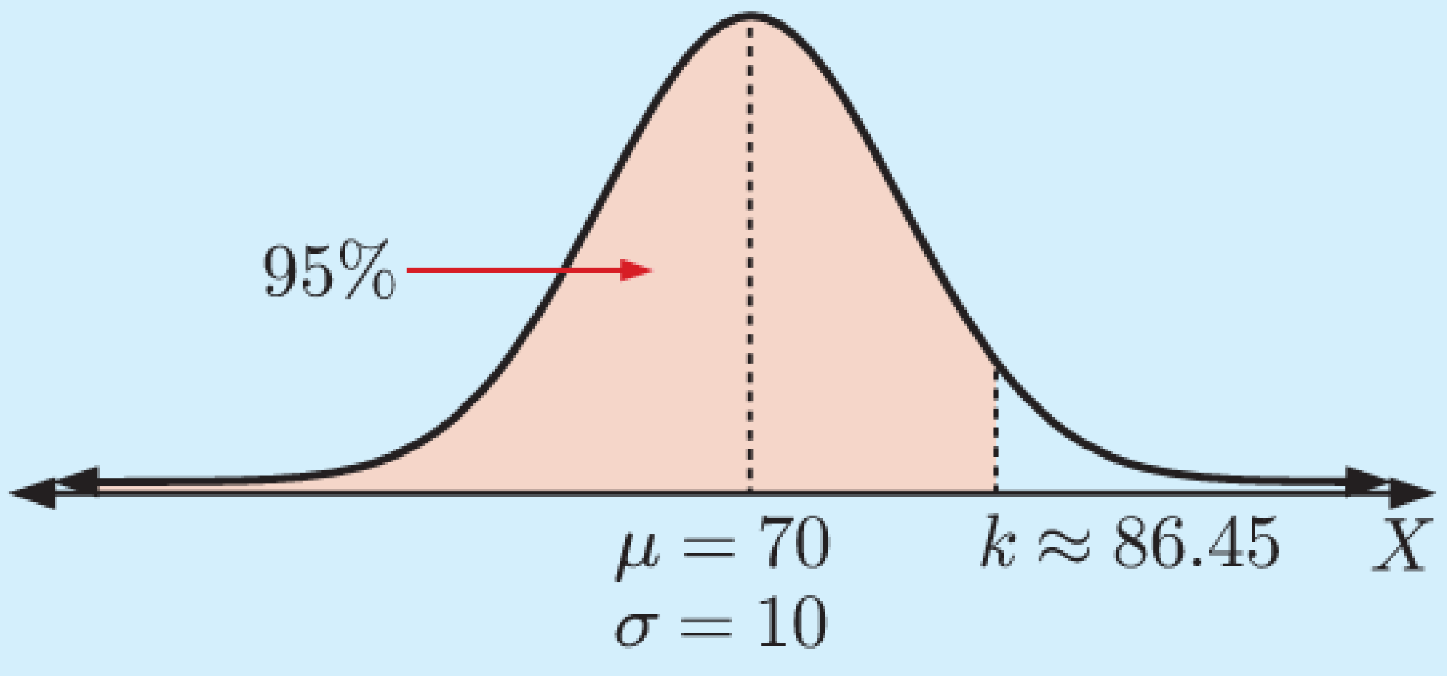


```
invNorm(0.95,70,10)          86.4485
```

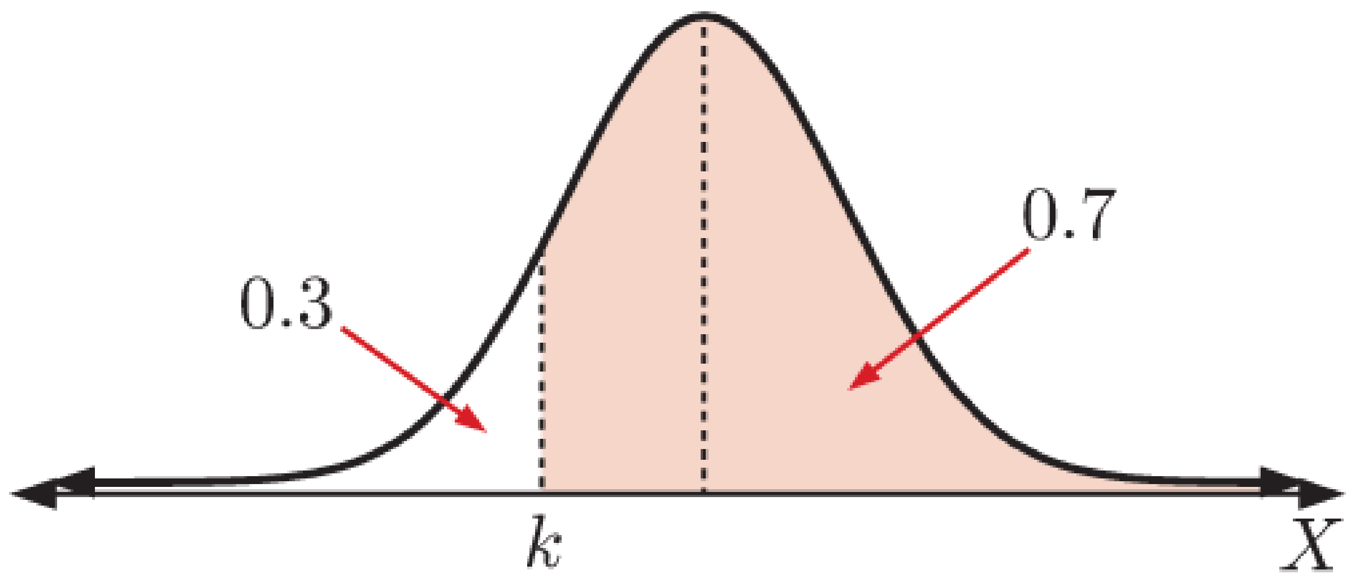
If $P(X < k) = 0.95$ then

$k \approx 86.45$

The 95% quantile corresponds to a shell width of 86.45 mm.



When using the **HP Prime**, **TI-84 Plus CE**, or **TI-*n*spire** calculators, we must always use the area to the *left* of k. Therefore, to find k such that $P(X > k) = 0.7$, we instead find k such that $P(X < k) = 1 - 0.7 = 0.3$.



EXERCISE 15C

1 Suppose X is normally distributed with mean 20 and standard deviation 3. Illustrate with a sketch and find k such that:

- **a** $P(X \leqslant k) = 0.3$
- **b** $P(X \leqslant k) = 0.9$
- **c** $P(X \leqslant k) = 0.5$
- **d** $P(X > k) = 0.2$
- **e** $P(X < k) = 0.62$
- **f** $P(X \geqslant k) = 0.13$

2 Given that $Z \sim N(0, 1^2)$, illustrate with a sketch and find k such that:

- **a** $P(Z \leqslant k) = 0.81$
- **b** $P(Z \leqslant k) = 0.58$
- **c** $P(Z \leqslant k) = 0.17$
- **d** $P(Z \geqslant k) = 0.95$
- **e** $P(Z \geqslant k) = 0.9$
- **f** $P(Z \geqslant k) = 0.41$

3 Suppose X is normally distributed with mean 30 and standard deviation 5, and $\mathrm{P}(X \leqslant a) = 0.57$.

- **a** Using a diagram, determine whether a is greater or less than 30.
- **b** Use technology to find a.
- **c** Without using technology, find:
  - **i** $\mathrm{P}(X \geqslant a)$
  - **ii** $\mathrm{P}(30 \leqslant X \leqslant a)$

4 Given that X is normally distributed with mean 15 and standard deviation 3, find k such that:

- **a** $\mathrm{P}(X < k) = 0.2$
- **b** $\mathrm{P}(X > k) = 0.1$
- **c** $\mathrm{P}(15 - k < X < 15 + k) = 0.9$

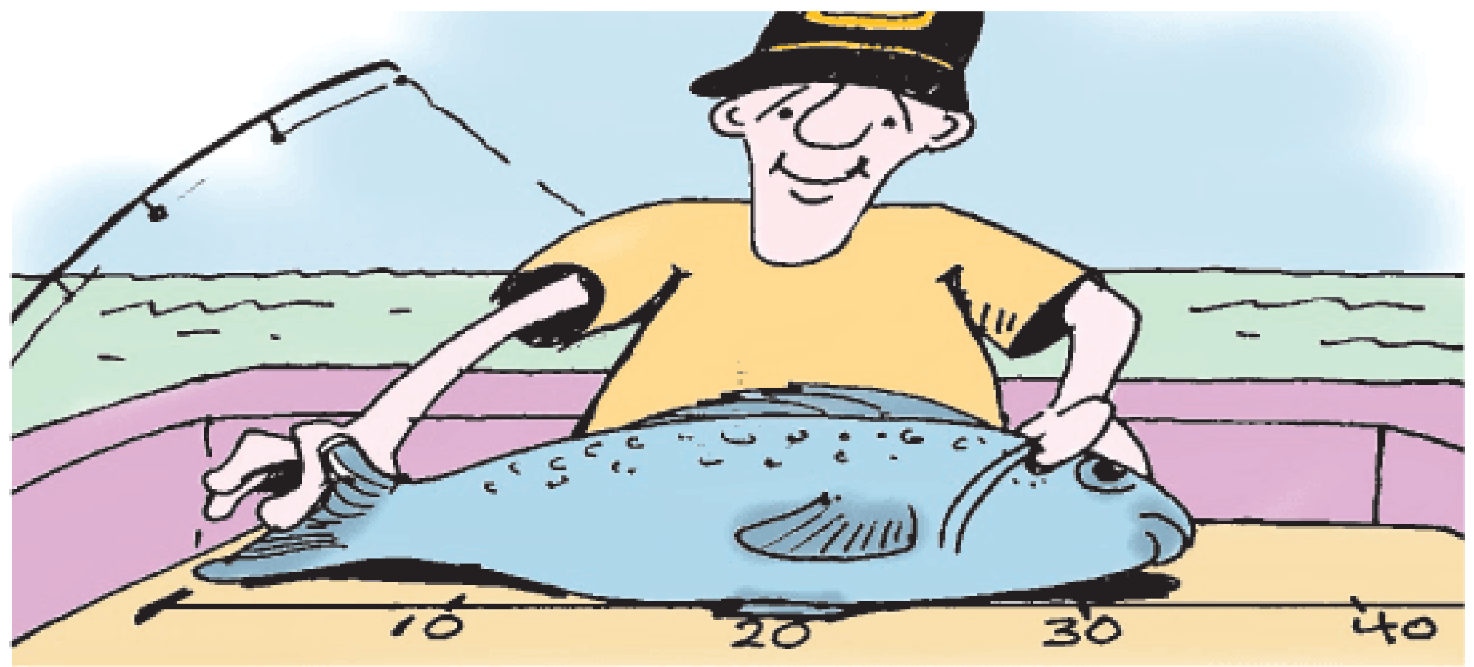
5 Suppose X is normally distributed with mean 80 and standard deviation 10.

- **a** Find $\mathrm{P}(X \leqslant 72)$.
- **b** Hence find k such that $\mathrm{P}(72 \leqslant X \leqslant k) = 0.1$.

6 Given that $X \sim \mathrm{N}(45, 8^2)$, find a such that:

- **a** $\mathrm{P}(35 \leqslant X \leqslant a) = 0.25$
- **b** $\mathrm{P}(a \leqslant X \leqslant 50) = 0.15$
- **c** $\mathrm{P}(a \leqslant X \leqslant 54) = 0.6$

7 The lengths of a fish species are normally distributed with mean 35 cm and standard deviation 8 cm. The fisheries department has decided that the smallest 10% of the fish are not to be harvested. What is the size of the smallest fish that can be harvested?



8 The lengths of screws produced by a machine are normally distributed with mean 75 mm and standard deviation 0.1 mm. 1% of the screws are rejected because they are too long. What is the length of the smallest screw to be rejected?

9 The marks X for a Mathematics examination are normally distributed with mean 57 and standard deviation 10. 60% of students passed the exam and 10% received an A grade.

- **a** Find:
  - **i** the passing mark k
  - **ii** the mark l required for an A.
- **b** Check that $\mathrm{P}(k \leqslant X \leqslant l) = 0.5$.

10 The volumes of cool drink in bottles filled by a machine are normally distributed with mean 503 mL and standard deviation 0.5 mL. 1% of the bottles are rejected because they are underfilled, and 2% are rejected because they are overfilled. They are otherwise kept for sale. Find, correct to 1 decimal place, the range of volumes in the bottles that are kept.

11 Abbey goes for a morning walk as long as the temperature is not too cold and not too hot. The morning temperatures are normally distributed with mean 20°C and standard deviation 5°C. Given that the lower limit of Abbey's walking temperatures is 11°C, and that she goes for a walk 95% of the time, find the upper limit of Abbey's walking temperatures.

GAME

Click on the icon to play a card game for the normal distribution.

CARD GAME

INVESTIGATION 3 THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

In the previous Chapter, we saw how the **binomial distribution** arises from considering the number of "successes" in a fixed number of independent trials of an experiment.

Suppose $X \sim \text{B}(n, p)$ is the number of successes in n independent trials, each with probability of success p. The probability of getting x successes is:

$$\mathbf{P}(\boldsymbol{X = x}) = \binom{n}{x} \boldsymbol{p^x (1-p)^{n-x}} \quad \text{where } x = 0, 1, 2, ..., n$$

You will have used this formula to calculate probabilities for binomial random variables in cases where the number of trials n is relatively small. As n increases, the probability becomes more difficult to calculate. This is because the binomial coefficient $\binom{n}{x}$ becomes very large.

What to do:

1 Click on the icon to access a demonstration which draws the probability distribution of $X \sim \text{B}(n, p)$.

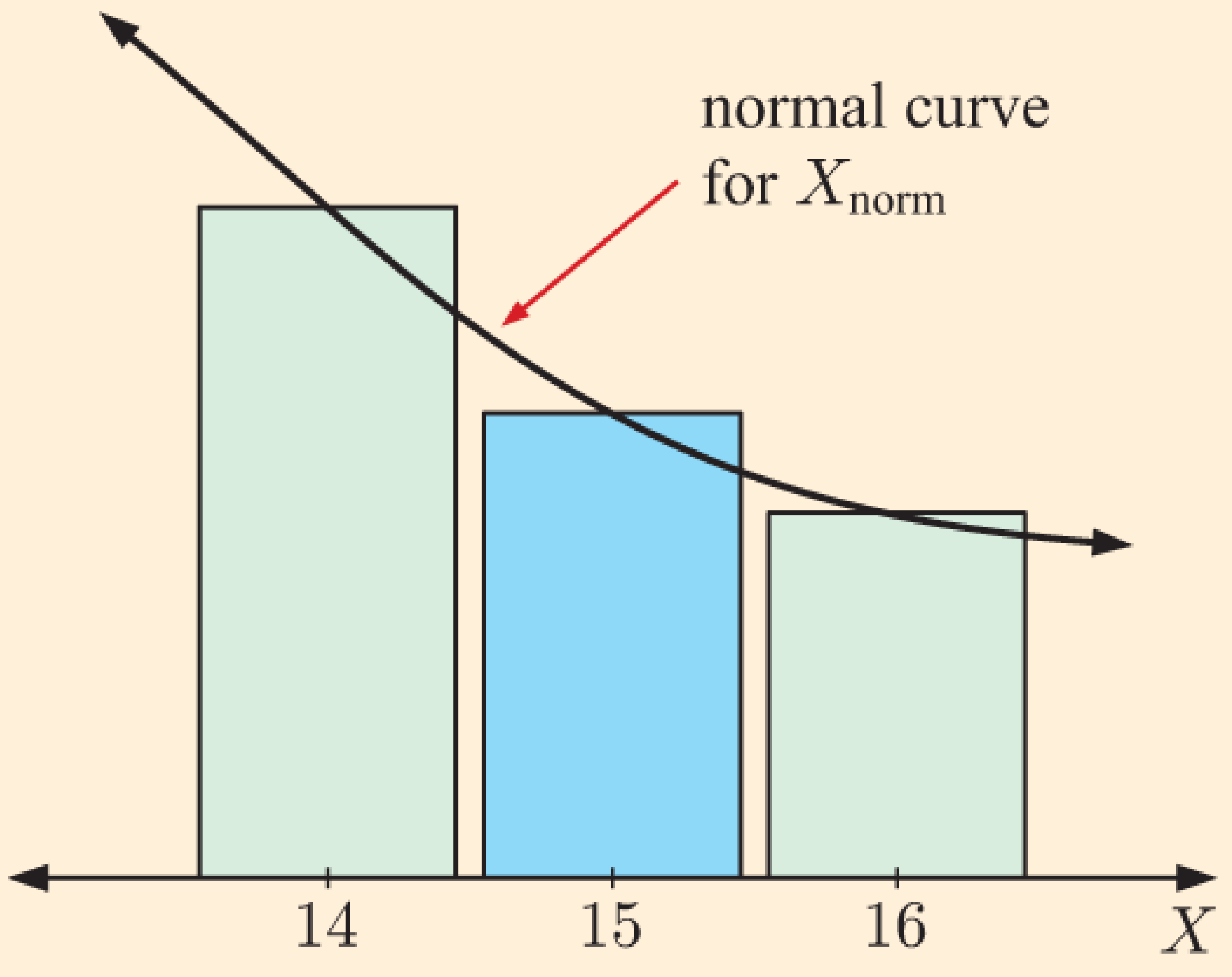
DEMO

- **a** Set $p = 0.5$ and use the sliders to change the value of n. Describe what happens to the distribution of X as n increases.
- **b** Repeat **a** for p equal to:
  - **i** 0.25
  - **ii** 0.1
  - **iii** 0.75
  - **iv** 0.9

  Comment on your observations.
- **c** Do you think that it would be *reasonable* to approximate the binomial distribution with a normal distribution? Explain your answer.
- **d** What should be the mean and standard deviation of this normal distribution?

2 Consider $X \sim \text{B}(50, 0.2)$ and its normal approximation $X_{\text{norm}} \sim \text{N}(\mu, \sigma^2)$.

- **a** Write down expressions for μ and σ.
- **b** Suppose we want to calculate the probability of 15 successes. The diagram alongside shows part of the probability distribution of X with the normal distribution curve of X_{norm} drawn over the top. Explain why $\text{P}(X = 15) \approx \text{P}(14.5 \leqslant X_{\text{norm}} \leqslant 15.5)$.
- **c** Describe how you would estimate the following using the normal approximation:
  - **i** $\text{P}(X \leqslant 10)$
  - **ii** $\text{P}(X < 25)$
  - **iii** $\text{P}(10 \leqslant X < 25)$



3 It is known that 2% of tyres manufactured by a company are unfit for sale. A quality inspector randomly sampled 500 tyres. Use a normal approximation to estimate the probability that at least 10 tyres in the sample will be unfit for sale.

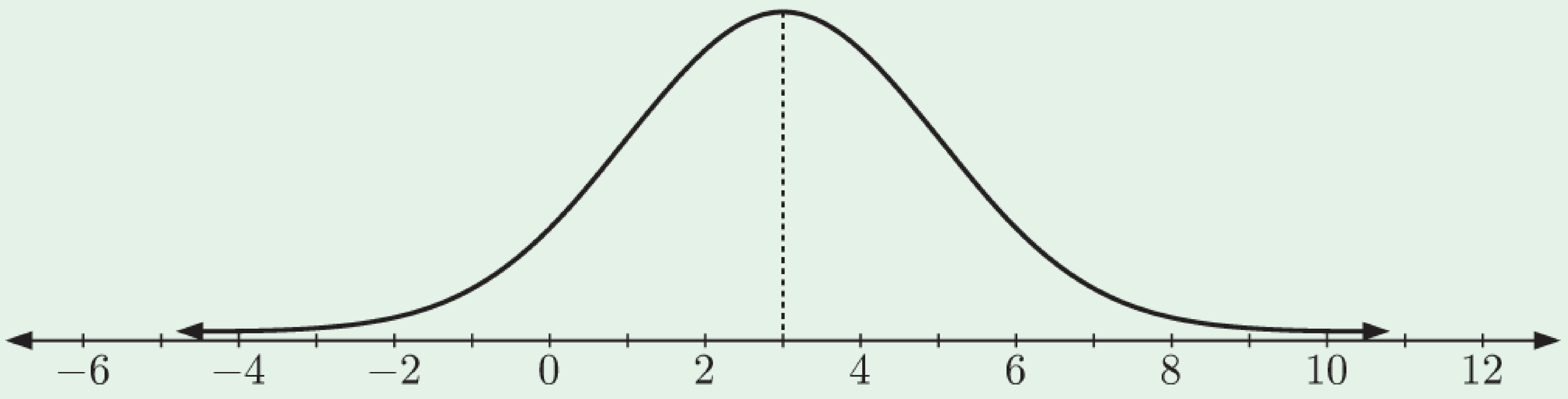
REVIEW SET 15A

1 Discuss whether the following variables will be normally distributed:

- **a** the time students take to read a novel
- **b** the amount spent on groceries at a supermarket.

2 The normal curve of the $\mathrm{N}(3,\ 2^2)$ distribution is shown below.

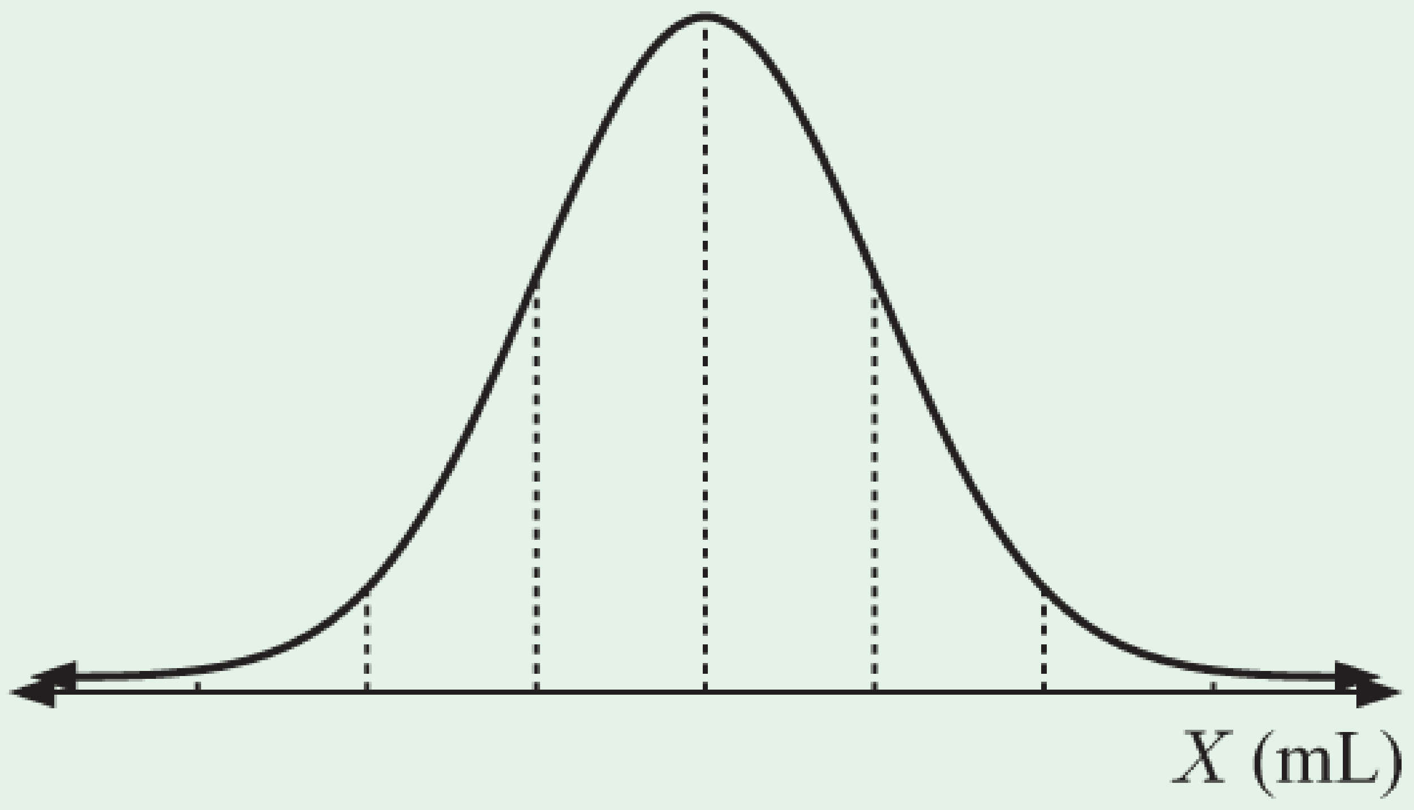
PRINTABLE GRAPH



On the same set of axes, sketch the normal curves for:

- **a** $\mathrm{N}(5,\ 2^2)$
- **b** $\mathrm{N}(1,\ 4^2)$
- **c** $\mathrm{N}(0,\ 1^2)$

3 The amount of juice Simon can squeeze from his lemons is normally distributed with mean 35 mL and standard deviation 5 mL.



- **a** Copy and complete this curve.
- **b** What percentage of the lemons will produce:
  - **i** between 25 mL and 35 mL of juice
  - **ii** at least 45 mL of juice?

4 The average height of 17 year old boys is normally distributed with mean 179 cm and standard deviation 8 cm. Calculate the percentage of 17 year old boys whose heights are:

- **a** more than 195 cm
- **b** between 171 cm and 187 cm
- **c** between 163 cm and 195 cm.

5 The weight of the edible part of a batch of Coffin Bay oysters is normally distributed with mean 38.6 grams and standard deviation 6.3 grams.

- **a** Find the percentage of oysters that weigh between 30 g and 40 g.
- **b** From a sample of 200 oysters, how many would you expect to weigh more than 50 g?

6 A random variable X is normally distributed with mean 20.5 and standard deviation 4.3. Find:

- **a** $\mathrm{P}(X \geqslant 22)$
- **b** $\mathrm{P}(18 \leqslant X \leqslant 22)$
- **c** k such that $\mathrm{P}(X \leqslant k) = 0.3$.

7 Let X be the weight in grams of bags of sugar filled by a machine. Bags less than 500 grams are considered underweight.

Suppose $X \sim \mathrm{N}(503,\ 2^2)$.

- **a** What percentage of bags are underweight?
- **b** If a quality inspector randomly selects 20 bags, what is the probability that 2 or fewer bags are underweight?

8 The daily energy intake of Canadian adults is normally distributed with mean 8700 kJ and standard deviation 1000 kJ.
Find the proportion of Canadian adults whose daily energy intake is:

- **a** greater than 8000 kJ
- **b** less than 7500 kJ
- **c** between 9000 and 10 000 kJ.

9 Suppose $X \sim \text{N}(30, 8^2)$. Illustrate with a sketch and find k such that:

- **a** $\text{P}(X \leqslant k) = 0.1$
- **b** $\text{P}(X \geqslant k) = 0.6$

10 The life of a Xenon-brand battery is normally distributed with mean 33.2 weeks and standard deviation 2.8 weeks.

- **a** Find the probability that a randomly selected battery will last at least 35 weeks.
- **b** For how many weeks can the manufacturer expect the batteries to last before 8% of them fail?

11 Suppose X is normally distributed with mean 25 and standard deviation 6. Find k such that:

- **a** $\text{P}(X \leqslant k) = 0.7$
- **b** $\text{P}(X \geqslant k) = 0.4$
- **c** $\text{P}(20 \leqslant X \leqslant k) = 0.3$

12 Machines A and B both produce nails whose lengths are normally distributed. The lengths of nails from machine A have mean 50.2 mm and standard deviation 1.1 mm. The lengths of nails from machine B have mean 50.6 mm and standard deviation 0.8 mm. Nails which are longer than 52 mm or shorter than 48 mm are rejected.

- **a** Find the probability of randomly selecting a nail that has to be rejected from:
  - **i** machine A
  - **ii** machine B.
- **b** A quality inspector randomly selects a nail from a randomly chosen machine. Find the probability that the nail was made by machine A *given* that it should be rejected.

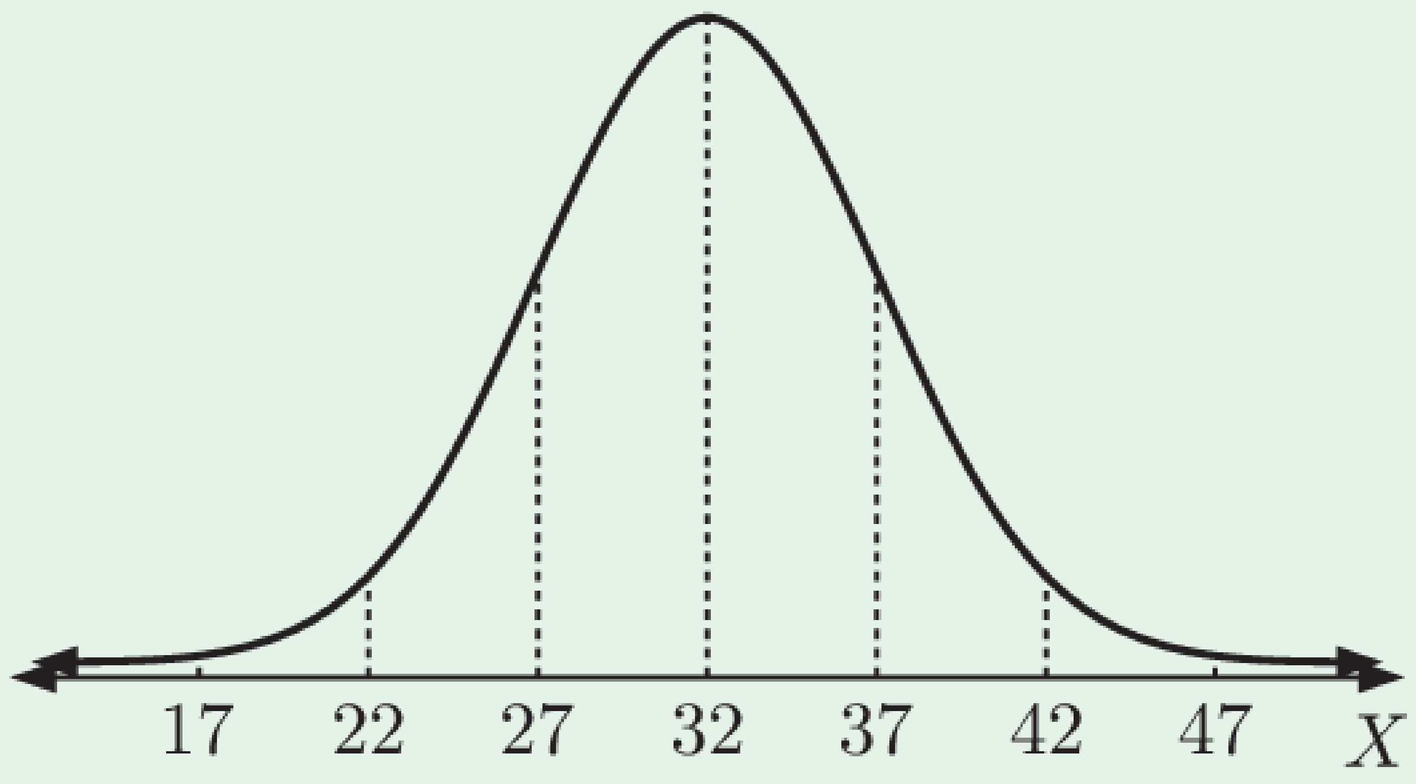
REVIEW SET 15B

1 Sketch the following normal distributions on the same set of axes.

Distribution	*Mean* (cm)	*Standard deviation* (cm)
A	22	2
B	20	4
C	25	7

2 The variable X is normally distributed with graph shown.

- **a** State the mean and standard deviation of X.
- **b** What percentage of values of X are:
  - **i** between 27 and 32
  - **ii** less than 37
  - **iii** greater than 42?



3 The random variable Z has the distribution $\text{N}(0, 1^2)$. Find the value of k for which $\text{P}(-k \leqslant Z \leqslant k) = 0.95$.

4 The contents of soft drink cans are normally distributed with mean 327 mL and standard deviation 4.2 mL.

- **a** Find the percentage of cans with contents:
  - **i** less than 318.6 mL
  - **ii** between 322.8 mL and 339.6 mL.
- **b** Find the probability that a randomly selected can contains between 327 mL and 331.2 mL.

5 The arm lengths of 18 year old females are normally distributed with mean 64 cm and standard deviation 4 cm.

- **a** Find the percentage of 18 year old females whose arm lengths are:
  - **i** between 61 cm and 73 cm
  - **ii** greater than 57 cm.
- **b** Find the probability that a randomly chosen 18 year old female has an arm length in the range 50 cm to 65 cm.
- **c** The arm lengths of 70% of the 18 year old females are more than x cm. Find the value of x.

6 Suppose X is normally distributed with mean 50 and standard deviation 7. Find:

- **a** $\text{P}(46 \leqslant X \leqslant 55)$
- **b** $\text{P}(X \geqslant 60)$
- **c** k such that $\text{P}(X > k) = 0.23$.

7 In a competition to see who could hold their breath underwater the longest, the times in the preliminary round were normally distributed with mean 150 seconds and standard deviation 12 seconds. If the top 15% of contestants went through to the finals, what time was required to advance?

8 For $X \sim \text{N}(12, 2^2)$, find a such that:

- **a** $\text{P}(X < a) = 0.07$
- **b** $\text{P}(X > a) = 0.2$
- **c** $\text{P}(a \leqslant X \leqslant 11) = 0.1$

9 The weights of suitcases at an airport are normally distributed with mean 17 kg and standard deviation 3.4 kg.

- **a** Find the probability that a randomly selected suitcase weighs between 10 kg and 15 kg.
- **b** 300 suitcases are presented for check-in over a one hour period. How many of these suitcases would you expect to be lighter than 20 kg?
- **c** 3.9% of the suitcases are rejected because they exceed the maximum weight limit. Find the maximum weight limit.

10 On an ostrich farm the weights of the birds are found to be normally distributed. The weights of the females have mean 78.6 kg and standard deviation 5.03 kg. The weights of the males have mean 91.3 kg and standard deviation 6.29 kg.

- **a** Find the probability that a randomly selected:
  - **i** male will weigh less than 80 kg
  - **ii** female will weigh less than 80 kg
  - **iii** female will weigh between 70 and 80 kg.
- **b** 20% of females weigh less than k kg. Find k.
- **c** The middle 90% of the males weigh between a kg and b kg. Find the values of a and b.
- **d** In one field there are 82% females and 18% males. One of these ostriches is selected at random.
  Calculate the probability that the ostrich weighs less than 80 kg.

11 The weight of an apple in an apple harvest is normally distributed with mean 300 grams and standard deviation 50 grams. Only apples with weights between 250 grams and 350 grams are considered fit for sale.

- **a** Find the percentage of apples fit for sale.
- **b** In a sample of 100 apples, what is the probability that at least 75 are fit for sale?

12 Giovanni and Beppe are both carrot farmers. The lengths of Giovanni's carrots are normally distributed with mean 22 cm and standard deviation 3.4 cm. The lengths of Beppe's carrots are also normally distributed, with mean 23.5 cm and standard deviation 4.2 cm.

- **a** Find the probability that a carrot is longer than 20 cm, given it comes from:
  - **i** Giovanni's farm
  - **ii** Beppe's farm.
- **b** A buyer randomly selects a carrot from each farmer's crop. Calculate the probability that *neither* carrot is longer than 20 cm.

Chapter 16

Hypothesis testing

Contents:

A Statistical hypotheses
B Student's t-test
C The two-sample t-test for comparing population means
D The χ^2 goodness of fit test
E The χ^2 test for independence

OPENING PROBLEM

Frank is a market gardener who grows tomatoes.

Last year, Frank weighed a sample of 50 tomatoes and found that the mean weight was 106.3 g with standard deviation 12.41 g.

This year, Frank used a new fertiliser to try to increase the weight of his crop. Frank collected a random sample of 65 tomatoes and found the mean weight of the sample to be 110.1 g with standard deviation 13.1 g.

110.1
ACME
FERTILIZER

Things to think about:

- **a** How can Frank use this information to determine whether the new fertiliser was effective?
- **b** How *significant* is Frank's evidence? Is it sufficient to conclude that the fertiliser was effective? Would it be more significant if the sample size was bigger?

We often hear claims about **population parameters** such as the **population mean** μ or a **population proportion** p.

For example, a manufacturer of insect repellent might claim that on average, their new product is effective for longer than 6 hours. In statistics, we call this a **statistical hypothesis**.

We can decide whether a statistical hypothesis is reasonable or justified using a formal procedure called a **hypothesis test**.

Hypothesis tests have three key components:

- **formulating** statistical hypotheses
- collecting data from a sample and **calculating statistics** to test our hypotheses
- **making decisions** about the population based on what we see in the sample.

A STATISTICAL HYPOTHESES

Suppose a claim is made that a population mean μ has the value μ_0. We call this the **null hypothesis** H_0, and we write

$$H_0\colon\ \mu = \mu_0.$$

This statement is assumed to be true unless we have enough evidence to reject it.

If H_0 is not rejected, we accept that the population mean is μ_0. So, the null hypothesis is a statement that there is *no difference* between μ and μ_0.

If H_0 is rejected, we accept that there *is a difference* between μ and μ_0. This statement is called the **alternative hypothesis** H_1.

ONE-TAILED AND TWO-TAILED ALTERNATIVE HYPOTHESES

Given the null hypothesis $H_0\colon\ \mu = \mu_0$, the alternative hypothesis could be:

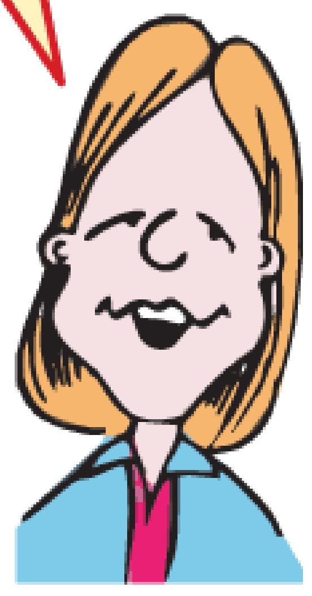
- $H_1\colon\ \mu > \mu_0$ **(one-tailed hypothesis)**
- $H_1\colon\ \mu < \mu_0$ **(one-tailed hypothesis)**
- $H_1\colon\ \mu \neq \mu_0$ **(two-tailed hypothesis**, as $\mu \neq \mu_0$ could mean $\mu > \mu_0$ or $\mu < \mu_0$).

Consider the insect repellent example on the previous page:

- If the manufacturer of the new brand wants evidence that the new product is *superior* in protection time, the hypotheses would be:
  H_0: $\mu = 6$ {the new product gives the same protection as the old ones}
  H_1: $\mu > 6$ {the new product protects for longer than the old ones}.

- If a competitor wants evidence that the new product has an *inferior* protection time, the hypotheses would be:
  H_0: $\mu = 6$ {the new product gives the same protection as the old ones}
  H_1: $\mu < 6$ {the new product protects for less time than the old ones}.

The null hypothesis H_0 always states that μ is **equal** to a specific value.



- If a market researcher studying all products on the market wants to show that the new product *differs* from the old ones, but is not concerned whether the protection time is more or less, the hypotheses would be:
  H_0: $\mu = 6$ {the new product gives the same protection as the old ones}
  H_1: $\mu \neq 6$ {the new product gives a different protection time compared with the old ones}.

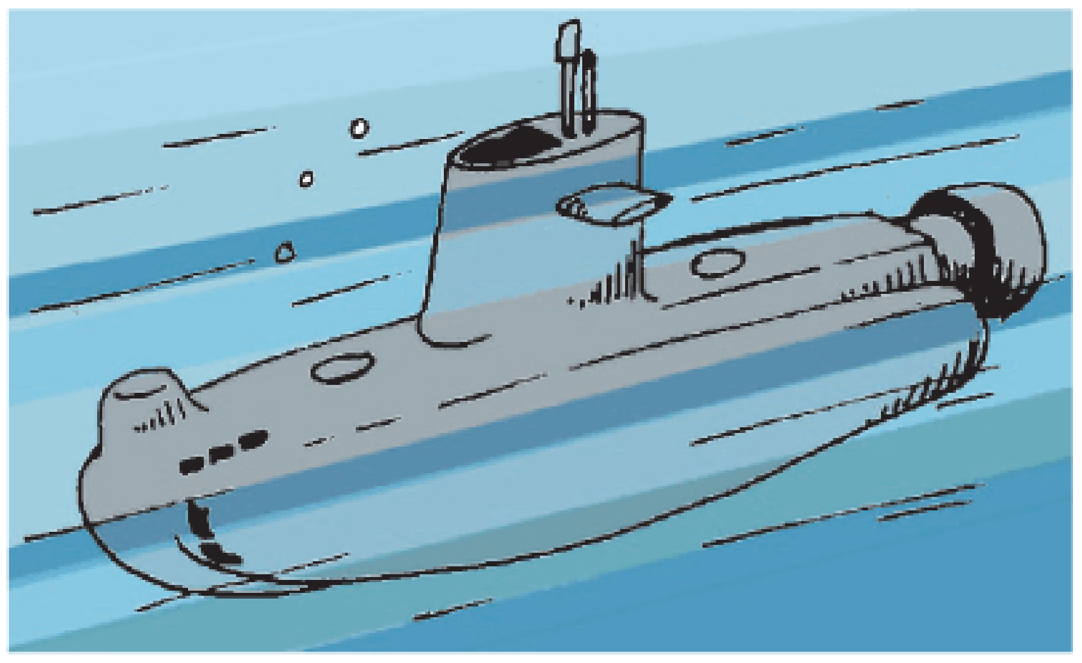
EXERCISE 16A

1 Current torch globes have a mean life of 80 hours. Globe Industries are considering mass production of a new globe they believe will last longer.

  a If Globe Industries wants to demonstrate that their new globe lasts longer, what set of hypotheses should they consider?

  b The new globe costs less to produce, so Globe Industries will adopt it unless it has an inferior lifespan to the old type. What set of hypotheses should they now consider?

2 The top speed of submarines currently produced by a manufacturer is 26.3 knots. When their engineers modify the design to reduce drag, they believe that the maximum speed will be increased. What set of hypotheses should they consider to test whether or not the new design is faster?



3 A machine is used to fill bags with 250 g of potato chips. A quality inspector wants to determine whether the machine is filling the bags with the *correct* weight of potato chips. What set of hypotheses should the quality inspector consider?

4 Whitex produce copy paper, and the weight of their copy paper is given as 80 g per m^2. The company wants to determine whether this information is correct. What set of hypotheses should be considered?

5 The average peak-hour travel time along a particular stretch of road is currently 27 minutes. To help reduce travel times, electronic signs displaying real-time information are erected. If the travel times improve, the signs will be more widely implemented. What set of hypotheses should be considered?

6 Brand A's muesli bars have 3 g of fat. Brand B claims that their muesli bars have 10% less fat than Brand A's muesli bars. Brand A wants evidence that Brand B's muesli bars have more fat than is claimed. What set of hypotheses should they consider?

B STUDENT'S t-TEST

In order to test our statistical hypotheses, we need *evidence* to base our decision on. The evidence comes from collecting data from a **sample** and then calculating **statistics**.

HISTORICAL NOTE

William Sealy Gosset (1876 - 1937) studied chemistry and mathematics at New College, Oxford University. In 1899 he moved to Dublin, Ireland, to work for the brewery of Arthur Guinness & Son. His desire was to improve the production process by selecting the best yielding varieties of barley. Through his study and work with **Karl Pearson** from University College, London, he devised a **test statistic**. His work was published in 1908 in the journal *Biometrika*, but using the name **Student** because of concern by Guinness that other brewers may use his work to their advantage.

Gosset's test statistic was revised by **Sir Ronald Aylmer Fisher** (1890 - 1962) who recognised the importance of Gosset's work. Fisher called the new statistic t, completing the name **Student's t-test**.

William Sealy Gosset

THE TEST STATISTIC OR t-STATISTIC

A **test statistic** summarises the information in a sample.

Consider a hypothesis test of H_0: $\mu = \mu_0$. If H_0 is true then we would expect the difference between the mean of the sample $\overline{x}$ and μ_0 to be close to 0. So a suitable test statistic should involve $\overline{x} - \mu_0$.

However, $\overline{x} - \mu_0$ alone does not take into account the *variation* of the data. If the standard deviation is also very small, a value of $\overline{x} - \mu_0$ close to 0 is likely to happen simply by chance alone. The test statistic should therefore also involve the sample standard deviation s.

Consider a hypothesis test of H_0: $\mu = \mu_0$.

Given a sample of size n with sample mean $\overline{x}$ and sample standard deviation s, the **test statistic** is:

$$\boldsymbol{t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}}$$

For example, consider the insect repellent example from the previous Section. Suppose a researcher took a random sample of 50 bottles of the new product and found that the mean protection time was $\overline{x} = 6.12$ hours with standard deviation $s = 15$ minutes $= 0.25$ hours.

In this case, the test statistic is $t = \dfrac{6.12 - 6}{\frac{0.25}{\sqrt{50}}} \approx 3.39$.

In examinations, you will not be required to calculate the test statistic by hand. You can click on the icon to obtain instructions for your **graphics calculator**.

GRAPHICS CALCULATOR INSTRUCTIONS

INVESTIGATION 1 THE DISTRIBUTION OF THE TEST STATISTIC

Every random sample will be different. The sample mean $\overline{x}$ and sample standard deviation will vary between samples, so the value of the test statistic t will be different for every sample.

In this Investigation, we will explore the *distribution* of the t-values for samples from a normal distribution.

What to do:

1 Click on this icon to access a simulation that generates random samples of size n from a normal distribution. The test statistic t is calculated for each sample.

SIMULATION

  a Set $n = 10$ and use the sliders to change the mean μ and standard deviation σ of the normal distribution. Describe the distribution of the t-values.

  b Use the slider to increase the value of n. What do you notice?

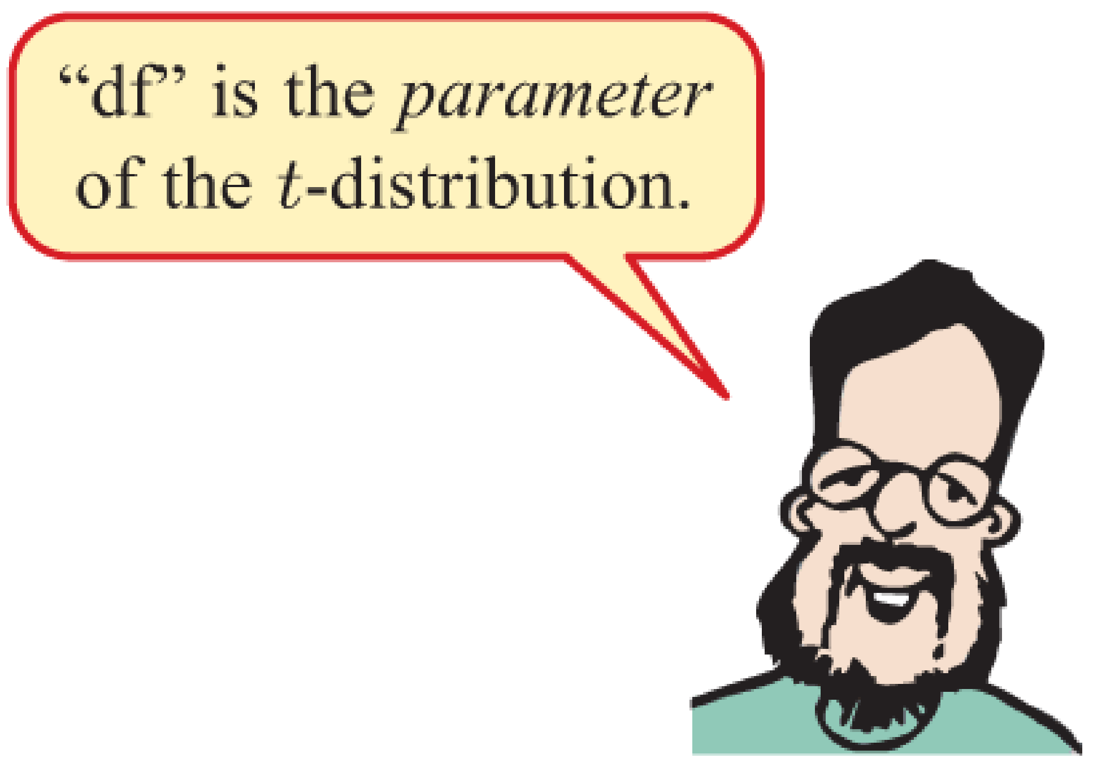
2 Describe how likely it is to obtain a t-value in the interval:

  a $-1 \leqslant t \leqslant 1$ **b** $-2 \leqslant t \leqslant 2$ **c** $-3 \leqslant t \leqslant 3$

The characteristics you observed in the **Investigation** might suggest that the distribution of the t-values is a normal distribution with mean 0. However, the distribution of the t-values actually has a different curve.

Suppose a population is approximately normally distributed. The distribution of t-values for samples of size n is a ***t*-distribution** with $n-1$ **degrees of freedom** (df).

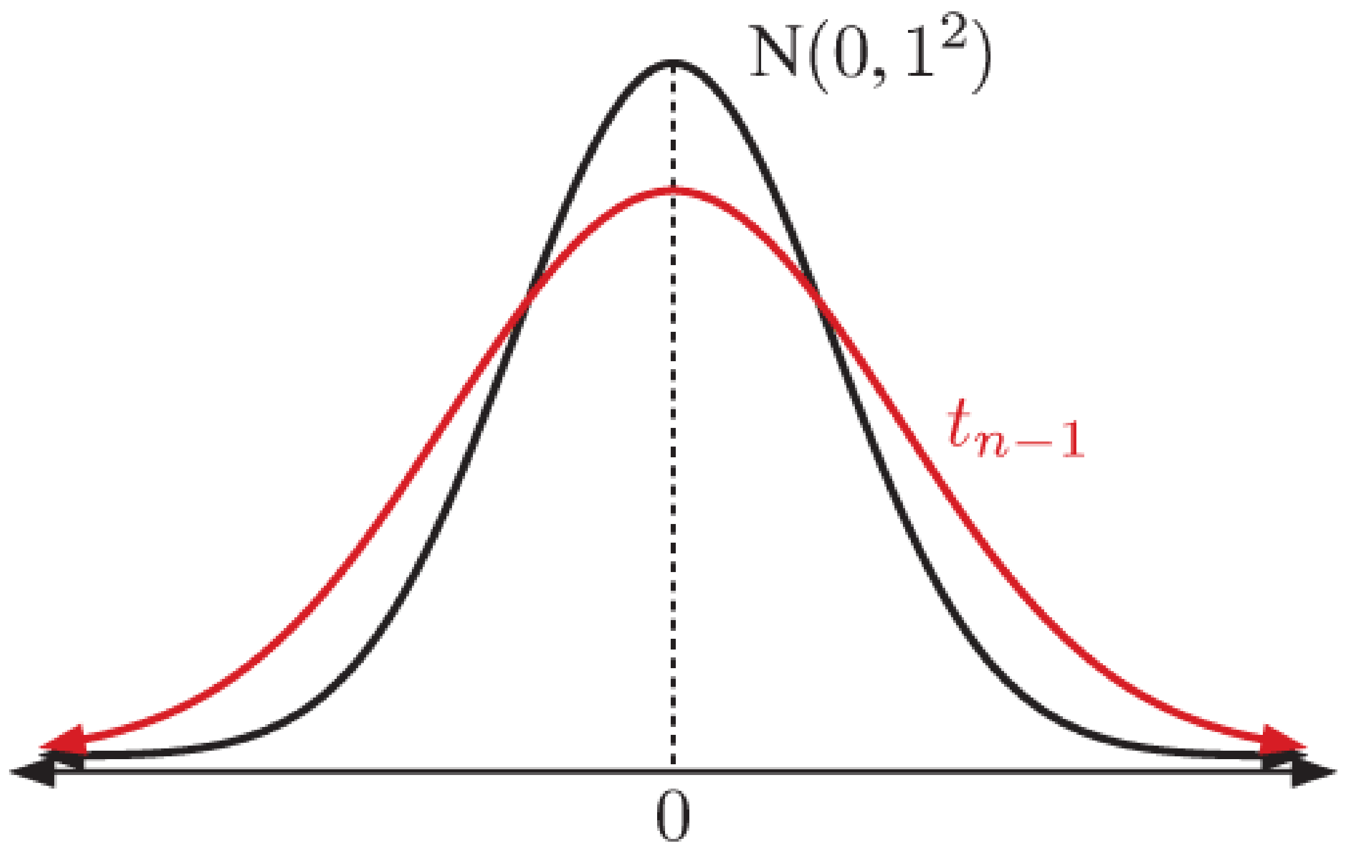
We write $\boldsymbol{T \sim t_{n-1}}$.


"df" is the *parameter* of the t-distribution.

The t-distribution is slightly "flatter" than the normal $\mathrm{N}(0,\ 1^2)$ distribution.

As the sample size and therefore the degrees of freedom increases, the shape of the t-distribution approaches that of $\mathrm{N}(0,\ 1^2)$.

THE t-DISTRIBUTION



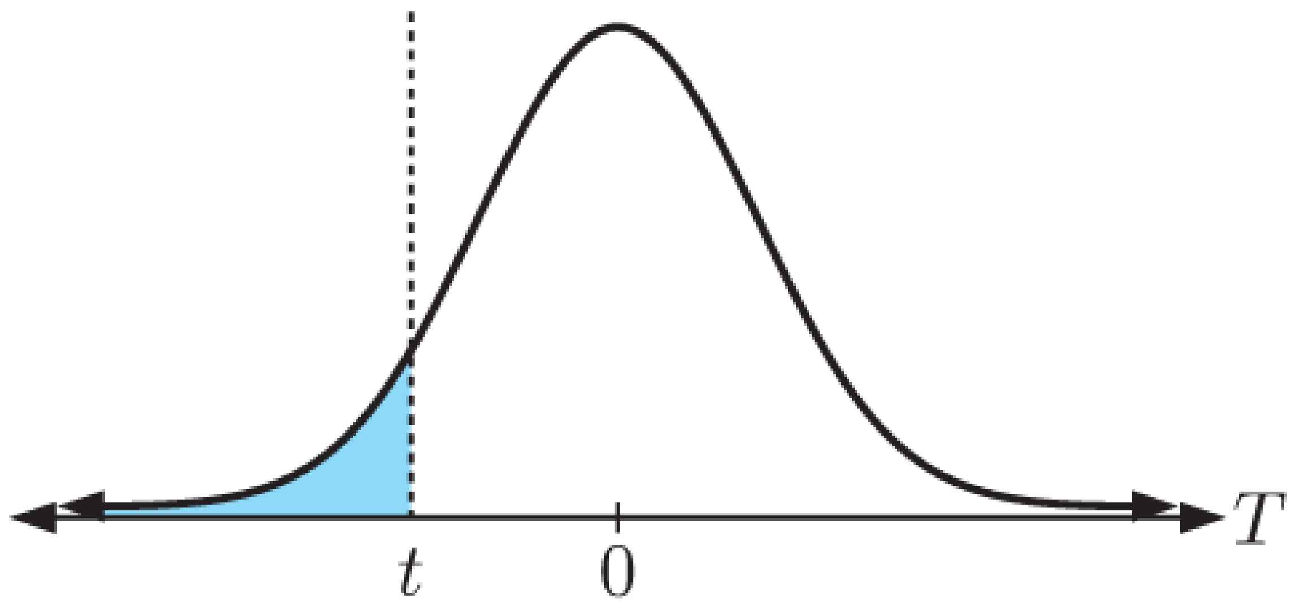
OBTAINING EVIDENCE

"Extreme" values of the test statistic are unlikely, so observing such a value is evidence against the null hypothesis. However, we need a measure of just *how* extreme a value is.

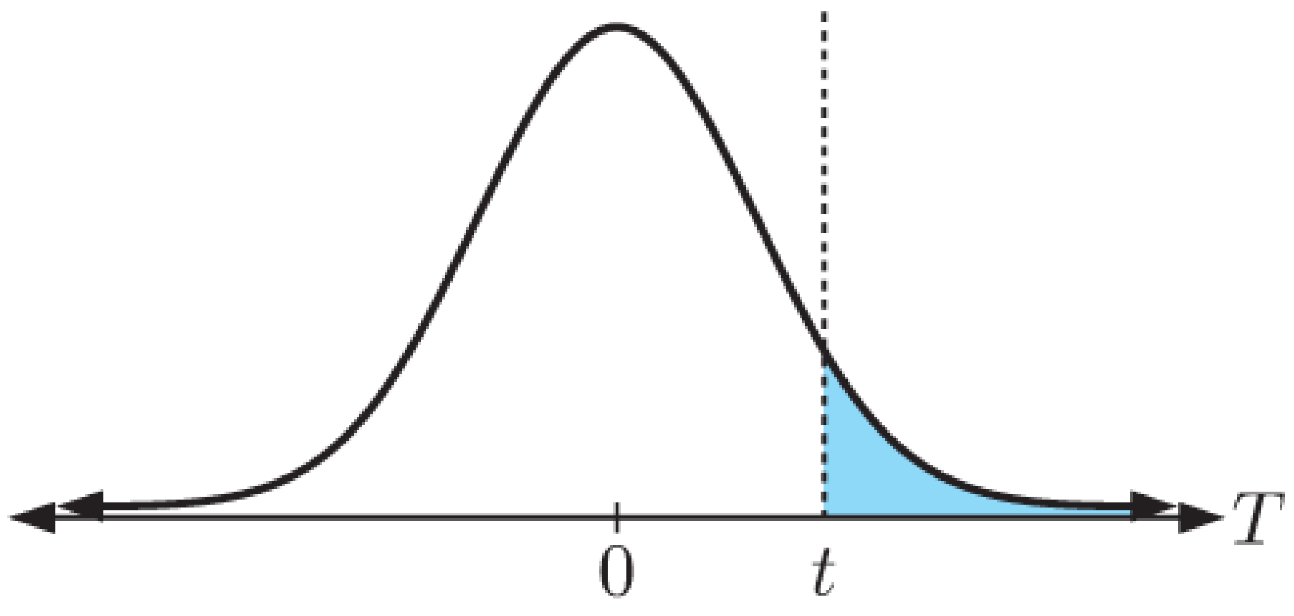
The ***p*-value** of a test statistic is the probability of that result being observed if H_0 is true.

The meaning of “extreme” depends on whether the alternative hypothesis is one-tailed or two-tailed:

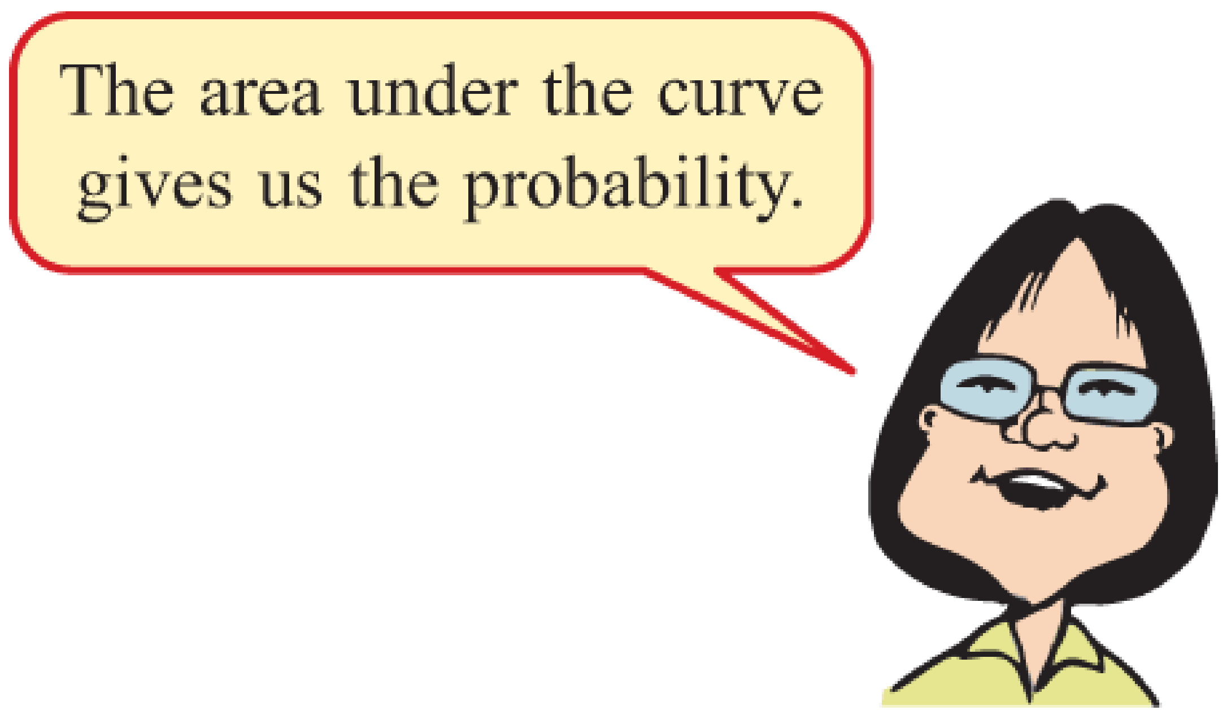
- If H_1 is a **one-tailed** alternative hypothesis, we use *one* tail of the t-distribution to calculate the p-value.
  - If H_1: $\mu < \mu_0$, we use the lower tail.
  - If H_1: $\mu > \mu_0$, we use the upper tail.



p**-value** $= \mathbf{P}(T \leqslant t)$



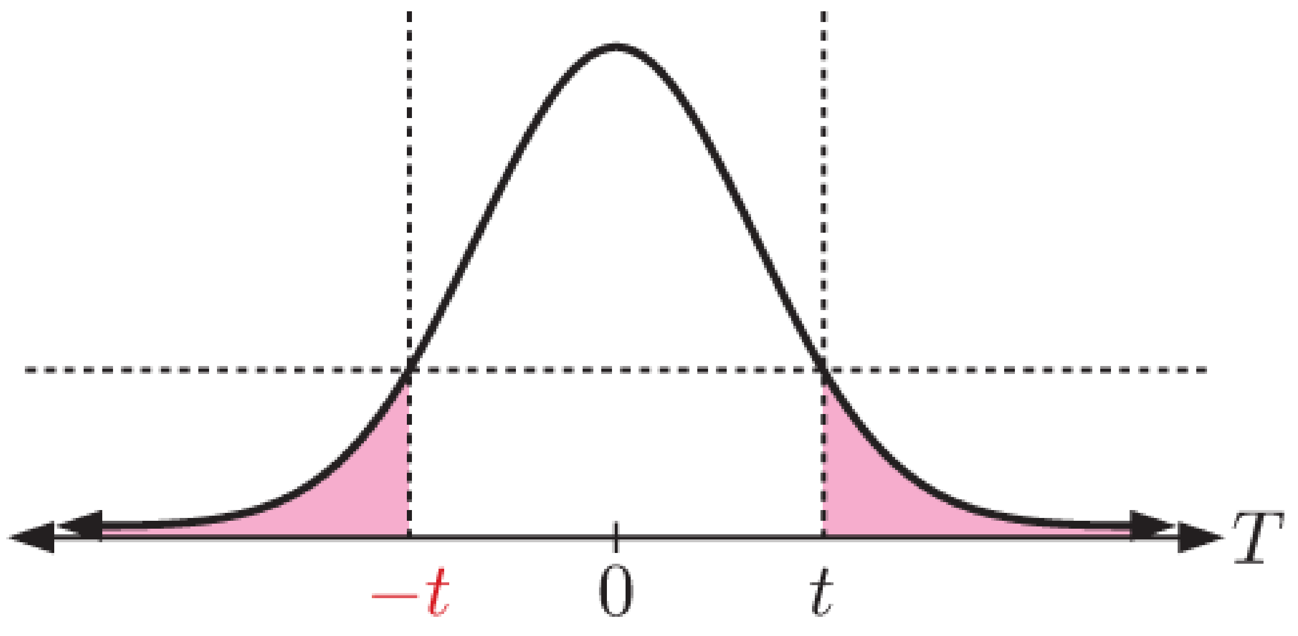
p**-value** $= \mathbf{P}(T \geqslant t)$


The area under the curve gives us the probability.

- If H_1 is the **two-tailed** alternative hypothesis H_1: $\mu \neq \mu_0$, then we must consider *both* tails of the t-distribution. We define “extreme” values as those which have probability less than or equal to that of the test statistic.

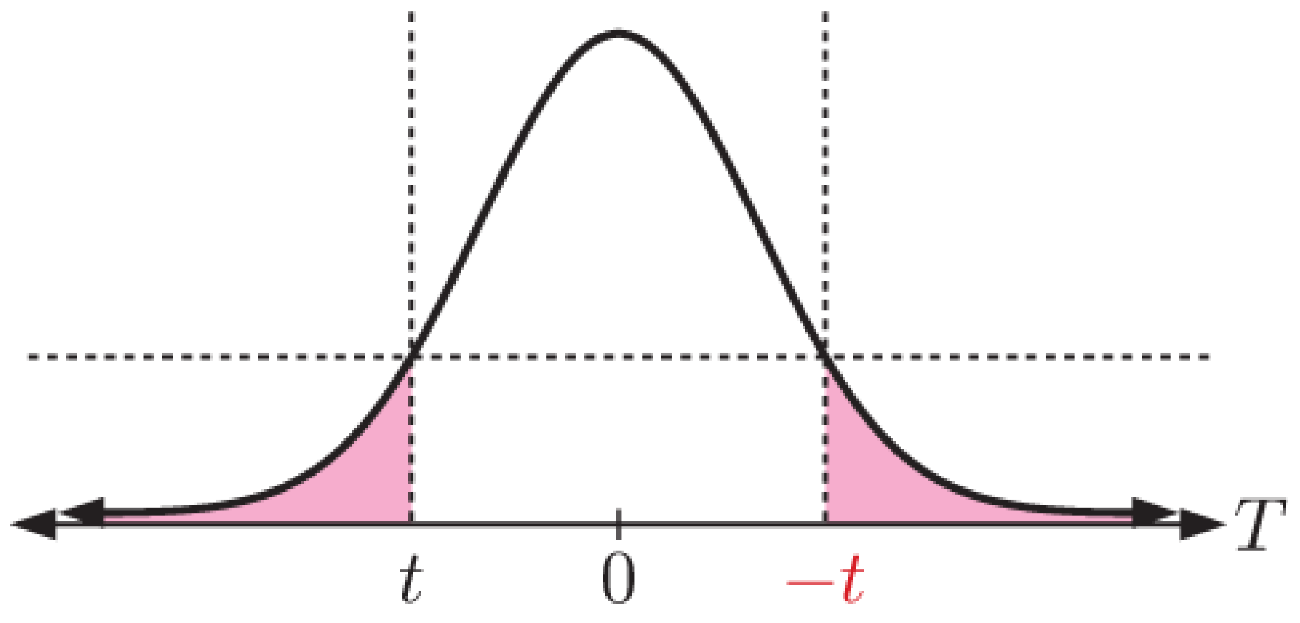
If $t \geqslant 0$,

$$\begin{aligned} p\text{-value} &= \mathrm{P}(T \geqslant t \ \text{ or } \ T \leqslant -t) \\ &= 2 \times \mathrm{P}(T \geqslant t) \qquad \{\text{symmetry}\} \end{aligned}$$



If $t < 0$,

$$\begin{aligned} p\text{-value} &= \mathrm{P}(T \geqslant -t \ \text{ or } \ T \leqslant t) \\ &= 2 \times \mathrm{P}(T \geqslant -t) \qquad \{\text{symmetry}\} \end{aligned}$$



So, for a two-tailed alternative hypothesis, p**-value** $= \mathbf{2 \times P}(T \geqslant |\,t\,|)$.

Click on the icon for instructions on how to calculate probabilities for the t-distribution.

GRAPHICS CALCULATOR INSTRUCTIONS

MAKING DECISIONS

Although “extreme” values are unlikely to occur, it is still *possible* to observe such values in a sample. We therefore need a rule which defines how much evidence is required to reject the null hypothesis.

The **significance level** α of a statistical hypothesis test is the largest p-value that would result in rejecting H_0. Any p-value less than or equal to α results in H_0 being rejected.

If a statistical hypothesis test has significance level α, the probability of *incorrectly* rejecting H_0 is α.

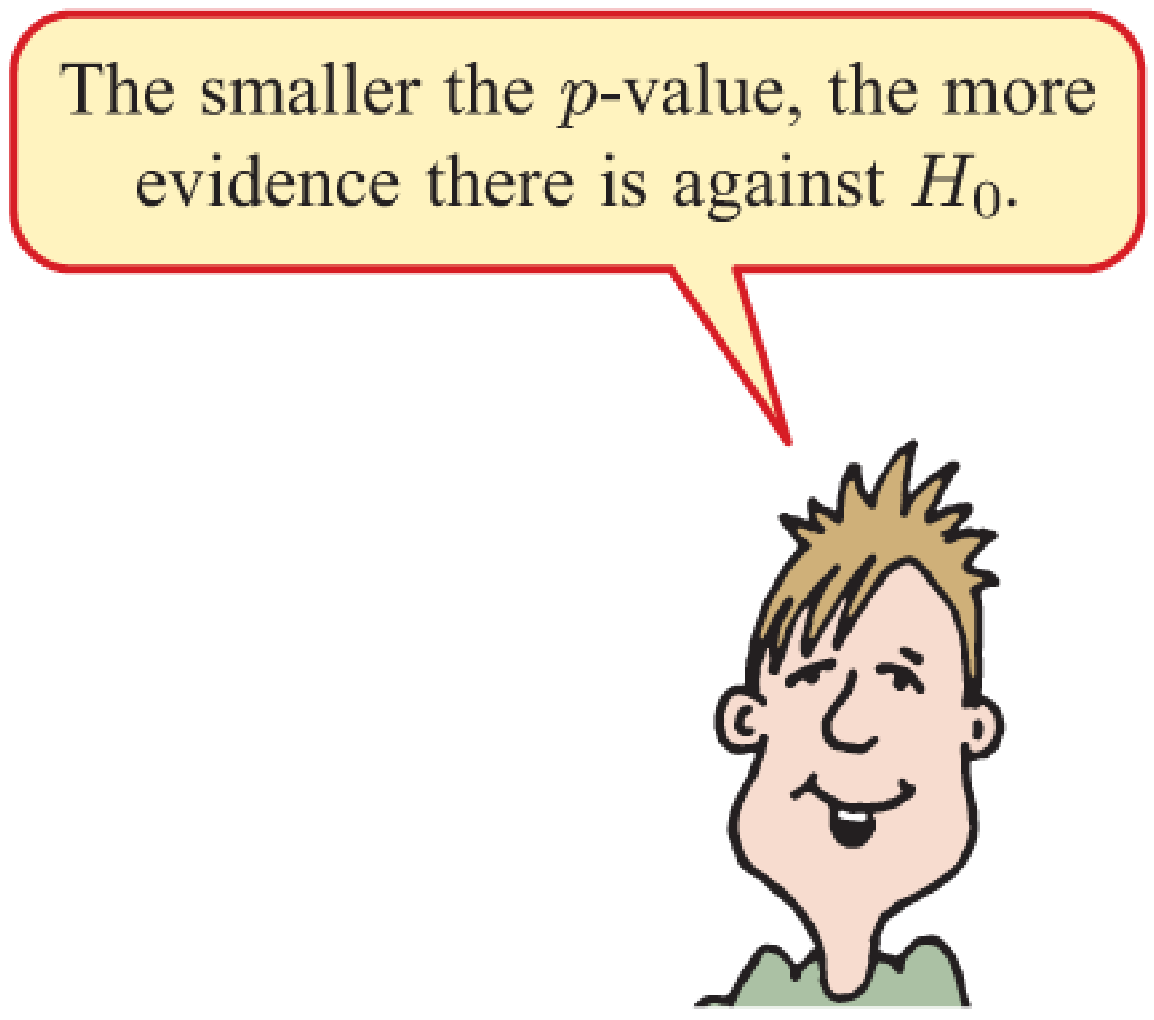
The significance level may be given as a decimal or as a percentage.

In our insect repellent example, suppose the researcher was concerned about any change in the average protection time. The researcher decides to test the hypothesis H_0: $\mu = 6$ against H_1: $\mu \neq 6$ at a significance level $\alpha = 0.01$.

Using the test statistic $t \approx 3.39$ and $T \sim t_{50-1}$,

$$\text{the } p\text{-value} \approx 2 \times \mathrm{P}(T \geqslant |\,3.39\,|)$$
$$\approx 0.001\,37$$

Since the p-value is less than the significance level α, the researcher has enough evidence to reject H_0.


The smaller the p-value, the more evidence there is against H_0.

Important:

We **must** choose a significance level **before** we test the hypothesis. Otherwise, we can be tempted to choose a significance level to give the test outcome that we desire. For example, it is *not appropriate* to calculate a p-value and then select a value of α so that H_0 will be rejected.

SUMMARY OF STEPS FOR STUDENT'S t-TEST FOR A POPULATION MEAN

Step 1: State the **null hypothesis** H_0: $\mu = \mu_0$ and **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the value of the **test statistic** $t = \dfrac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$.

Step 4: Calculate the ***p*-value** using $T \sim t_{n-1}$ as follows:

- If H_1: $\mu > \mu_0$, p-value $= \mathrm{P}(T \geqslant t)$.
- If H_1: $\mu < \mu_0$, p-value $= \mathrm{P}(T \leqslant t)$.
- If H_1: $\mu \neq \mu_0$, p-value $= 2 \times \mathrm{P}(T \geqslant |\,t\,|)$.

Step 5: Reject H_0 if p-value $< \alpha$.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Example 1 **Self Tutor**

The manager of a restaurant chain goes to a seafood wholesaler and inspects a large catch of over 50 000 prawns. It is known that the population is normally distributed. He will buy the catch if the mean weight exceeds 55 grams per prawn.

A random sample of 60 prawns is taken. The sample mean weight is 56.2 grams with standard deviation 4.2 grams.

Conduct a one-tailed hypothesis test with significance level $\alpha = 0.05$ to determine whether the manager should purchase the catch.

Step 1: Let μ be the population mean weight per prawn.
The hypotheses that should be considered are:
H_0: $\mu = 55$ {the mean weight is 55 grams per prawn}
H_1: $\mu > 55$ {the mean weight exceeds 55 grams per prawn}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 56.2$ grams, $s = 4.2$ grams, $n = 60$

The value of the test statistic is

$$t = \frac{56.2 - 55}{\frac{4.2}{\sqrt{60}}} \approx 2.21$$

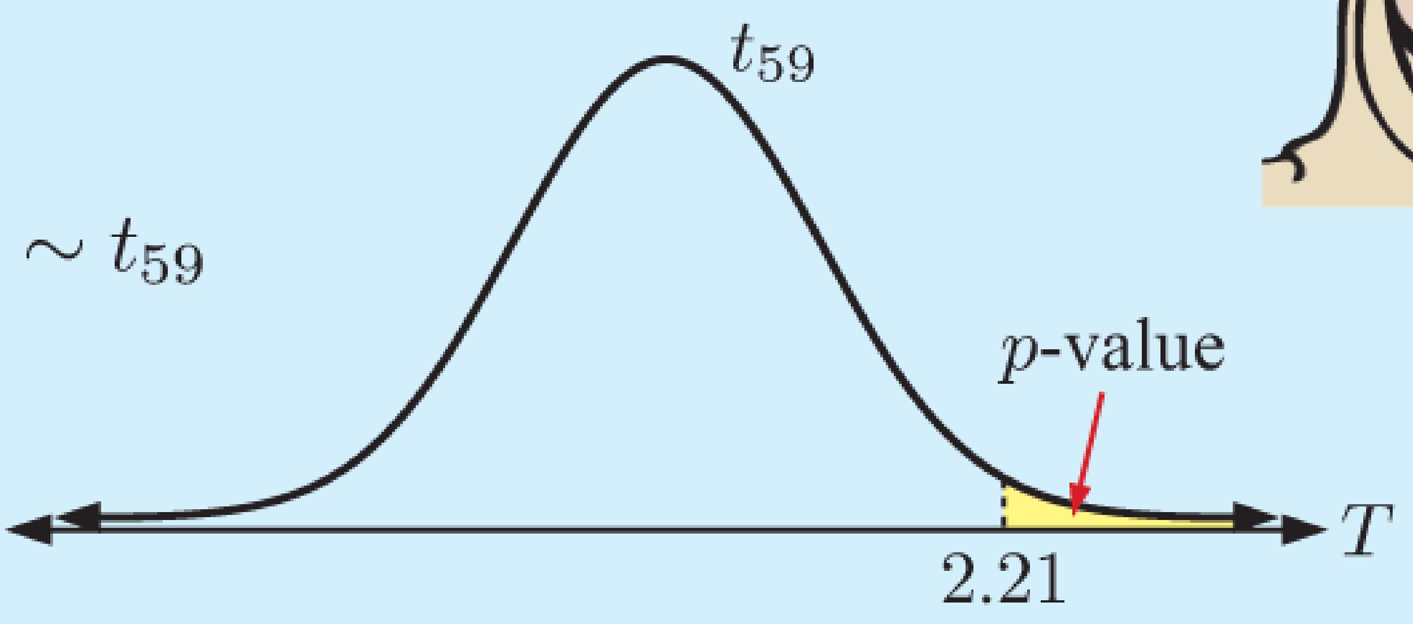
In examinations you can calculate t using your calculator.

Step 4: Since H_1: $\mu > 55$,

p-value $= \mathrm{P}(T \geqslant t)$ where $T \sim t_{59}$

$\approx \mathrm{P}(T \geqslant 2.21)$

≈ 0.0154



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have accepted H_1, we conclude that the mean weight exceeds 55 grams per prawn. The manager should purchase the catch.

Example 2 **Self Tutor**

The fat content (in grams) of 30 randomly selected pasties at the local bakery was recorded:

15.1	14.8	13.7	15.6	15.1	16.1	16.6	17.4	16.1	13.9
17.5	15.7	16.2	16.6	15.1	12.9	17.4	16.5	13.2	14.0
17.2	17.3	16.1	16.5	16.7	16.8	17.2	17.6	17.3	14.8

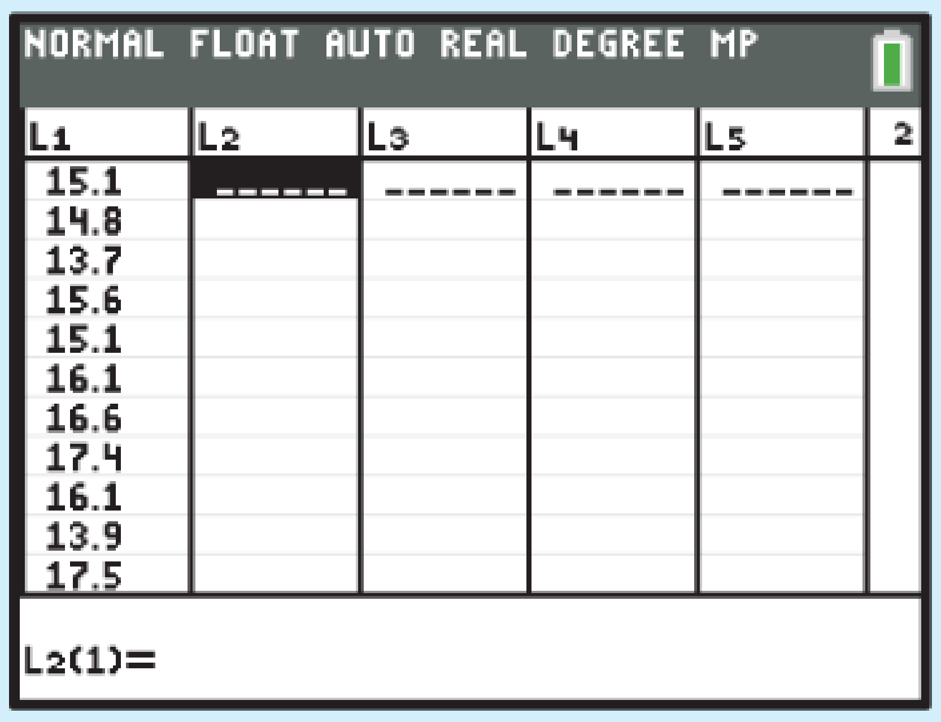
For a mean fat content of pasties made at this bakery μ, conduct a two-tailed t-test of H_0: $\mu = 16$ grams on a 10% level of significance.

Step 1: H_0: $\mu = 16$ {the mean fat content is 16 grams}

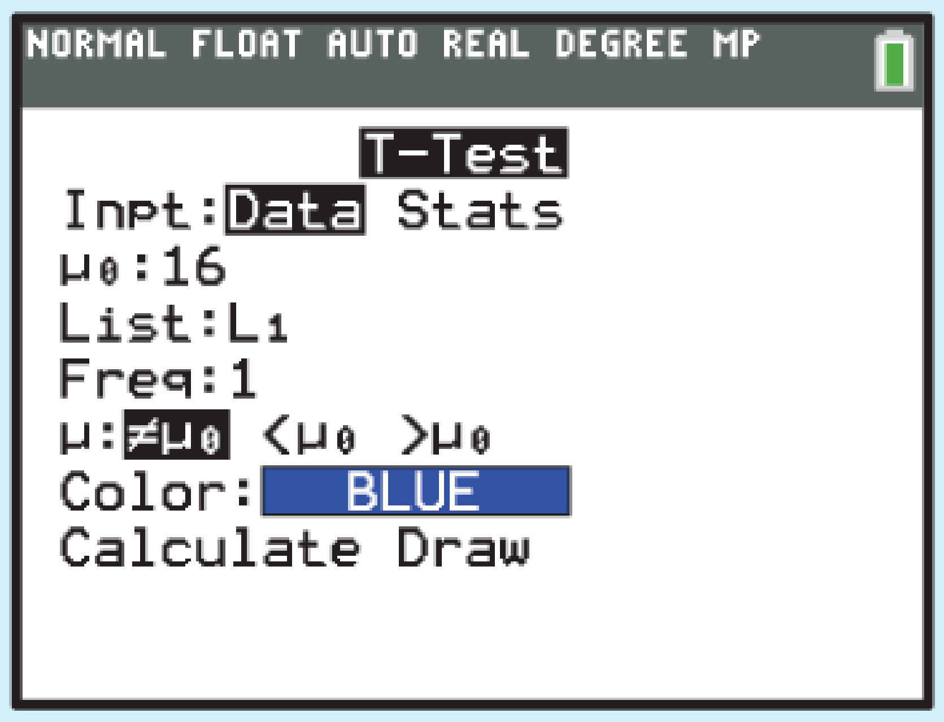
H_1: $\mu \neq 16$ {the mean fat content is *not* 16 grams}

Step 2: The significance level is $\alpha = 0.1$.

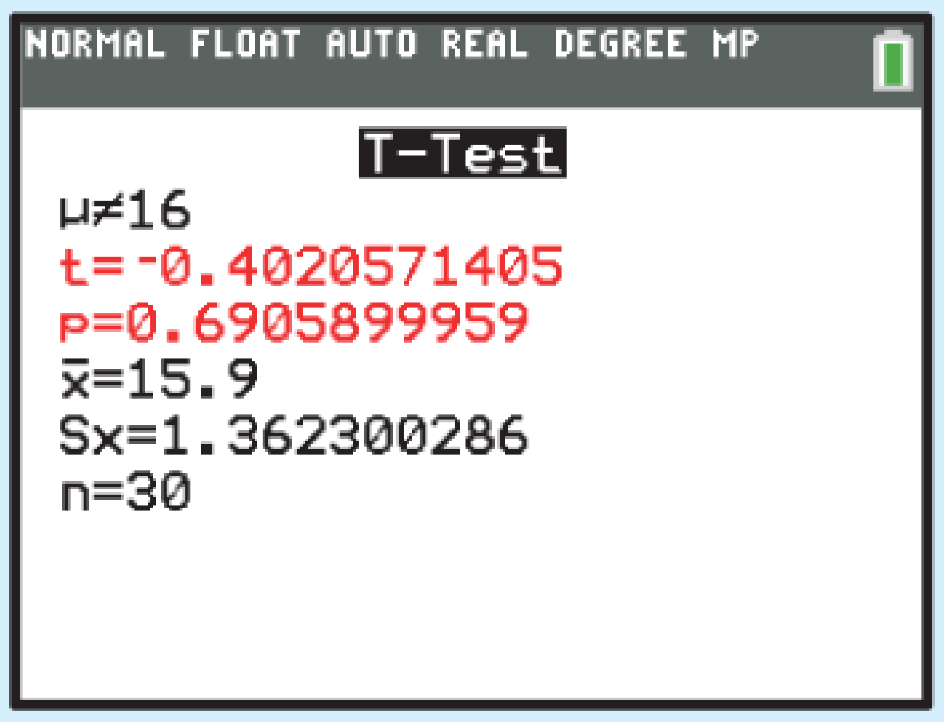
Steps 3 and 4:



```
NORMAL FLOAT AUTO REAL DEGREE MP
L1    L2     L3     L4     L5     2
15.1  ------ ------ ------ ------
14.8
13.7
15.6
15.1
16.1
16.6
17.4
16.1
13.9
17.5
L2(1)=
```

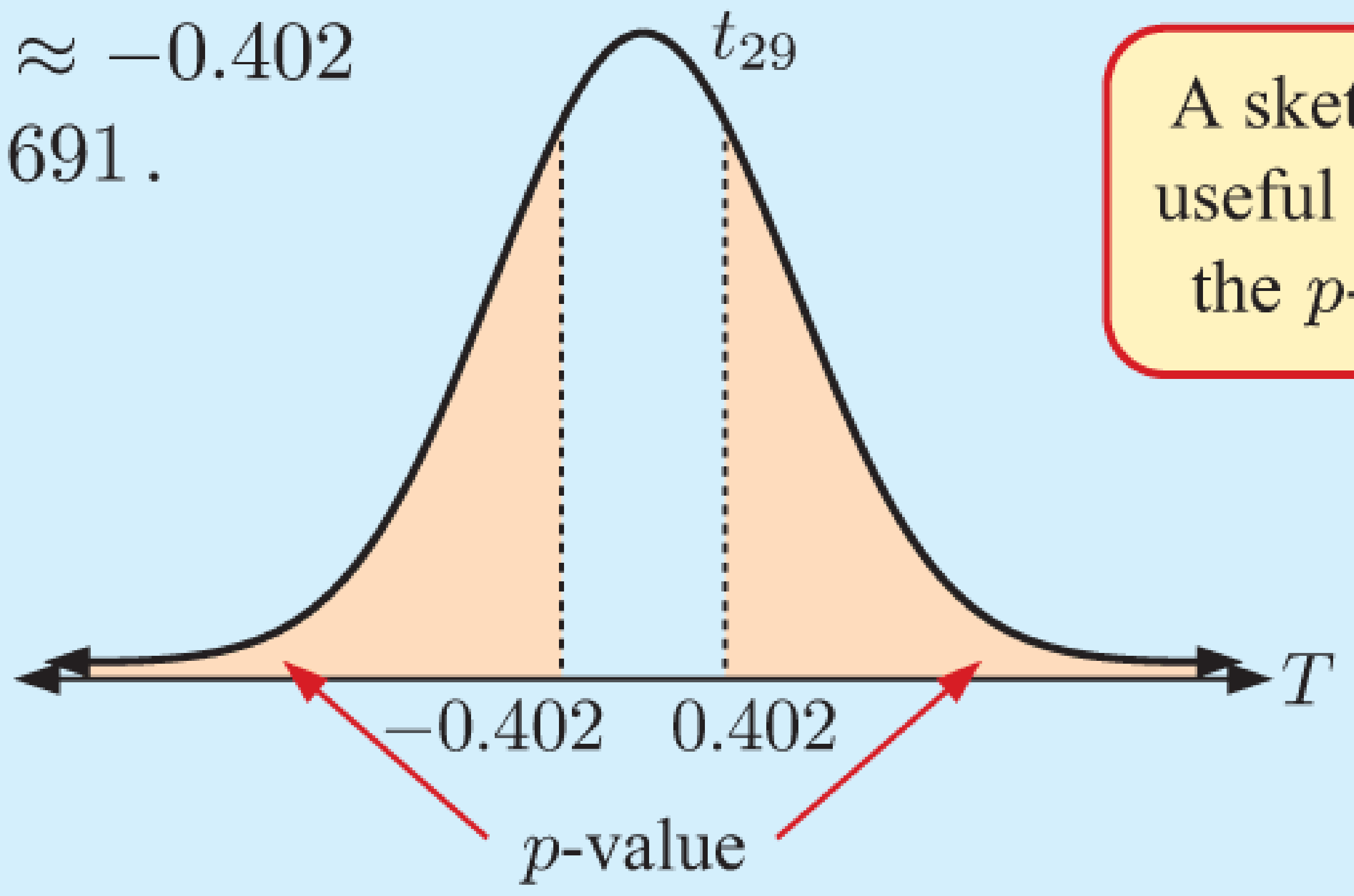


```
NORMAL FLOAT AUTO REAL DEGREE MP
T-Test
Inpt:Data Stats
μ0:16
List:L1
Freq:1
μ:≠μ0 <μ0 >μ0
Color: BLUE
Calculate Draw
```

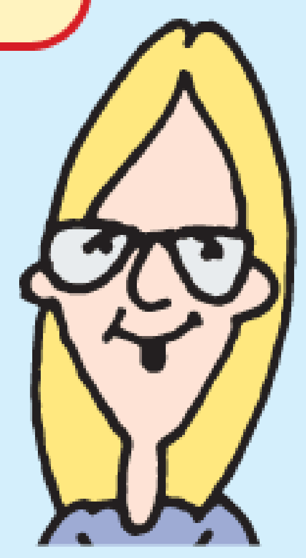


```
NORMAL FLOAT AUTO REAL DEGREE MP
T-Test
μ≠16
t=-0.4020571405
p=0.6905899959
x̄=15.9
Sx=1.362300286
n=30
```

Using technology, $t \approx -0.402$ and the p-value ≈ 0.691.



A sketch of the curve is useful to remind us what the p-value represents.



Step 5: Since p-value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 10% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude that the mean fat content is appreciably different from 16 grams.

EXERCISE 16B

1 A population has known standard deviation $\sigma = 3.97$. A sample of size 36 is taken and the sample mean $\overline{x} = 23.75$. We are required to test the hypothesis H_0: $\mu = 25$ against H_1: $\mu < 25$.

a Find: **i** the test statistic **ii** the p-value.

b What decision should be made at a 5% level?

2 A statistician believes that a population has a mean μ that is greater than 80. To test this he takes a random sample of 200 measurements, and finds the sample mean is 83.1 and the sample standard deviation is 12.9. He then performs a hypothesis test with significance level $\alpha = 0.01$.

a Write down the null and alternative hypotheses.

b Find the value of the test statistic.

c Calculate the p-value.

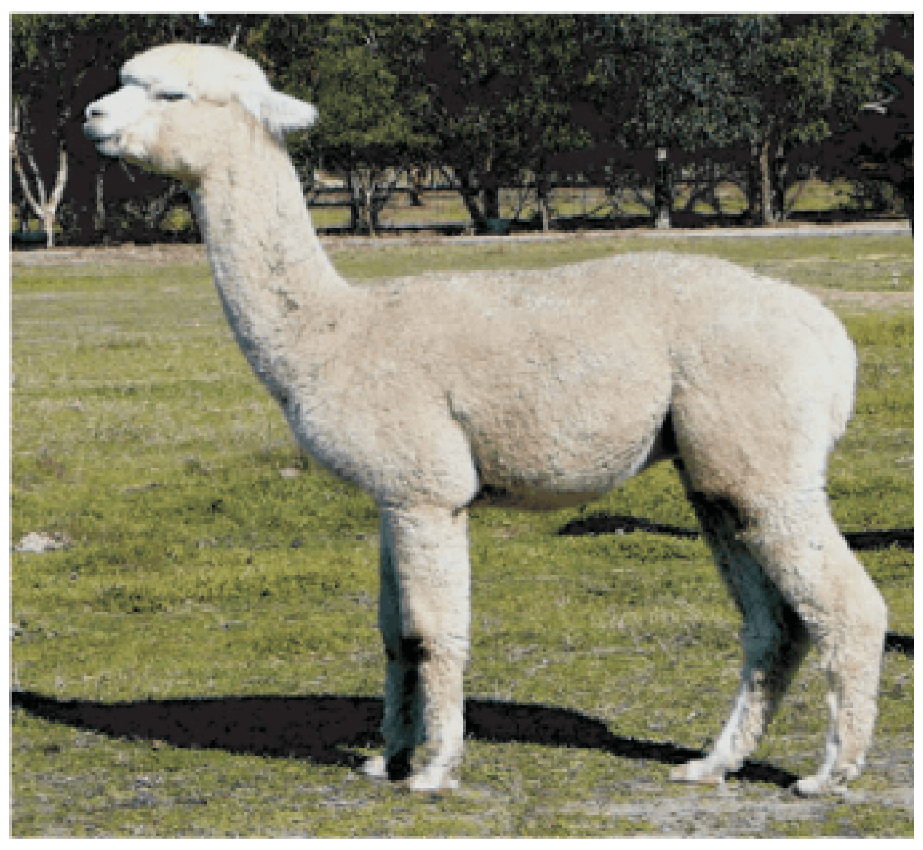
d Make a decision to reject or not reject H_0.

e State the conclusion for the test.

3 Bags of salted cashew nuts state their net contents is 100 g. A customer claims that the bags have been lighter in recent purchases, so the factory quality control manager decides to investigate. He samples 40 bags and finds that their mean weight is 99.4 g with standard deviation 1.6 g.

Perform a hypothesis test at the 5% level of significance to determine whether the customer's claim is valid.

4 An alpaca breeder wants to produce fleece which is extremely fine. In 2015, his herd had mean fineness 20.3 microns. In 2019, a sample of 80 alpacas from the herd was randomly selected, and the mean fineness was 19.2 microns with standard deviation 2.89 microns. Perform a two-tailed hypothesis test at the 5% level of significance to determine whether the herd fineness has changed.



5 The length of screws produced by a machine is known to be normally distributed. The machine is supposed to produce screws with mean length $\mu = 2.00$ cm. A quality controller selects a random sample of 15 screws. She finds that the mean length of the 15 screws is $\overline{x} = 2.04$ cm with sample standard deviation $s = 0.09$ cm. Does this justify the need to adjust the machine on a 2% level of significance?

6

A machine packs sugar into 1 kg bags. It is known that the masses of the bags of sugar are normally distributed. A random sample of eight filled bags was taken and the masses of the bags measured to the nearest gram. Their masses in grams were:

1001, 998, 999, 1002, 1001, 1003, 1002, 1002.

Perform a test at the 1% level, to determine whether the machine under-fills the bags.

7 A market gardener claims that the carrots in his field have a mean weight of more than 50 grams. A prospective buyer will purchase the crop if the market gardener's claim is true. To test this she pulls 20 carrots at random, and finds that their individual weights in grams are:

57.6	34.7	53.9	52.5	61.8	51.5	61.3	49.2	56.8	55.9
57.9	58.8	44.3	58.3	49.3	56.0	59.5	47.0	58.0	47.2

a Explain why it is reasonable to assume that the carrots' weights are normally distributed.

b Determine whether the buyer will purchase the crop using a 5% level of significance.

INVESTIGATION 2 MULTIPLE TESTING AND STATISTICAL FALLACY

In many applications of hypothesis testing, it is common to conduct multiple identical or very similar hypothesis tests simultaneously. For example, in genetics an experiment called a **DNA microarray** is used to measure expression levels of thousands of genes, each with their own set of hypotheses to test.

In this Investigation, we will explore the effects of conducting multiple hypothesis tests on the interpretation of individual outcomes.

What to do:

1 A normally distributed population has mean μ and standard deviation $\sigma = 5$.
Consider the following hypotheses for this population: H_0: $\mu = 2$
H_1: $\mu \neq 2$

Click on the icon to run a computer simulation which generates samples of size 10 from the $N(2, 5^2)$ distribution. The above hypotheses are tested for each sample at a significance level of α, and a p-value is calculated.

SIMULATION

a Write down the formula used to calculate the test statistic given a sample mean $\overline{x}$ and sample standard deviation s.

b Set $\alpha = 0.05$. Copy and complete the following table by generating m samples and counting the number of times H_0 is rejected.

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20		
50		
100		
500		
1000		
5000		
10 000		

c Repeat **b** for:
i $\alpha = 0.1$ **ii** $\alpha = 0.025$ **iii** $\alpha = 0.01$.
Comment on your results.

2 For the simulation in **1**, explain why:

a H_0 is true in every sample that the simulation generates

b $P(\text{incorrectly reject } H_0) = \alpha$ for each sample

c the *expected number* of samples where H_0 is incorrectly rejected is $m\alpha$.

3 Sabeen is a psychologist. She is writing a journal article about the effect of diet cola on a person's ability to concentrate. To account for possible confounding factors, Sabeen divides her data into 10 different age groups for each gender. For each age group and gender, she conducts a hypothesis test and obtains a p-value.

Sabeen's results are shown in the table below:

Age group	10 - 14	15 - 19	20 - 24	25 - 29	30 - 34
Male	0.296	0.143	0.305	0.378	0.169
Female	0.814	0.022	0.125	0.301	0.432

Age group	35 - 39	40 - 44	45 - 49	50 - 54	55+
Male	0.699	0.221	0.078	0.790	0.423
Female	0.987	0.643	0.448	0.672	0.789

a Which age group and gender do you think Sabeen is most likely to report on in her journal article? Explain your answer.

b Sabeen's colleague Mysha repeats Sabeen's experiment. She samples people exclusively from the group you identified in **a**. Do you think Mysha is likely to replicate Sabeen's results for this group? Explain your answer.

THEORY OF KNOWLEDGE

Sir Ronald Aylmer Fisher was an English statistician and biologist. He was known for his work in both agriculture and statistics, combining the disciplines with his work in classical statistics and significance testing.

In 1952 Fisher published a book titled *Statistical Methods for Research Workers* which is best known for the following statement about p-values:

> "*The value for which* $p = 0.05$, *or* 1 *in* 20 [....] *it is convenient to take this point as a limit in judging whether a deviation is to be considered significant or not.*"

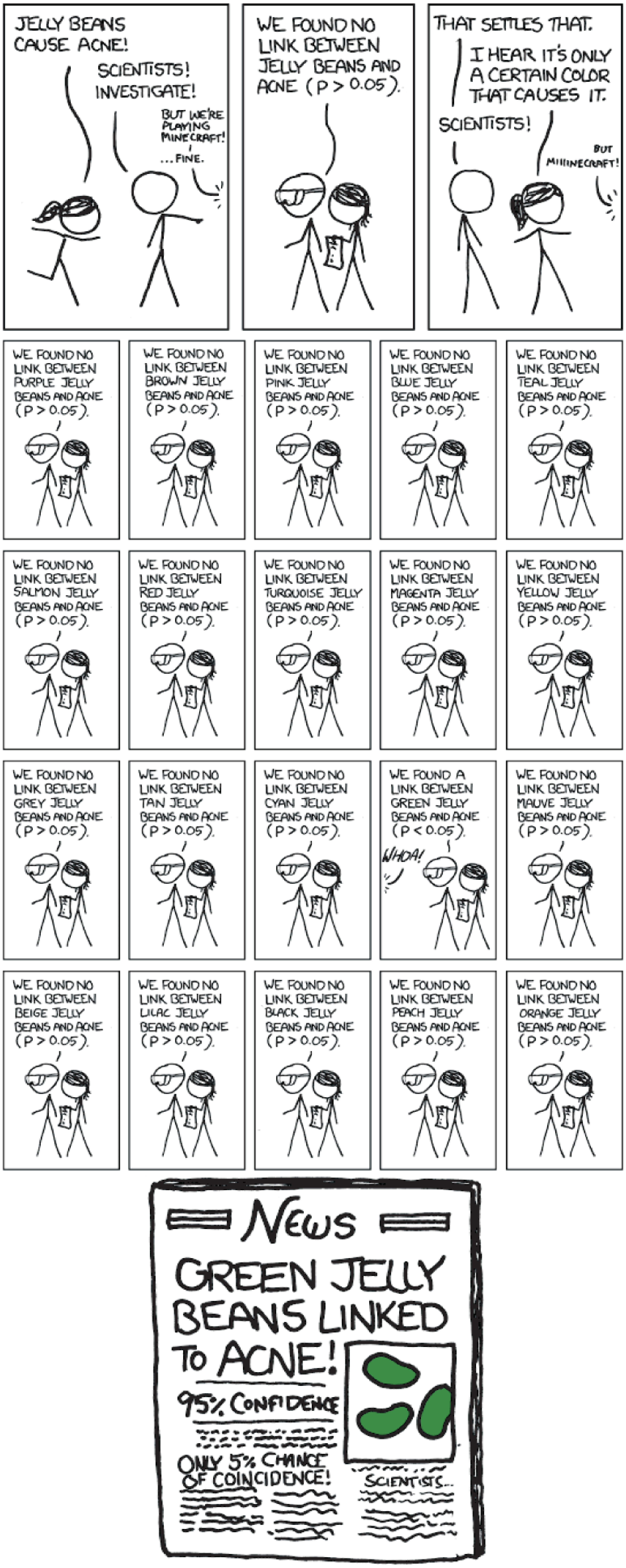
Today, a significance level of $\alpha = 0.05$ is still widely quoted in scientific journals when testing for significance.

Sir Ronald Fisher

1 If someone tells you that they are 95% confident in something, do you normally stop to consider the 5% chance that the person is wrong? Do you think you *should*?

2 Suppose a researcher is writing a report for a medical journal.

- **a** Is it realistic to expect the researcher to be 100% confident in their findings?
- **b** What is *your* expectation of how confident a researcher should be, in order that they publish their results?
- **c** Do you think it is important that the researcher tells you how confident they are?


JELLY BEANS CAUSE ACNE!
SCIENTISTS! INVESTIGATE!
BUT WE'RE PLAYING MINECRAFT!
... FINE.
WE FOUND NO LINK BETWEEN JELLY BEANS AND ACNE (P > 0.05).
THAT SETTLES THAT.
I HEAR IT'S ONLY A CERTAIN COLOR THAT CAUSES IT.
SCIENTISTS!
BUT MIIIINECRAFT!
WE FOUND NO LINK BETWEEN PURPLE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BROWN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PINK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLUE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TEAL JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN SALMON JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN RED JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TURQUOISE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN MAGENTA JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN YELLOW JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN GREY JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN CYAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND A LINK BETWEEN GREEN JELLY BEANS AND ACNE (P < 0.05).
WHOA!
WE FOUND NO LINK BETWEEN MAUVE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BEIGE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN LILAC JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLACK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PEACH JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN ORANGE JELLY BEANS AND ACNE (P > 0.05).
NEWS
GREEN JELLY BEANS LINKED TO ACNE!
95% CONFIDENCE
ONLY 5% CHANCE OF COINCIDENCE!
SCIENTISTS...

C THE TWO-SAMPLE t-TEST FOR COMPARING POPULATION MEANS

Consider Frank's tomatoes in the **Opening Problem** on page **382**. We could test the effectiveness of the fertiliser by letting the mean of last year's sample be μ_0 in a t-test of this year's sample.

However, there are two problems with this approach:

- In general, the sample mean will not be exactly equal to the population mean.
- The variation of last year's sample is ignored.

If we are given the data for **two** samples, we use the **two-sample *t*-test** instead.

Suppose we have two populations with means μ_1 and μ_2 that we wish to compare.

Our null hypothesis would state that the means are equal:

$$H_0\colon\ \mu_1 = \mu_2$$

or equivalently, the *difference* between the means is zero:

$$H_0\colon\ \mu_1 - \mu_2 = 0 \qquad \text{or} \qquad H_0\colon\ \mu_2 - \mu_1 = 0$$

The formula for the test statistic t and the distribution used to calculate the p-values are beyond the scope of this course, but we can use technology to calculate t and the p-value.

GRAPHICS CALCULATOR INSTRUCTIONS

The two-sample t-test follows the same general procedure as the single sample version on page **387**.

Example 3 **Self Tutor**

A pharmaceutical company has developed a new medication to help lower cholesterol levels.

To test the effectiveness of the new medication, volunteers were separated into two groups. 15 patients were given the old medication and 17 patients were given the new medication. The *reduction* in cholesterol levels after 4 weeks was measured for both samples. The results are shown below:

	Sample mean	*Sample standard deviation*
Old medication	0.351	0.058
New medication	0.497	0.077

Conduct a two-sample t-test to determine whether the new medication is *better* than the old medication using a 5% level of significance.

Step 1: Let μ_1 be the population mean cholesterol level reduction of the old medication group, and μ_2 be the population mean cholesterol level reduction of the new medication group.

We could have also written
H_0: $\mu_1 - \mu_2 = 0$
H_1: $\mu_1 - \mu_2 < 0$

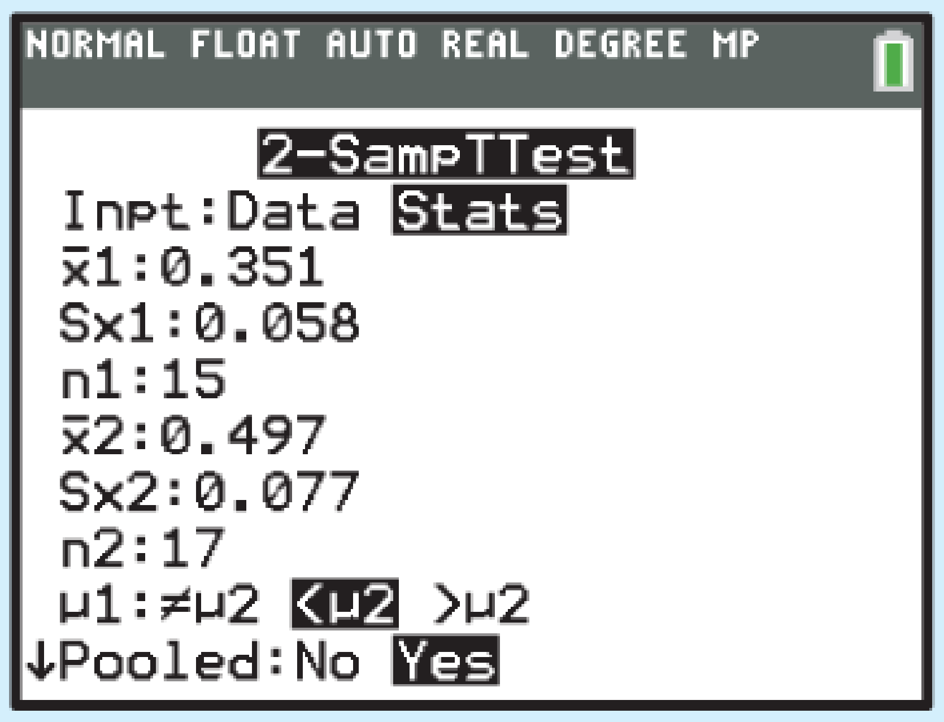
The hypotheses that should be considered are:

$H_0\colon\ \mu_1 = \mu_2$ {new medication is no better than the old medication}

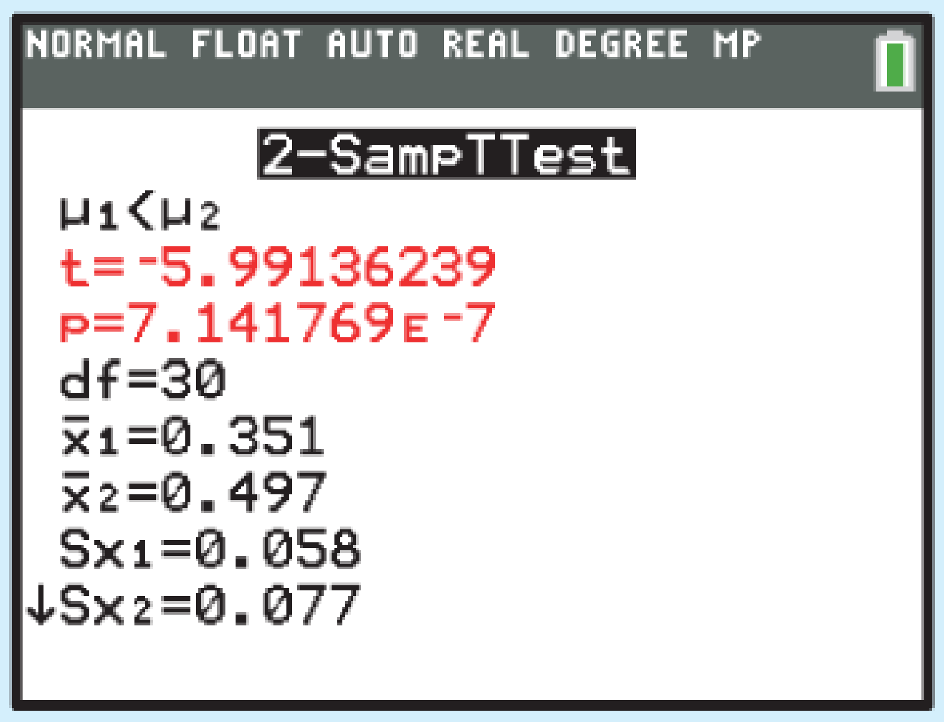
$H_1\colon\ \mu_1 < \mu_2$ {new medication is better than the old medication}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: Using technology, the value of the test statistic is $t \approx -5.99$.



```
NORMAL FLOAT AUTO REAL DEGREE MP
      2-SampTTest
 Inpt:Data Stats
 x̄1:0.351
 Sx1:0.058
 n1:15
 x̄2:0.497
 Sx2:0.077
 n2:17
 μ1:≠μ2 <μ2 >μ2
↓Pooled:No Yes
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
      2-SampTTest
 μ1<μ2
 t=-5.99136239
 p=7.141769E-7
 df=30
 x̄1=0.351
 x̄2=0.497
 Sx1=0.058
↓Sx2=0.077
```

Step 4: From the screenshots above, the p-value $\approx 7.14 \times 10^{-7}$.

Step 5: Since the p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Step 6: Since we have accepted H_1, we conclude that the new medication is better than the old medication.

POOLED VERSUS UNPOOLED TESTS

You should have noticed that your calculator will give you the option of performing a "pooled" or "unpooled" test.

In a pooled test, it is assumed that the variances of the two populations are the same. The samples are therefore combined or *pooled* to estimate this common variance when calculating t and the p-value.

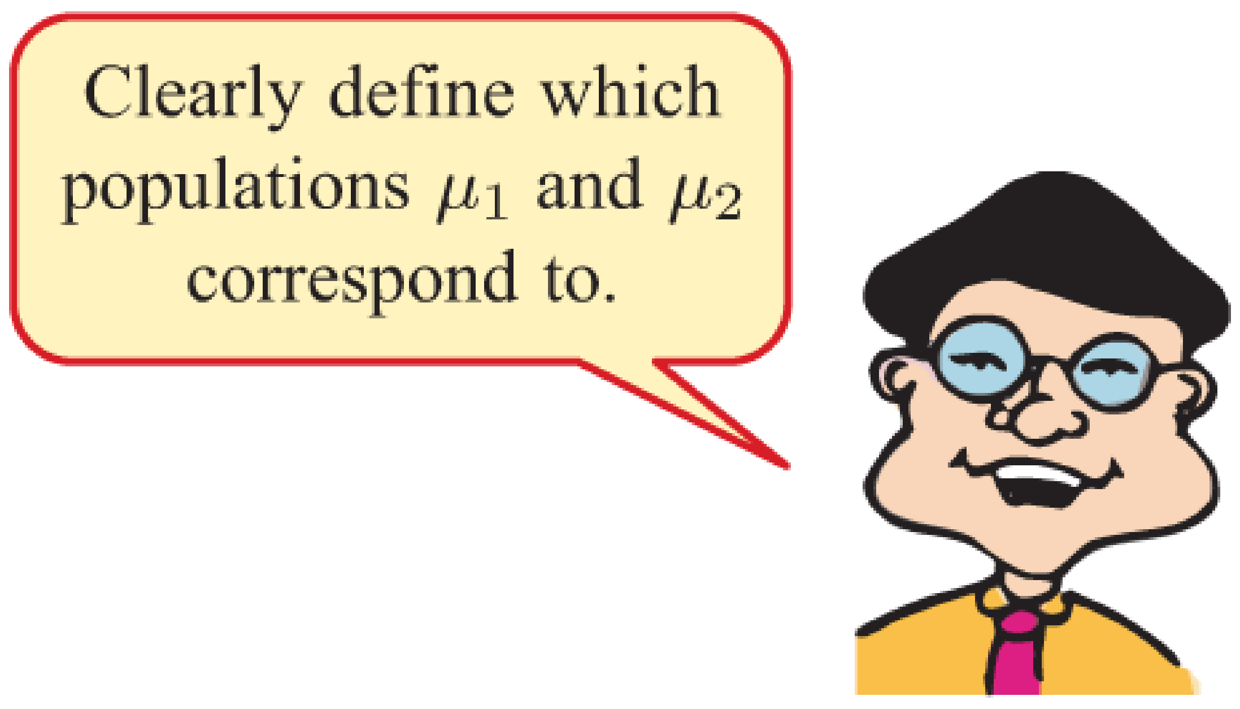
In an unpooled test, the variances are assumed to be different. We therefore use a different formula to calculate t and the p-value.

In this course you are expected to assume **equal variances** and hence use the **pooled two-sample *t*-test**.

EXERCISE 16C

1 Consider Frank's tomatoes in the **Opening Problem** on page **382**.

- **a** Write down the hypotheses that Frank should consider.
- **b** Conduct a two-sample t-test to determine whether the fertiliser was effective on a 1% level of significance.


Clearly define which populations μ_1 and μ_2 correspond to.

2 A researcher claims that on average, high school students sleep less than middle school students. The researcher recorded the sleeping times of 49 middle school students and 55 high school students.
For the middle school students, the mean daily time asleep was 8.0 hours with standard deviation 0.2 hours.
For the high school students, the mean daily time asleep was 7.9 hours with standard deviation 0.5 hours.
Conduct a two-sample t-test to test the researcher's claim on the 5% level.

3 Two groups of seedlings were grown with different brands of compost: A and B. After a period of time, the height (in cm) of each seedling was measured.

Brand A:	12.1	14.6	10.1	8.7	13.2	15.1	16.5	14.6
Brand B:	12.3	15.2	9.9	9.5	13.4	14.9	17.0	14.8

The manufacturer of *Brand B* guarantees it will improve the growth of seedlings more than *Brand A* compost. Conduct a two-sample t-test at a 5% level, to determine whether the claim is valid.

4 A mathematics tutor claims to significantly increase students' test results with a week of tutoring. To test this claim, 12 students were tested prior to receiving tutoring, and their results recorded. However, the students were not given the answers or their results. After a week of tutoring, the students repeated the test to see whether they had improved. The results were:

Before tutoring:	15	17	25	11	28	20	23	34	27	14	26	26
After tutoring:	20	16	25	18	28	19	26	37	31	13	27	20

Conduct a two-sample t-test at a 5% level of significance, to test the tutor's claim.

5 The following data show the times (in seconds) that Jesiah and Billy recorded for the 100 m sprint over the last fortnight:

Jesiah:	12.3	13.0	11.3	11.5	14.3	12.0	14.5
	13.7	12.9	11.9	12.1	12.6	11.3	12.7
Billy:	10.4	12.0	11.7	13.2	11.2	11.1	11.1
	12.0	11.4	12.5	12.1	12.0		

a Conduct a two-sample t-test to determine whether there is a significant difference between the runners' times on the 5% level.

b Which runner do you think is faster? Explain your answer.

D THE χ^2 GOODNESS OF FIT TEST

The hypothesis tests we have studied so far have considered a population mean.

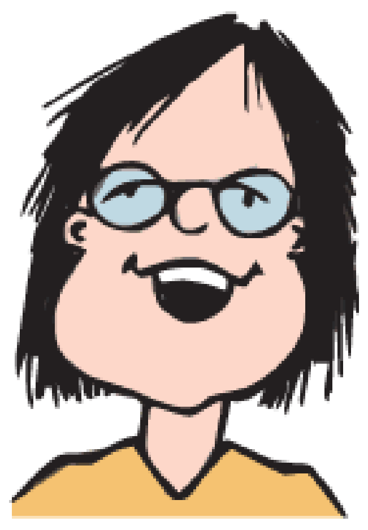
We can also consider claims about **population proportions**.

For example, suppose Rico rolled a die 60 times and obtained the rolls in the table. Since the relative frequencies or the *proportions* of the outcomes are quite different, Rico claims that his die is *biased*.

In this Section, we will study the **χ^2 (chi-squared) goodness of fit test** and see how it can be used to test hypotheses about population proportions.

Number	*Frequency*
1	20
2	10
3	5
4	8
5	7
6	10

The Greek letter χ is written as "chi" and pronounced "ki".



THE HYPOTHESES

For Rico's die, suppose we let p_1 be the probability of rolling a 1, p_2 be the probability of rolling a 2, and so on.

Our null hypothesis H_0 is that the die is fair. If this is the case, then each number is *equally likely* to occur on each roll. We therefore have H_0: $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{6}$, ..., $p_6 = \frac{1}{6}$.

Our alternative hypothesis H_1 is that the die is *not* fair. If this is the case, then at least one outcome has a probability which is different from the others. We therefore have
H_1: at least one of p_1, p_2, ..., $p_6 \neq \frac{1}{6}$.

Consider a scenario with k categories. Let p_i be the population proportion of individuals in category i, where $p_1 + p_2 + ... + p_k = 1$.

The **hypotheses** in a χ^2 goodness of fit test have the form:

$$H_0\colon\ p_1 = p_{01},\ \ p_2 = p_{02},\ \ ....,\ \ \text{and}\ \ p_k = p_{0k}$$

$$H_1\colon\ \text{at least one of}\ \ p_i \neq p_{0i}$$

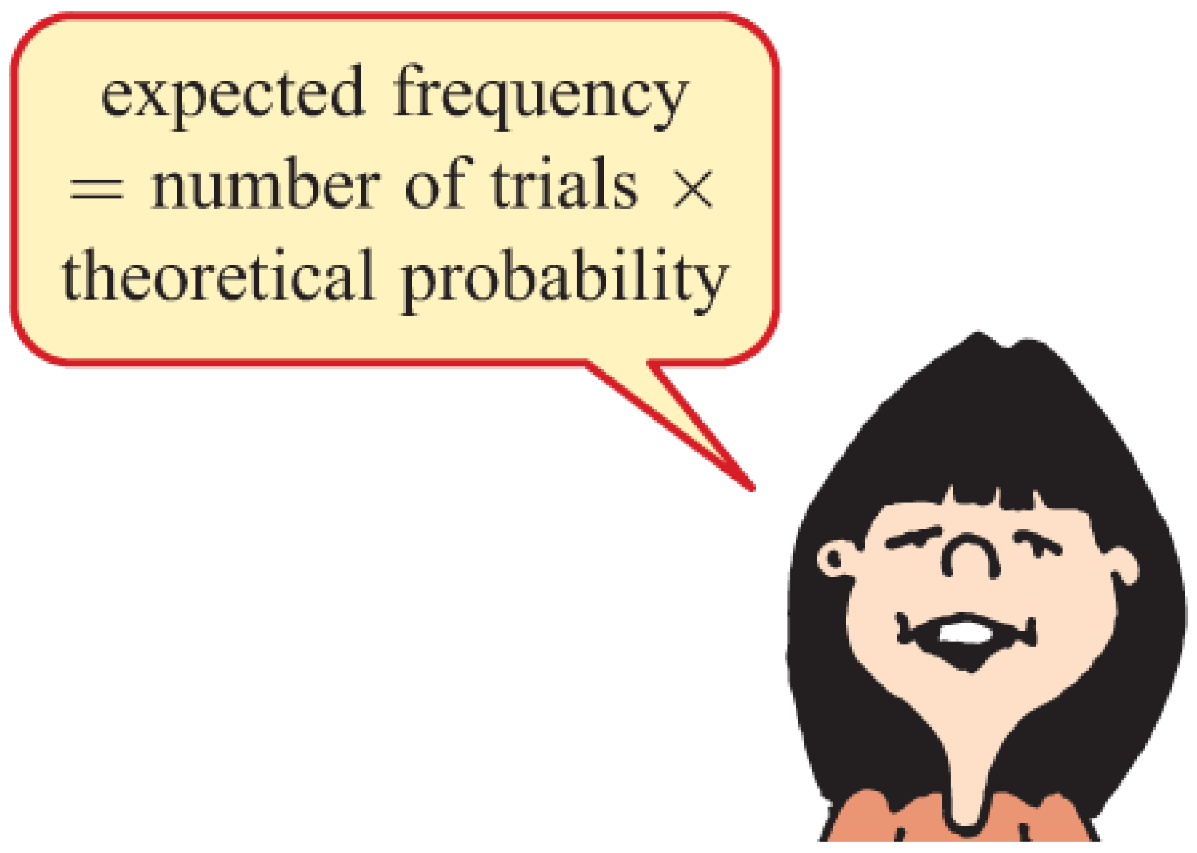
where p_{0i} is the population proportion of category i under the null hypothesis.

In the die rolling example on the previous page, $p_{01}, p_{02}, ..., p_{06}$ are all equal to $\frac{1}{6}$. However, in general the population proportions under H_0 do not have to all be the same.

THE TEST STATISTIC

In our study of probability we calculated the number of times we *expect* an event to occur given its theoretical probability.

For example, if Rico's die was fair then we would expect to see $60 \times \frac{1}{6} = 10$ of each number.



We are therefore interested in how the *observed* frequencies differ from their expected values.

The **test statistic** for a χ^2 goodness of fit test is:

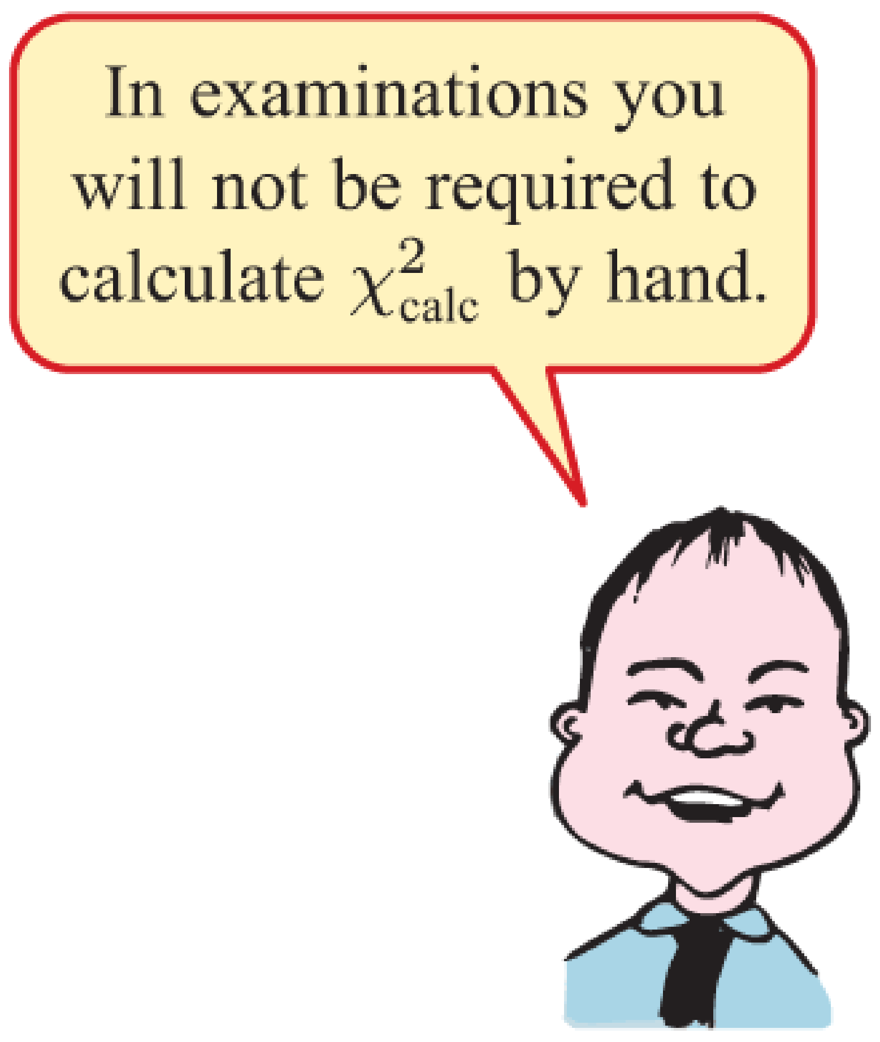
$$\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$$

where f_{obs} is an **observed** frequency

f_{exp} is an **expected** frequency.

For the die rolling example, we can calculate the test statistic χ^2_{calc} with the help of a table:

Number	f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
1	20	10	10	100	10
2	10	10	0	0	0
3	5	10	-5	25	2.5
4	8	10	-2	4	0.4
5	7	10	-3	9	0.9
6	10	10	0	0	0
				Total	13.8


In examinations you will not be required to calculate χ^2_{calc} by hand.

So, $\chi^2_{\text{calc}} = 13.8$.

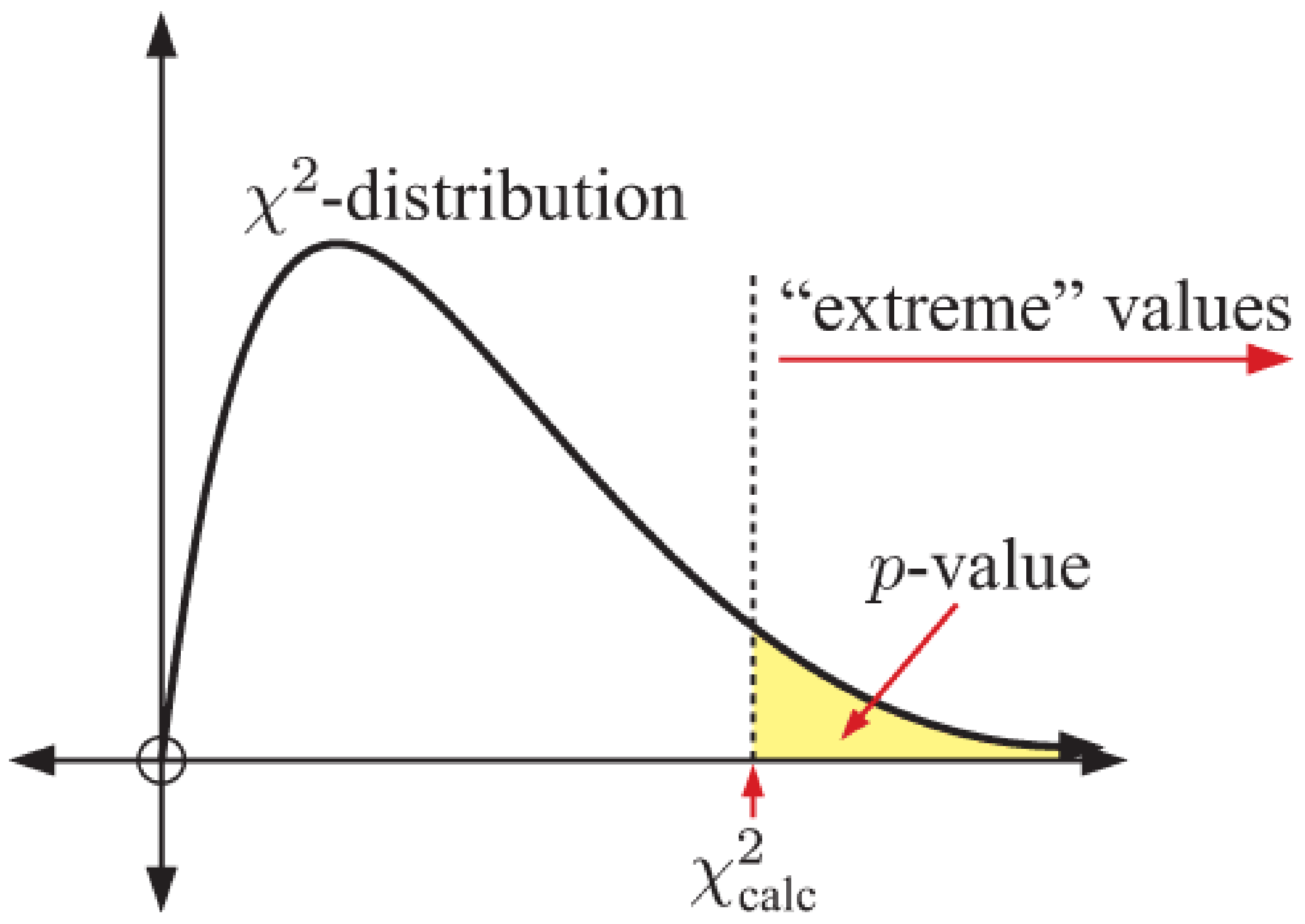
THE p-VALUE

In order to make a decision on whether or not to reject H_0 based on χ^2_{calc}, we need to calculate a ***p*-value** and compare it to the **significance level** α of the test.

For a t-test, we calculated the p-value as the probability of observing a value that is as "extreme" as the test statistic. Depending on the form of the alternative hypothesis H_1, this would correspond to the upper tail, lower tail, or both tails of the t-distribution.

In the goodness of fit test, if H_0 was not true then the difference between the observed frequencies and expected frequencies would be large. Hence the value of χ^2_{calc} would also be large. So, for the χ^2 goodness of fit test:

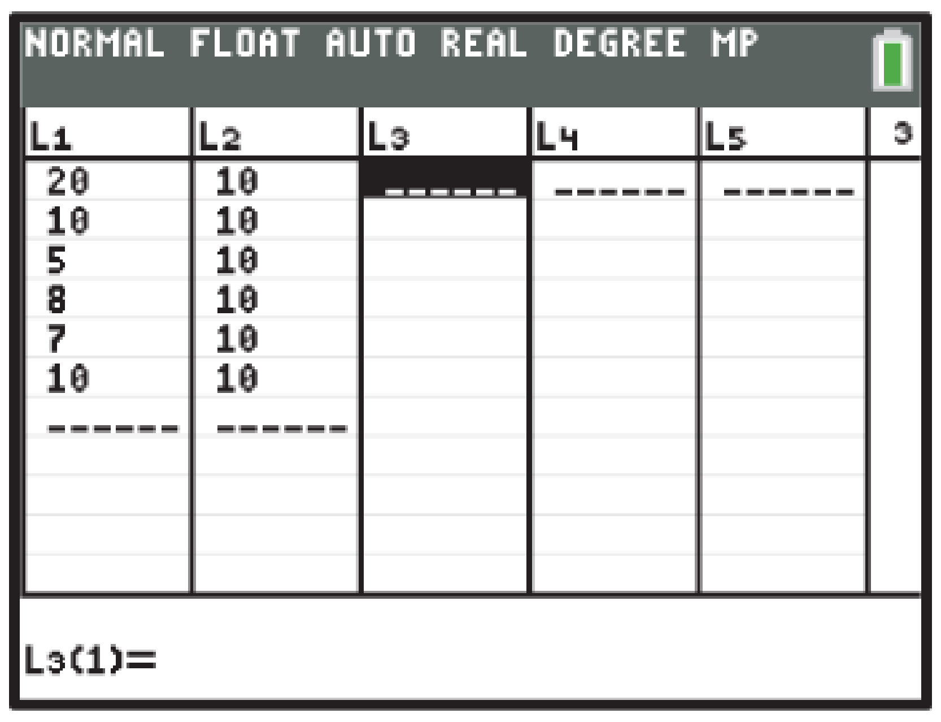
***p*-value** = probability of observing a value **greater than or equal to** χ^2_{calc}.



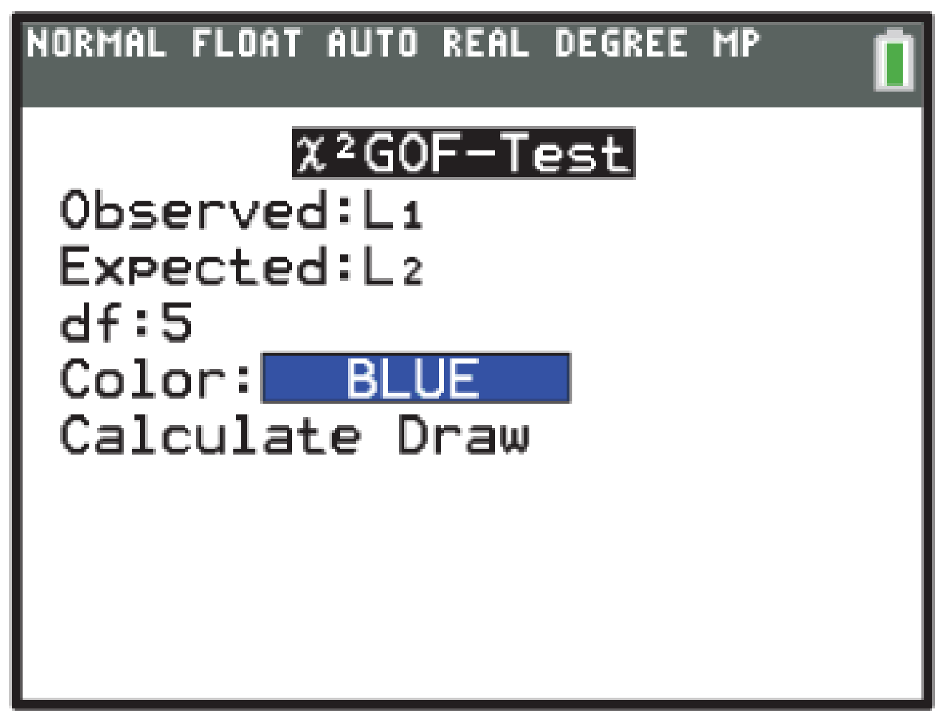
You can use your calculator to evaluate p-values.

GRAPHICS CALCULATOR INSTRUCTIONS

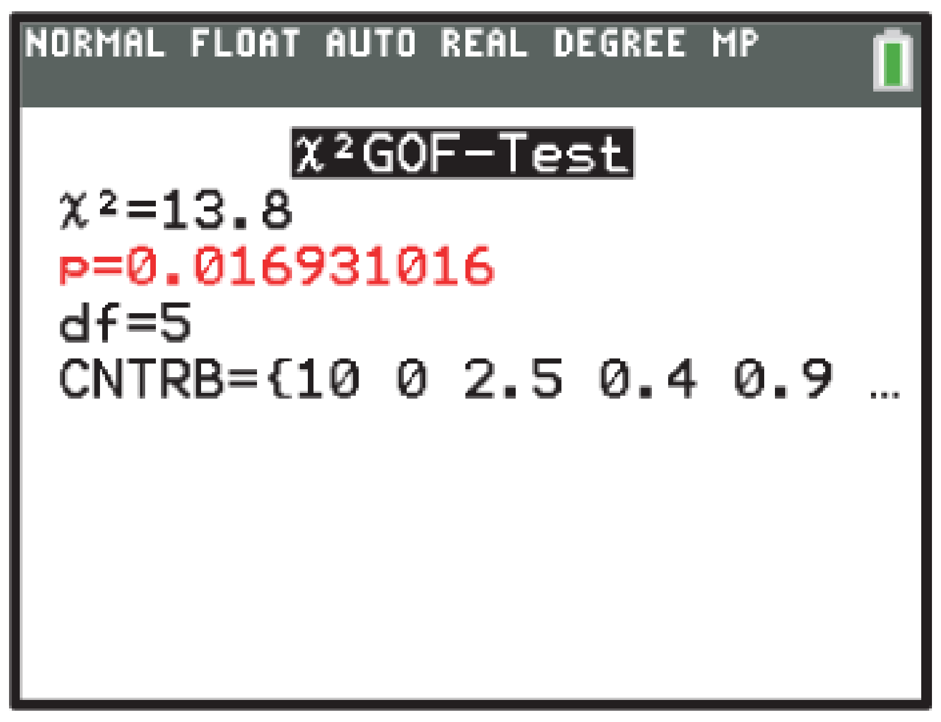
For the die rolling example, we obtain:



```
NORMAL FLOAT AUTO REAL DEGREE MP
L1      L2      L3      L4      L5      3
 20      10     ------  ------  ------
 10      10
 5       10
 8       10
 7       10
 10      10
 ------  ------
L3(1)=
```



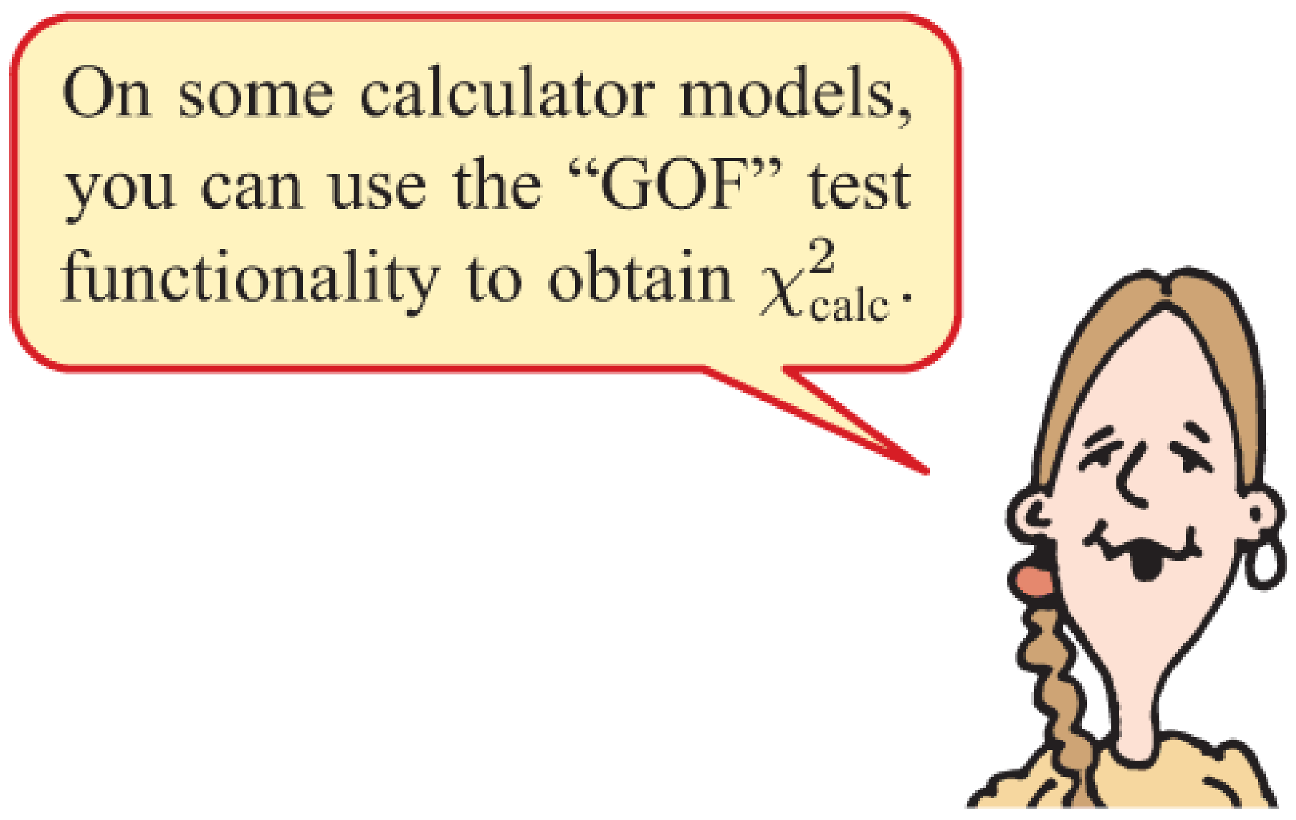
```
NORMAL FLOAT AUTO REAL DEGREE MP
        χ²GOF-Test
 Observed:L1
 Expected:L2
 df:5
 Color:  BLUE
 Calculate Draw
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
        χ²GOF-Test
 χ²=13.8
 p=0.016931016
 df=5
 CNTRB={10 0 2.5 0.4 0.9 …
```

So, the p-value ≈ 0.0169. For a test of H_0 on a significance level of $\alpha = 0.05$, this is enough evidence to reject H_0 in favour of H_1.

We therefore conclude that Rico's die is biased.


On some calculator models, you can use the "GOF" test functionality to obtain χ^2_{calc}.

DEGREES OF FREEDOM

In finding the p-value with your calculator, you should have noticed that you also need a value for "df". Like in the t-test, this stands for "**degrees of freedom**".

In a χ^2 goodness of fit test, the value of df is the number of values that are *free to vary*.

For example, consider the population proportions of 3 categories p_1, p_2, and p_3. Since they are proportions, they must add to 1. We can therefore write any one of the proportions in terms of the other two. For example, $p_1 = 1 - p_2 - p_3$.

So, only two of the proportions are *free to vary*.

For a χ^2 goodness of fit test, **df = number of categories − 1**.

SUMMARY OF THE χ^2 GOODNESS OF FIT TEST

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the value of the **test statistic**: $\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$.

Step 4: Use technology to calculate the ***p*-value**, using $\text{df} = \text{number of categories} - 1$.

Step 5: Reject H_0 if $p\text{-value} < \alpha$.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Notice the similarity between the steps of the goodness of fit test and the t-test on page **387**.

Example 4

Self Tutor

The table alongside shows the grades received by university students taking a second year computer science course.

In the following semester, a new coordinator is appointed for the course. The new coordinator is concerned by the high number of High Distinctions and Distinctions awarded, and wants to determine if the course should be adjusted.

It is considered usual if 5% of students receive a High Distinction, 10% receive a Distinction, 15% receive a Credit, 40% receive a Pass, and the remaining 30% receive a Fail.

Grade	*Frequency*
High Distinction	16
Distinction	21
Credit	21
Pass	59
Fail	34
Total	151

Conduct a χ^2 goodness of fit test to determine whether the course should be adjusted with a 5% level of significance.

Step 1: Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of students who receive a High Distinction, Distinction, Credit, Pass, and Fail respectively.

The hypotheses that should be tested are:

$$H_0:\ p_1 = 0.05,\ \ p_2 = 0.1,\ \ p_3 = 0.15,\ \ p_4 = 0.4,\ \ p_5 = 0.3$$
$$H_1:\ \text{at least one of}\ \ p_1 \neq 0.05,\ \ p_2 \neq 0.1,\ \ \ldots.,\ \ \text{or}\ \ p_5 \neq 0.3\,.$$

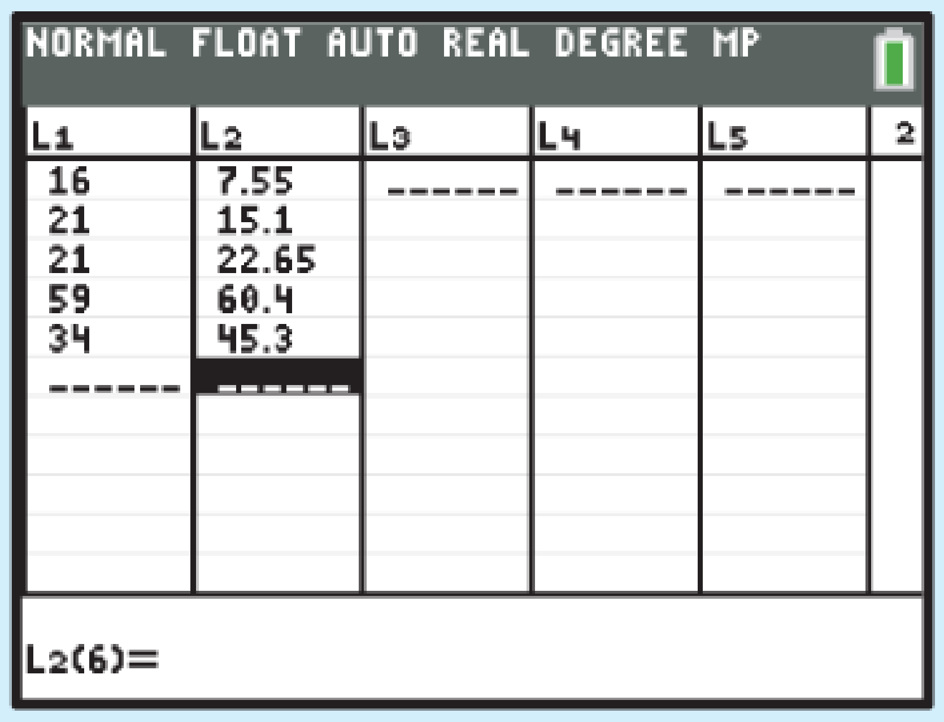
Step 2: The significance level is $\alpha = 0.05$.

Step 3:

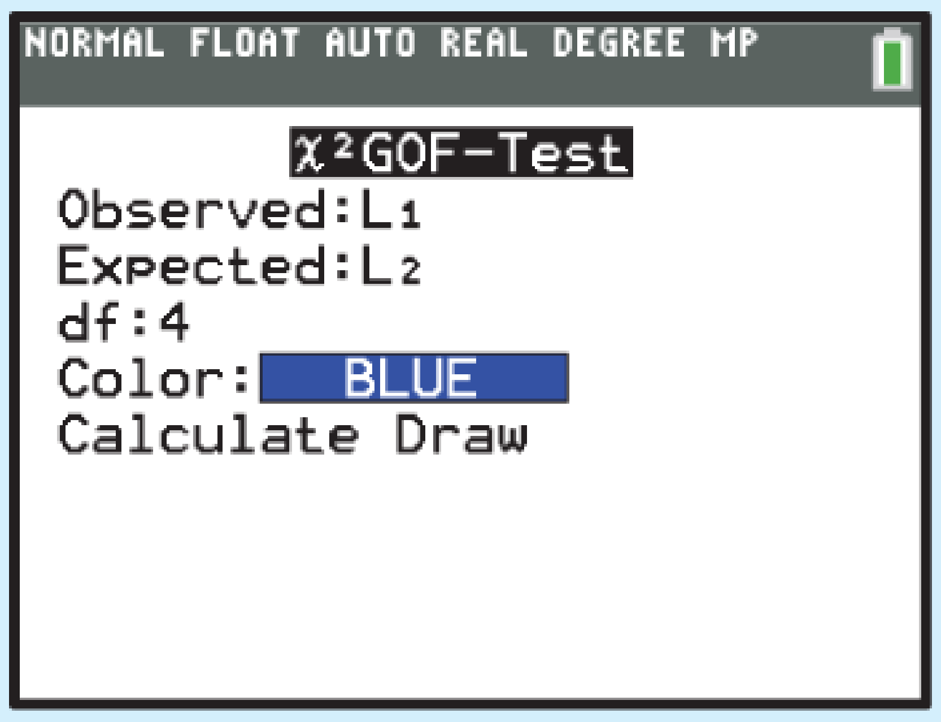
Grade	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
High Distinction	16	$151 \times 0.05 = 7.55$	≈ 9.4573
Distinction	21	$151 \times 0.1 = 15.1$	≈ 2.3053
Credit	21	$151 \times 0.15 = 22.65$	≈ 0.1202
Pass	59	$151 \times 0.4 = 60.4$	≈ 0.0325
Fail	34	$151 \times 0.3 = 45.3$	≈ 2.8188
		Total	≈ 14.734

So, $\chi^2_{\text{calc}} \approx 14.7$.

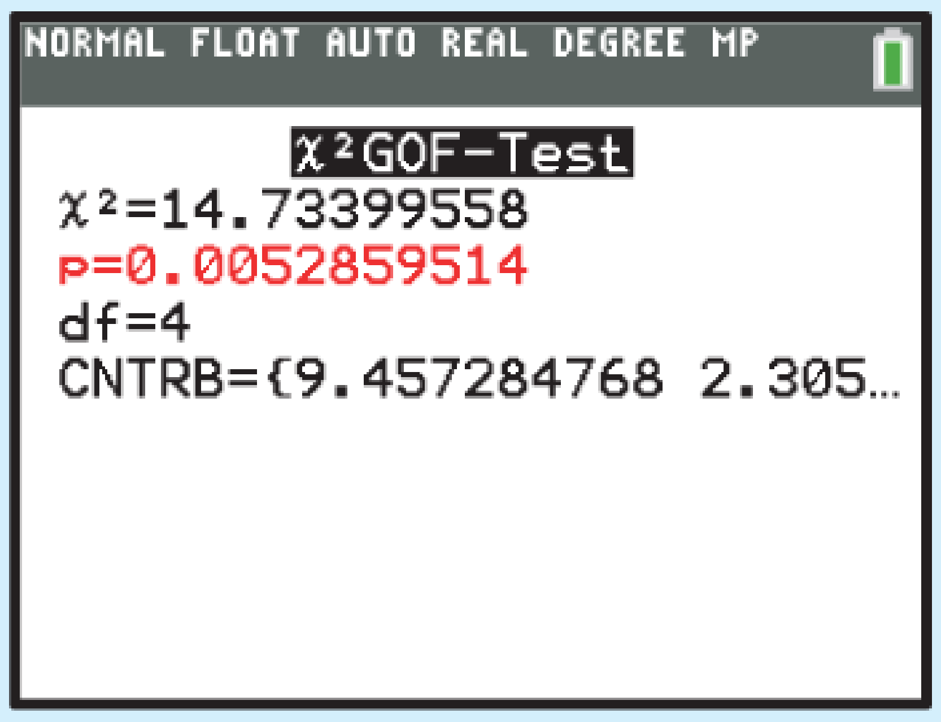
Step 4: df $= 5 - 1 = 4$



```
NORMAL FLOAT AUTO REAL DEGREE MP
L1     L2     L3     L4     L5     2
16     7.55   ------ ------ ------
21     15.1
21     22.65
59     60.4
34     45.3
------ ------
L2(6)=
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
χ²GOF-Test
Observed:L1
Expected:L2
df:4
Color: BLUE
Calculate Draw
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
χ²GOF-Test
χ²=14.73399558
p=0.0052859514
df=4
CNTRB={9.457284768 2.305…
```

Using technology, p-value $\approx 0.005\,29$.

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Step 6: Since we have accepted H_1, we conclude that the course should be adjusted.

LIMITATIONS (EXTENSION)

In order for χ^2 to be distributed appropriately, the sample size n must be sufficiently large. Generally, n is sufficiently large if none of the expected frequencies is less than 5.

In cases where there are expected frequencies less than 5, we can **combine** similar categories to get more reliable results.

For example, when investigating the number of serious sports injuries treated by a hospital, the following expected frequencies are calculated. Since the expected frequencies of two age groups are less than 5, the data for some categories is combined.

Original table

Age group	*Expected frequency*
10 - 15	14.9
15 - 20	14.1
20 - 30	15.6
30 - 40	4.4
40 - 50	0.9
50+	0.2

Combined

Age group	*Expected frequency*
10 - 15	14.9
15 - 20	14.1
20 - 30	15.6
30+	$4.4 + 0.9 + 0.2 = 5.5$

EXERCISE 16D.1

1 When a coin was tossed 96 times, 54 heads were observed. A χ^2 goodness of fit test is to be conducted to determine if the coin is biased with a 5% level of significance.

- **a** Write down the hypotheses that should be considered.
- **b** How many tails were observed?
- **c** Calculate the *expected* frequencies of heads and tails assuming the coin is fair.
- **d** Use a table to help you calculate χ^2_{calc}.
- **e** State the number of degrees of freedom.
- **f** Use technology to find the p-value.
- **g** Is there enough evidence to conclude that the coin is biased?

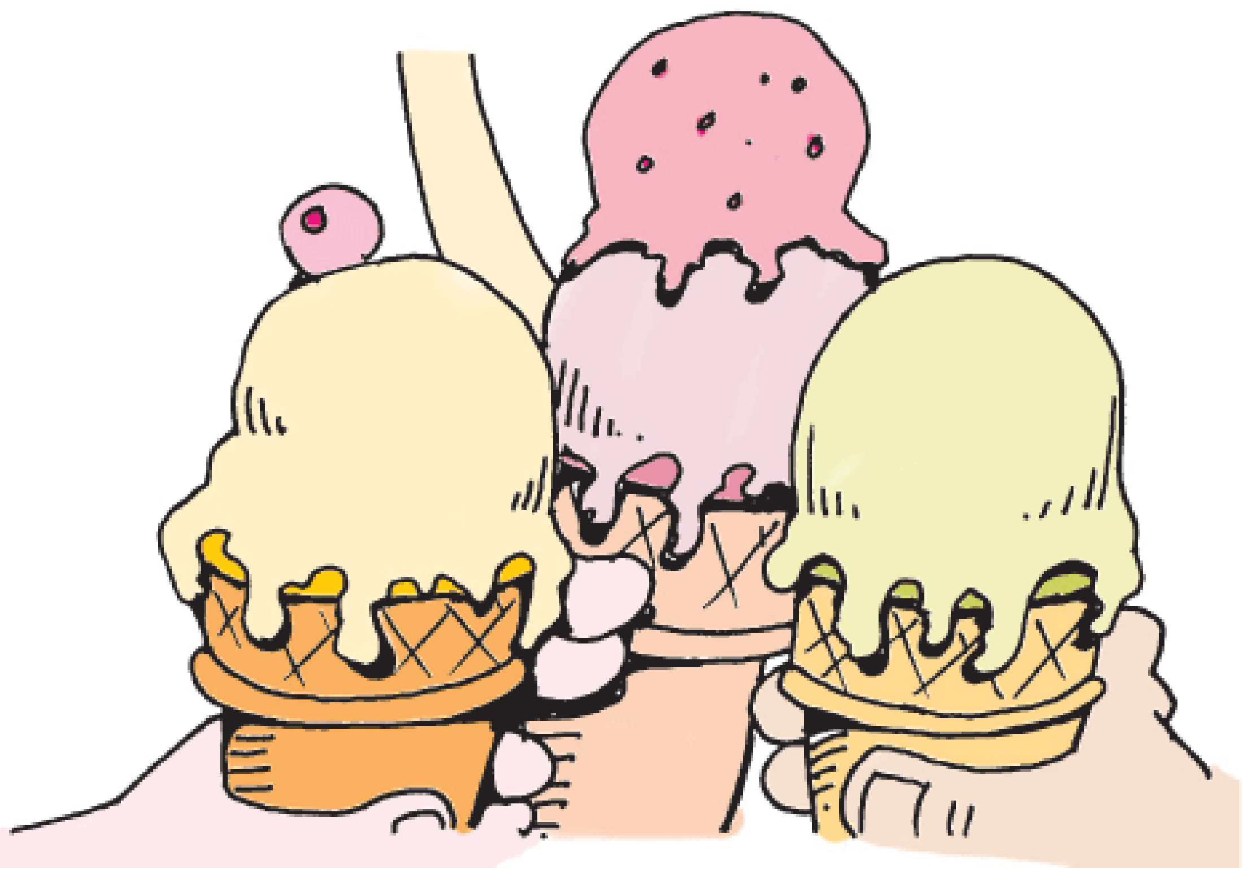
2 In the last election 4 years ago, 54% of voters voted for party A, 30% of voters voted for party B, and the rest voted for party C.

When a polling agency conducted a survey asking 300 voters which party they are going to vote for in the upcoming election, the following results were obtained:

Party	A	B	C
Voters	141	105	54

Conduct a χ^2 goodness of fit test with a 1% level of significance to determine whether there has been a change in the proportions of voters supporting each party since the last election.

3 Brian owns a chocolate café. He wants to start offering ice cream in addition to his chocolate menu items. He initially makes the same amount of chocolate, strawberry, vanilla, honeycomb, and choc-chip ice cream, assuming that the flavours will be equally liked.



The number of sales of each ice cream flavour are shown in the table alongside.

Conduct a χ^2 goodness of fit test with a 10% level of significance to determine whether Brian should change the amount of each ice cream flavour that he makes.

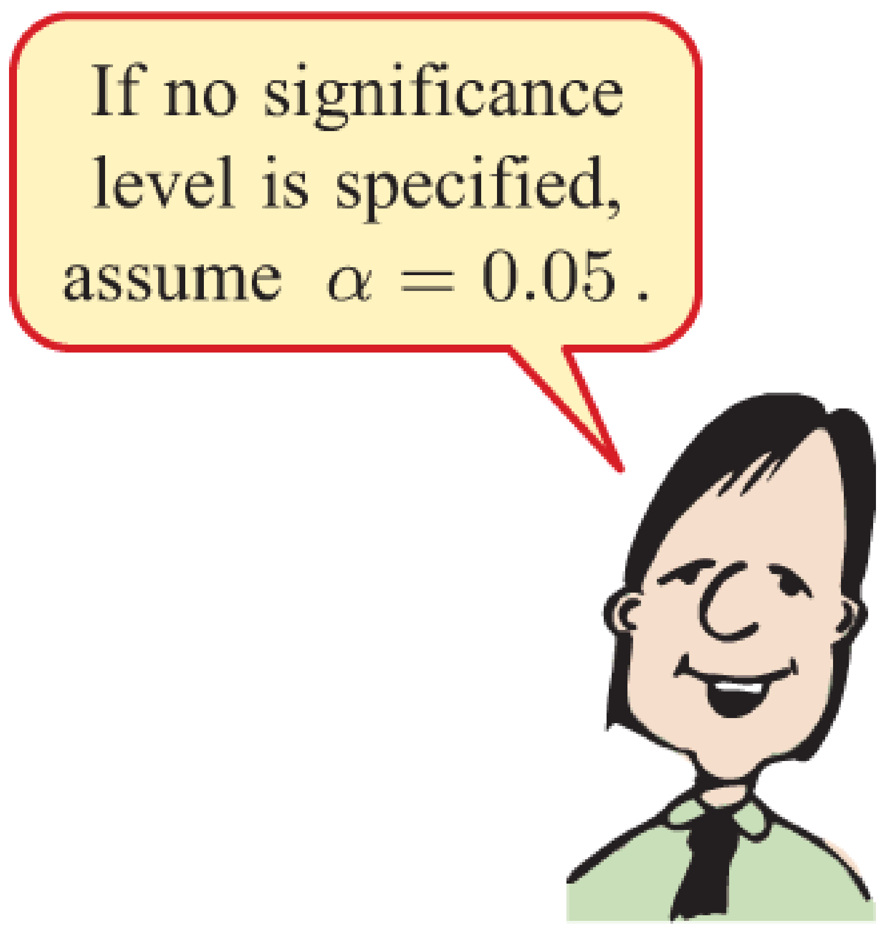
Flavour	*Sales*
chocolate	54
strawberry	48
vanilla	35
honeycomb	28
choc-chip	40
Total	205

4 In 2001, 71.2% of people living in London identified as being White, 12.1% were Asian/Asian British, 10.9% were Black/Black British, 3.2% were of mixed ethnicity, and 2.6% were of other ethnicities.

The table alongside shows the number of people recorded for each ethnic group in the 2011 UK Census.

Conduct a χ^2 goodness of fit test to determine whether there was a significant change in London's demographics between 2001 and 2011.

Ethnic group	*Frequency*
White	4 887 435
Asian/Asian British	1 511 546
Black/Black British	1 088 640
Mixed	405 279
Other	281 041
Total	8 173 941


If no significance level is specified, assume $\alpha = 0.05$.

5 In Australia, the NAPLAN tests are used to gauge the literacy and numeracy skills of students. The students are allocated into "bands" based on the score they obtain.

The table below shows the results for Year 9 students at a particular school, and the national percentages for each band:

Band	*School frequency*	*National percentage*
10	5	7.9%
9	9	16.7%
8	55	29.8%
7	53	29.7%
6	23	13.5%
5 and below	5	2.4%
Total	150	100%

a Use the national percentages to calculate the expected frequency for each band.

b Conduct a χ^2 goodness of fit test with a 1% significance level to determine whether there is a substantial difference between the school's results and the rest of the nation.

c Explain why you may wish to combine the "Band 6" and "Band 5 and below" categories.

d Combine the "Band 6" and "Band 5 and below" categories and repeat **b**. Comment on your results.

ACTIVITY 1 ASSESSING THE GOODNESS OF FIT OF A PROBABILITY MODEL

The name "χ^2 goodness of fit test" comes from its use in seeing how well a probability distribution (or model) fits the data.

In **Chapter 15** we saw how the binomial distribution can be used to model discrete data.

In this Activity we will use the goodness of fit test to consider whether a binomial distribution is an appropriate model for the number of correct answers in a multiple choice quiz.

What to do:

Consider a multiple choice quiz with 10 questions. Each question has 4 choices, only 1 of which is correct.

The quiz is given to 150 people, and the number of correct answers for the participants is summarised in the table alongside.

Number of correct answers	*Frequency*
0	7
1	34
2	52
3	36
4	12
5 or more	9

1 **a** Suppose a person answers a question by a random guess. State the probability that they will answer it correctly.

b Suppose a person answers all questions by randomly guessing. Let X be the number of questions they answer correctly.

- **i** Explain why X is a binomial random variable.
- **ii** State the parameters of the binomial distribution.

2 Now that we have a model, we can determine what we can *expect* if the model is appropriate.

   a Write down the probability distribution of X in a table like the one below:

x	0	1	2	3	4	5 or more
$\mathrm{P}(X = x)$						

   b Use the probability distribution to calculate the expected frequencies of each value of x.

   c Why do you think the outcomes greater than or equal to 5 have been combined into a single "5 or more" category?

   d Perform a χ^2 goodness of fit test with a 5% level of significance to determine whether the model is appropriate.

3 When we formulated the model, we assumed that people would randomly guess the answers.

   a Do you think this assumption is *realistic*?

   b List any other assumptions that might have been made.

4 Now suppose that each person is able to reject one answer from each question which they are sure is incorrect. They will then randomly guess from the remaining options.

   a State the distribution for Y, the number of questions which they answer correctly.

   b Perform a χ^2 goodness of fit test with a 5% level of significance to determine whether the model is appropriate.

5 How could the goodness of fit test be used to consider other probability distributions such as the normal distribution?

CRITICAL VALUES

So far we have used p-values in our hypothesis tests to make decisions about our hypotheses. The significance level gives us a threshold for rejecting the null hypothesis.

We can consider a threshold for the **test statistic** in a similar way.

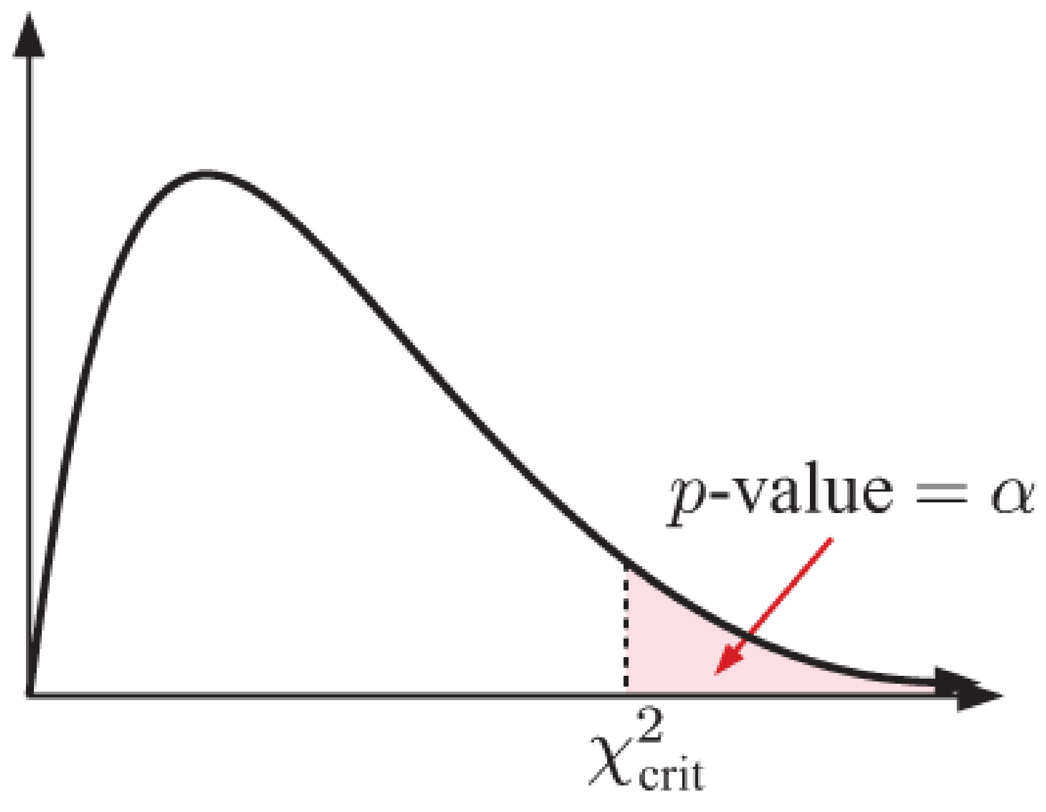
In a χ^2 goodness of fit test with significance level α, the **critical value** of the test is the value of the test statistic χ^2_{calc} that yields $p\text{-value} = \alpha$.

The critical value is denoted χ^2_{crit}.

The critical value χ^2_{crit} is shown in the diagram alongside.

Any value of χ^2_{calc} that is *greater* than χ^2_{crit} will yield a smaller p-value.

Since the p-value of χ^2_{crit} is α, any value of $\chi^2_{\text{calc}} \geqslant \chi^2_{\text{crit}}$ results in H_0 being rejected.



In a χ^2 goodness of fit test with critical value χ^2_{crit}, the null hypothesis should be rejected if $\boldsymbol{\chi^2_{\text{calc}} \geqslant \chi^2_{\text{crit}}}$.

This inequality is called the **rejection inequality**.

The critical value χ^2_{crit} depends on the degrees of freedom (df) and the significance level α of the test.

The table alongside shows the critical value for various values of df and α.

Click on the icon for a more detailed table of critical values.

CRITICAL VALUES

Degrees of freedom (df)	*Significance level* α		
	10%	5%	1%
1	2.71	3.84	6.63
2	4.61	5.99	9.21
3	6.25	7.81	11.34
4	7.78	9.49	13.28
5	9.24	11.07	15.09
6	10.64	12.59	16.81
7	12.02	14.07	18.48
8	13.36	15.51	20.09
9	14.68	16.92	21.67
10	15.99	18.31	23.21

χ^2 GOODNESS OF FIT TEST PROCEDURE WITH CRITICAL VALUES

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the value of the **test statistic**: $\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$.

Step 4: Find the **critical value** χ^2_{crit} where df $=$ number of categories $-$ 1.

Step 5: Reject H_0 if $\chi^2_{\text{calc}} \geqslant \chi^2_{\text{crit}}$.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

EXERCISE 16D.2

1 A goodness of fit test is to be used to test the following hypotheses:

H_0: $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{5}$, ..., $p_5 = \frac{1}{5}$

H_1: at least one of $p_1, p_2, ..., p_5 \neq \frac{1}{5}$ with a 5% level of significance.

The value of the test statistic is $\chi^2_{\text{calc}} = 10.3$.

a Write down the critical value χ^2_{crit} for the test.

b Hence determine whether H_0 should be rejected.

2 Lucy bought a large packet of Chewy Chews lollies. She counted the number of lollies of each colour it contained. Her results are shown in the table.

The manufacturer of Chewy Chews claims that they make equal quantities of each colour.

Lucy wants to test the manufacturer's claim with a goodness of fit test using a 10% level of significance.

Colour	*Frequency*
red	12
yellow	17
green	20
blue	16
Total	65

a Write down the hypotheses that Lucy should test.

b Calculate χ^2_{calc}.

c Find the critical value χ^2_{crit} for the test.

d Does Lucy have enough evidence to reject the manufacturer's claim?

e Verify your answer by examining the p-value.

3 An internet service provider (ISP) conducted a survey asking their customers how satisfied they were with their service. They found that:

- 5% responded "very satisfied"
- 25% responded "satisfied"
- 41% responded "neutral"
- 20% responded "dissatisfied"
- 9% responded "very dissatisfied".

To improve customer satisfaction, the ISP made several changes to the internet plans they offered. They then repeated the same survey 6 months later. The results are shown in the table opposite.

Conduct a χ^2 goodness of fit test using a critical value and a 1% level of significance to determine whether the ISP's changes were effective.

Response	*Frequency*
very satisfied	25
satisfied	78
neutral	77
dissatisfied	36
very dissatisfied	17
Total	233

ACTIVITY 2 MENDEL'S DATA

Gregor Johann Mendel (1822 - 1884) is often credited as being "the father of modern genetics".

One of his most important contributions was the use of recessive and dominant genes to describe inherited traits.

Amongst his research, he conducted experiments involving over 28 000 plants, the majority of which were pea plants. The table below shows the results of one such experiment.

Gregor Johann Mendel

Type of pea	*Frequency*
Yellow round seeds	315
Green round seeds	108
Yellow wrinkled seeds	101
Green wrinkled seeds	32
Total	556

According to Mendel's model, he expected the ratio

$$\text{yellow round} : \text{green round} : \text{yellow wrinkled} : \text{green wrinkled} = 9 : 3 : 3 : 1$$

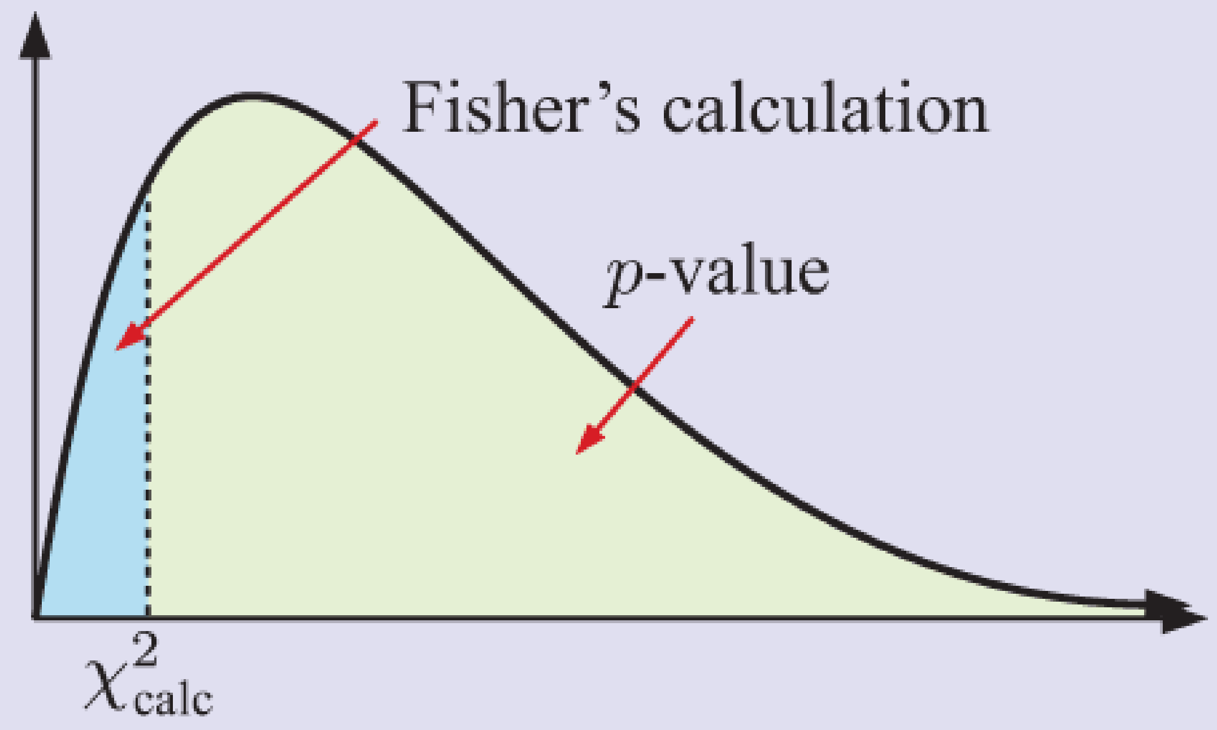
What to do:

1. Calculate the expected frequency of each type of pea.
2. In the context of a χ^2 goodness of fit test, show that $\chi^2_{\text{calc}} \approx 0.470$ and $p\text{-value} \approx 0.925$.
3. What can we conclude based on these results?

THEORY OF KNOWLEDGE

In 1936, **Sir Ronald Aylmer Fisher** analysed Mendel's experiments described in **Activity 2**.

He considered the probability that Mendel's results were *consistent* with his expectations, and found that the probability of observing a test statistic *less* than $\chi^2_{\text{calc}} \approx 0.470$ was ≈ 0.075.



In fact, Fisher observed results like this for all of Mendel's experiments. By combining all of Mendel's data, Fisher found that the probability of getting data as good as Mendel's was about 4 in 100 000. Fisher concluded that Mendel had manipulated the data to obtain the results he desired.

1. Does Fisher's finding invalidate the importance of Mendel's contribution to biology and genetics?

Today, the manipulation of data and "data mining" is a major problem in research. Most academic publications report findings which are "statistically significant", as these findings are more likely to yield more interesting results and lead to further research.

2. Will all claims in academic publications on statistical data necessarily be true? You might want to consider your findings in **Investigation 2**.
3. Discuss the role that statistical interpretation plays in research ethics.

The MMR vaccine controversy was caused by a fraudulent paper published in 1998 which claimed a causal relationship between the MMR vaccine and autism in children. The paper has since been retracted after thorough investigation. However, because of the widespread misconceptions that it has caused, it has been cited as "perhaps the most damaging medical hoax of the last 100 years".

4. Research the details of the MMR vaccine controversy. In particular, consider how the authors collected and used the data cited in the original paper.
5. What other things should a statistician be mindful of when analysing data?
6. Who do you think was responsible for the damage caused by the MMR controversy? Was it the authors of the paper, the media, or the general public?

E THE χ^2 TEST FOR INDEPENDENCE

This table shows the results of a sample of 400 randomly selected adults classified according to *gender* and *regular exercise*.

We call this a 2×2 **contingency table**.

	Regular exercise	*No regular exercise*	*Sum*
Male	110	106	216
Female	98	86	184
Sum	208	192	400

We may be interested in how the variables *gender* and *regular exercise* are related. The variables may be **dependent**, for example females may exercise more regularly than males. Alternatively, the variables may be **independent**, which means the gender of a person has no effect on whether they exercise regularly.

In this Section, we will apply what we have learned about the χ^2 goodness of fit test to determine whether two variables from the same sample are independent.

THE HYPOTHESES

We have previously seen that if two events A and B are independent, then

$$\mathrm{P}(A \text{ and } B) = \mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B).$$

For example, if the variables *gender* and *regular exercise* are independent, then

$$\begin{aligned}\mathrm{P}(\textit{male} \cap \textit{regular exercise}) &= \mathrm{P}(\textit{male}) \times \mathrm{P}(\textit{regular exercise}) \\ &= \tfrac{216}{400} \times \tfrac{208}{400}\end{aligned}$$

In a contingency table with r rows and c columns, there are $r \times c$ possible combinations, and hence $r \times c$ probabilities to consider.

Instead of writing all these probabilities down when we formulate the hypotheses, we can equivalently write:

H_0: the variables are independent

H_1: the variables are dependent

CALCULATING THE TEST STATISTIC

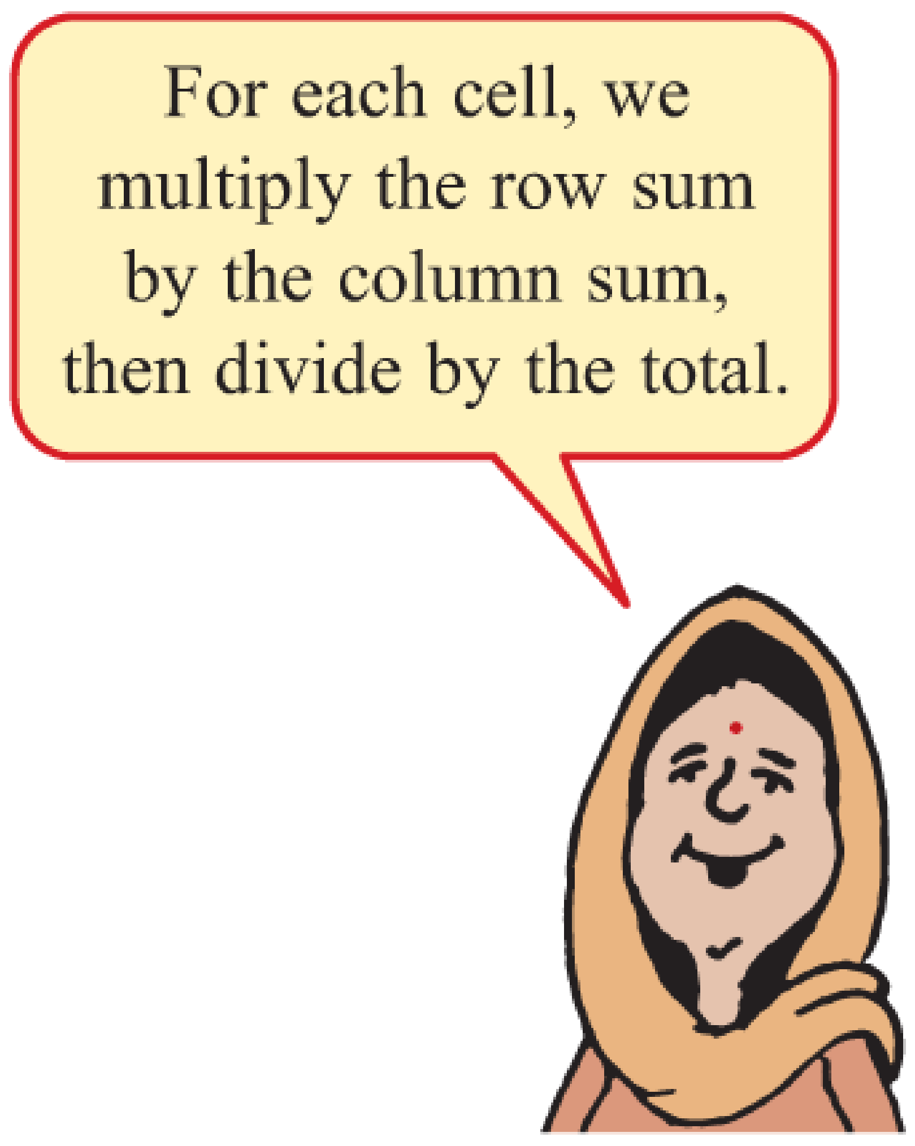
Having established a theoretical probability for each possible outcome, we can calculate **expected frequencies** and hence the test statistic χ^2_{calc}.

For example, in a sample of 400 adults, we would expect

$$400 \times \left(\frac{216}{400} \times \frac{208}{400}\right) = \frac{216 \times 208}{400} = 112.32 \text{ to be male and exercise regularly.}$$

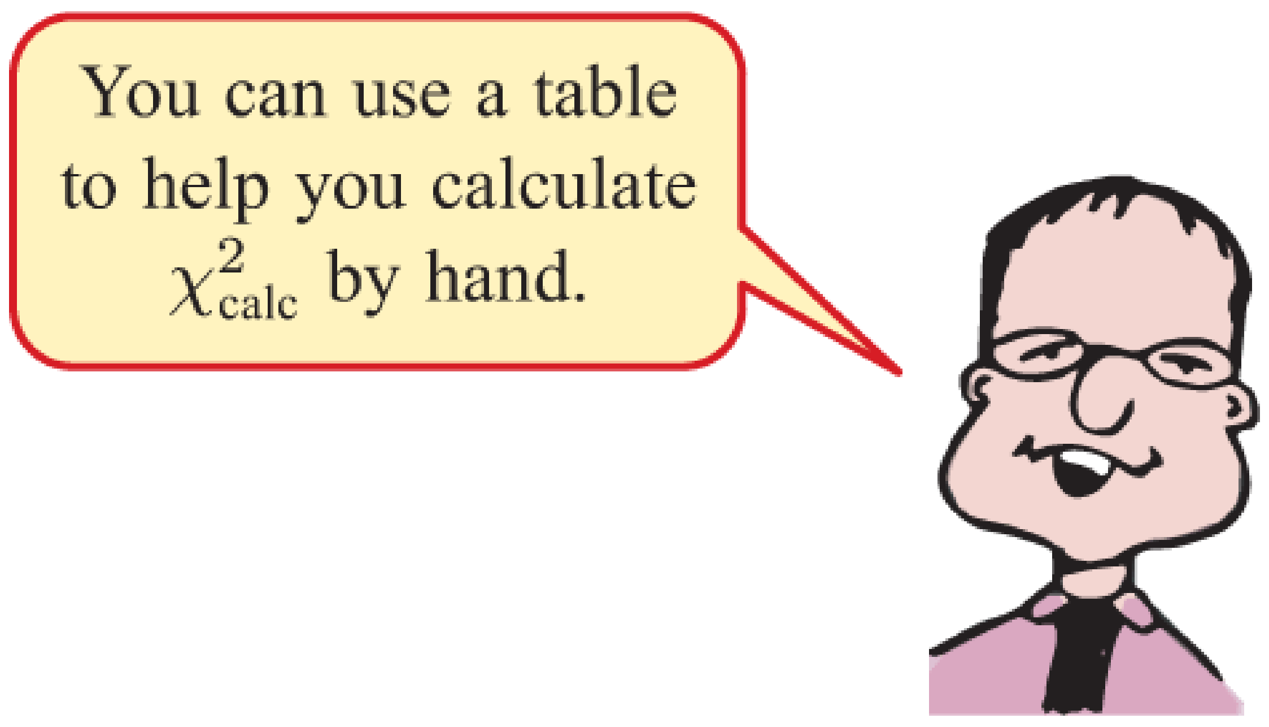
Performing similar calculations for each possible outcome, we can complete an **expected frequency table**.

	Regular exercise	*No regular exercise*	*Sum*
Male	$\frac{216 \times 208}{400} = 112.32$	$\frac{216 \times 192}{400} = 103.68$	216
Female	$\frac{184 \times 208}{400} = 95.68$	$\frac{184 \times 192}{400} = 88.32$	184
Sum	208	192	400


For each cell, we multiply the row sum by the column sum, then divide by the total.

Using the formula for χ^2_{calc} on page **396**:

$$\begin{aligned}\chi^2_{\text{calc}} &= \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}} \\ &= \frac{(110 - 112.32)^2}{112.32} + ... + \frac{(86 - 88.32)^2}{88.32} \\ &\approx 0.2170\end{aligned}$$


You can use a table to help you calculate χ^2_{calc} by hand.

DEGREES OF FREEDOM

In the previous Section we saw that the number of **degrees of freedom (df)** is the number of values which are free to vary.

Consider the 2×2 contingency table alongside, with the sum values given.

	A_1	A_2	*Sum*
B_1			12
B_2			8
Sum	15	5	20

The value in the top left corner is free to vary. It can take many possible values, one of which is 9. However, once we set this value, the remaining values are *not* free to vary, as they are determined by the row and column sums.

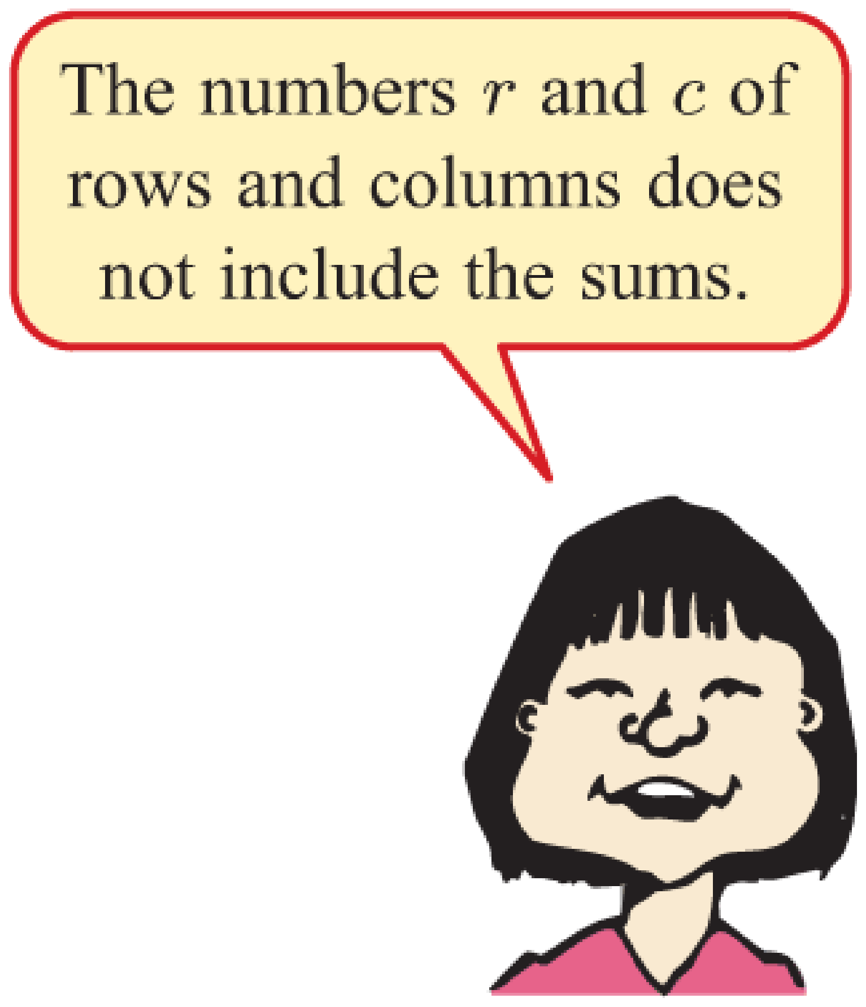
	A_1	A_2	*Sum*
B_1	9	3	12
B_2	6	2	8
Sum	15	5	20

So, the number of degrees of freedom is 1, which is $(2-1) \times (2-1)$.

In a 3×3 contingency table, we can choose $(3-1) \times (3-1) = 4$ values before the remaining values are not free to vary.

	C_1	C_2	C_3	*Sum*
D_1				12
D_2				8
D_3				13
Sum	13	9	11	33

	C_1	C_2	C_3	*Sum*
D_1	5	3	4	12
D_2	2	4	2	8
D_3	6	2	5	13
Sum	13	9	11	33


The numbers r and c of rows and columns does not include the sums.

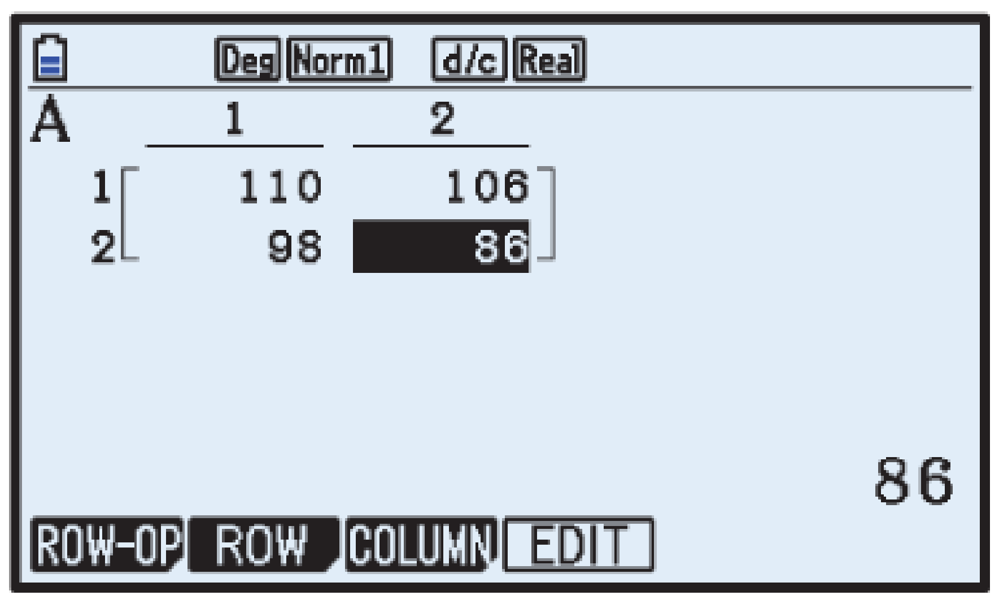
For a contingency table which has r rows and c columns,
df $= (r-1)(c-1)$.

THE p-VALUE

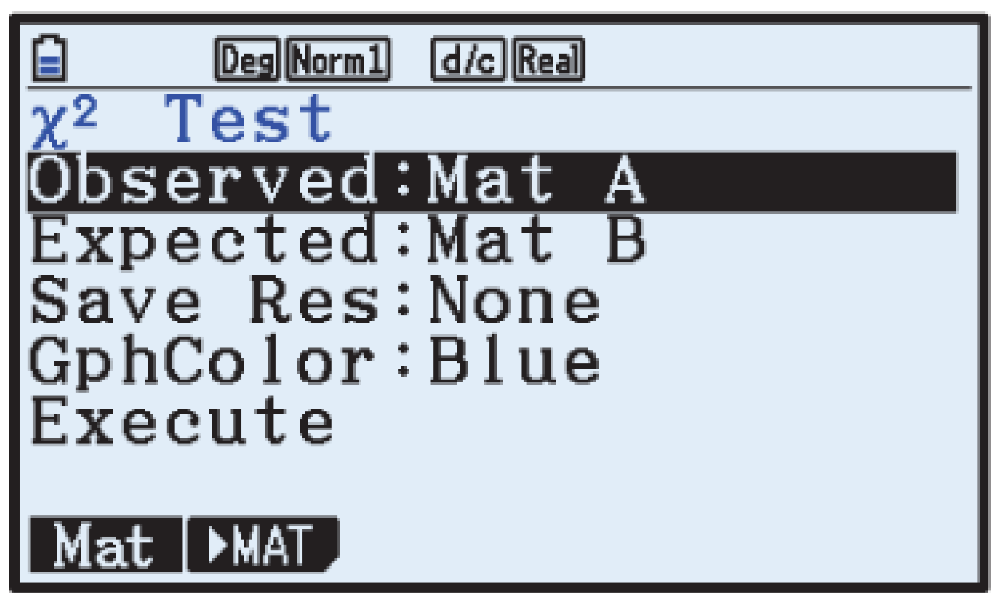
Like we did for the χ^2 goodness of fit test, we use technology to calculate χ^2_{calc} and the **p-value**.

Your **graphics calculator** likely has a special function for the χ^2 test for independence.

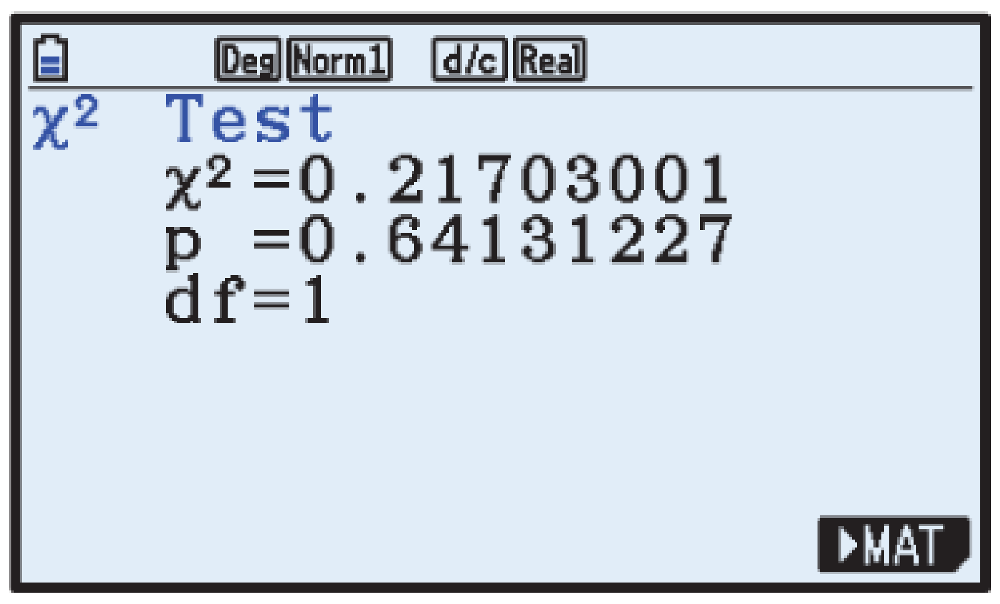
GRAPHICS CALCULATOR INSTRUCTIONS



```
A     1      2
  1[ 110    106]
  2[  98     86]
                  86
```



```
χ² Test
Observed:Mat A
Expected:Mat B
Save Res:None
GphColor:Blue
Execute
```



```
χ² Test
   χ²=0.21703001
   p =0.64131227
   df=1
```

Click on this icon to obtain a printable flow chart to help you remember which hypothesis test to use.

HYPOTHESIS TESTING FLOW CHART

CRITICAL VALUES

We can use the same table of critical values on page **403** to find the critical value χ^2_{crit} of the test.

However, for a χ^2 test for independence, remember that the degrees of freedom $= (r-1)(c-1)$.

SUMMARY OF THE χ^2 TEST FOR INDEPENDENCE

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1, which have the form:
H_0: the variables are independent
H_1: the variables are dependent.

Step 2: State the **significance level** α.

Step 3: Calculate $\mathbf{df = (r-1)(c-1)}$ where r and c are the number of rows and columns of the contingency table respectively.

Step 4: Construct the **expected frequency table** and calculate the value of the **test statistic**:
$$\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}.$$

Step 5: Use technology to calculate the ***p*-value**. *or* Find the critical value χ^2_{crit} for the test.

Step 6: Reject H_0 if p-value $< \alpha$ *or* if $\chi^2_{\text{calc}} \geqslant \chi^2_{\text{crit}}$.

Step 7: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Example 5 **Self Tutor**

A survey was given to randomly chosen high school students from years 9 to 12 on possible changes to the school's canteen. The results are shown in this contingency table.

	Year group			
	9	10	11	12
Change	7	9	13	14
No change	14	12	9	7

At a 5% significance level, test whether a student's *canteen preference* depends on their *year group*.

Step 1: H_0: *year group* and *canteen preference* are independent.
H_1: *year group* and *canteen preference* are not independent.

Step 2: The significance level is $\alpha = 0.05$.

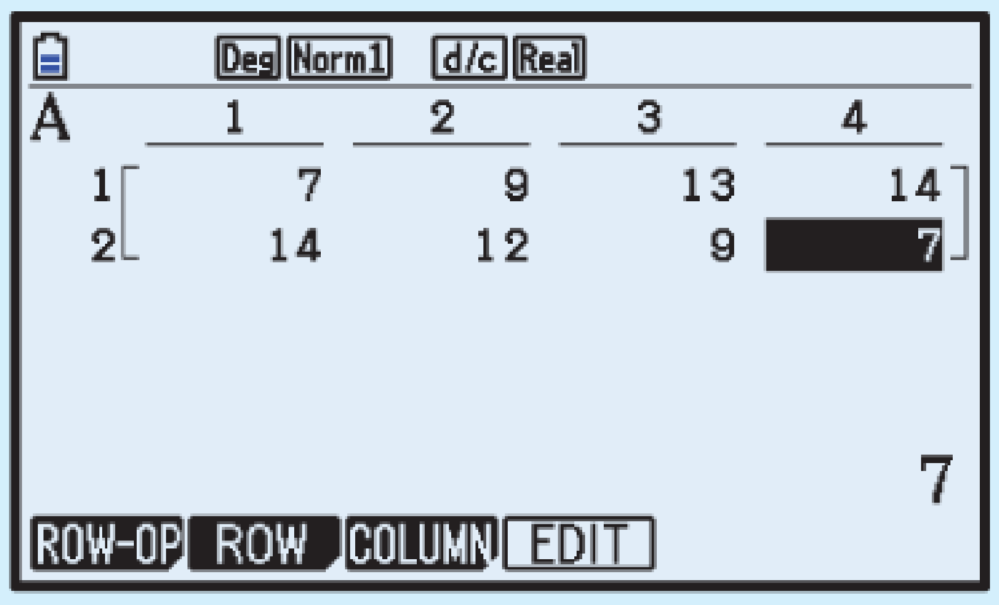
Step 3: $\text{df} = (2-1)(4-1) = 3$

Step 4: The 2×4 contingency table is:

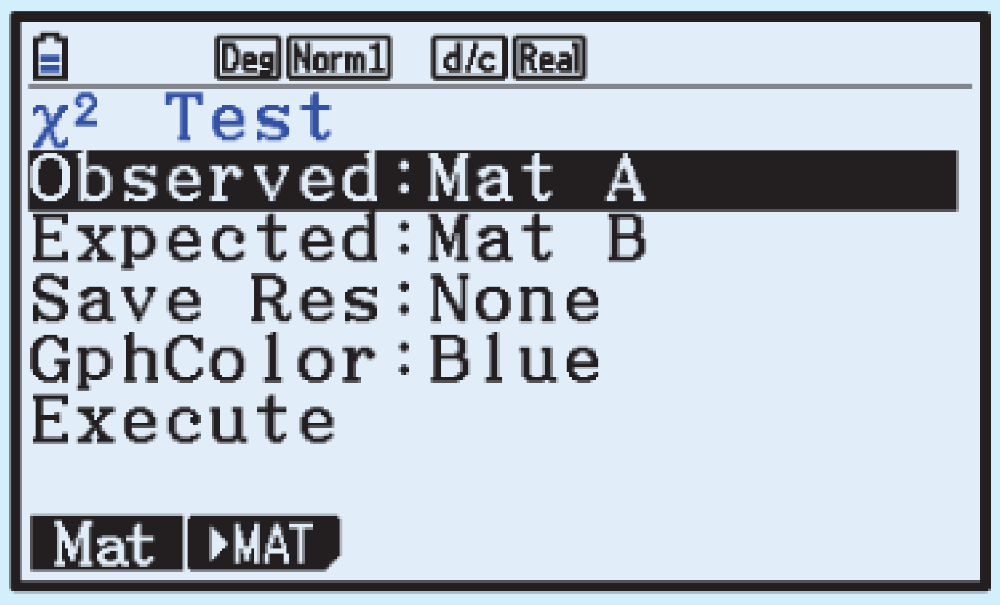
	Year group				
	9	10	11	12	*Sum*
C	7	9	13	14	43
C'	14	12	9	7	42
Sum	21	21	22	21	85

The expected frequency table is:

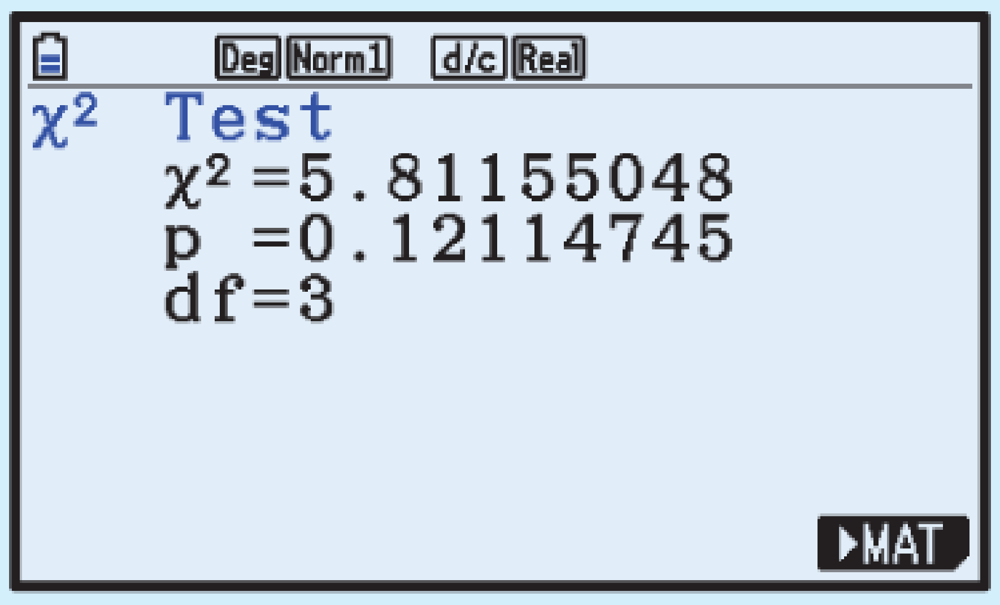
	Year group			
	9	10	11	12
C	10.6	10.6	11.1	10.6
C'	10.4	10.4	10.9	10.4



```
A    1      2      3      4
1 [  7      9     13     14 ]
2 [ 14     12      9      7 ]
                              7
```



```
χ² Test
Observed:Mat A
Expected:Mat B
Save Res:None
GphColor:Blue
Execute
```



```
χ² Test
 χ²=5.81155048
 p =0.12114745
 df=3
```

Using technology, $\chi^2_{\text{calc}} \approx 5.81$.

Step 5: From the screenshots above, p-value ≈ 0.121.

Step 6: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 7: We conclude that the variables *year group* and *canteen preference* are independent.

EXERCISE 16E.1

1 Construct an expected frequency table for the contingency table:

a

	Drove to work	*Cycled to work*	*Public transport*	*Sum*
Male				44
Female				36
Sum	46	14	20	80

b

	Junior school	*Middle school*	*High school*	*Sum*
Plays sport	35	59	71	165
Does not play sport	23	27	35	85
Sum	58	86	106	250

c

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*	*Sum*
Sunburnt	3	5	13	
Not sunburnt	36	17	1	
Sum				

2 Consider the contingency table:

	Pass Maths test	*Fail Maths test*	*Sum*
Male	24	26	50
Female	36	14	50
Sum	60	40	100

a Construct the corresponding expected frequency table.

b Interpret the value in the top left corner of your expected frequency table.

c Calculate χ^2_{calc} by copying and completing this table:

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
24				
26				
36				
14				
			Total	

In examinations you can calculate χ^2 using technology.

3 This contingency table shows the responses of a randomly chosen sample of adults regarding the person's *weight* and whether they have *diabetes*.

Conduct a χ^2 test for independence at a 5% level to determine whether there is a link between *weight* and suffering from *diabetes*.

	Weight		
	light	*medium*	*heavy*
Diabetic	11	19	26
Non-diabetic	79	68	69

4 The table alongside shows the way in which a random sample of people intend to vote in the next election.

	Age of voter		
	18 to 35	36 to 59	60+
Party A	85	95	131
Party B	168	197	173

a For a 10% significance level, what is the critical value χ^2_{crit}?

b Conduct a χ^2 test for independence at a 10% level to determine whether there is any association between the *age of a voter* and the *party they wish to vote for*.

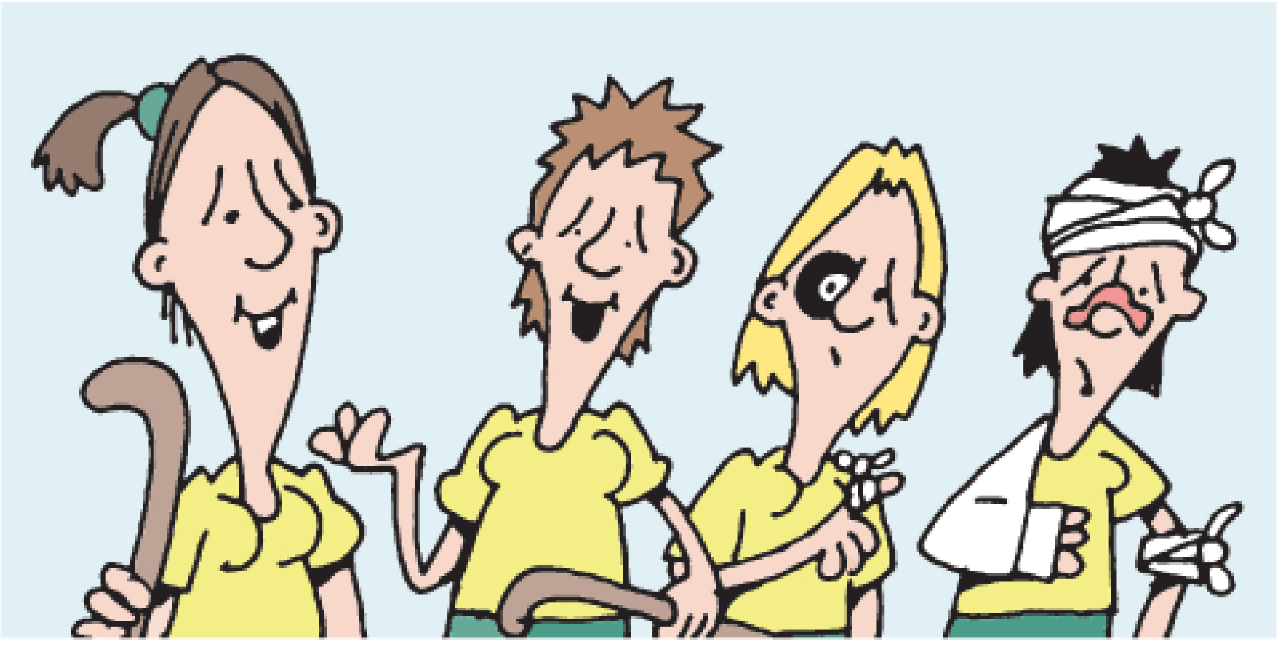
5 The guests staying at a hotel are asked to provide their *reason for travelling*, and to *rate* the hotel on a scale from Poor to Excellent. The results are shown in the table:

		Rating			
		Poor	*Fair*	*Good*	*Excellent*
Reason for travelling	*Business*	27	25	20	8
	Holiday	9	17	24	30

a Show that, at a 5% significance level, the variables *reason for travelling* and *rating* are dependent.

b By examining the contingency table, describe how a guest's *rating* is affected by their *reason for travelling*.

6 Hockey player Julie wondered whether the position you played affected your likelihood of being injured. She asked a random sample of hockey players what position they played, and what injuries they had sustained in the last year.



		Position			
		Forward	*Midfielder*	*Defender*	*Goalkeeper*
Injury type	*No injury*	23	18	24	7
	Mild injury	14	34	23	11
	Serious injury	10	16	13	7

Test, at a 10% significance level, whether the variables *position* and *injury type* are independent.

7 Consider the contingency table alongside:

		Own a pet?	
		Yes	*No*
Age	0 - 19	5	3
	20 - 29	32	22
	30 - 49	42	58
	50+	39	34

a Construct the expected frequency table.

b Are any of the expected frequencies less than 5?

c Combine the data so that none of the cells have an expected frequency less than 5.

d For the combined data, test at a 5% level whether there is a link between *age* and *owning a pet*.

8 The following table is a result of a major investigation considering *intelligence level* and *cigarette smoking.*

Smoking habits	*Intelligence level:* *Low*	*Average*	*High*	*Very high*
Non smoker	279	386	96	2
Medium level smoker	123	201	58	5
Heavy smoker	100	147	64	2

a Construct the expected frequency table.

b Test at a 1% level whether there is a link between *intelligence level* and *cigarette smoking*.

c Combine appropriate columns so that none of the expected frequencies is less than 5.

d Perform this test again at a 1% level. Is your conclusion the same as in **b**?

YATES' CONTINUITY CORRECTION (EXTENSION)

The χ^2 test for independence may be unreliable if the number of degrees of freedom is 1. This occurs when we have a 2×2 contingency table.

To improve the reliability of the test in this case, we can apply **Yates' continuity correction**. We use a modified formula to find χ^2_{calc} as follows:

If df = 1, we use $$\boldsymbol{\chi^2_{\text{calc}} = \sum \frac{(|\,f_{\text{obs}} - f_{\text{exp}}\,| - 0.5)^2}{f_{\text{exp}}}}$$

where $|\,f_{\text{obs}} - f_{\text{exp}}\,|$ is the **absolute value** or **modulus** of $f_{\text{obs}} - f_{\text{exp}}$.

When conducting the χ^2 test with Yates' continuity correction, we cannot use technology to calculate χ^2_{calc} and the p-value. We must therefore calculate χ^2_{calc} by hand and use the critical value to make a decision about H_0 instead.

Example 6 **Self Tutor**

80 people were surveyed about whether they enjoy surfing and skiing. The results are shown alongside.

Test, at a 1% level, whether there is an association between *enjoying surfing* and *enjoying skiing.*

Enjoy skiing?	*Enjoy surfing?* *Yes*	*No*
Yes	17	15
No	8	40

Step 1: H_0: The variables *enjoying surfing* and *enjoying skiing* are independent.
H_1: The variables *enjoying surfing* and *enjoying skiing* are not independent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $\text{df} = (2-1)(2-1) = 1$

Step 4: The 2×2 contingency table is:

Enjoy skiing?	*Enjoy surfing?* *Yes*	*No*	*Sum*
Yes	17	15	32
No	8	40	48
Sum	25	55	80

The expected frequency table is:

Enjoy skiing?	*Enjoy surfing?* *Yes*	*No*
Yes	10	22
No	15	33

We find χ^2_{calc} using Yates' continuity correction:

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5$	$(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2$	$\dfrac{(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2}{f_{\text{exp}}}$
17	10	7	6.5	42.25	4.225
15	22	-7	6.5	42.25	1.920
8	15	-7	6.5	42.25	2.817
40	33	7	6.5	42.25	1.280
				Total	10.242

So, $\chi^2_{\text{calc}} \approx 10.2$

Step 5: At a $\alpha = 0.01 = 1\%$ level with $\text{df} = 1$, the critical value $\chi^2_{\text{crit}} = 6.63$.

Step 6: Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}} = 6.63$, we have enough evidence to reject H_0 in favour of H_1 on a 1% level of significance.

Step 7: Since we have accepted H_1, we conclude that *enjoying surfing* and *enjoying skiing* are dependent.

EXERCISE 16E.2

1 Horace claims that he can predict the outcome of a coin toss. To test this, he tosses a coin 200 times, and tries to guess the outcome of each toss. The results are shown alongside.

		Result	
		Heads	*Tails*
Guess	*Heads*	54	50
	Tails	41	55

a Construct the expected frequency table.

b Use Yates' continuity correction to find χ^2_{calc}.

c At a 5% level with $\text{df} = 1$, $\chi^2_{\text{crit}} \approx 3.84$. Test whether Horace's *guess* and the *result* are independent.

d Comment on the validity of Horace's claim.

2 The practical tests for a motorbike licence differ between France and Germany. An inquiry into the two systems yielded the following results for randomly selected candidates.

A χ^2 test at a 10% significance level is used to investigate whether the result of a motorbike test is independent of the country where it took place.

		Result	
		Pass	*Fail*
Country	*France*	56	29
	Germany	176	48

a Construct the expected frequency table.

b Write down the critical value χ^2_{crit}.

c Using Yates' continuity correction, find χ^2_{calc} for this data.

d What conclusion can be drawn from this χ^2 test?

REVIEW SET 16A

1 Buses serving route 033F are scheduled to arrive at stop 2 at 9:45 am each day. The bus company wants to determine whether the buses serving this route have been arriving late to stop 2. Write down the set of hypotheses that the bus company should consider.

2 A null hypothesis H_0 and an alternative hypothesis H_1 are tested at a 10% significance level. A p-value ≈ 0.0794 was obtained for this particular hypothesis test.

- **a** Interpret the p-value.
- **b** What does the significance level represent?
- **c** Is there enough evidence to reject H_0 in this case?

3 Quickshave produces disposable razorblades. They claim that the mean number of shaves before a blade has to be thrown away is 13. A researcher wishing to test the claim asks 30 men to supply data on how many shaves they get from one Quickshave blade. The researcher found the sample mean was 12.8 and the sample standard deviation was 1.6.

Perform a t-test with a 5% level of significance to test the manufacturer's claim.

4 Rosario owns an apricot orchard. Last year, the mean weight of his apricots was 90 grams. Rosario fears that severe droughts this year may have reduced the weight of his apricots. To address his concerns, Rosario randomly selected 20 apricots from his current harvest and recorded their weights in grams:

88	72	93	71	86	94	70	99	86	80
92	93	88	78	83	72	79	75	78	84

- **a** Write down the hypotheses that Rosario should test.
- **b** Are Rosario's concerns justified at the 1% level?

5 Joe and Ruben are friends who like to fish in their free time. Joe is very competitive and claims that he is a better fisherman than Ruben.

The table below summarises the number of fish that Joe and Ruben have caught during their last 12 fishing trips:

	Mean	*Standard deviation*
Joe	$\overline{x}_1 = 10.9$	$s_1 \approx 3.34$
Ruben	$\overline{x}_2 = 10.25$	$s_2 \approx 2.26$

- **a** Write down the hypotheses that Joe should use to test his claim.
- **b** Use a two-sample t-test with a 5% level of significance to test Joe's claim.

6 Kelly recorded the time spent shopping, in minutes, by a sample of customers in two neighbouring supermarkets.

Supermarket A:	12	28	13	7	22	19	4	13	6	11		
Supermarket B:	14	35	32	21	14	8	2	16	24	27	19	42

Conduct a two-sample t-test to determine whether there is a significant difference between the time spent shopping by customers at the supermarkets. Use a 10% level of significance.

7 When a menswear store opens, the sizes of its shirts are distributed according to the percentages below.

Size	Small	Medium	Large	X-Large	XX-Large
Percentage (%)	10	20	35	25	10

In the first week, the store sells 70 shirts. The number sold of each size is shown in this table:

Size	Small	Medium	Large	X-Large	XX-Large
Number sold	4	7	22	24	13

To determine whether the sales are consistent with the distribution of the sizes stocked by the store, a χ^2 goodness of fit test is performed at a 5% level of significance.

a Write down the:

i null hypothesis

ii number of degrees of freedom.

b Calculate the p-value for the test.

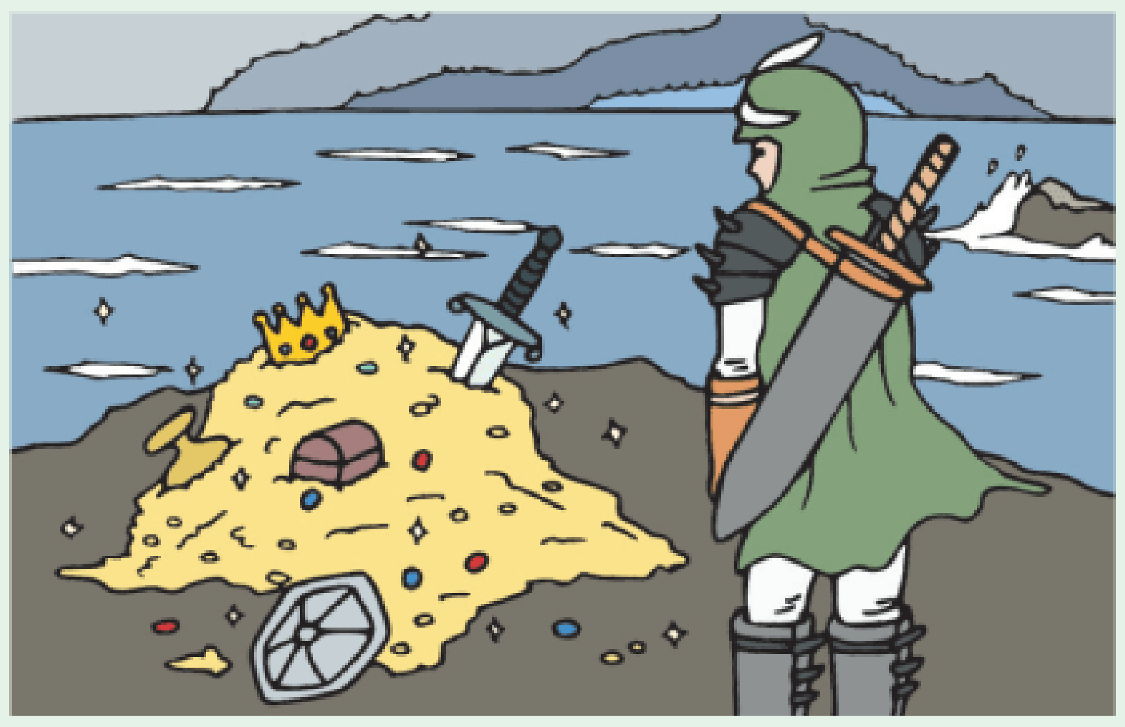
c Decide whether the store should change the distribution of shirt sizes that it stocks.

8 A guide for a particular role playing game lists the following percentages for the rarity of items obtained when a "loot box" is opened.

Item rarity	*Percentage chance*
super rare	5%
rare	10%
uncommon	25%
common	60%

Emmanuel thinks that the guide is incorrect. To test his suspicions, he opens 250 loot boxes. His results are shown in the table below.

Item rarity	*Frequency*
super rare	5
rare	17
uncommon	76
common	152
Total	250



a Calculate the expected frequency for each item rarity.

b Conduct a χ^2 goodness of fit test with a 1% level of significance to determine whether Emmanuel's suspicions are justified.
Use a critical value to make your decision.

9 The table alongside shows the responses to a survey about whether the city speed limit should be increased.
Conduct a χ^2 test for independence at a 10% level to determine whether there is any association between the *age of a driver* and their *opinion* on the speed limit.

	Age of driver		
	18 to 30	31 to 54	55+
Increase	234	169	134
No increase	156	191	233

REVIEW SET 16B

1 Quickchick grow chickens to sell to a supermarket chain. However, the buyers believe that the supplied chickens are lighter than the minimum advertised weight of 1.2 kg.
What set of hypotheses should the buyers consider to investigate their concerns?

2 A particular χ^2 test has 6 degrees of freedom (df) and significance level $\alpha = 0.05$.

- **a** State the critical value χ^2_{crit} for this test.
- **b** The test statistic obtained for this test was $\chi^2_{\text{calc}} \approx 5.71$. Is there sufficient evidence to reject the null hypothesis?

CRITICAL VALUES

3 As part of a routine health check of its employees, a company wants to check whether their systolic blood pressure is too high. High systolic blood pressure is generally diagnosed when a blood pressure test is more than 140 mm Hg.
A doctor measures the systolic blood pressure of a random sample of 35 company employees. She finds that the sample mean is 143.7 mm Hg with standard deviation 11.2 mm Hg.
Are the company's concerns justified on a 5% level of significance?

4 The average distance Arthur can hit a golf ball is 115 metres.
After spending time with a professional, Arthur measured the distance of 30 drives. The results in metres were as follows:

100	126	93	171	131	94	136	144	138	110
168	132	100	49	156	119	119	150	146	139
149	145	122	56	140	118	115	73	105	133

Is there sufficient evidence at the 5% level to claim that Arthur has improved?

5 To decide the credit limit of a prospective credit card holder, a bank gives points based on factors such as employment, income, home and car ownership, and general credit history.
The points totals of randomly selected people living in the suburbs Maple Grove and Berkton are shown below:

Maple Grove:	14	11	13	13	15	12	12	12	10	11	11	11	12	13
	14	13	11	12	14	14	14	13	15	14	11	10	11	16
	11	12	12	10	11	10	10	12	13	13	13	12		
Berkton:	11	10	12	14	11	12	14	11	11	14	13	13	14	13
	12	12	14	11	12	11	12	12	11	12	13	11	12	13
	12	12	10	10	12	11	11	9	14	10	13	13	10	12

Conduct a two-sample t-test with a 10% level of significance to determine whether there is a difference between the average points total of the two suburbs.

6 The following scores are the final IB examination results for students who took a revision course and students who did not:

Revision course:	32	39	31	35	40	33	34	33	34	35	39	33	32	35
No revision course:	28	31	30	23	33	31	36	36	38	35				

- **a** Find the mean and standard deviation of each sample.
- **b** Conduct a two-sample t-test with a 10% level of significance to determine whether the revision course was effective.

7 A toy company claims to manufacture glass, agate, alabaster, and onyx marbles in the ratio $4:2:2:1$. The marbles are sold in bags of 50.

Aggie bought a bag of marbles and counted the number of each type. Her results are shown in the table.

Test the manufacturer's claim using a goodness of fit test with a 5% significance level.

Type	*Frequency*
glass	19
agate	16
alabaster	13
onyx	2

8 Consider the contingency table alongside. Test whether the variables P and Q are independent using a χ^2 test:

- **a** at a 5% level of significance
- **b** at a 1% level of significance.

	Q_1	Q_2	Q_3	Q_4
P_1	19	23	27	39
P_2	11	20	27	35
P_3	26	39	21	30

9 The following table shows the results from an investigation considering *level of education* and *business success*.

		Education level			
		High school	*Graduate certificate*	*Undergraduate degree*	*Postgraduate*
Business success	*No success*	35	30	41	25
	Low success	28	41	26	29
	Success	35	24	41	56
	High success	52	38	63	72

At a 1% level with df $= 9$, the critical value is 21.67. Test at a 1% level whether there is a link between *education level* and *business success*.

Chapter 17

Voronoi diagrams

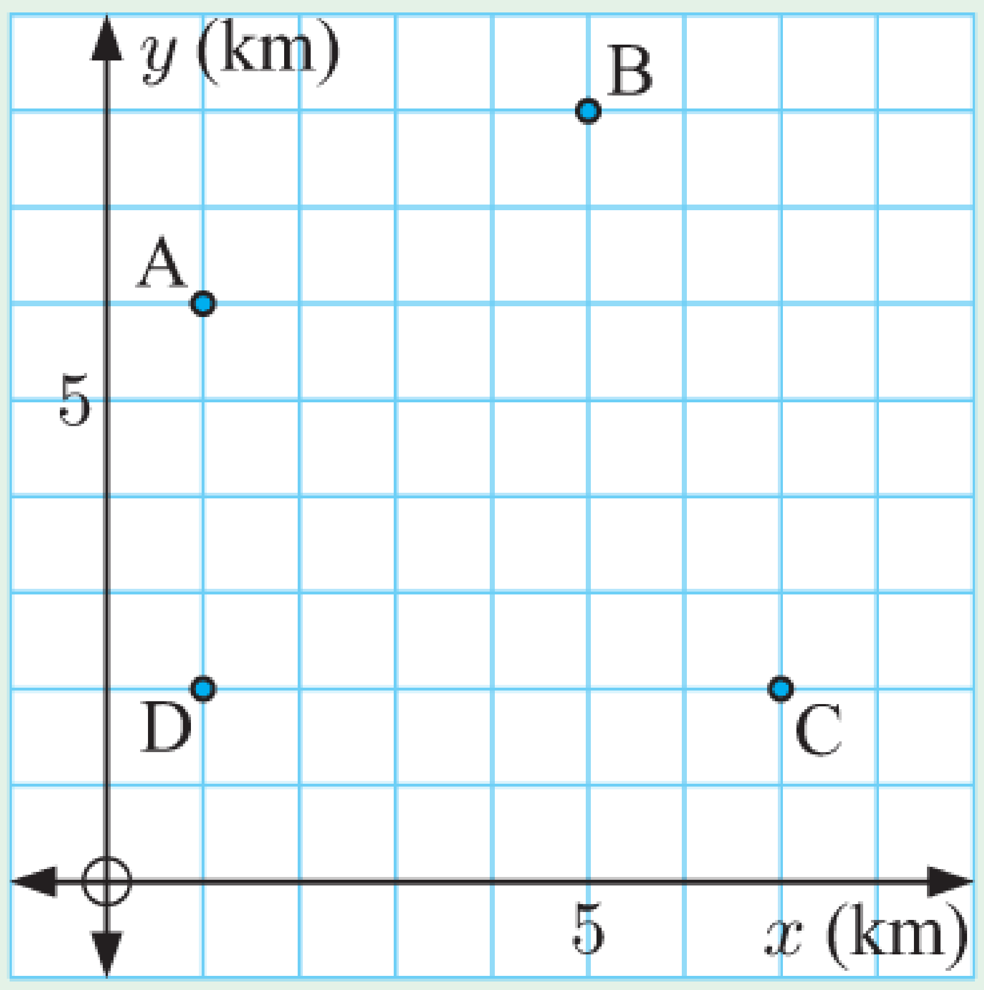
Contents:

A Voronoi diagrams
B Constructing Voronoi diagrams
C Adding a site to a Voronoi diagram
D Nearest neighbour interpolation
E The Largest Empty Circle problem

OPENING PROBLEM

This map shows the locations of the four fire stations in a city.

When a fire starts in the city, it is important that the nearest fire station is alerted.



Things to think about:

a Which fire station is closest to:
  i (6, 3) **ii** (4, 6)?

b The Station Master at fire station D wants to know the region of the city that is closest to his station. What does this region look like?

c How could we improve the map to make it easier to determine the closest fire station to any given location?

d A new fire station is to be built within the city grid. To maximise the efficiency of the fire stations, the new station should be built as far as possible from existing stations. Where should the new station be built?

To solve problems like the **Opening Problem** we can draw a **Voronoi diagram**. Voronoi diagrams have wide-ranging applications in science, engineering, city planning, health, and meteorology.

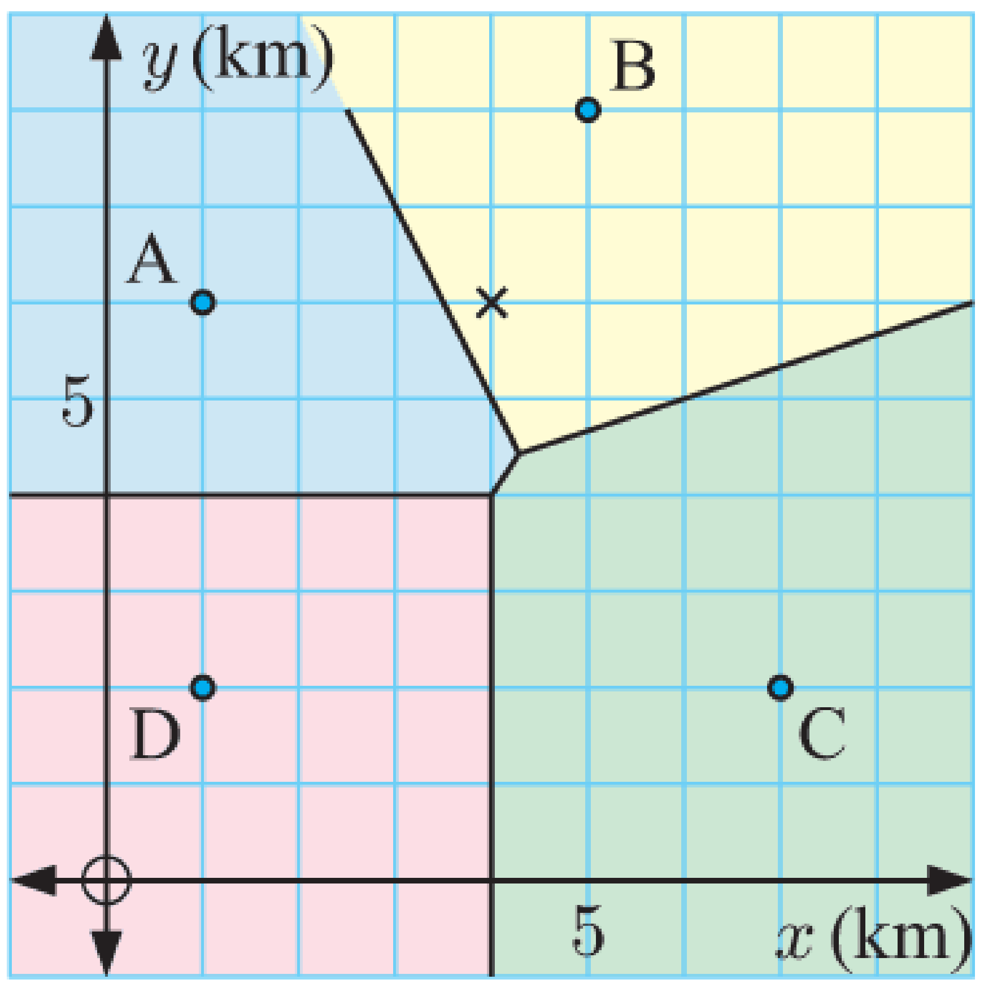
HISTORICAL NOTE

Georgy Feodosevich Voronoy (1868 - 1908) was a mathematician from Pyriatyn (formerly the Russian Empire, now Ukraine). He studied at Saint Petersburg University, and is credited with inventing Voronoi diagrams.

Georgy Voronoy

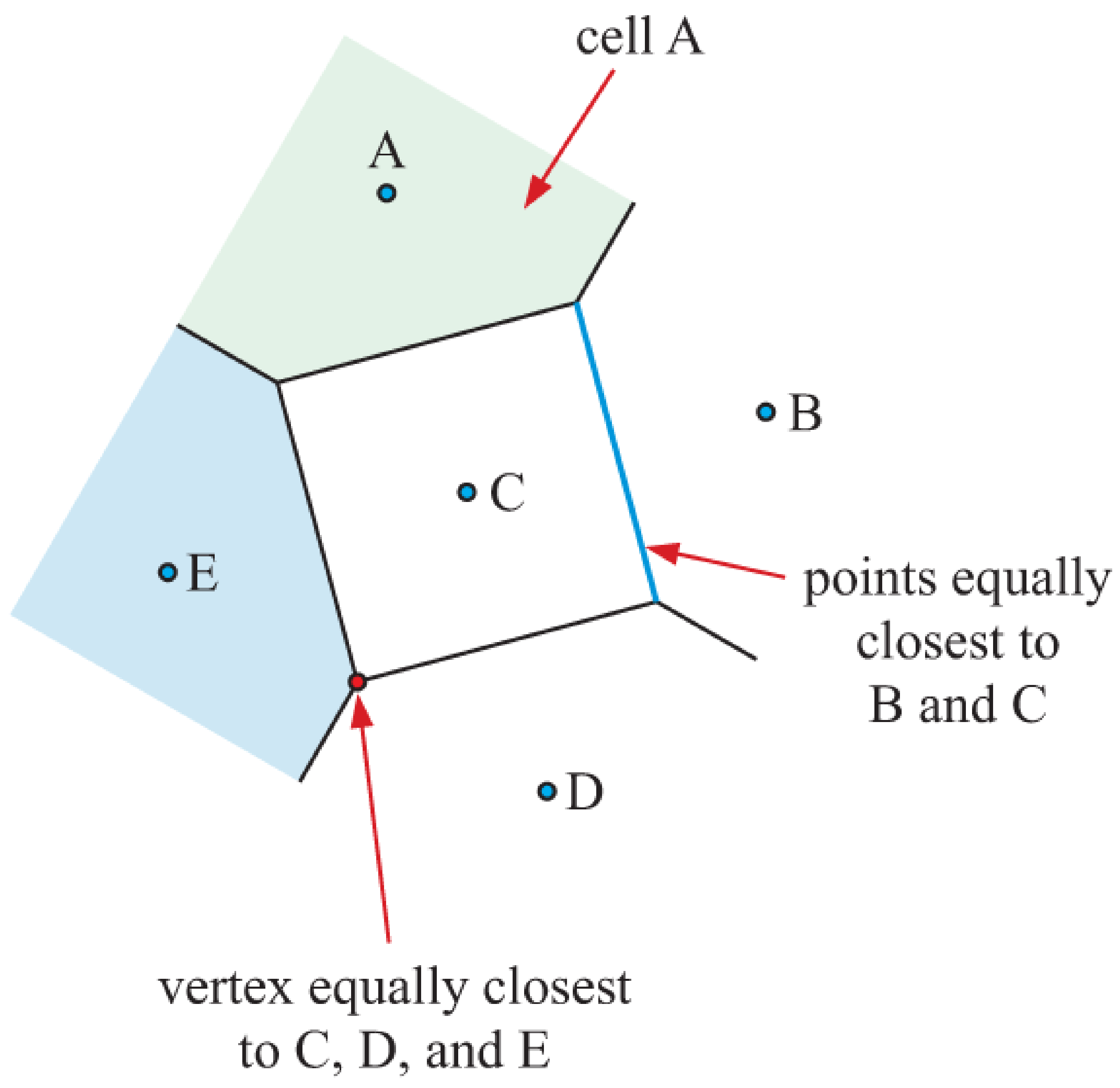
A VORONOI DIAGRAMS

Consider the map of fire stations in the **Opening Problem**. For each station, there is a region which contains all the points that are closer to that station than to any other station. The **Voronoi diagram** alongside illustrates these regions. Notice that each region contains one fire station. This is the closest fire station to every point in the region. For example, (4, 6) lies in the region containing station B, so station B is the closest station to (4, 6).



In a Voronoi diagram:

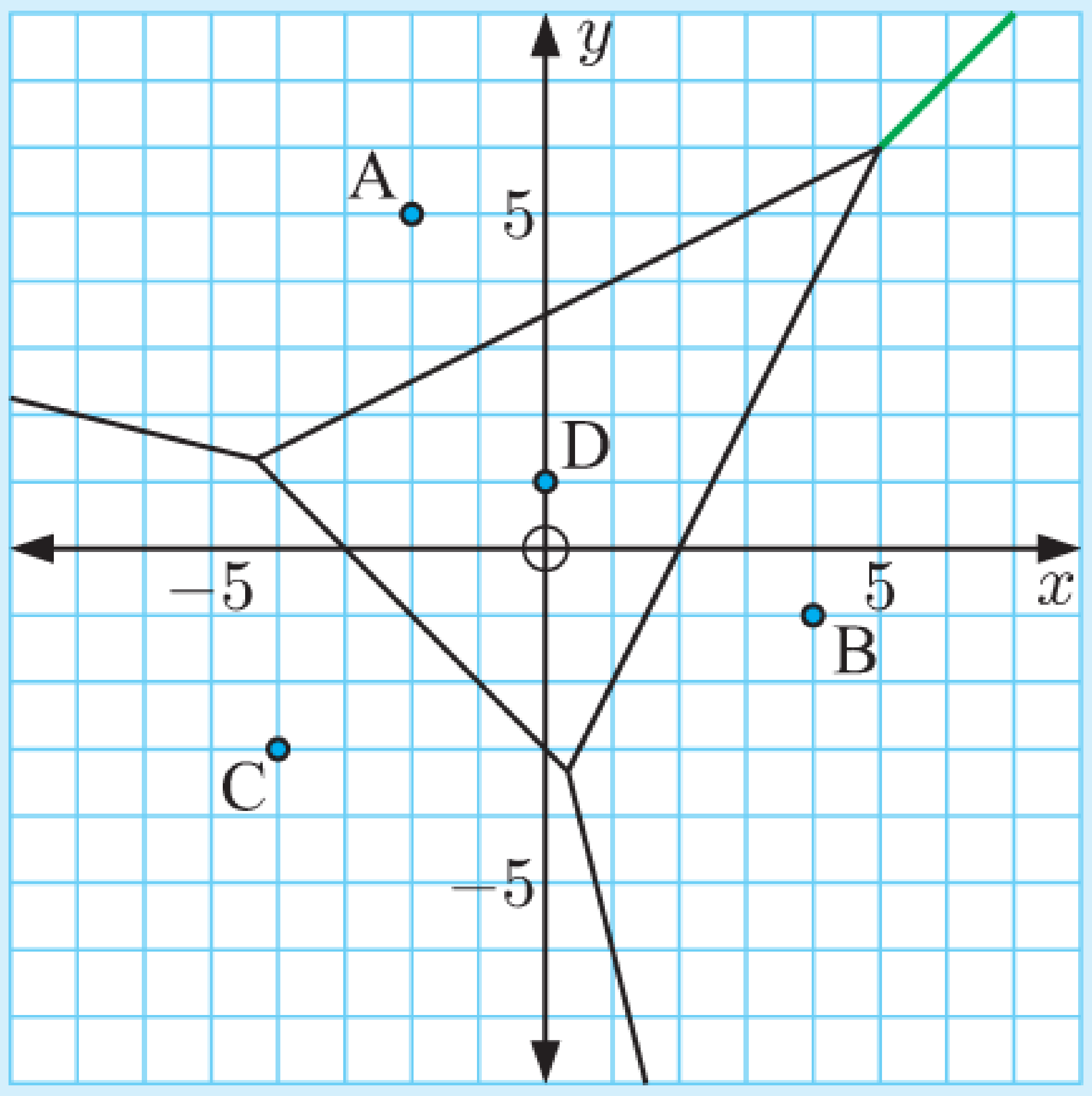
- Important locations are called **sites**. These are the fire stations in the example on the previous page.
- Each site is surrounded by a region or **cell** which contains the points which are closer to that site than to any other site. The cells are labelled according to the site which they contain. For example, the cell which contains site A is called cell A.
- The lines which separate the cells are called **edges**. Each point on an edge is equally closest to the two sites whose cells are adjacent to that edge. For example, any point which lies on the blue edge is equally closest to sites B and C.
- The points at which the edges meet are called **vertices**. Each vertex is equally closest to the sites whose cells meet at that vertex. For example, the red vertex is equally closest to sites C, D, and E.


cell A
A
B
C
E
points equally closest to B and C
D
vertex equally closest to C, D, and E

Example 1

Self Tutor

Consider this Voronoi diagram for the sites A, B, C, and D.



a How many cells does the diagram contain?

b How many vertices does the diagram contain?

c Identify the site(s) closest to:

i $(2, -3)$ **ii** $(1, 5)$ **iii** $(-1, -2)$

d What can we say about points which lie on the green edge?

a The diagram contains 4 cells, one for each site.

b The diagram contains 3 vertices.

c **i** $(2, -3)$ lies in cell B, so it is closest to site B.

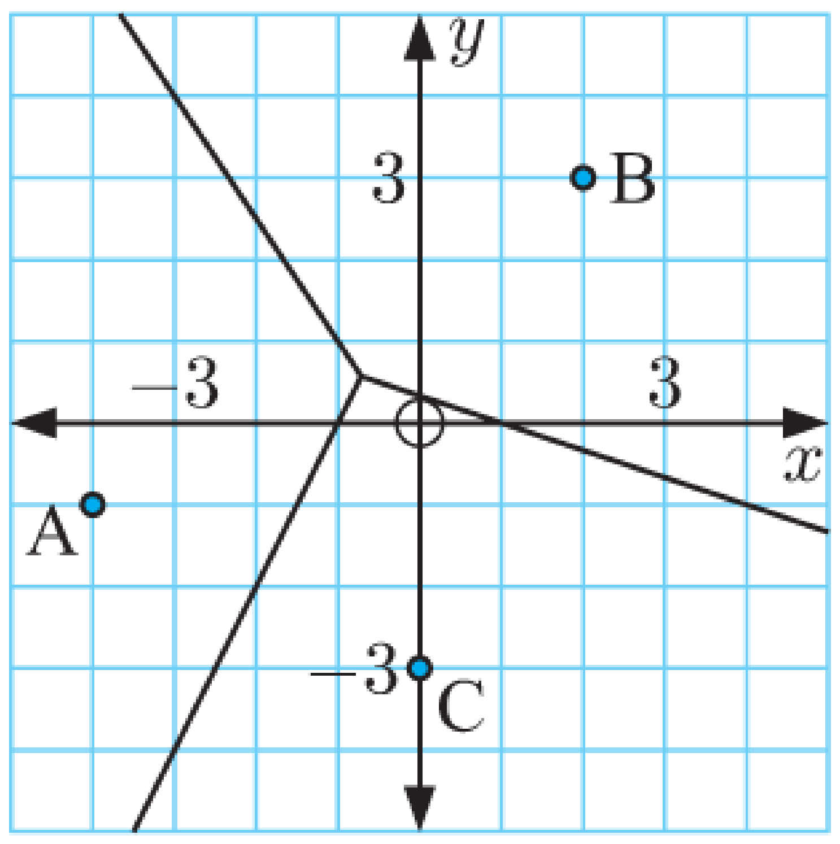
ii $(1, 5)$ lies in cell A, so it is closest to site A.

iii $(-1, -2)$ lies on the edge adjacent to cells C and D, so it is equally closest to sites C and D.

d The green edge is adjacent to cells A and B, so points which lie on this edge are equally closest to sites A and B.

EXERCISE 17A

1 Consider this Voronoi diagram for the sites A, B, and C.

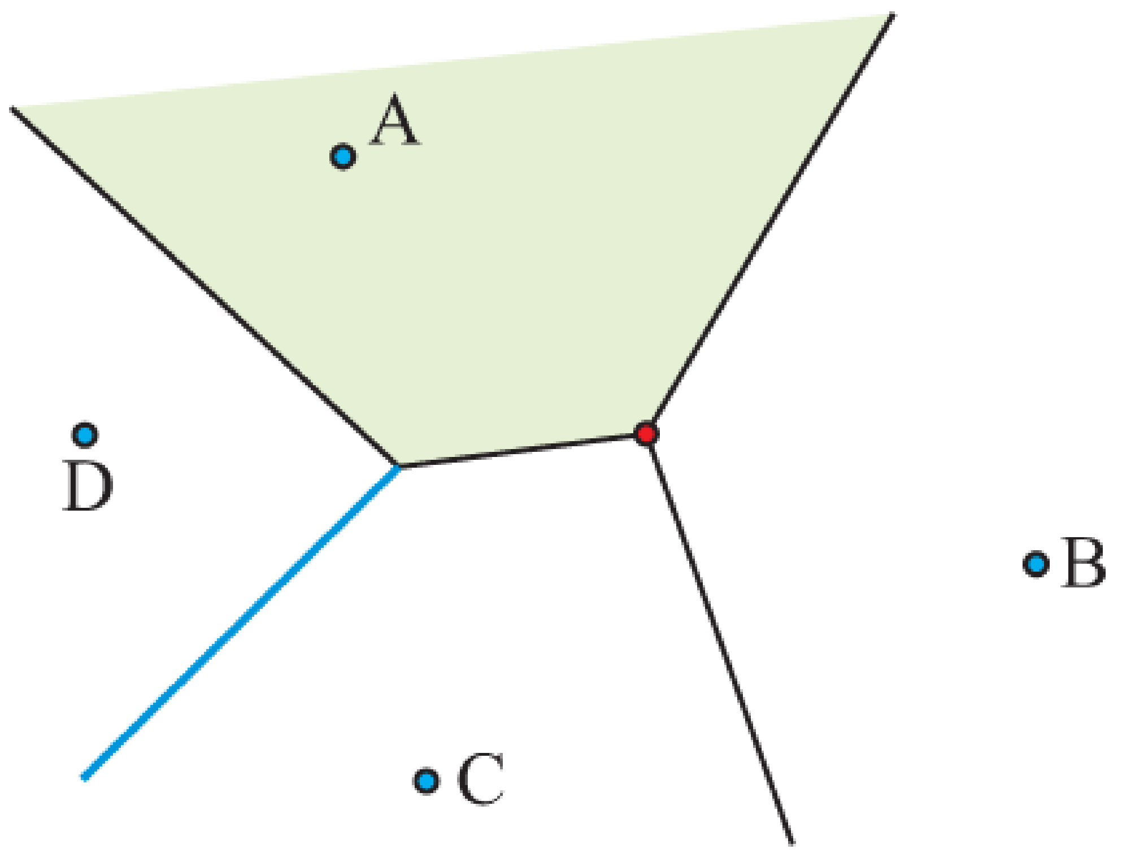


a State the number of:
  - **i** cells
  - **ii** edges
  - **iii** vertices.

b The point P has coordinates $(-1, 2)$.
  - **i** Which site is closest to P? Explain your answer.
  - **ii** Verify your answer by calculating the distance between P and each site.

c The point Q has coordinates $(-1, 0)$ and lies on an edge of the Voronoi diagram.
  - **i** Identify the site(s) closest to Q. Explain your answer.
  - **ii** Verify your answer by calculating the distance between Q and each site.

2 This Voronoi diagram has been constructed for the sites A, B, C, and D.
What can you say about:

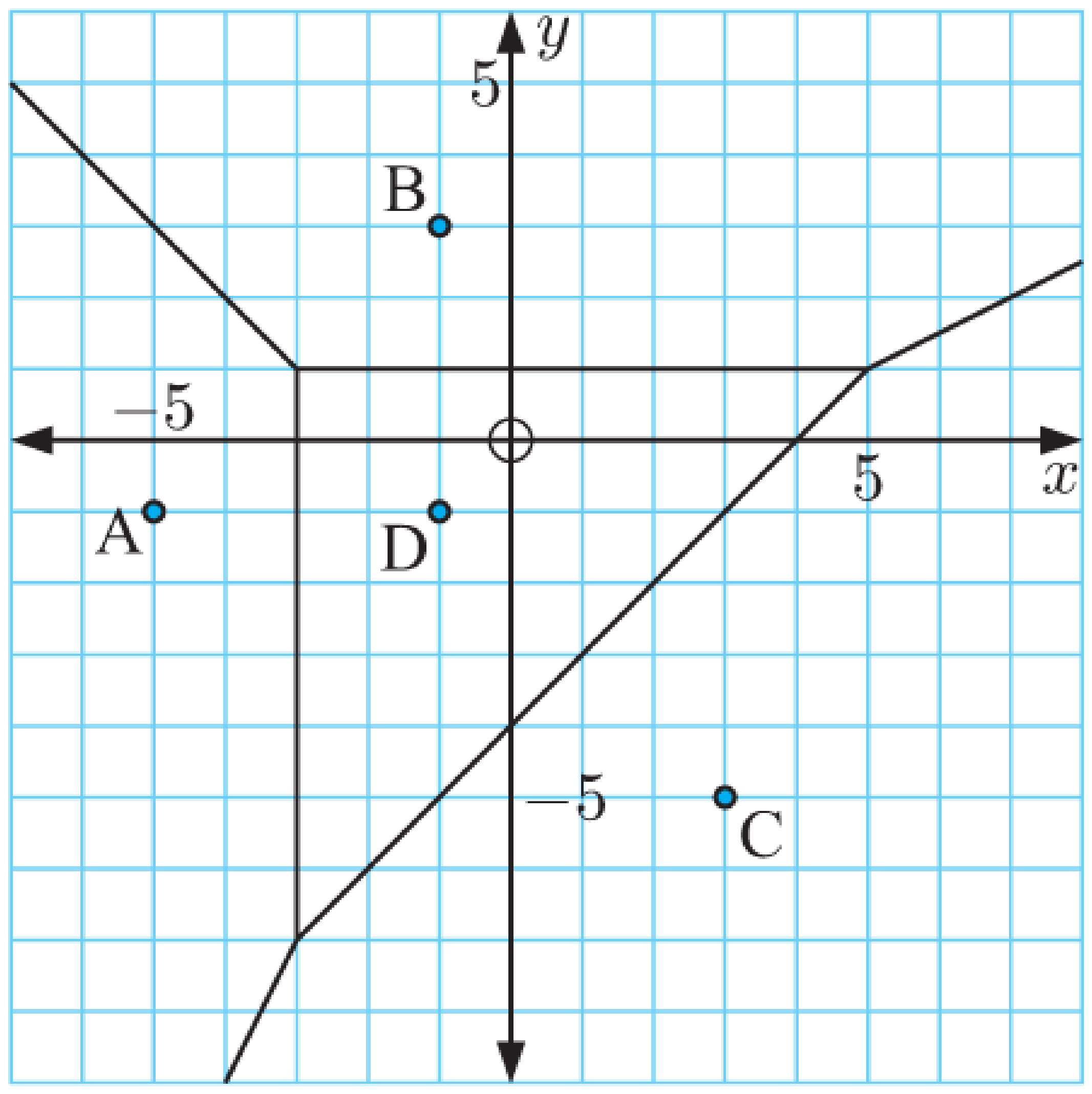


a the points which lie in the green cell

b the points which lie on the blue edge

c the red vertex?

3 Consider this Voronoi diagram for the sites A, B, C, and D.



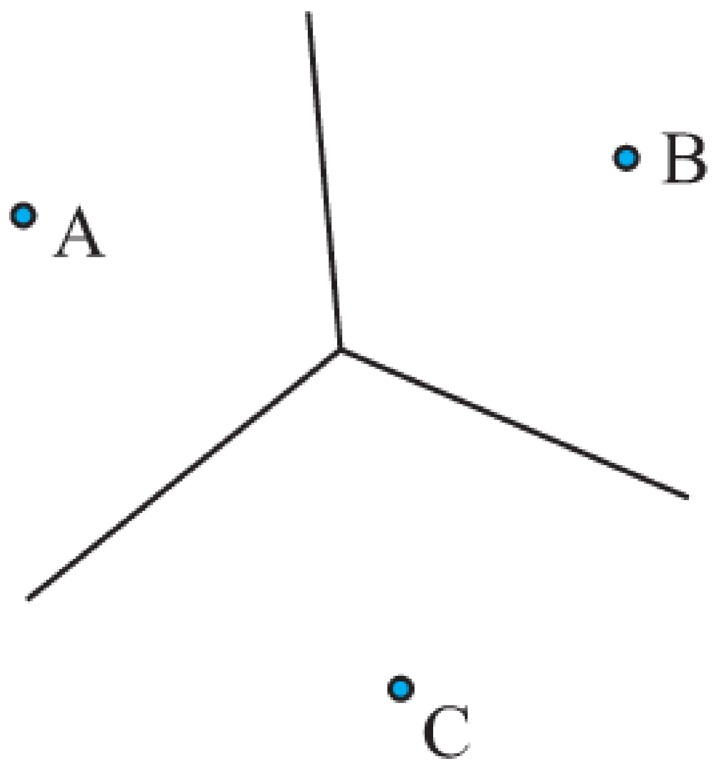
a Identify the site which is closest to:
  - **i** $(2, 3)$
  - **ii** $(-1, -4)$
  - **iii** $(6, 0)$
  - **iv** $(-4, -3)$

b Verify that:
  - **i** $(-3, 0)$ is equidistant from A and D
  - **ii** $(-3, 2)$ is also equidistant from A and D.

c Explain why $(-3, 0)$ lies on an edge of the Voronoi diagram, but $(-3, 2)$ does not.

d Find the area of cell D.

4 In the Voronoi diagram alongside, suppose point P_A lies in cell A, and P_B lies in cell B.
Discuss whether the following statements are necessarily true:



a P_A is closer to A than to any other site.

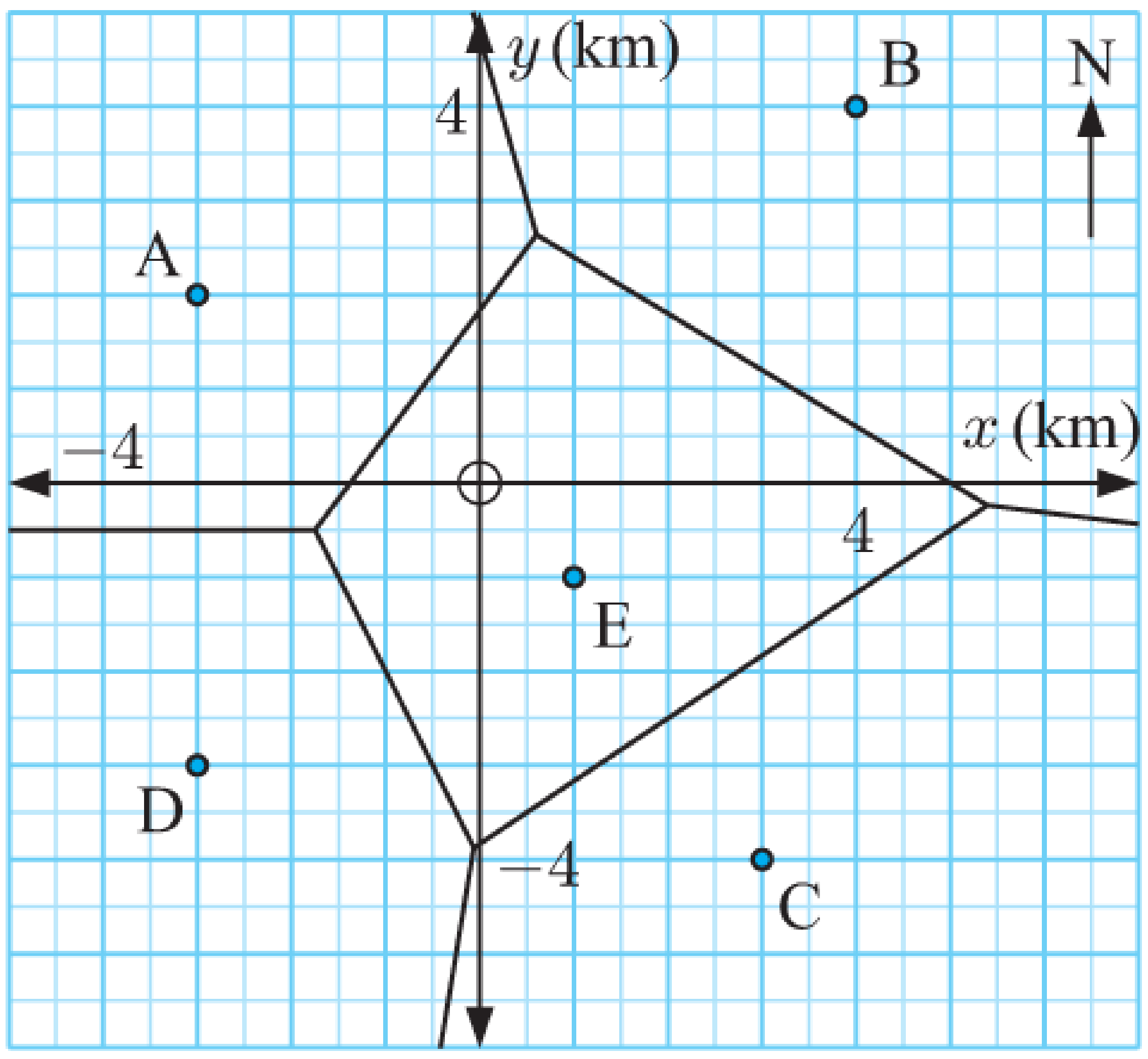
b P_B is closer to B than to C.

c A is closer to P_A than to P_B.

d B is closer to P_B than to C.

5 Let X be a site on a Voronoi diagram, and P be a point within the interior of cell X. Suppose a circle is drawn with centre P, passing through X. Explain why this circle cannot contain any other sites.

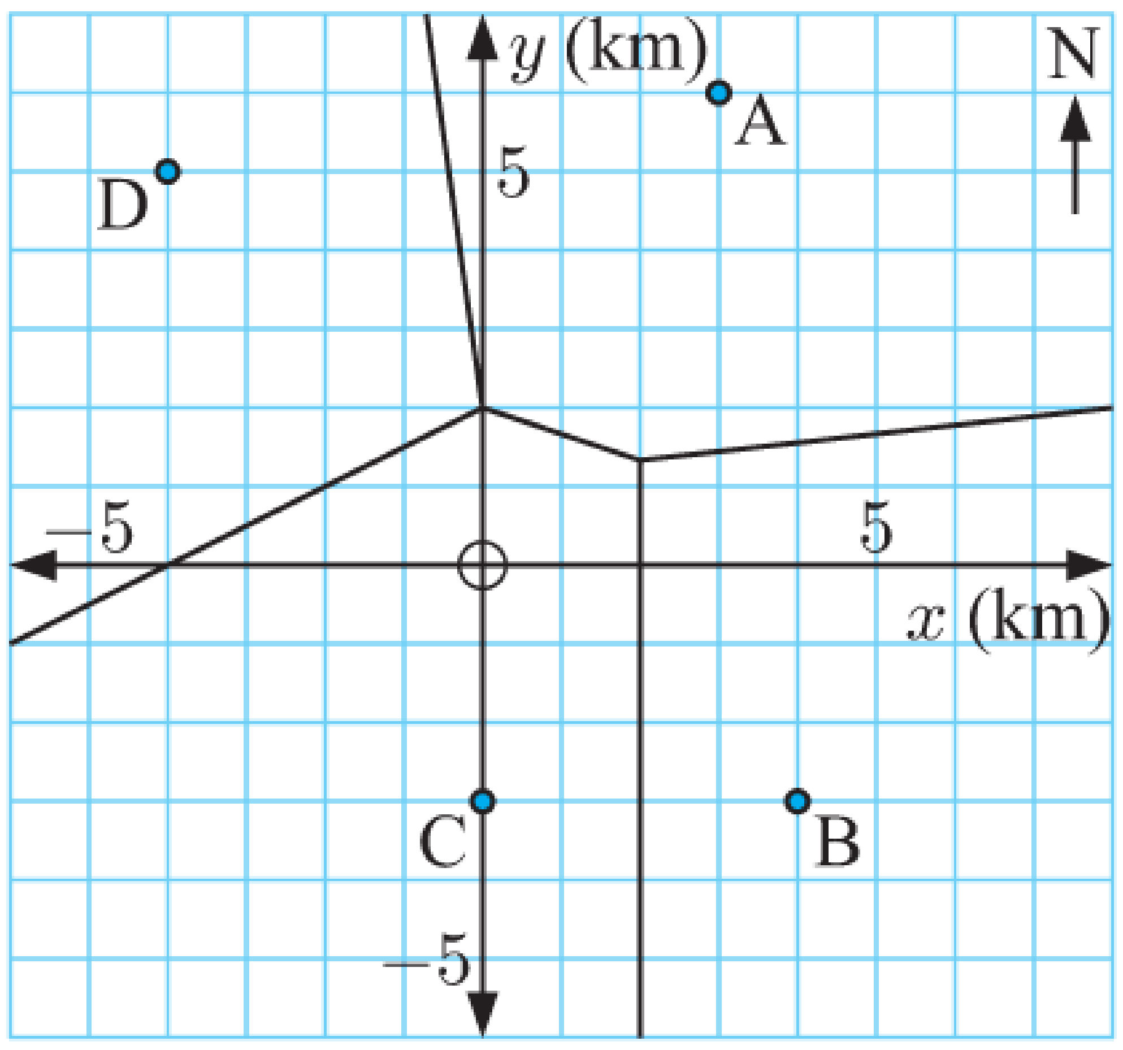
6 This Voronoi diagram shows the post offices in a small city. Parcels which cannot be delivered to a location are held at the nearest post office for collection.



- **a** Determine the holding post office for a parcel sent to:
  - **i** (0, 0)
  - **ii** (4, 1)
  - **iii** (−2, 0)
  - **iv** (3, −2)
- **b** A company leases two buildings in the city. One building is 2.5 km north of the other. One building is equally closest to post offices D and E, and the other is equally closest to A and E.

  Determine the locations of the buildings.

7 This Voronoi diagram shows the public schools in a city. The city is divided into catchment zones, so that students are sent to their nearest public school.

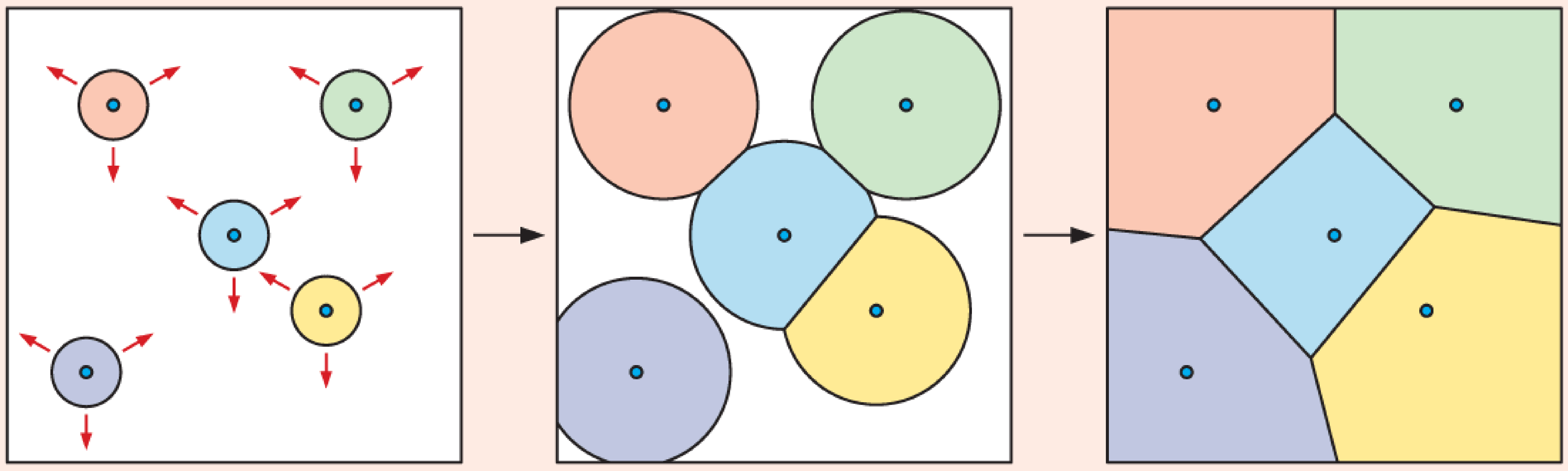


- **a** Identify the nearest public school for a student who lives at:
  - **i** (−3, −2)
  - **ii** (6, 2)
  - **iii** (−2, 4)
  - **iv** (5, 1)
- **b** Bailey's family lives at (−1, 4).
  - **i** Which school's catchment zone does Bailey live in?
  - **ii** Suppose Bailey's family moves to a new home 1 km east of their old home. Show that this will move Bailey into a new catchment zone.
  - **iii** Verify by direct calculation that Bailey's new school is now closer to Bailey than his old school.
- **c** Lizzy's house is equally closest to schools A, C, and D.
  - **i** Where does Lizzy live?
  - **ii** How far does Lizzy live from each of these schools?

8 Let V be a vertex of a Voronoi diagram, and X be the site in a cell adjacent to that vertex. Suppose a circle is drawn with centre V, passing through X. Explain why this circle must pass through at least two other sites.

RESEARCH VORONOI DIAGRAMS IN NATURE

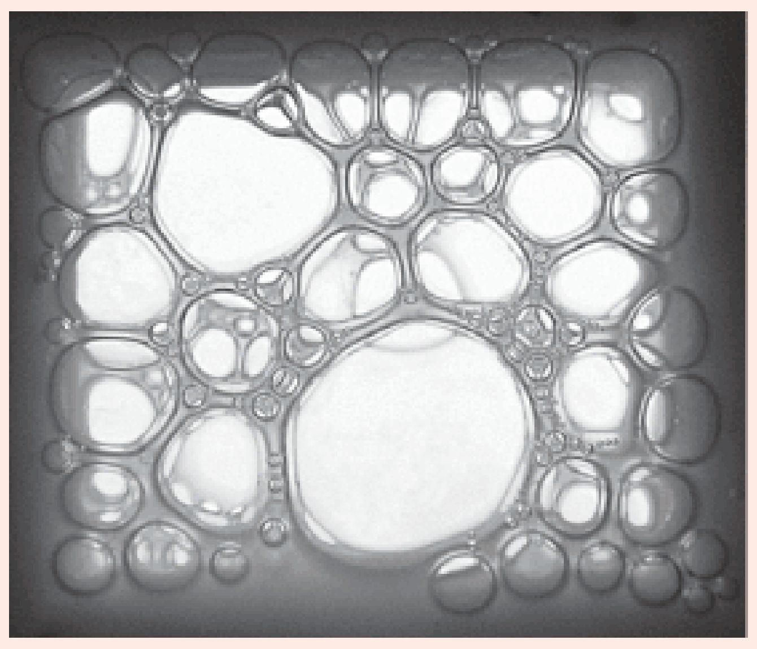
Consider a set of sites. Suppose we create circles centred at each site, which expand until they meet another circle. The regions formed by this process are the cells of the Voronoi diagram for the sites.

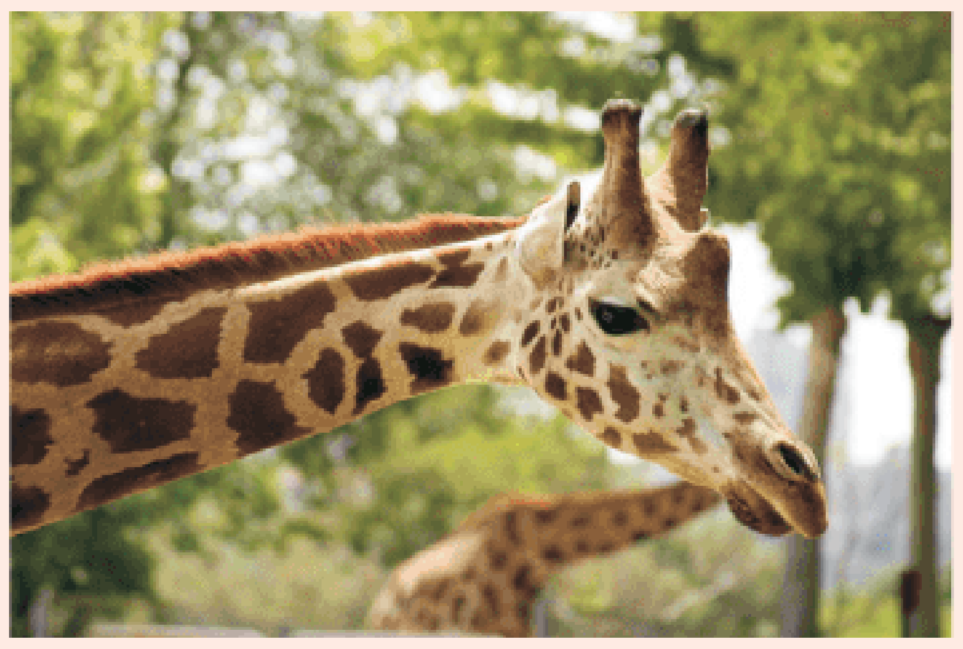


ANIMATION

For this reason, we often observe Voronoi diagrams in the world around us. For example, when soap bubbles form, each cell expands until it meets another membrane.

Voronoi diagrams can also be found in the veins of leaves, and the markings of giraffes.



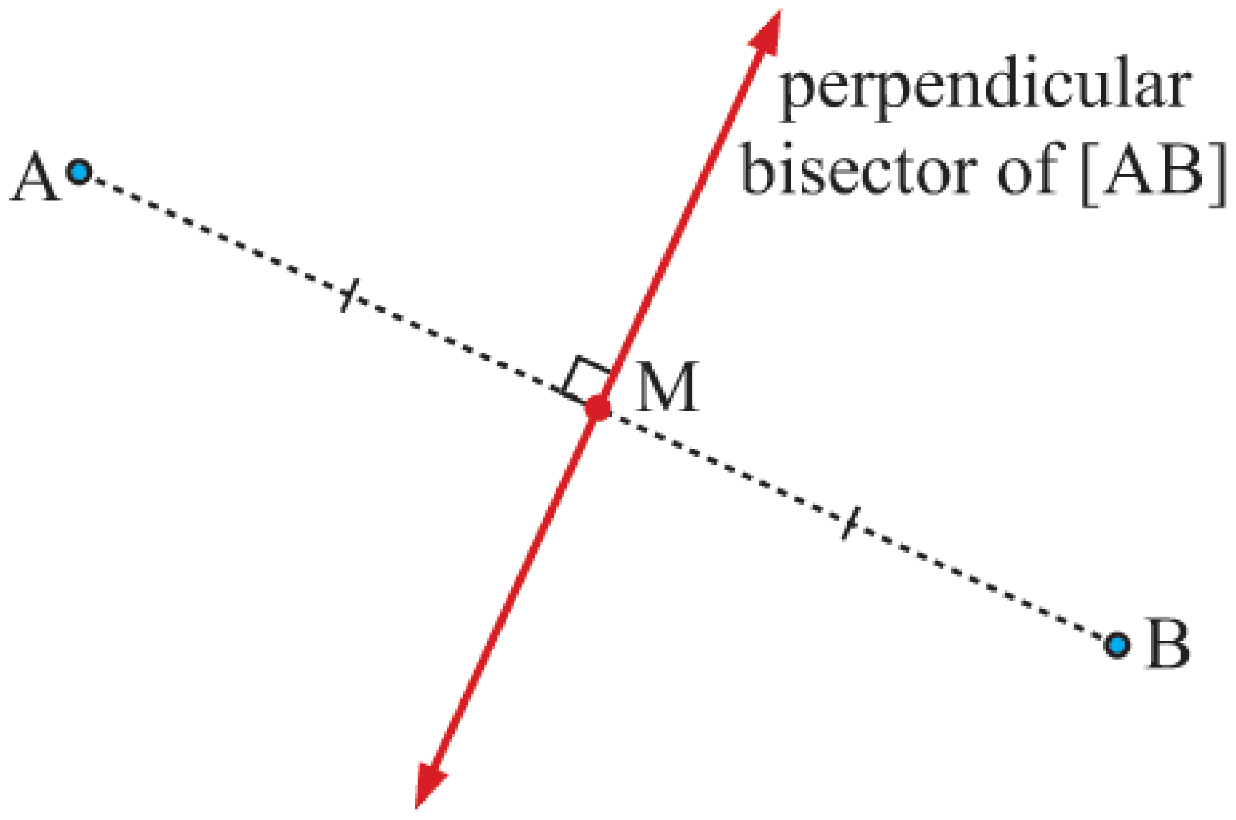


Research other occurrences of Voronoi diagrams in nature. For each occurrence, try to explain why this pattern may have arisen.

B CONSTRUCTING VORONOI DIAGRAMS

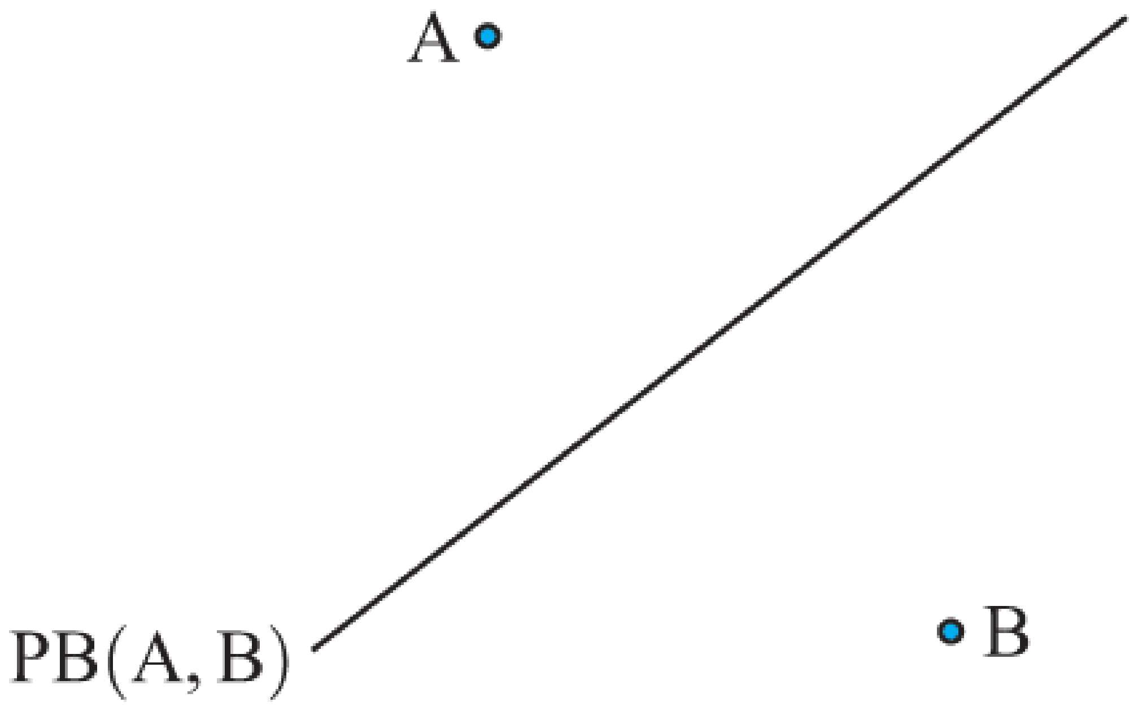
In **Chapter 1** of the **Core Topics SL** book, we saw that:

- The **perpendicular bisector** of a line segment [AB] is the line perpendicular to [AB] which passes through its midpoint.
- All points on the perpendicular bisector are equidistant from A and B.
- The perpendicular bisector divides the plane into two regions. On one side of the line are points that are closer to A than to B, and on the other side are points that are closer to B than to A.



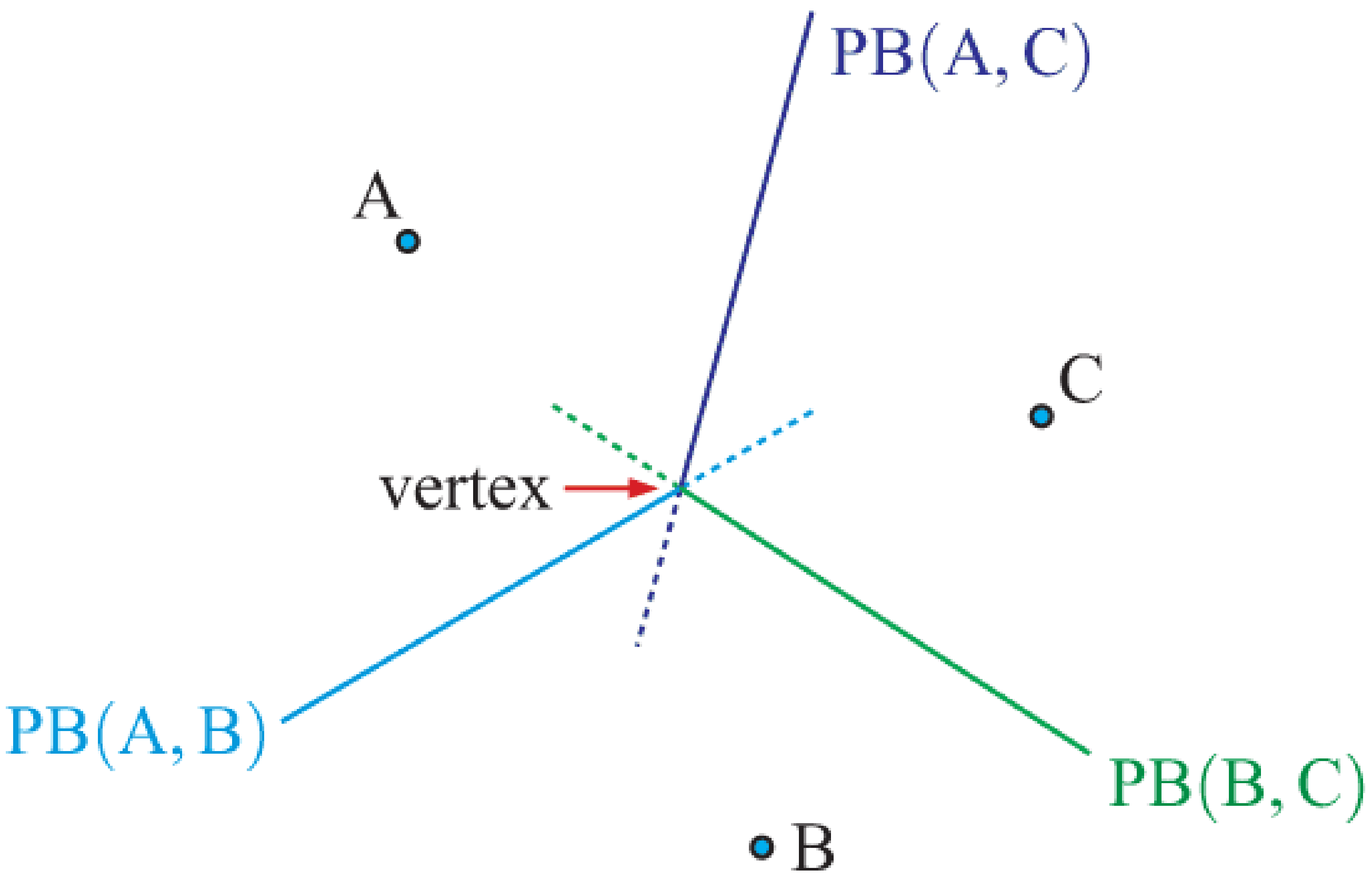
This means that perpendicular bisectors are extremely useful for constructing Voronoi diagrams.

For a plane with two sites A and B, the Voronoi diagram simply consists of the perpendicular bisector of [AB], which we will call PB(A, B).



For a plane with three sites A, B, and C, we draw the perpendicular bisectors of [AB], [AC], and [BC]. Provided A, B, and C are not collinear, these lines will meet at a single point equidistant from A, B, and C. This point is a vertex of the Voronoi diagram.

However, notice that only a section of each perpendicular bisector is included as a Voronoi edge. For example, the points on the dotted blue section of PB(A, B) are *not* part of the Voronoi edge. This is because, although they are equidistant from sites A and B, their closest site is site C.



Example 2

Self Tutor

Construct a Voronoi diagram for the sites:

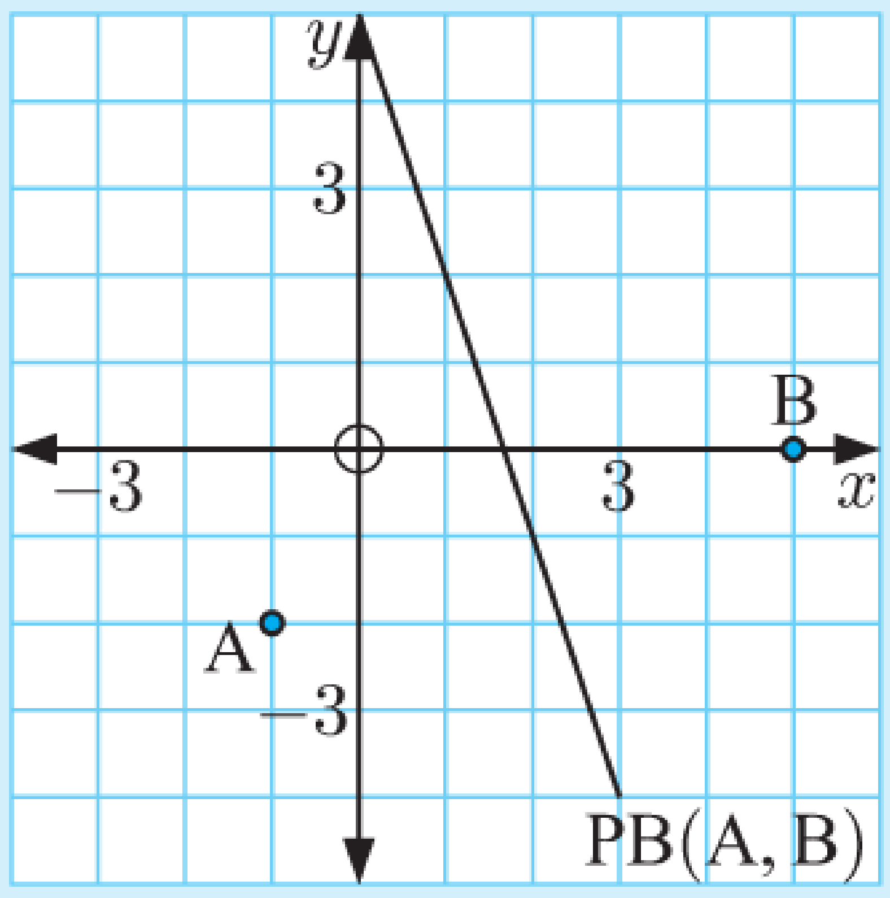
a A$(-1, -2)$ and B$(5, 0)$ **b** A$(-5, 2)$, B$(3, -4)$, and C$(-3, -2)$

a The midpoint of [AB] is $\left(\dfrac{-1+5}{2}, \dfrac{-2+0}{2}\right)$

or $(2, -1)$.

The gradient of [AB] is $\dfrac{0--2}{5--1} = \dfrac{2}{6} = \dfrac{1}{3}$.

So, PB(A, B) has gradient -3 and passes through $(2, -1)$.



b The midpoint of [AB] is $\left(\dfrac{-5+3}{2}, \dfrac{2+-4}{2}\right)$ or $(-1, -1)$.

The gradient of [AB] is $\dfrac{-4-2}{3--5} = \dfrac{-6}{8} = -\dfrac{3}{4}$.

So, PB(A, B) has gradient $\frac{4}{3}$ and passes through $(-1, -1)$.

The midpoint of [AC] is $\left(\dfrac{-5+-3}{2}, \dfrac{2+-2}{2}\right)$ or $(-4, 0)$.

The gradient of [AC] is $\dfrac{-2-2}{-3--5} = \dfrac{-4}{2} = -2$.

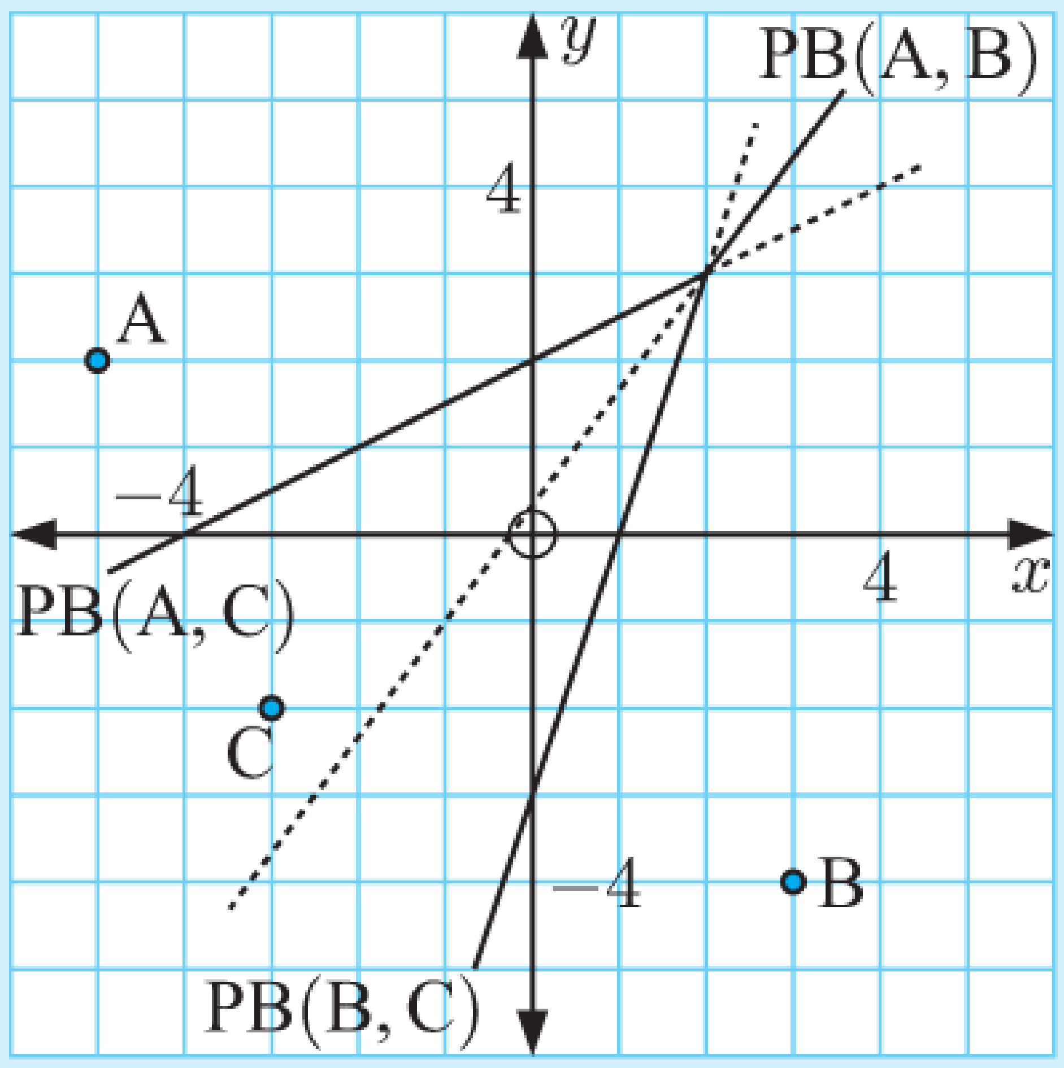
So, PB(A, C) has gradient $\frac{1}{2}$ and passes through $(-4, 0)$.

The midpoint of [BC] is $\left(\dfrac{3+-3}{2}, \dfrac{-4+-2}{2}\right)$ or $(0, -3)$.

The gradient of [BC] is $\dfrac{-2--4}{-3-3} = \dfrac{2}{-6} = -\dfrac{1}{3}$.

So, PB(B, C) has gradient 3 and passes through $(0, -3)$.

To draw the Voronoi diagram we plot the three sites on a set of axes. We can draw the perpendicular bisectors as dashed lines, then make solid only the parts which form the Voronoi edges.



If you have trouble determining which part of the perpendicular bisector to include as an edge, remember that all the edges surrounding a cell are perpendicular bisectors involving the corresponding site. For example, in part **b** of the Example above, cell C is formed by PB(A, C) and PB(B, C).

EXERCISE 17B

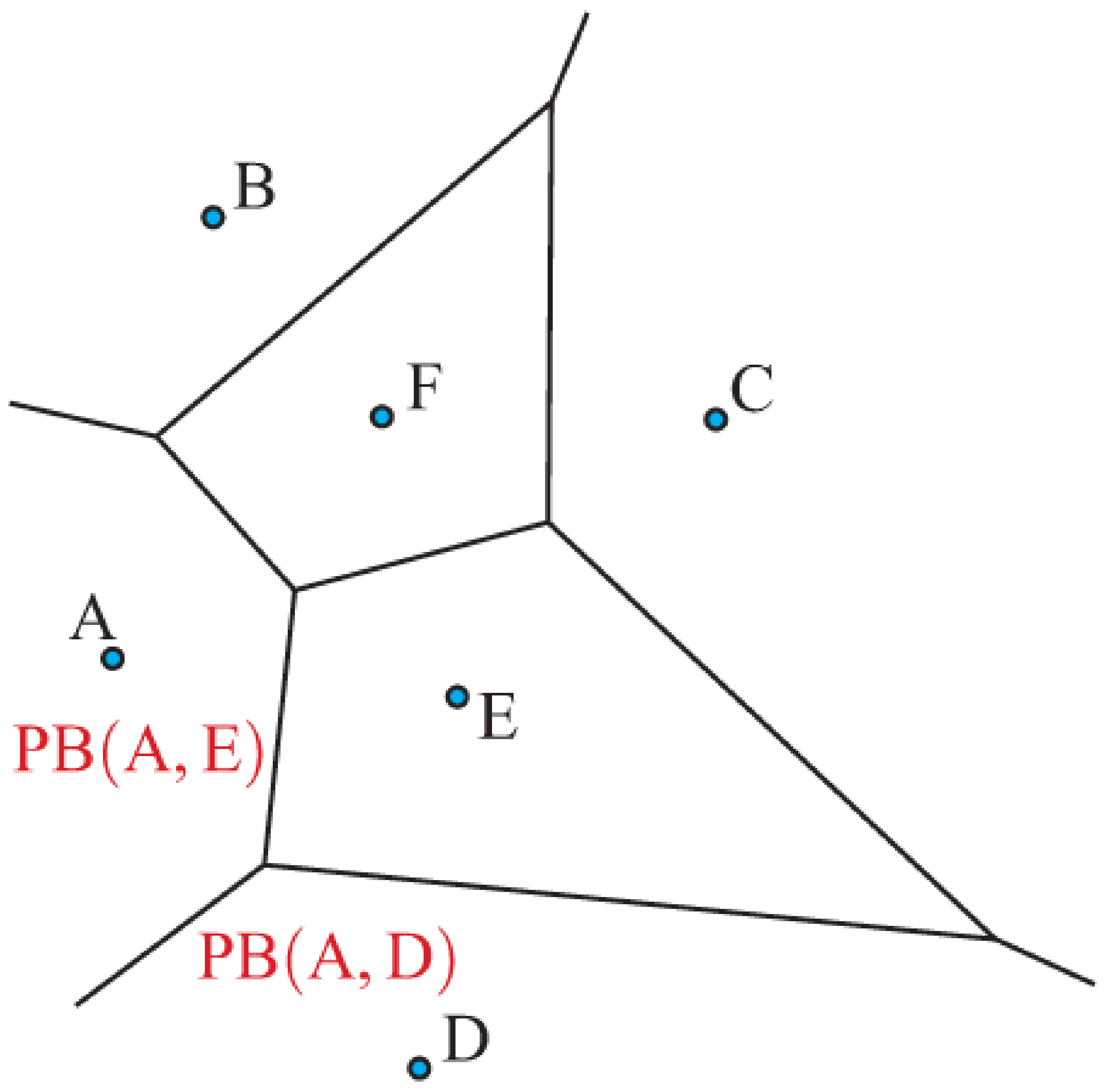
1 Construct a Voronoi diagram for the sites:

- **a** A(2, 6) and B(6, 4)
- **b** A(−3, −4) and B(1, 6).

2
- **a** Construct a Voronoi diagram for the sites A(−2, 5) and B(4, 3).
- **b** Find the equation of the Voronoi edge.
- **c** Verify that the point (−2, −5):
  - **i** lies on the Voronoi edge
  - **ii** is equidistant from A and B.
- **d** Use your Voronoi diagram to identify the site closest to:
  - **i** (0, 0)
  - **ii** (3, 6)
  - **iii** (−4, −5)

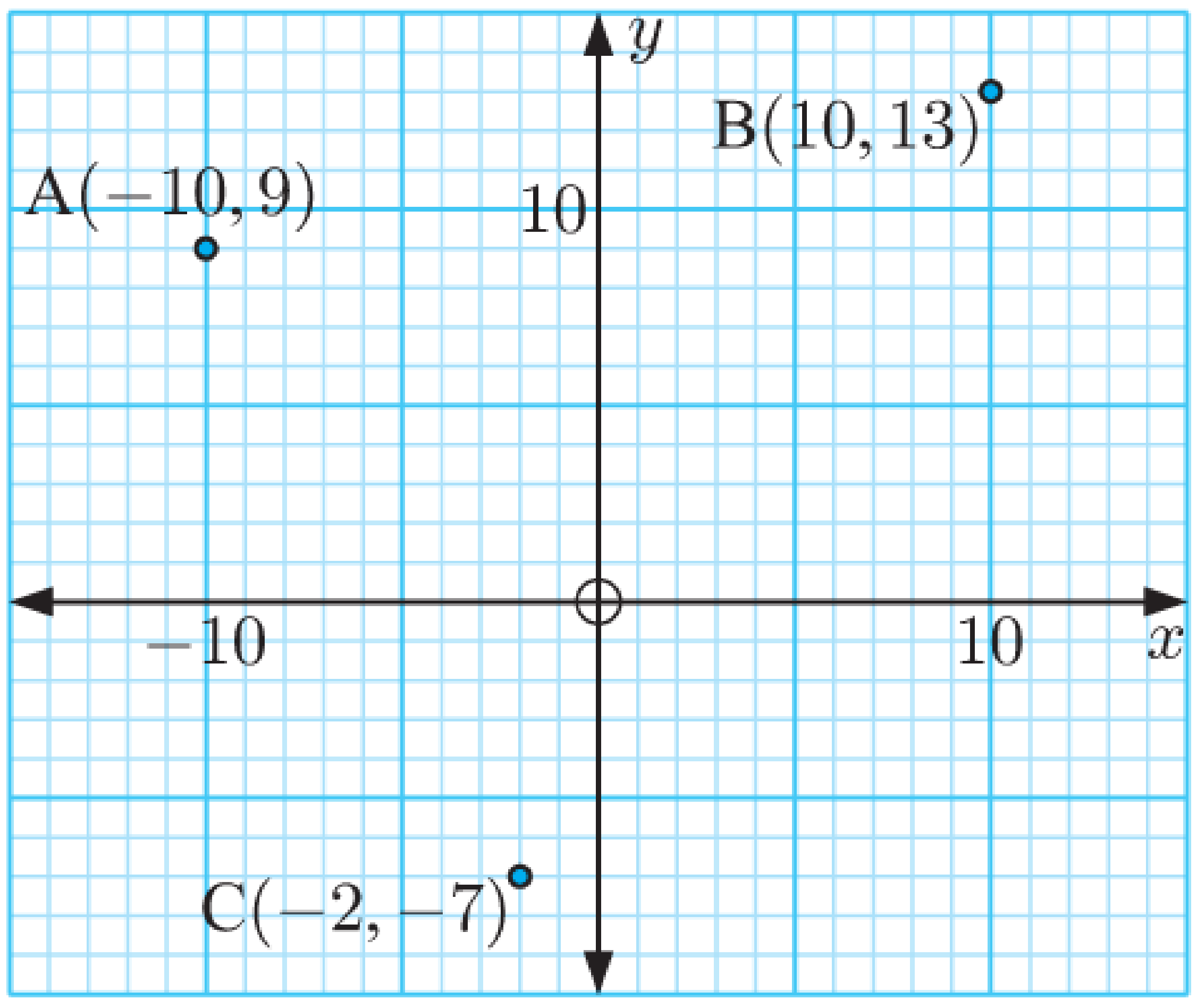
3 Copy this Voronoi diagram, and label each of the remaining edges.

PRINTABLE DIAGRAMS



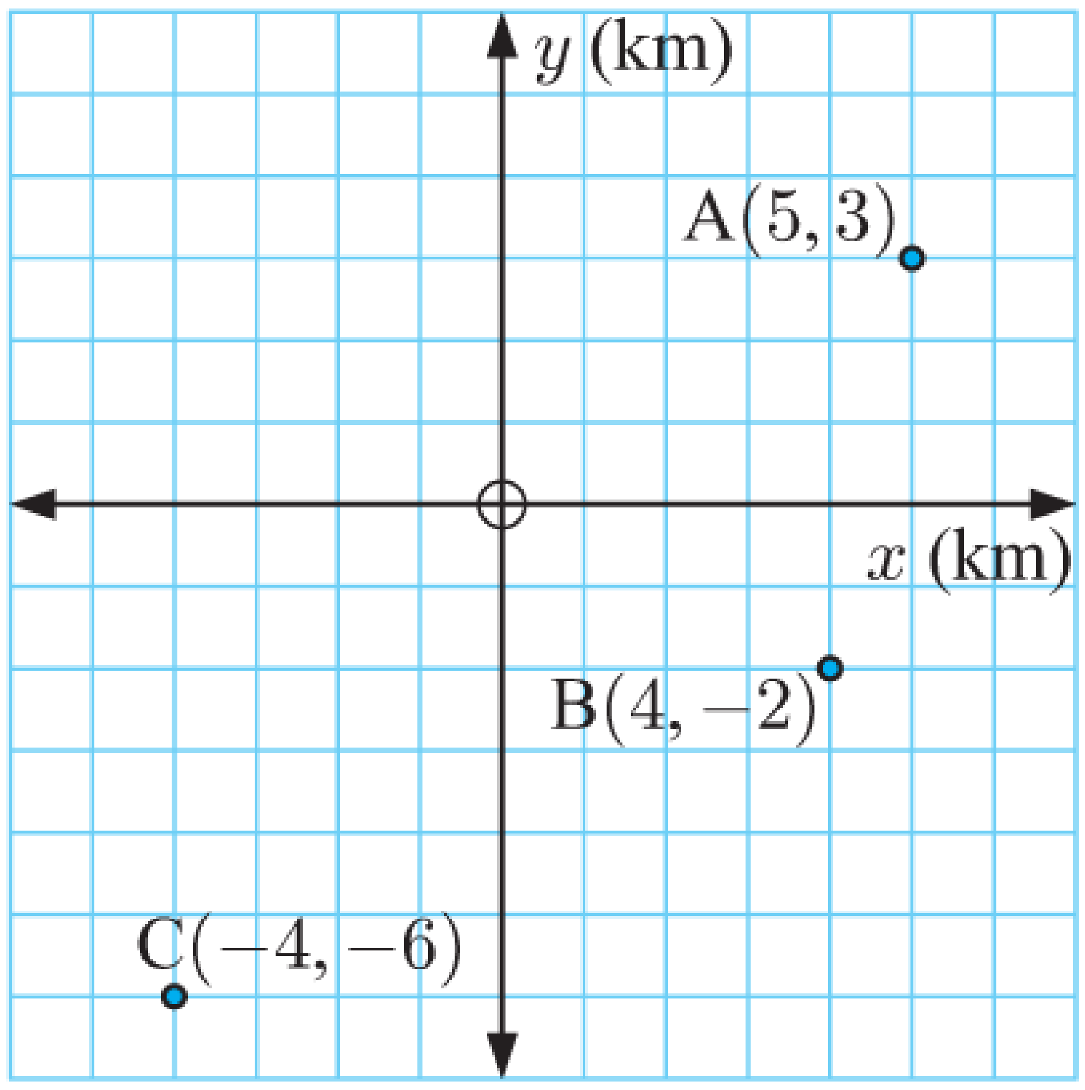
4 Construct a Voronoi diagram for the sites:

- **a** A(4, 7), B(8, 3), and C(0, −5)
- **b** A(−1, 4), B(5, 2), and C(−5, −4).



5
- **a** Construct a Voronoi diagram for the sites shown.
- **b** Find the equation of each edge of your Voronoi diagram.
- **c** Find the coordinates of the vertex of your Voronoi diagram. Verify that this point is equidistant from A, B, and C.
- **d** Use your Voronoi diagram to identify the site closest to:
  - **i** (−2, 8)
  - **ii** (5, 5)
  - **iii** (2, −3)

6



This map shows the locations of the Pablo's Pizza stores in a city. When customers ring Pablo's Pizza, they are automatically transferred to their nearest store.

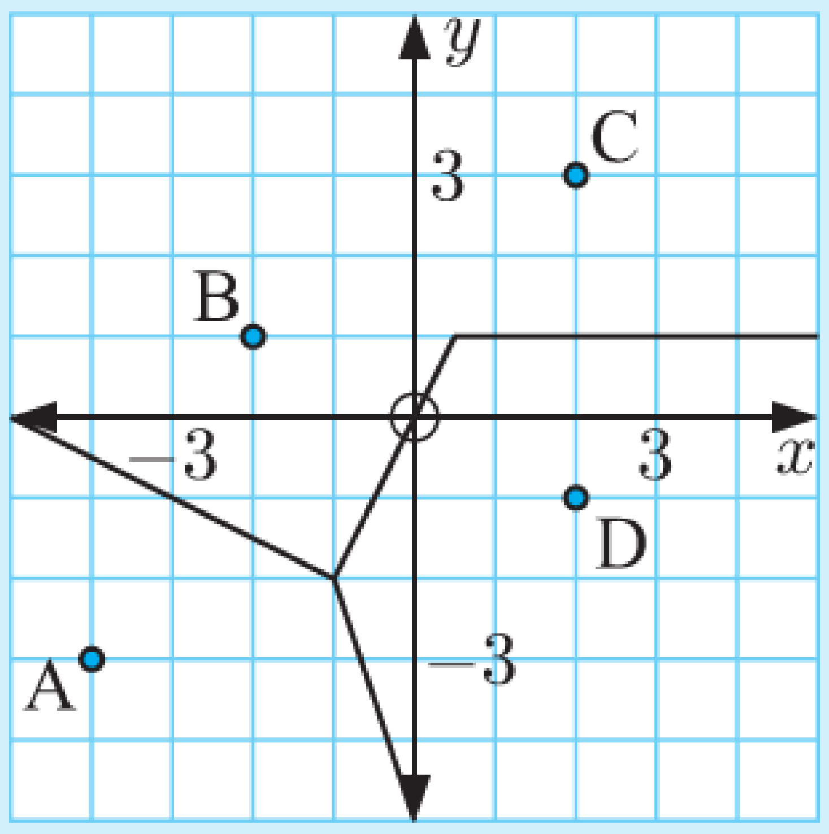
- **a** Construct a Voronoi diagram for these sites.
- **b** Which store is closest to a customer who rings from:
  - **i** (3, 1)
  - **ii** (−2, 2)
  - **iii** (−5, −1)?
- **c** Amanda is equidistant from all three stores.
  - **i** Where does Amanda live?
  - **ii** How far is Amanda from each store?

Example 3

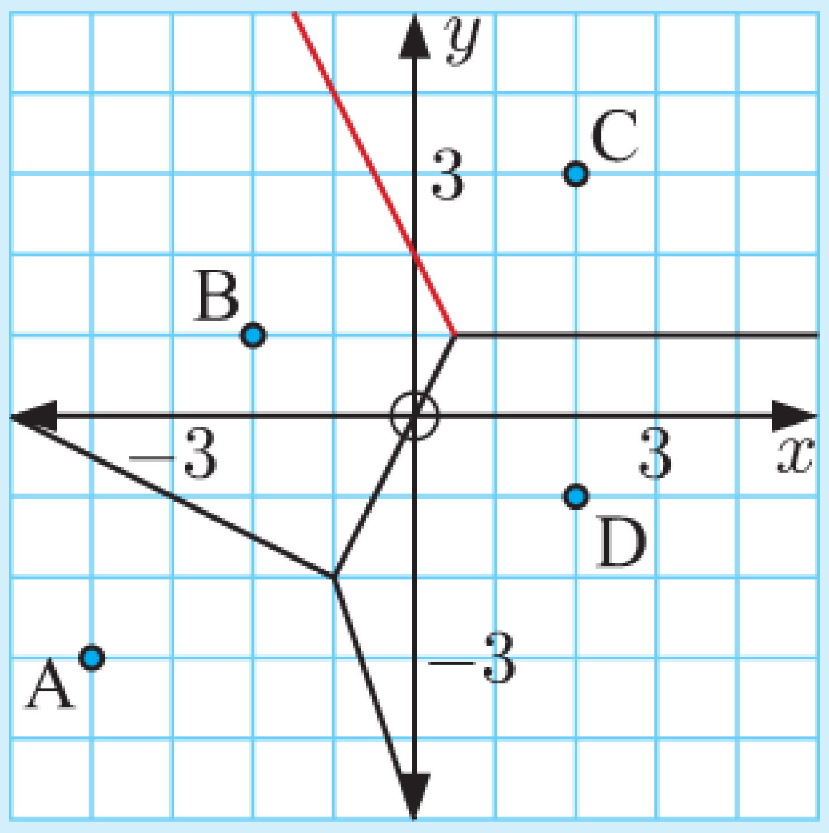
Self Tutor

This Voronoi diagram is missing an edge.

Complete the diagram, and find the equation of the missing edge.



Sites B(−2, 1) and C(2, 3) are currently in the same cell, so the missing edge must be the perpendicular bisector of [BC].



The midpoint of [BC] is $\left(\frac{-2+2}{2}, \frac{1+3}{2}\right)$

or (0, 2).

The gradient of [BC] is $\frac{3-1}{2--2} = \frac{2}{4} = \frac{1}{2}$.

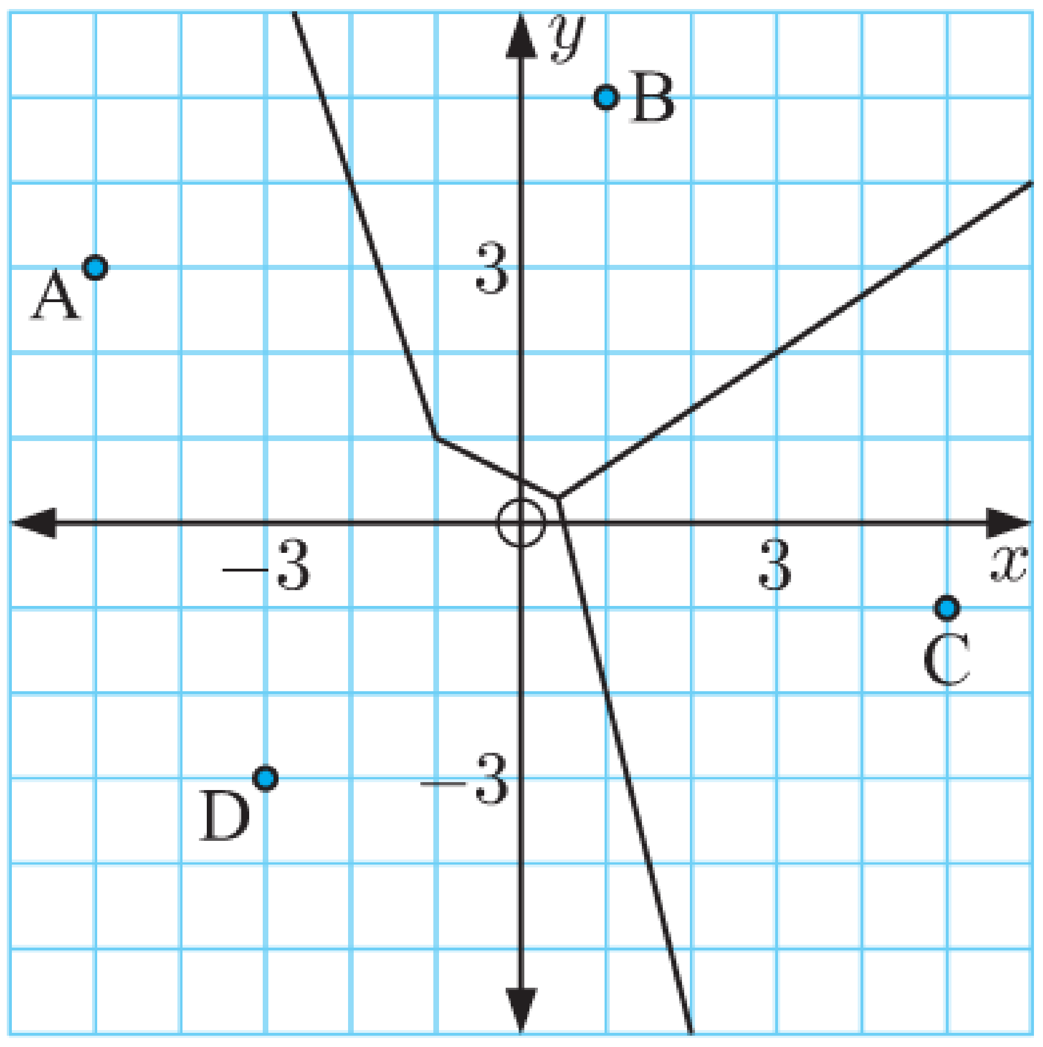
So, PB(B, C) has gradient −2.

$\therefore$ its equation is $2x + y = 2(0) + 2$

or $2x + y = 2$

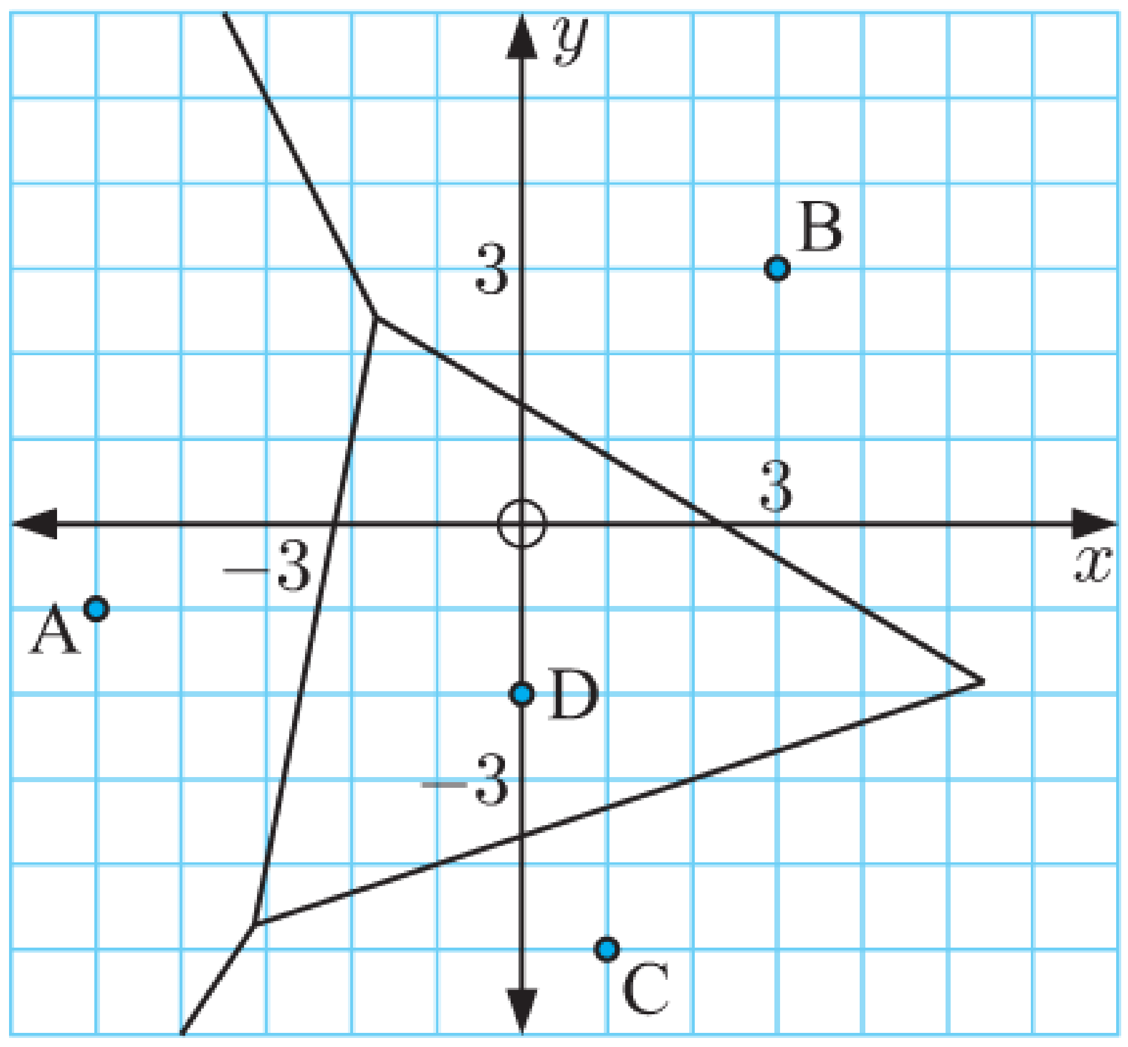
7 **a** Explain why this Voronoi diagram must have an edge missing.

b Copy and complete the diagram, and find the equation of the missing edge.

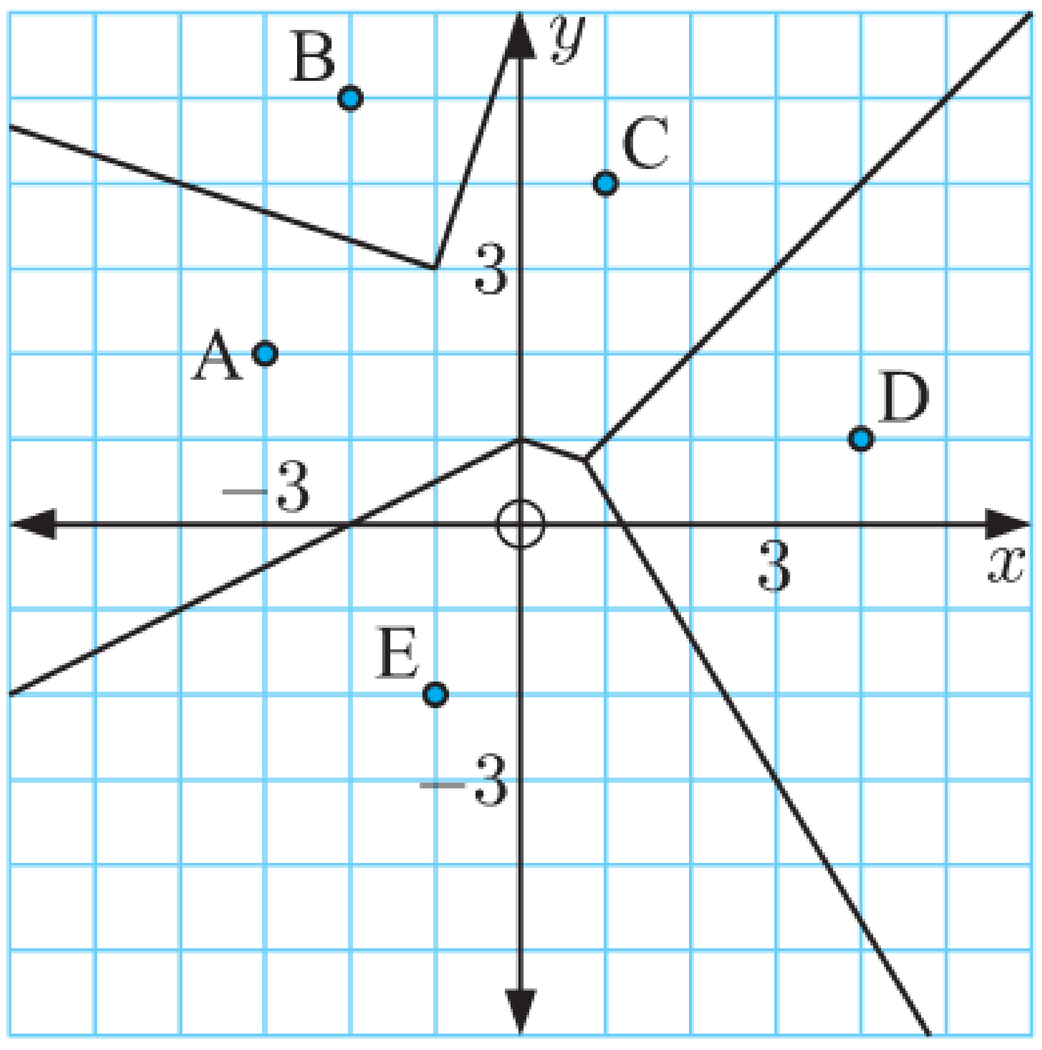


8 These Voronoi diagrams are missing an edge. Complete each diagram, and find the equation of each missing edge. Give your equations in the form $ax + by + d = 0$.

a



b



9 **a** How many edges are missing from this Voronoi diagram?

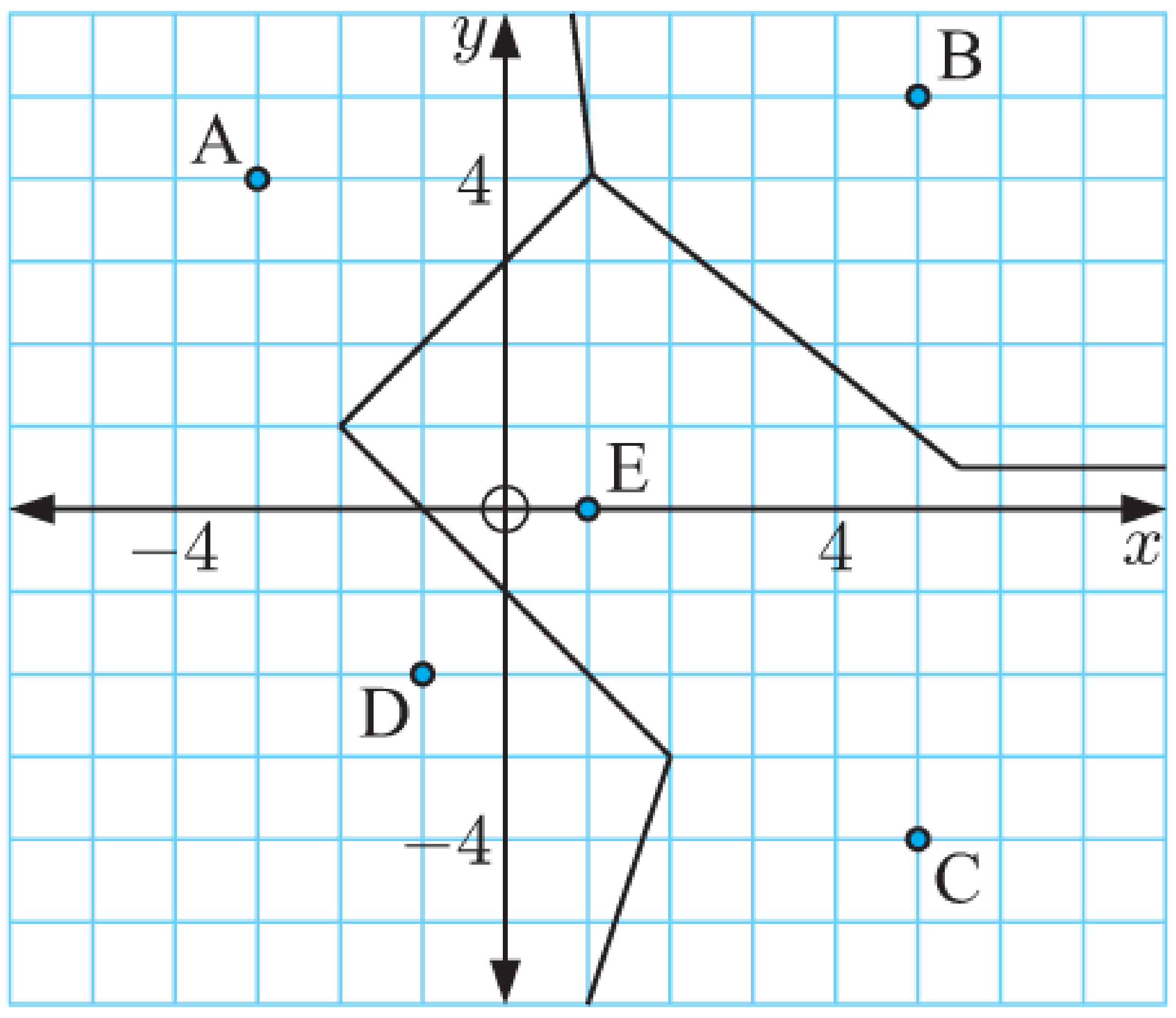
b Complete the diagram, and find the equation of each missing edge.

c Identify the site which is closest to:

i $(-4, 1)$ **ii** $(2, 3)$

d Identify the closest sites to:

i $(4, -1)$ **ii** $(-2, 1)$



10 This map shows the camping sites in a national park.

a Find the equation of the blue edge.

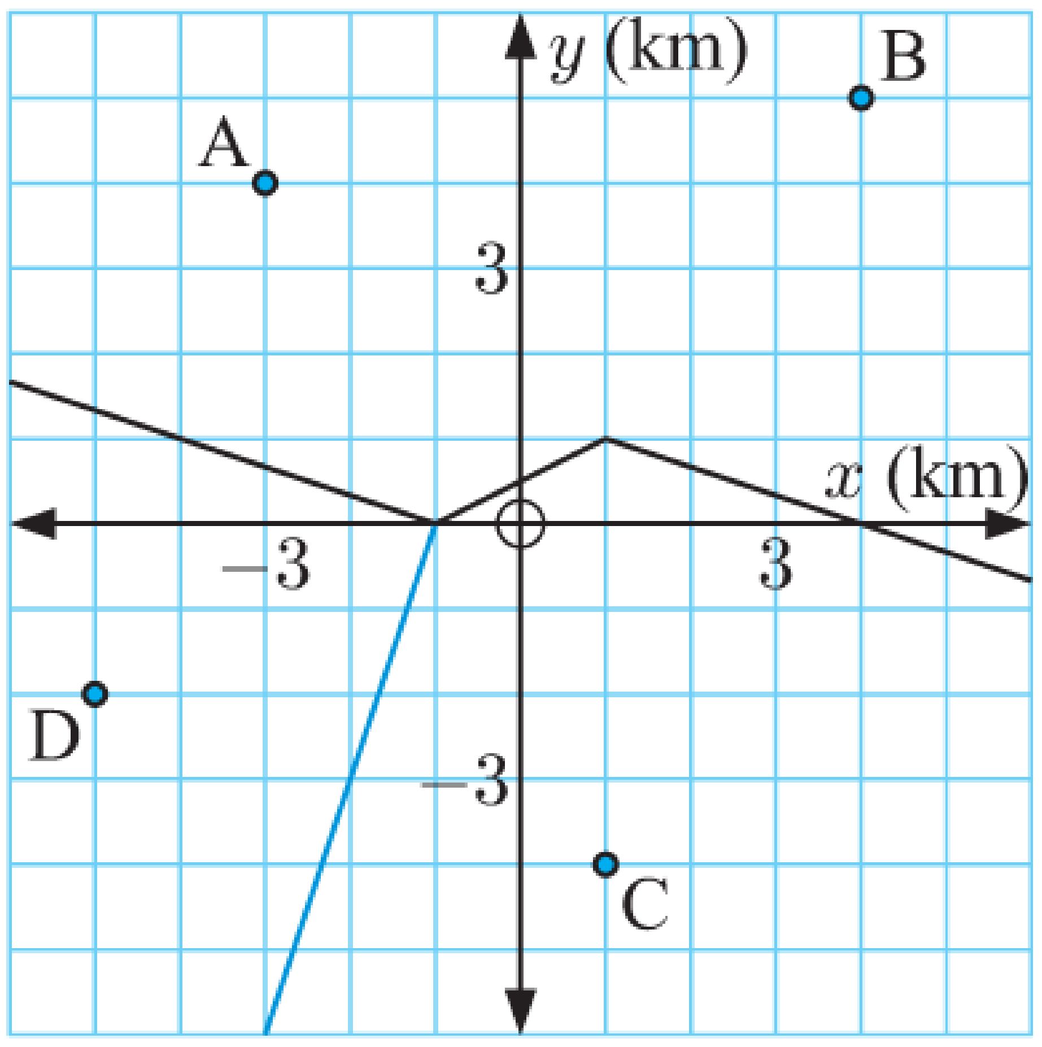
b Complete the diagram, and find the equation of the missing edge.

c Julie is hiking at $(1, 3)$.

i Which campsite is she nearest to?

ii How far is she from this campsite?

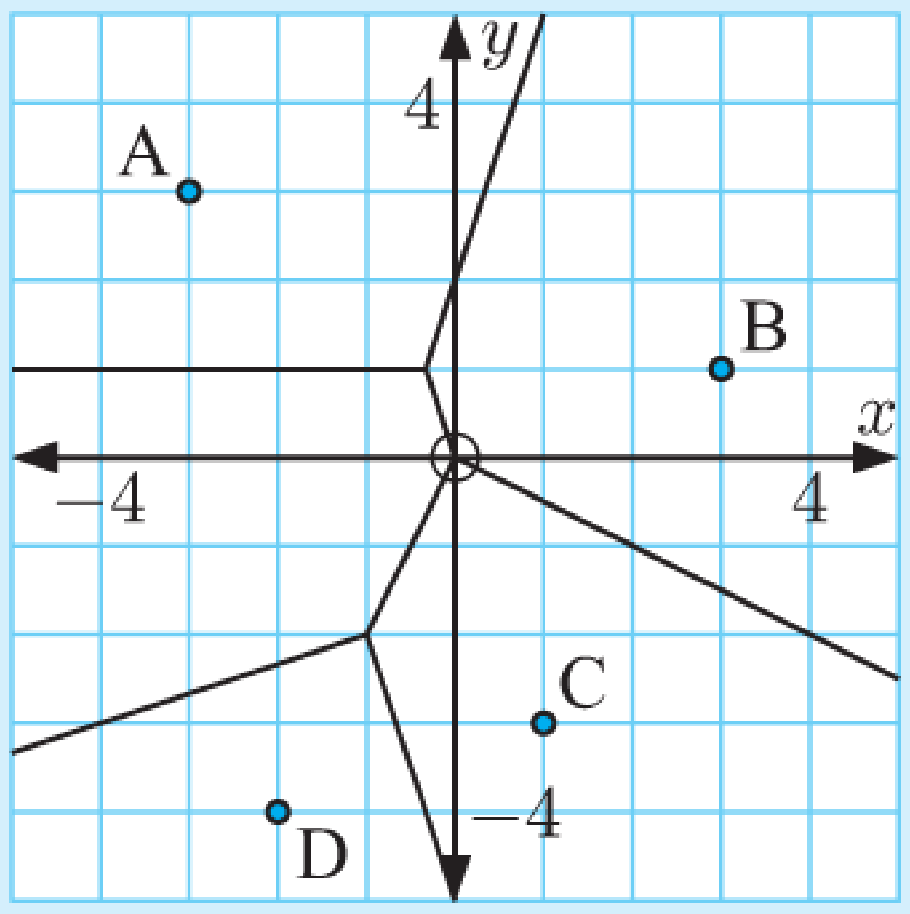
d Simon is equally closest to campsites C and D. If he hiked 1 km north, his closest campsite would be A. Mark Simon's possible positions on your diagram.



Example 4

Self Tutor

Find the coordinates of the missing site X in this Voronoi diagram.

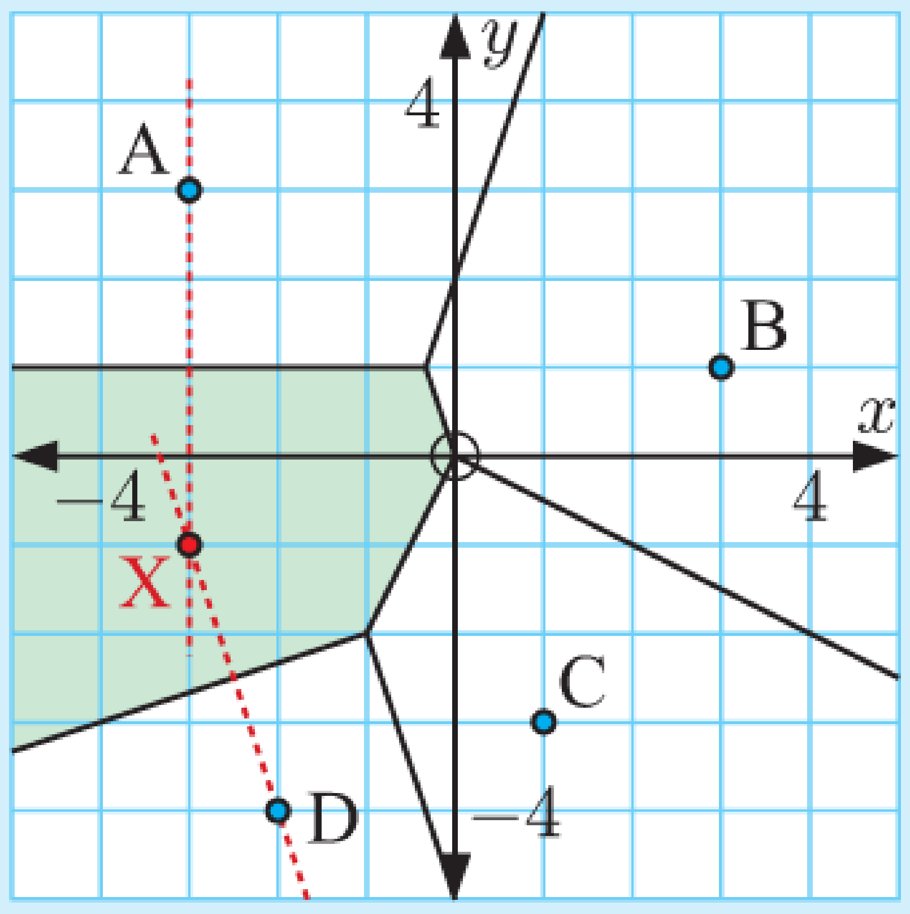


The missing site X must lie in the shaded cell, as this cell currently has no site.

Now PB(A, X) is horizontal, and PB(D, X) has gradient $\frac{1}{3}$.

$\therefore$ [AX] is vertical and [DX] has gradient -3.

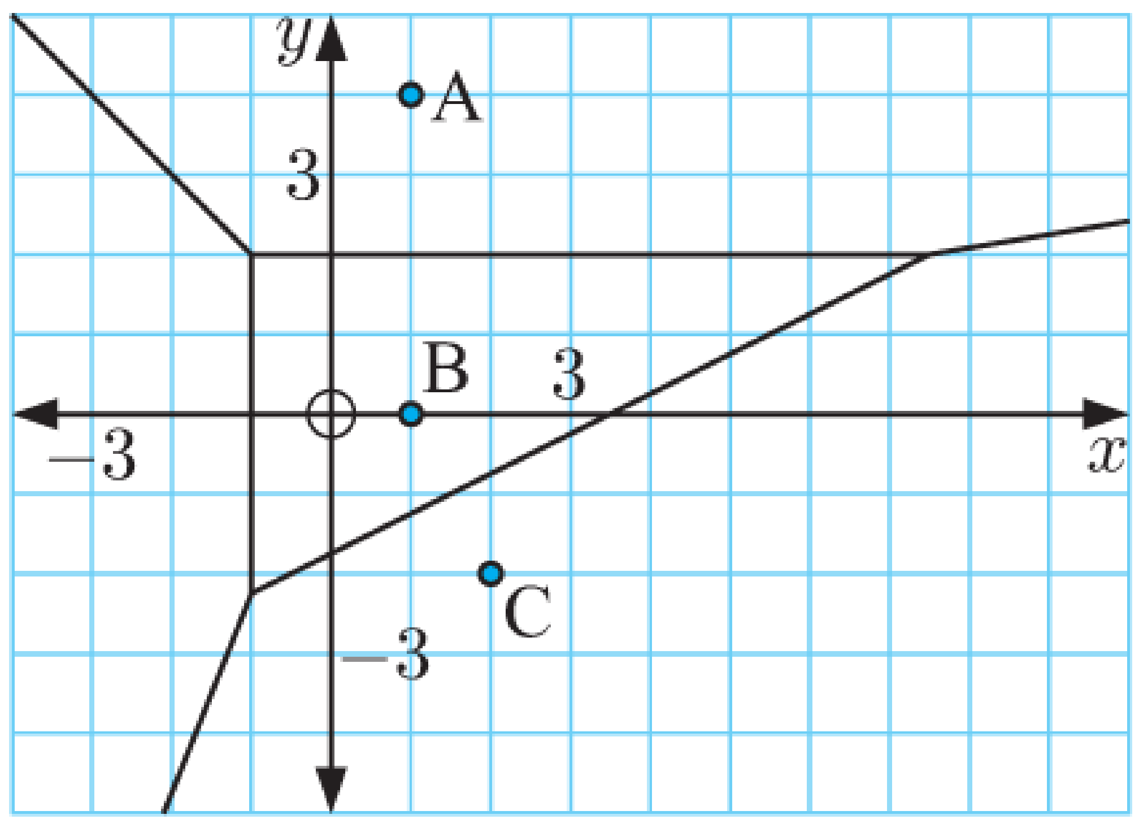
If we draw lines (AX) and (DX) through A and D respectively, their intersection must be point X. We observe that X has coordinates $(-3, 1)$.



11 **a** Explain why the Voronoi diagram alongside must have a site missing.

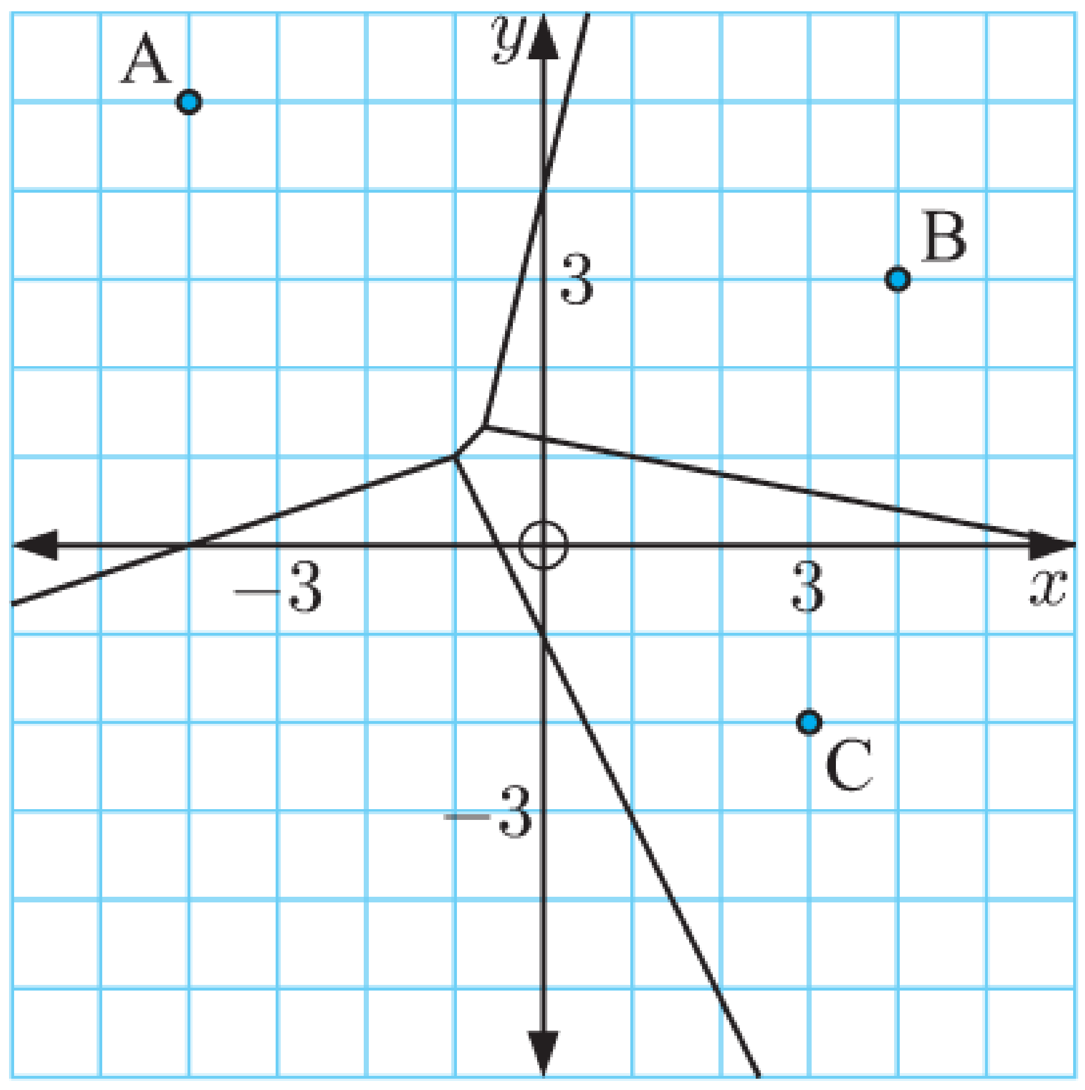
b Explain why the missing site must lie on the x-axis.

c Find the coordinates of the missing site.

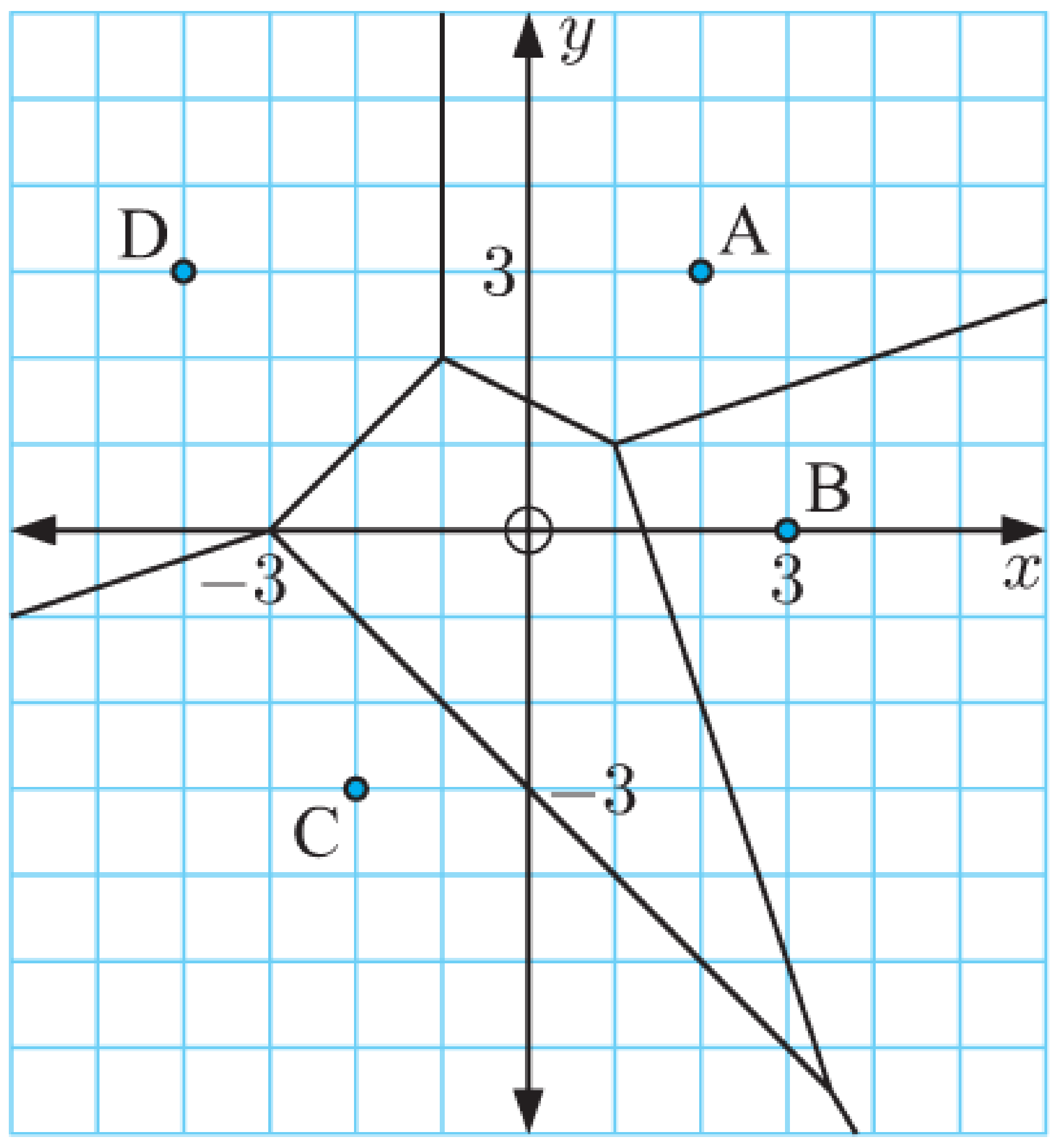


12 Find the coordinates of the missing site X in each Voronoi diagram:

a



b



13 This Voronoi diagram shows the vet clinics in a particular district.

a Copy the diagram, and shade the cell with the missing vet clinic.

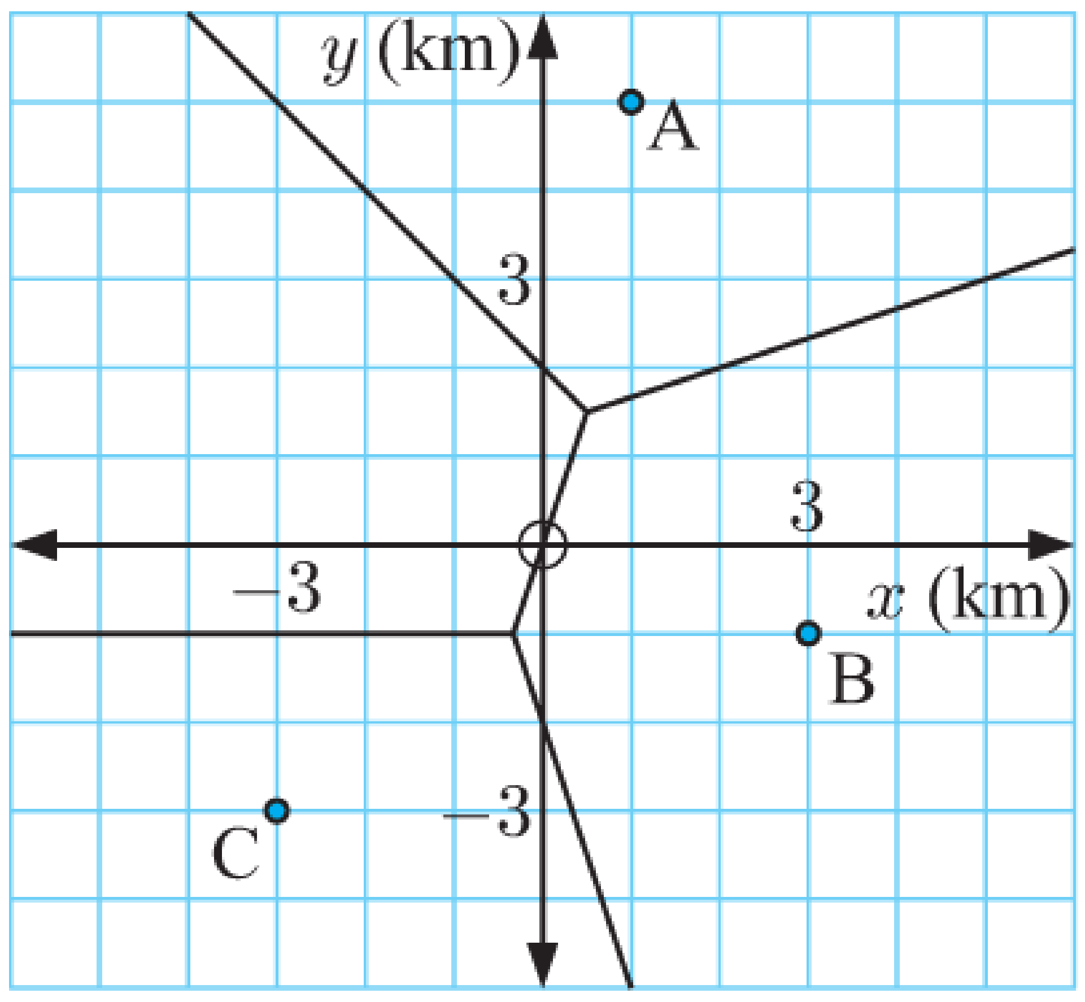
b By considering distances from $(0, -1)$, explain why the missing vet clinic is not at $(-2, 1)$.

c Find the coordinates of the missing vet clinic D.

d Find the vet clinic which is closest to:

i $(-1, 4)$ **ii** $(1, -5)$

e Larissa lives equally closest to vet clinics A and D. Find the shortest possible distance that Larissa's house could be from these vet clinics.

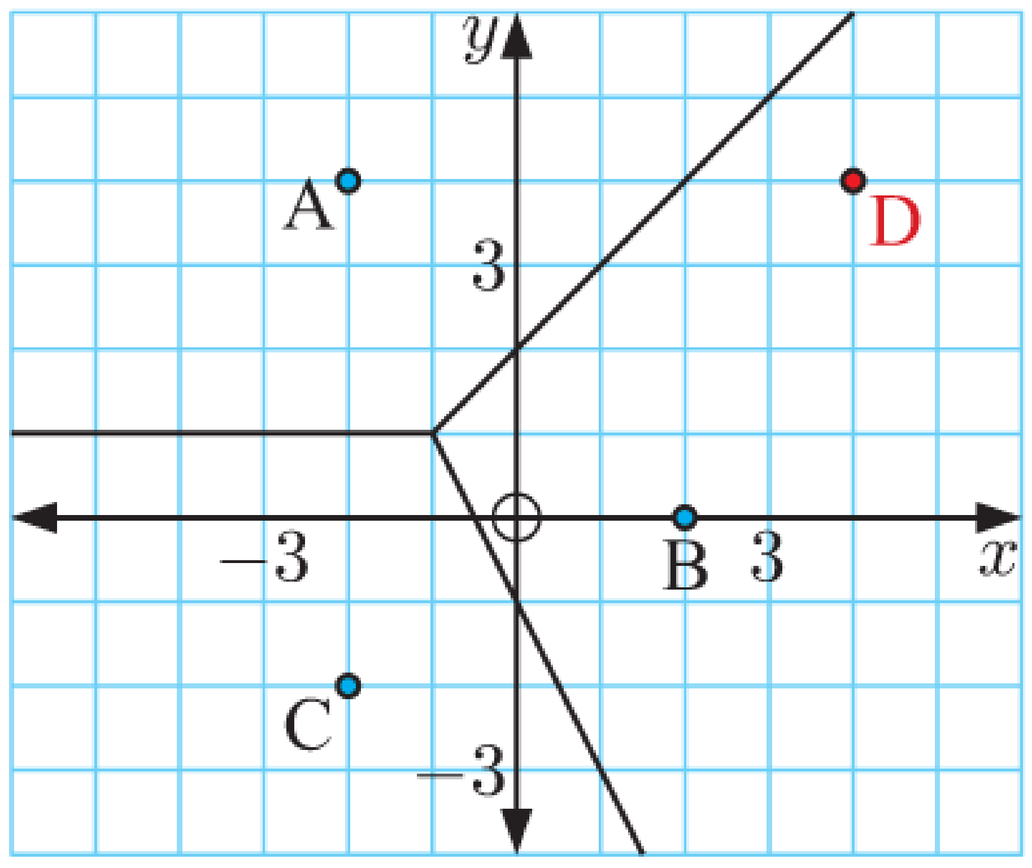


C ADDING A SITE TO A VORONOI DIAGRAM

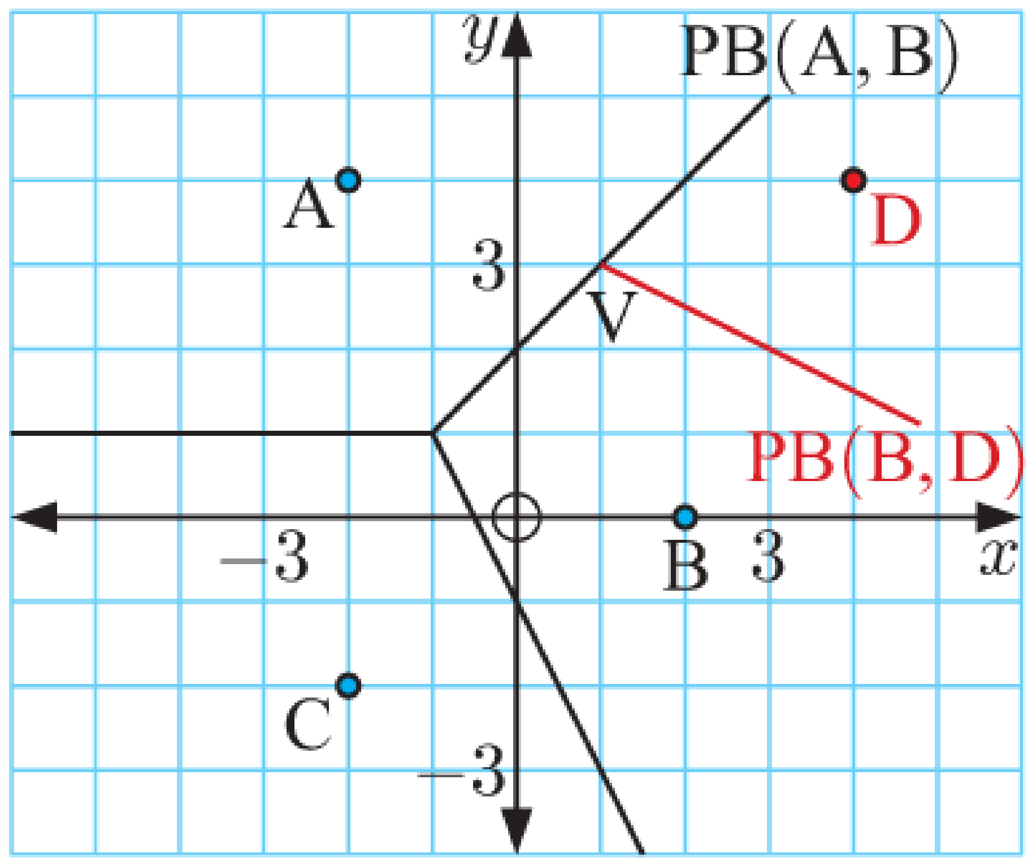
This Voronoi diagram shows the hospitals A, B, and C in a city.

A new hospital is being built at site D marked in red. When it is completed, the Voronoi diagram must be updated to include a cell corresponding to site D.

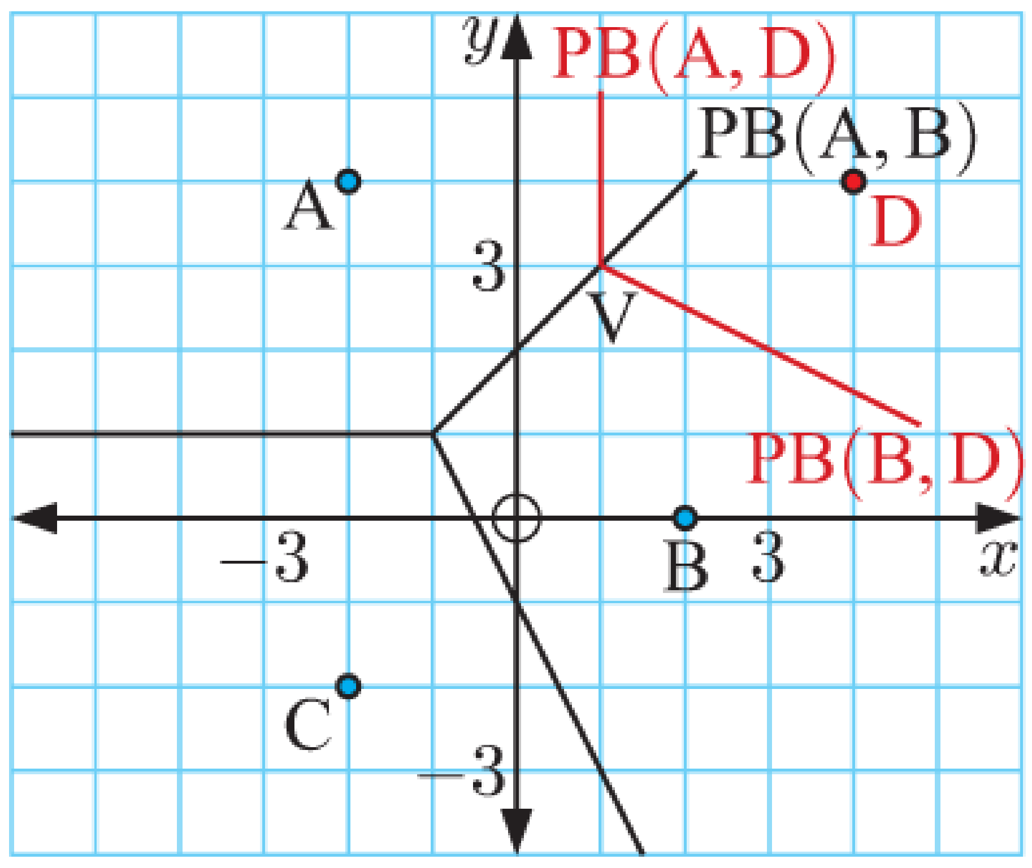
The new site currently lies in cell B, so some areas which were previously closest to Hospital B will now be closest to Hospital D.



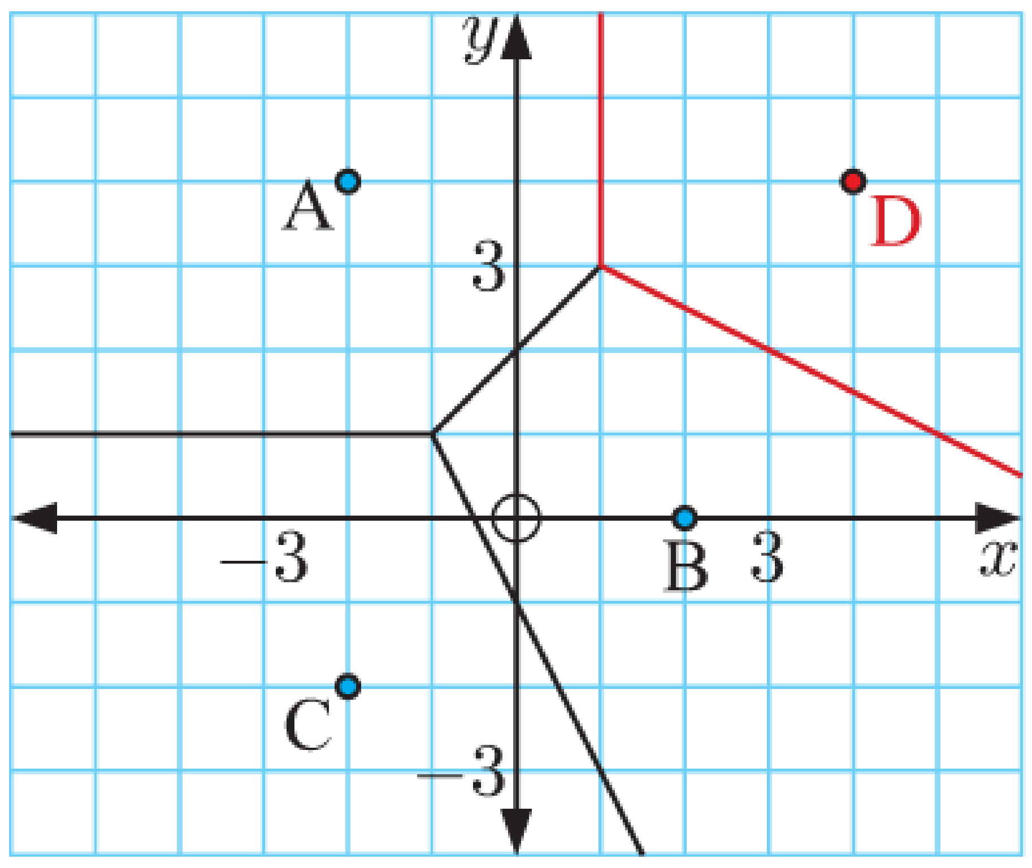
To establish the border between the new cell B and cell D, we draw the perpendicular bisector of [BD]. This line meets the edge PB(A, B) at point V.



V is equidistant from A, B, and D, so is a vertex of the new Voronoi diagram. We therefore need to add PB(A, D) to the diagram, starting at V as shown. This line does not meet any other existing edges, so no more new vertices are created. This tells us that the construction of cell D is complete.



Finally, we remove the part of the existing edge PB(A, B) which now lies within cell D. Notice that the cells for Hospital A and Hospital B were affected by the introduction of Hospital D. Hospital C is relatively further away from Hospital D, so cell C was not affected.



In general, to add the cell for a new site X to an existing Voronoi diagram with sites $P_1, P_2, P_3, ..., P_n$, we follow these steps:

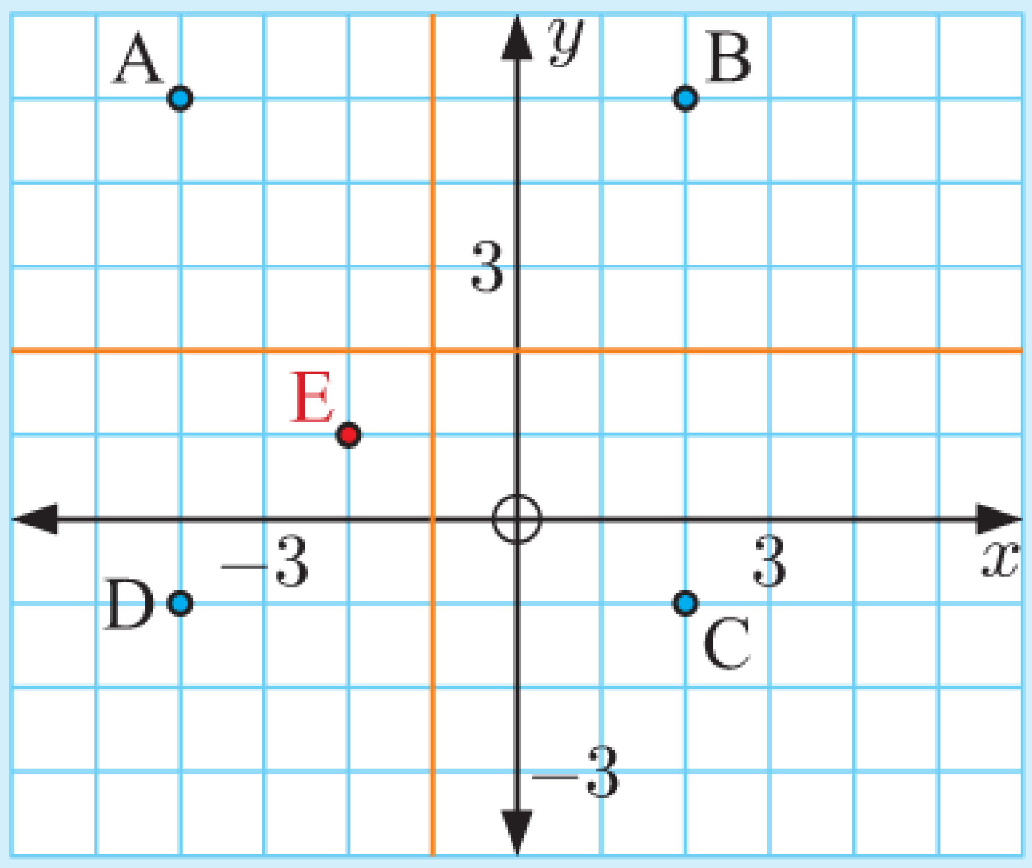
Step 1: Identify the site P_i whose cell contains the new site X. Construct $PB(P_i, X)$ within this cell. At any point where this line meets an existing edge, create a new vertex.

Step 2: For each site P_j whose cell is adjacent to a new vertex, construct $PB(P_j, X)$ within that cell through the vertex. Continue to create new vertices as in *Step 1*. Repeat this process until no more new vertices are created. At this time cell X is complete.

Step 3: Remove any segments of edges from the original Voronoi diagram which now lie within cell X.

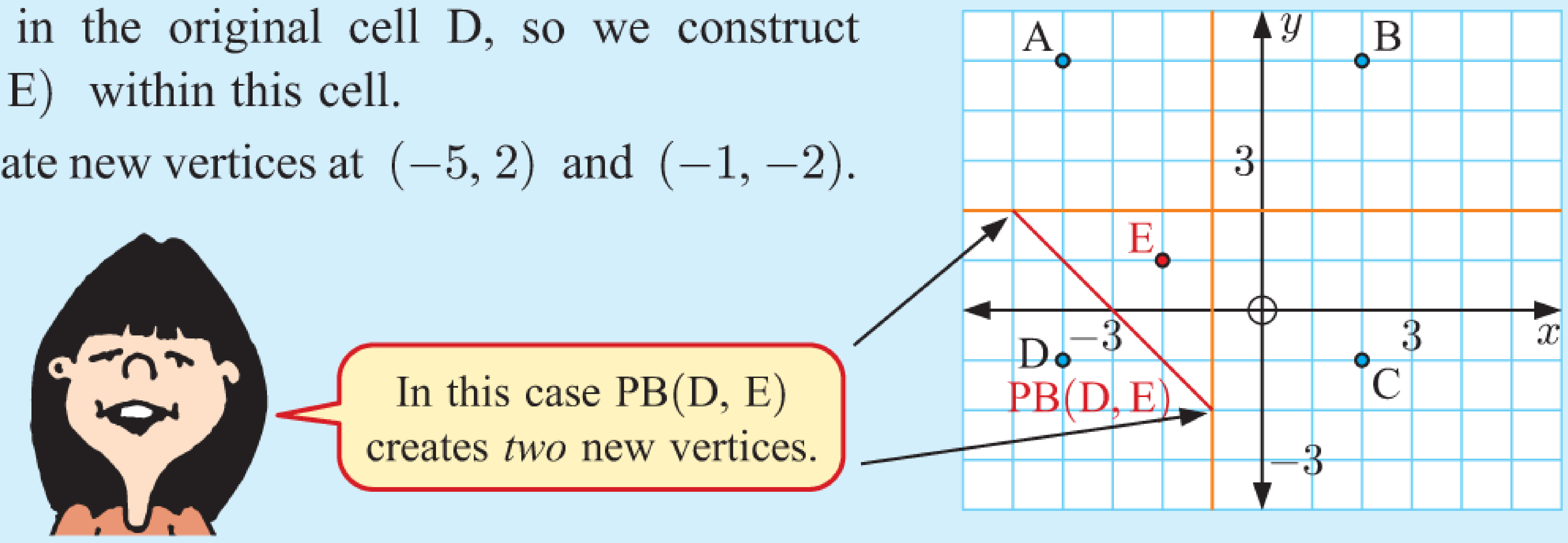
Example 5 **Self Tutor**

Redraw this Voronoi diagram with an additional site at E(−1, 1).



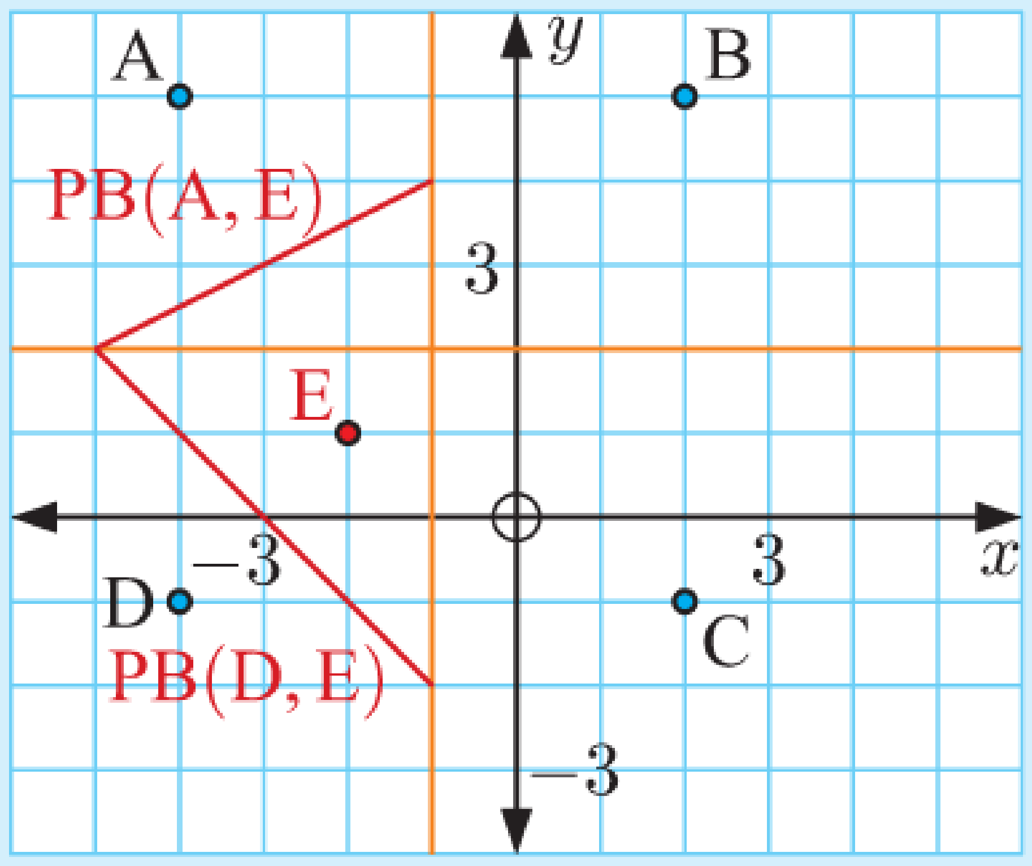
Step 1: E lies in the original cell D, so we construct PB(D, E) within this cell.

We create new vertices at $(-5, 2)$ and $(-1, -2)$.


In this case PB(D, E) creates *two* new vertices.
A
B
y
3
E
-3
3
x
D
C
PB(D, E)
-3

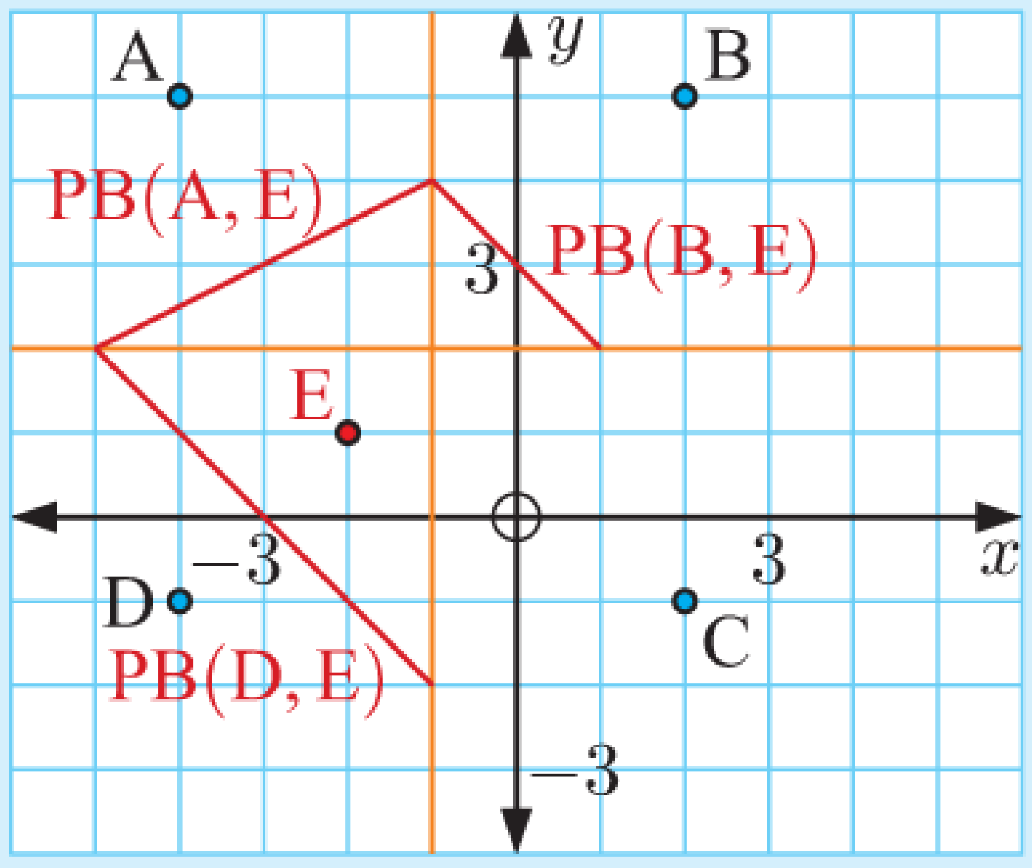
Step 2: Cell A is adjacent to the vertex $(-5, 2)$, so we construct PB(A, E) from $(-5, 2)$ through cell A.

This creates a new vertex at $(-1, 4)$.


A
B
y
PB(A, E)
3
E
-3
3
x
D
C
PB(D, E)
-3

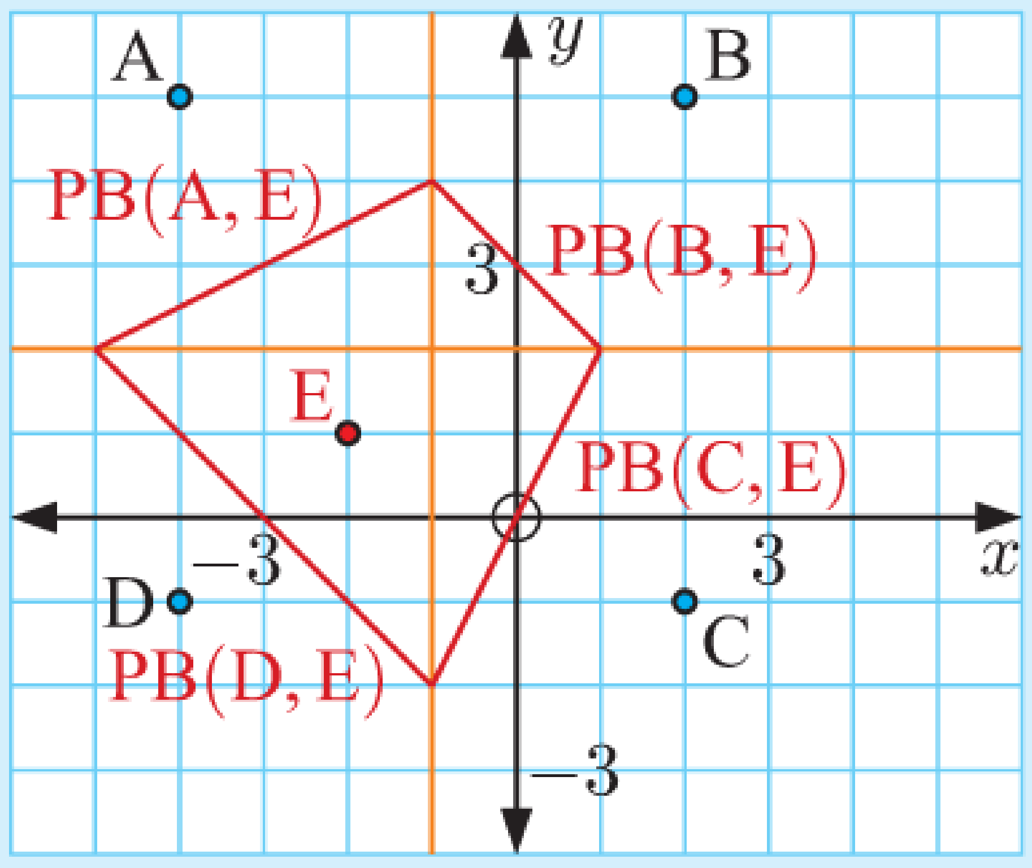
Cell B is adjacent to the vertex $(-1, 4)$, so we construct PB(B, E) from $(-1, 4)$ through cell B.

This creates a new vertex at $(1, 2)$.

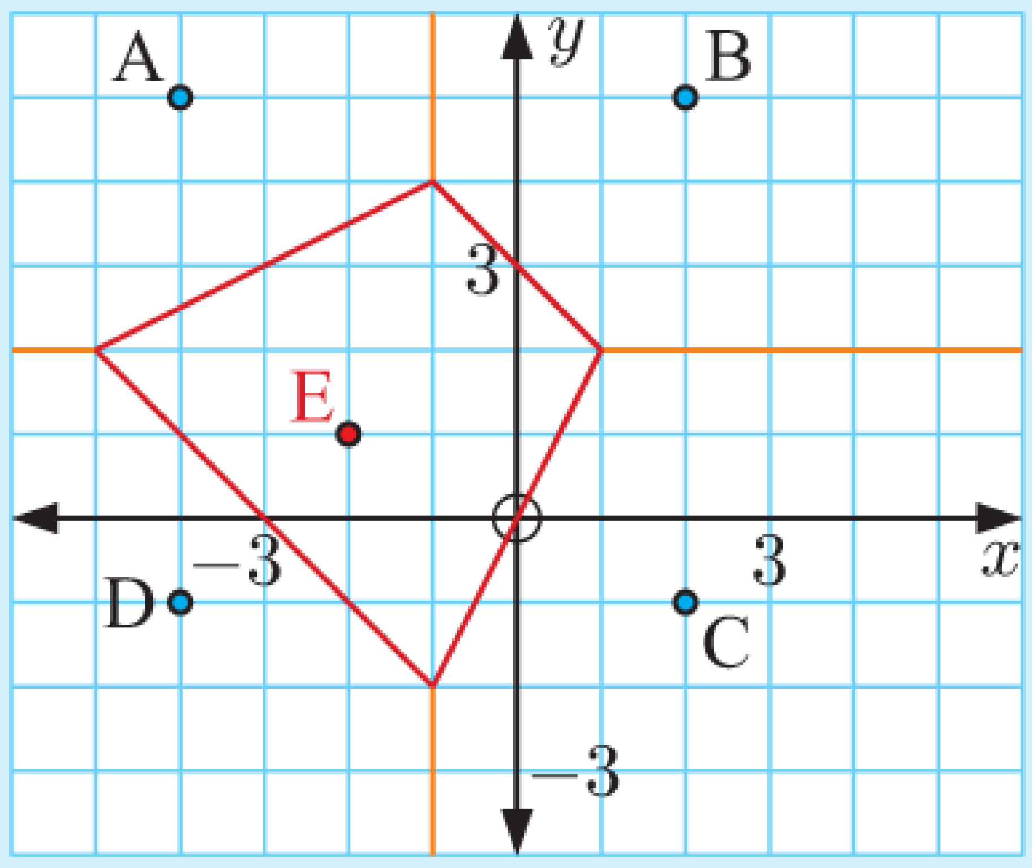

A
B
y
PB(A, E)
3
PB(B, E)
E
-3
3
x
D
C
PB(D, E)
-3

Cell C is adjacent to the vertex $(1, 2)$, so we construct PB(C, E) from $(1, 2)$ through cell C.

This connects us back to the new vertex $(-1, -2)$.


A
B
y
PB(A, E)
3
PB(B, E)
E
PB(C, E)
-3
3
x
D
C
PB(D, E)
-3

Step 3: We remove the segments of edges from the original Voronoi diagram which now lie within cell E.

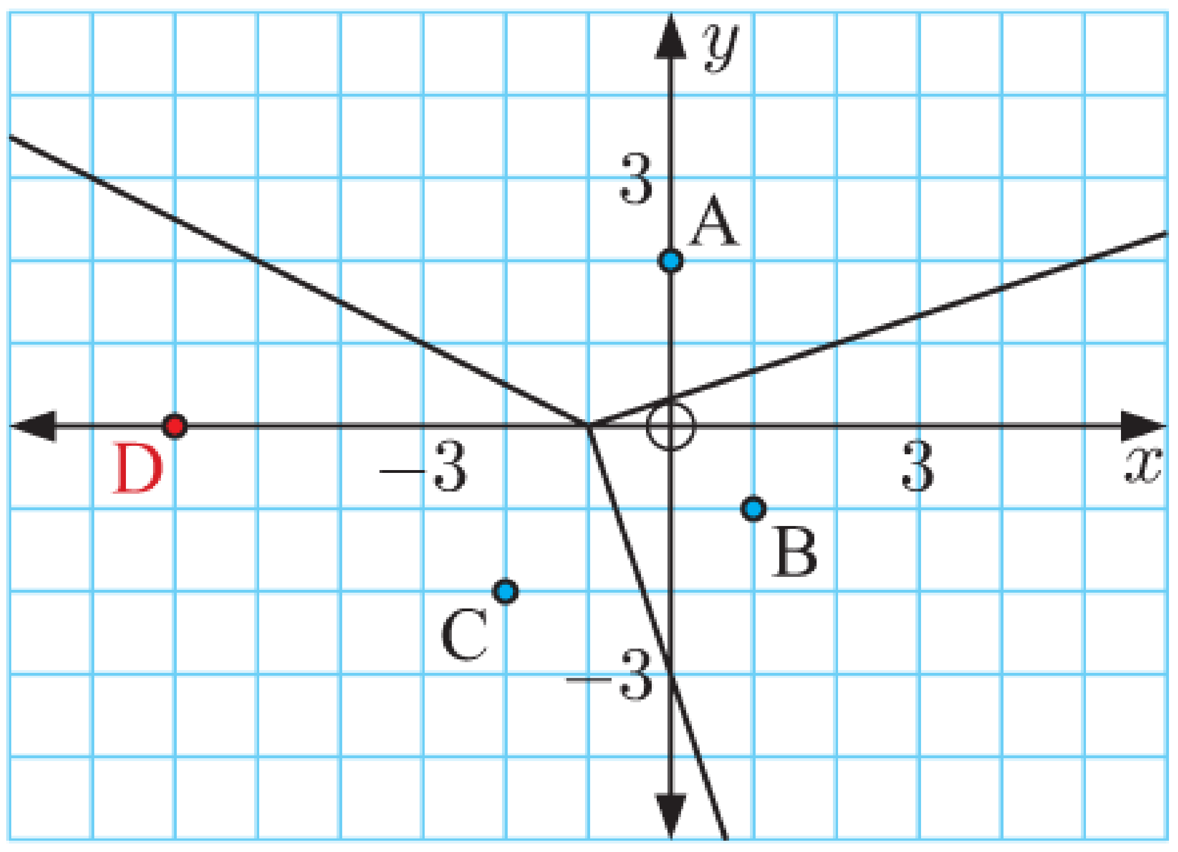

A
B
y
3
E
-3
3
x
D
C
-3

EXERCISE 17C

1 Site D is to be added to the Voronoi diagram shown.

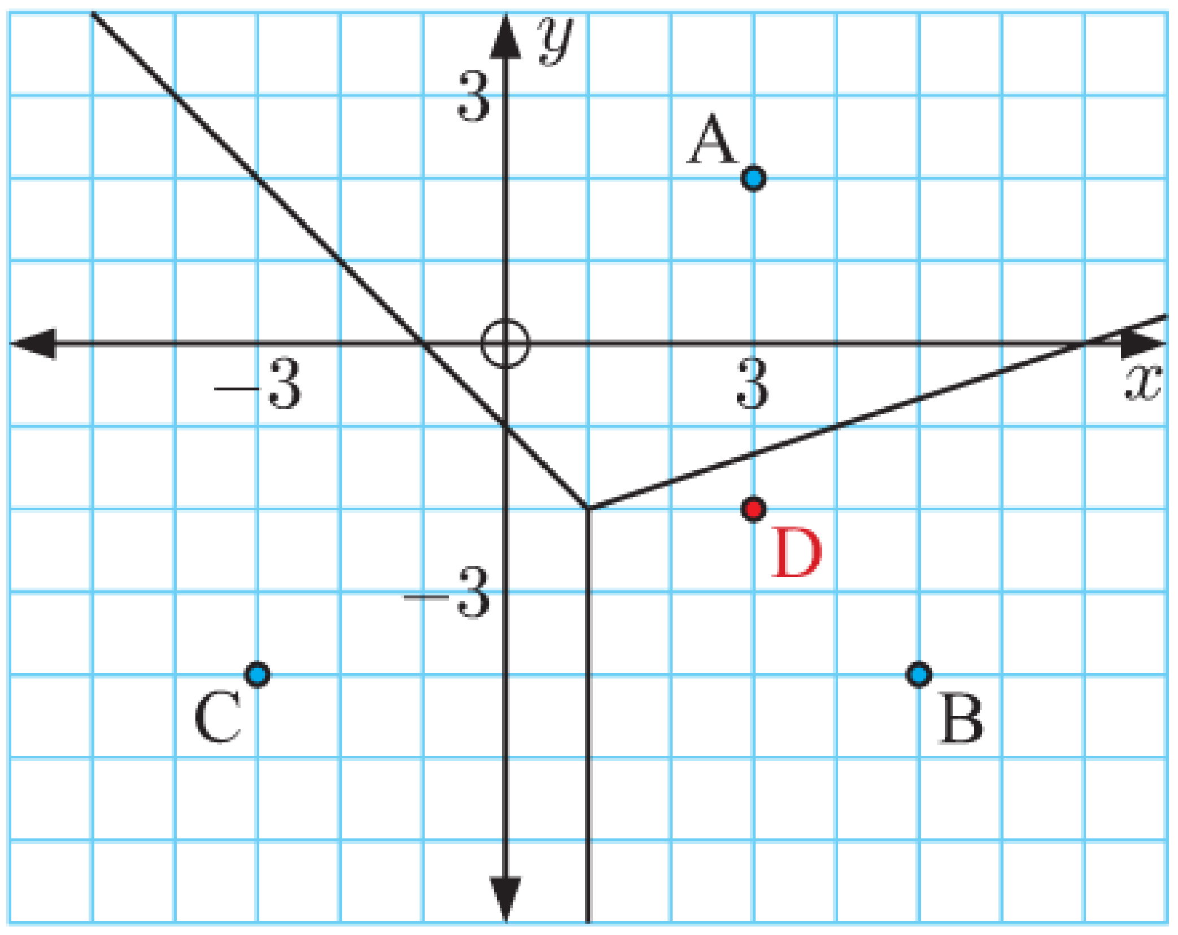
- **a** In which existing cell does site D lie?
- **b** Which of the existing cells do you think will be affected by the introduction of site D? Explain your answer.
- **c** Redraw the Voronoi diagram with site D added.

PRINTABLE DIAGRAMS



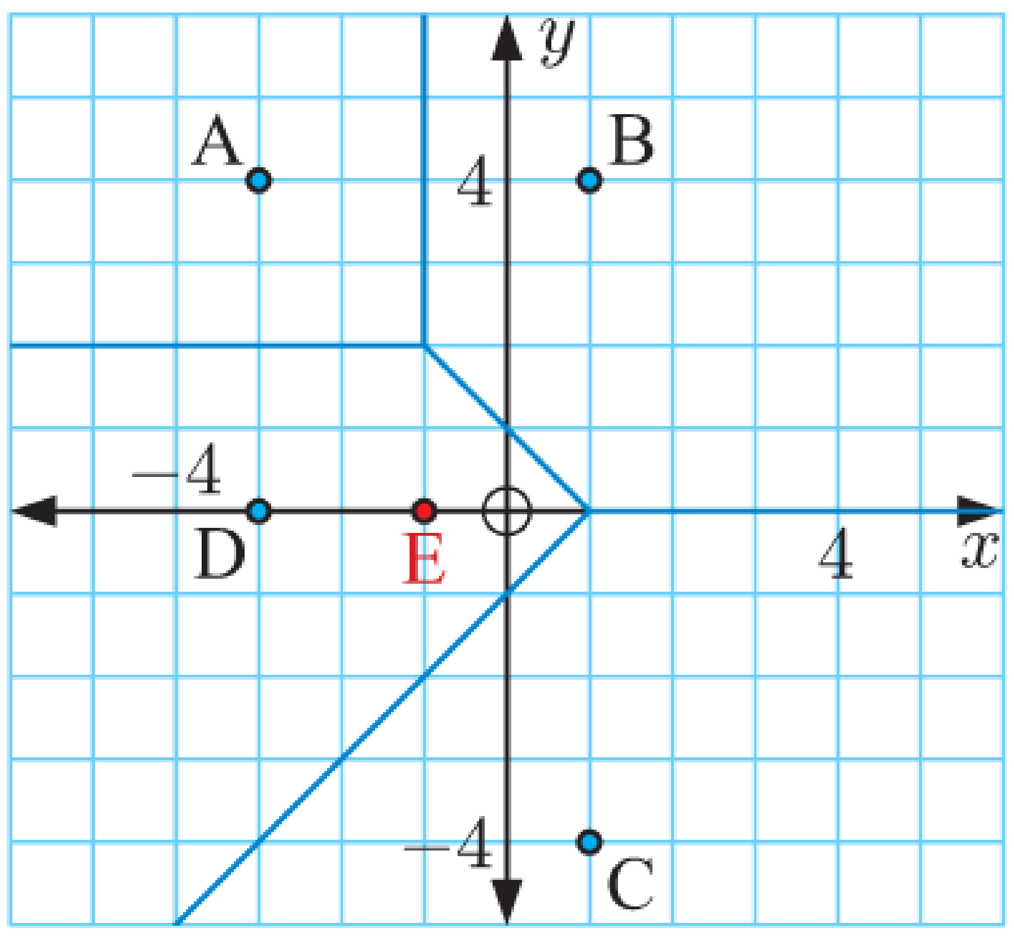
2 Site D is to be added to the Voronoi diagram shown.

- **a** Explain why you would expect this addition to affect all of the existing cells.
- **b** Redraw the Voronoi diagram with site D added.
- **c** Find the area of cell D.

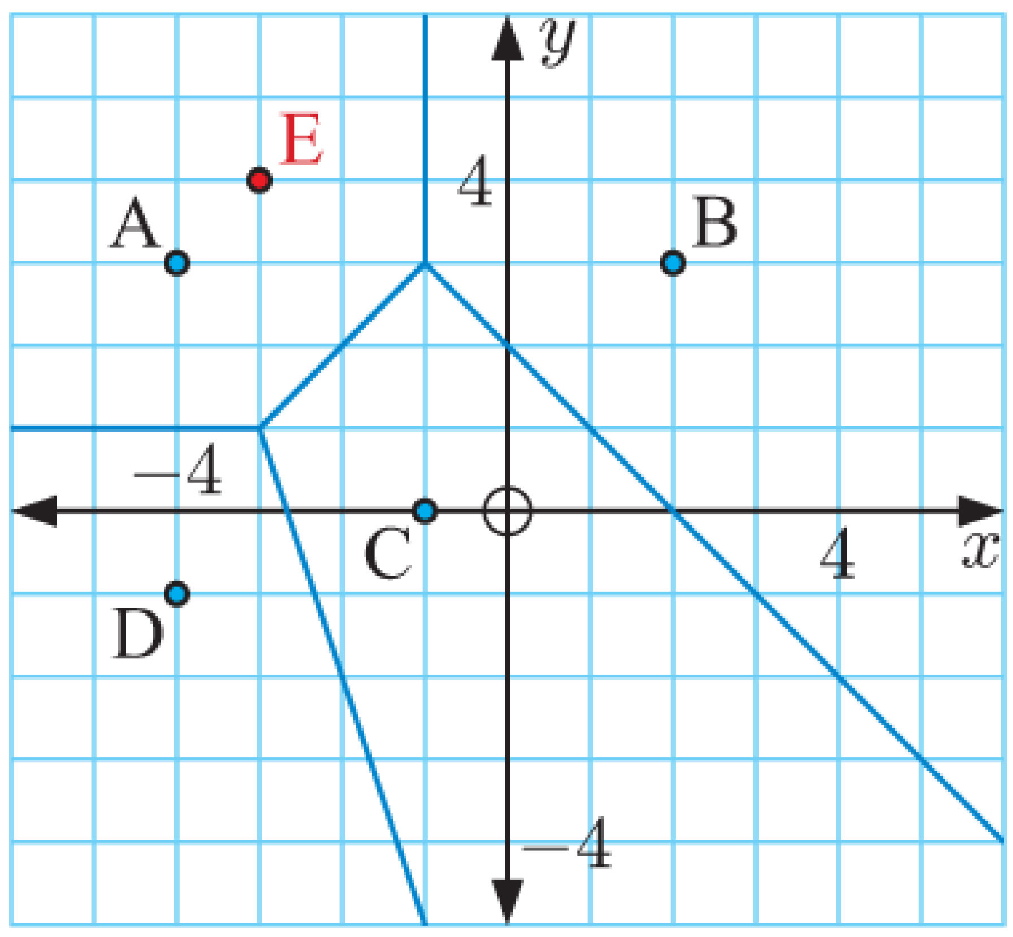


3 Redraw the following Voronoi diagrams with the site E added:

a



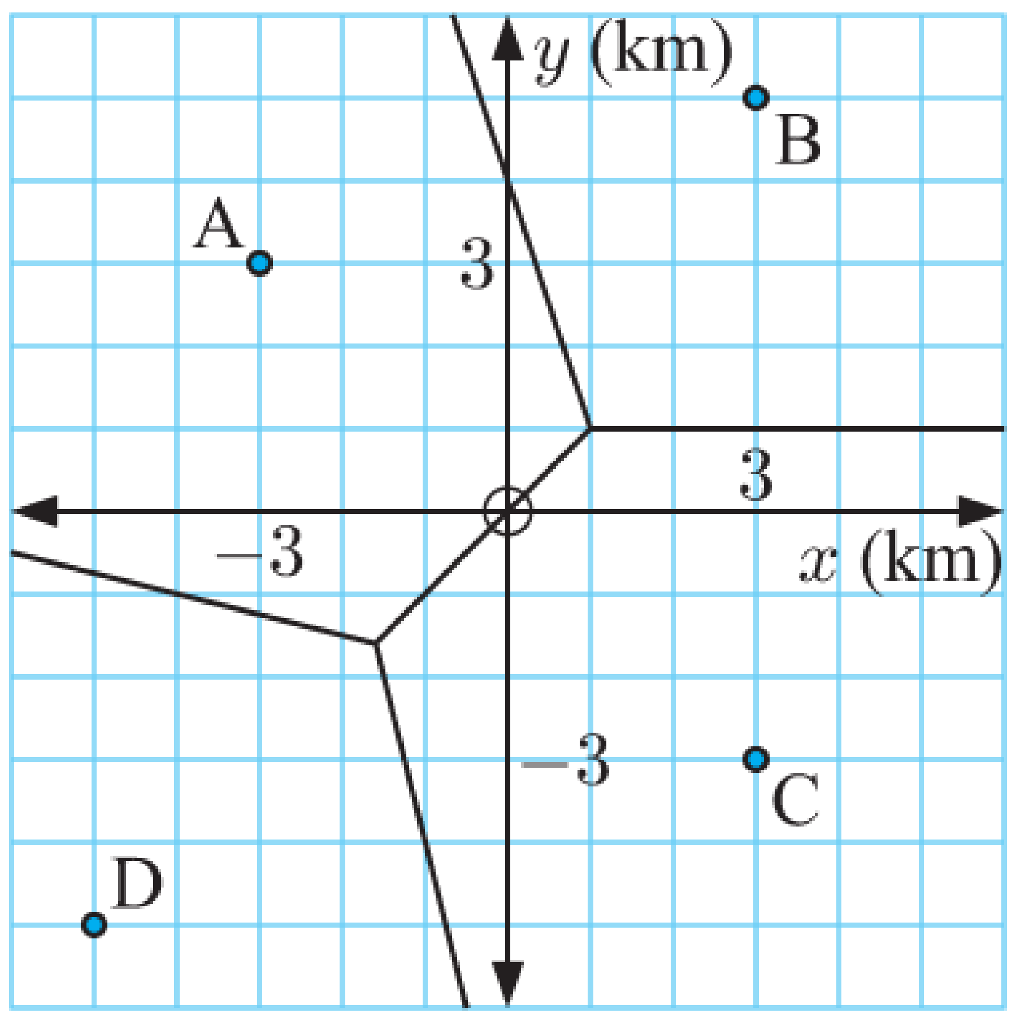
b



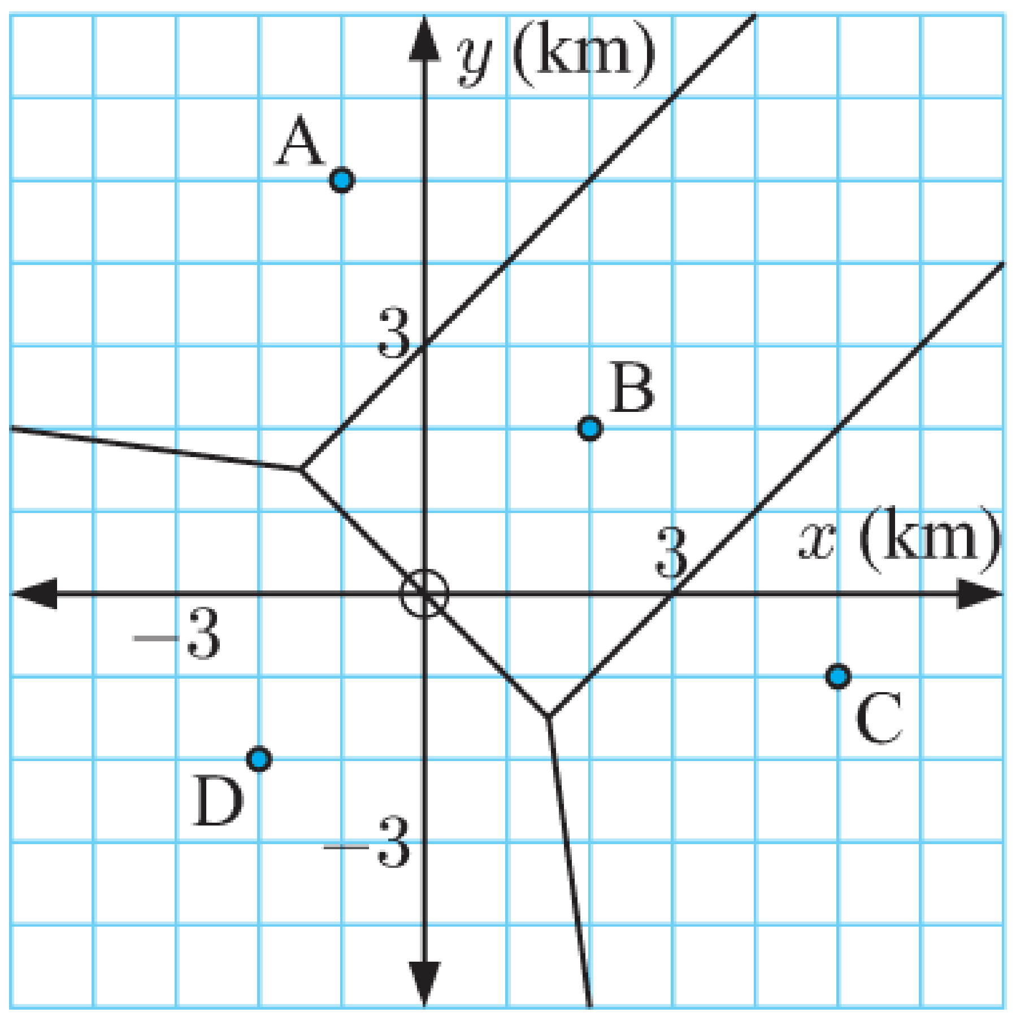
4 For each set of sites, construct a Voronoi diagram for A, B, and C, then add D:

- **a** A$(-3, 1)$, B$(1, 3)$, C$(1, -1)$, D$(0, -2)$
- **b** A$(-3, 3)$, B$(-3, -3)$, C$(1, 1)$, D$(3, -1)$

5 This map shows the automatic teller machines (ATMs) in a suburb.



- **a** Identify the ATM which is closest to:
  - **i** $(-2, -1)$
  - **ii** $(5, 2)$
- **b** A new ATM is installed at E$(1, -1)$.
  - **i** Redraw the Voronoi diagram to include the new ATM.
  - **ii** Are there any residents whose nearest ATM has changed from D to E as a result of the new ATM? Explain your answer.
  - **iii** Morris is equally closest to ATMs B, C, and E. Determine Morris' location.



6 This Voronoi diagram shows the location of polling booths in a particular electorate.

a Which polling booth is closest to:

i (−3, 2) **ii** (1, −4)?

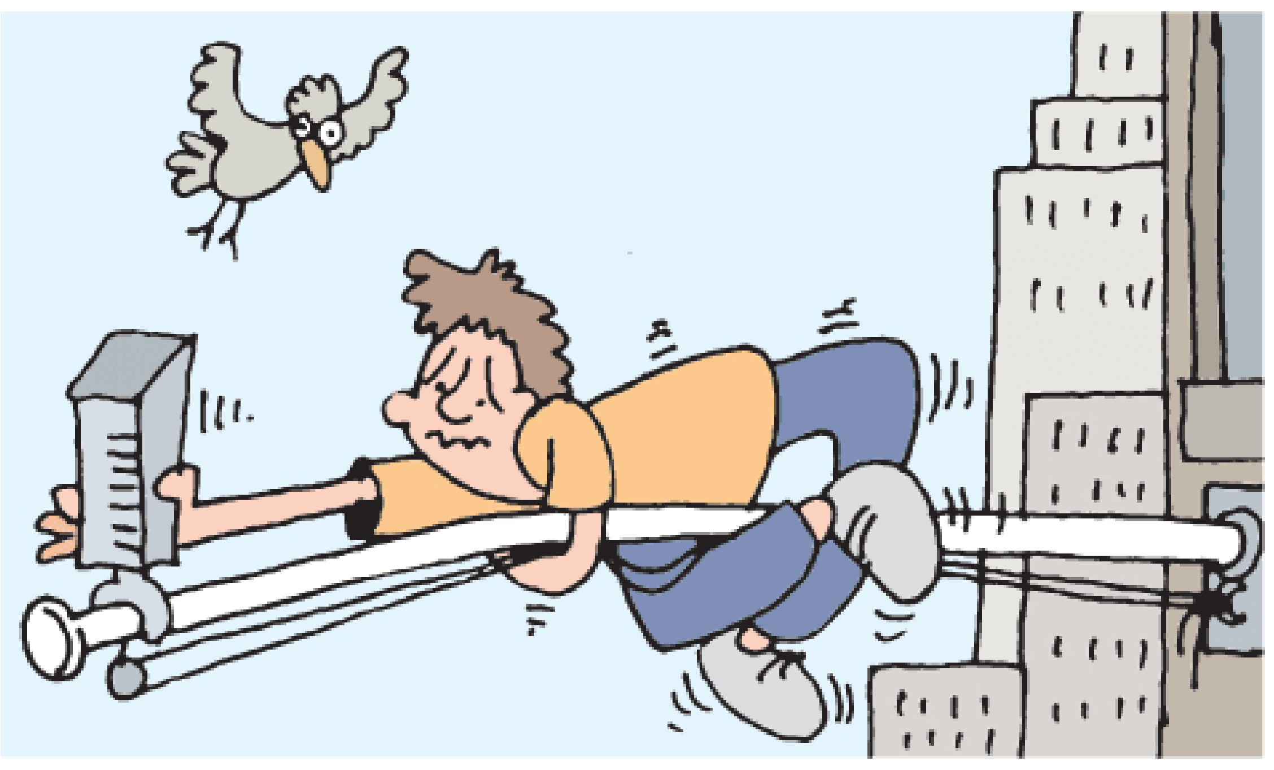
b The polling booth at B was heavily congested at the previous election, so a new polling booth will be set up at (0, 2) for the next election.

i Redraw the Voronoi diagram to include the new polling booth.

ii Which cell will be unaffected by the addition of the new polling booth?

iii Find the area of the cell that is now closest to the new polling booth.

D NEAREST NEIGHBOUR INTERPOLATION

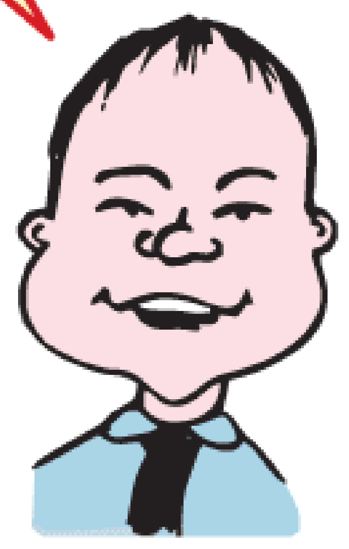


Suppose Ted is interested in the rainfall in his city. He has set up rain gauges at several locations to measure the rainfall. He wants to use these measurements to *estimate* the rainfall in other parts of the city.

On a 2-dimensional grid, the process of using values of a variable at known points to estimate the variable's value at other points is called **interpolation**.

Nearest neighbour interpolation is a simple method of interpolation. To estimate the value of a variable at any point, we use the variable's value at the *nearest* known data point.

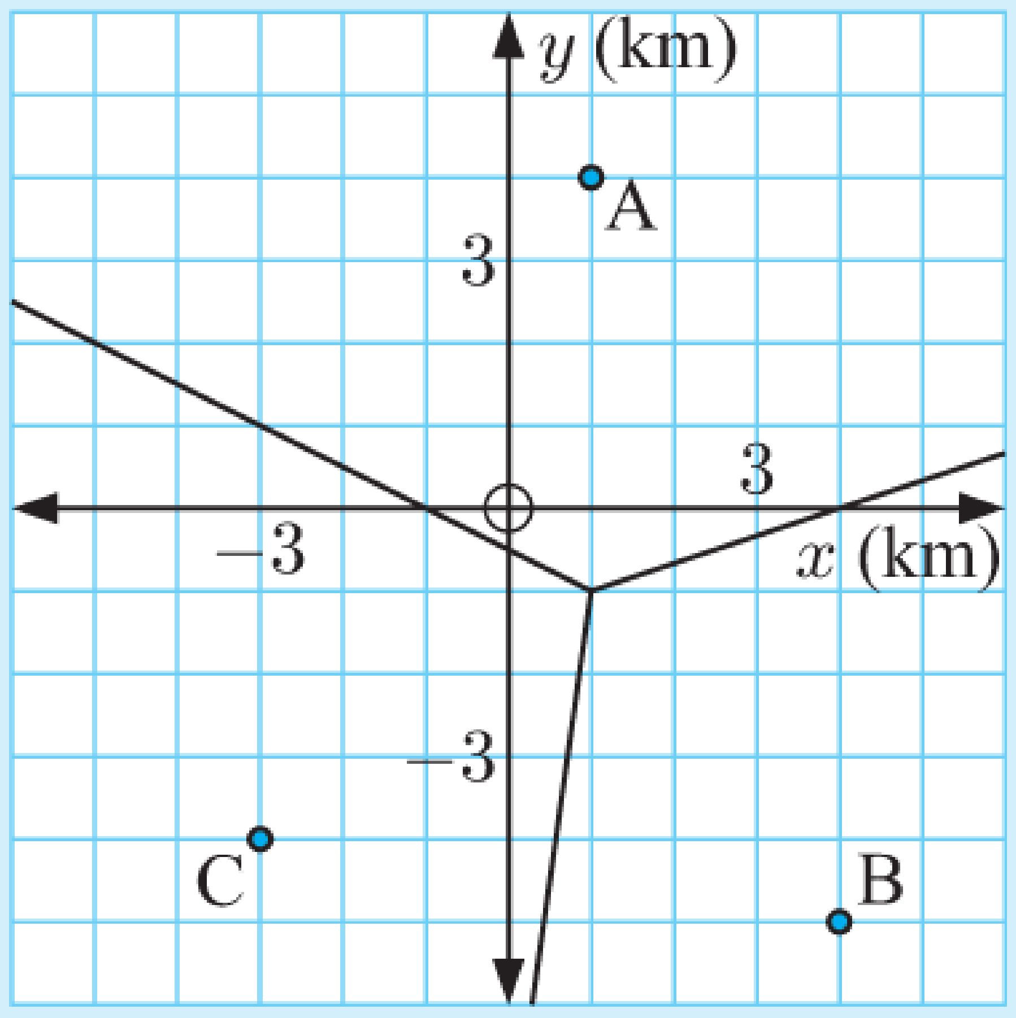
Nearest neighbour interpolation is used in many fields, including image processing and 3D rendering.



By constructing a Voronoi diagram with the known data points as sites, we can quickly identify the nearest known data point to any given point.

If the given point lies on an edge or at a vertex, we take the average of the closest known data points.

Example 6 **Self Tutor**



Ted measured the rainfall in his city at three locations A, B, and C, marked on the map. The results are shown below.

Location	*Rainfall* (mm)
A	12
B	7
C	15

Use nearest neighbour interpolation to estimate the rainfall at:

a (−4, 1) **b** (3, −2) **c** (1, −1)

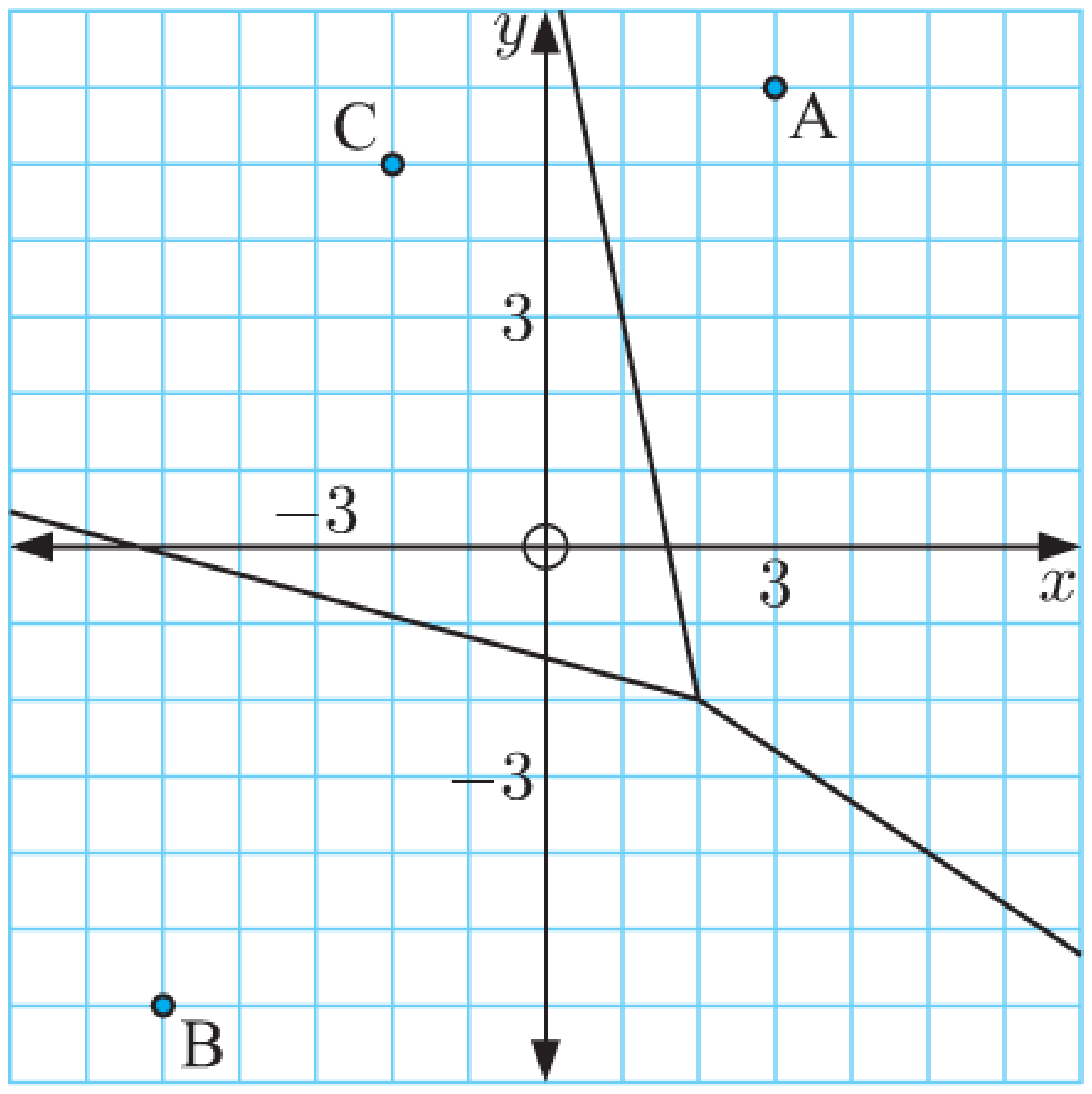
a $(-4, 1)$ is closest to C, so we estimate 15 mm of rainfall at $(-4, 1)$.

b $(3, -2)$ is closest to B, so we estimate 7 mm of rainfall at $(3, -2)$.

c $(1, -1)$ is the vertex equidistant from A, B, and C, so we estimate $\frac{12+7+15}{3} \approx 11.3$ mm of rainfall at $(1, -1)$.

EXERCISE 17D

1 Weather stations measure the temperature in a city at the three locations A, B, and C shown. Their measurements at 3 pm are shown below:



Location	*Temperature* (°C)
A	28.4
B	25.6
C	27.3

Use nearest neighbour interpolation to estimate the 3 pm temperature at:

a $(1, 0)$ **b** $(-3, -1)$ **c** $(5, -2)$

2 John measured the elevation at four locations in a park.

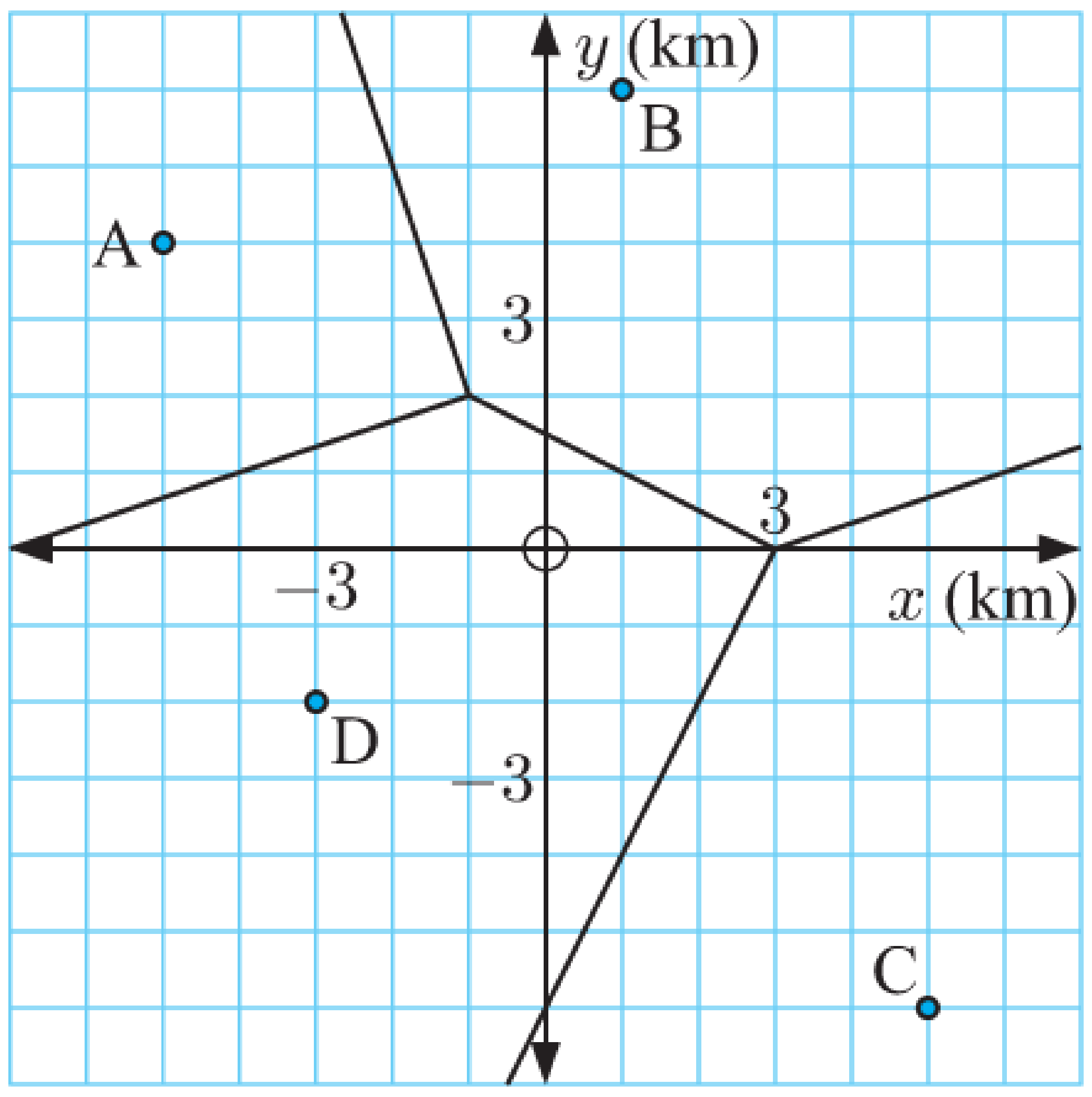
a Construct a Voronoi diagram for these locations.

b Estimate the elevation at:

i $(0, 1)$ **ii** $(-4, 2)$ **iii** $(3, -4)$

Location	*Elevation* (m)
A$(-4, 4)$	57
B$(-2, 0)$	48
C$(2, 4)$	55
D$(2, -6)$	36

3 On a snowy day, the snowfall was measured in four locations across a city.



Location	*Snowfall* (inches)
A	7
B	5.5
C	12.2
D	9.3

a Estimate the snowfall received at:

i $(-1, -4)$ **ii** $(2, 1)$ **iii** $(1, -4)$

b A further report indicated that 10.6 inches of snow fell at $E(-1, -6)$.

i Redraw the Voronoi diagram with location E added.

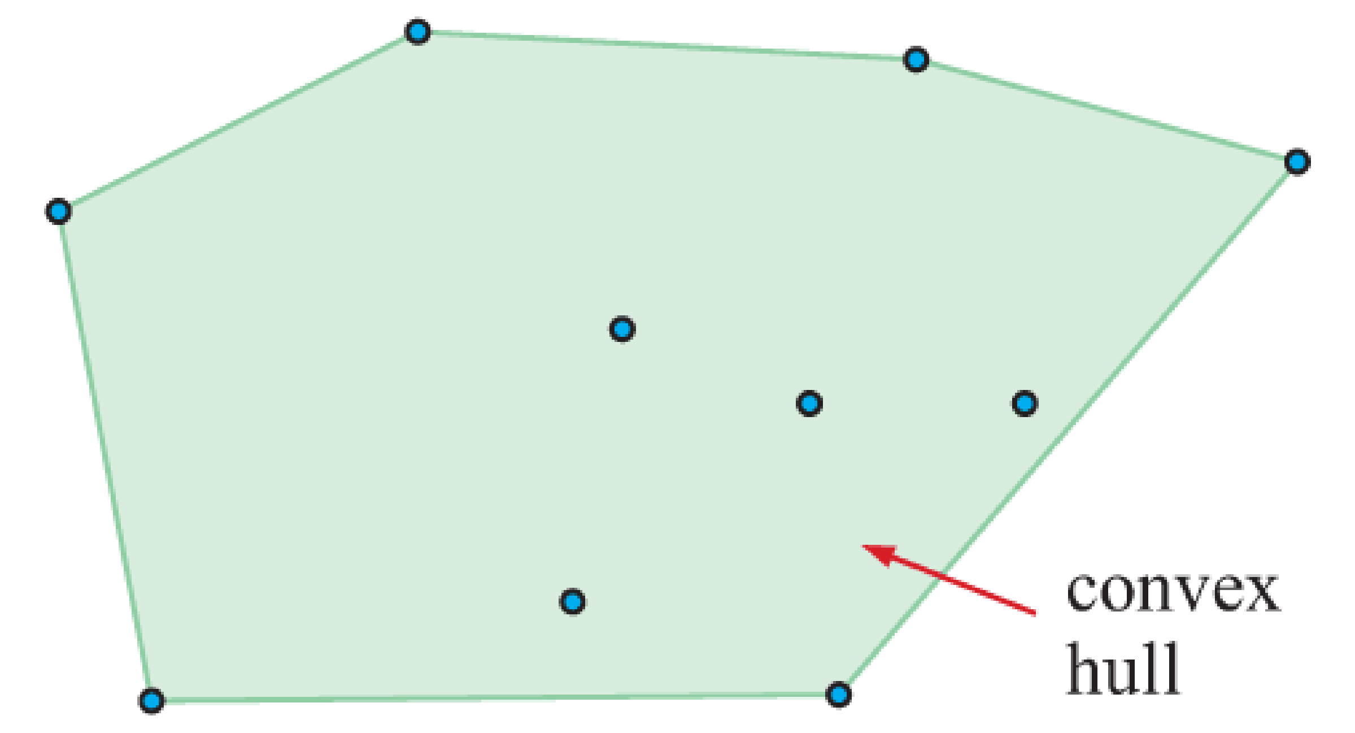
ii Does this change any of your estimates in **a**?

DISCUSSION

What are the advantages and disadvantages of nearest neighbour interpolation? Can you think of a more accurate way to perform the interpolation?

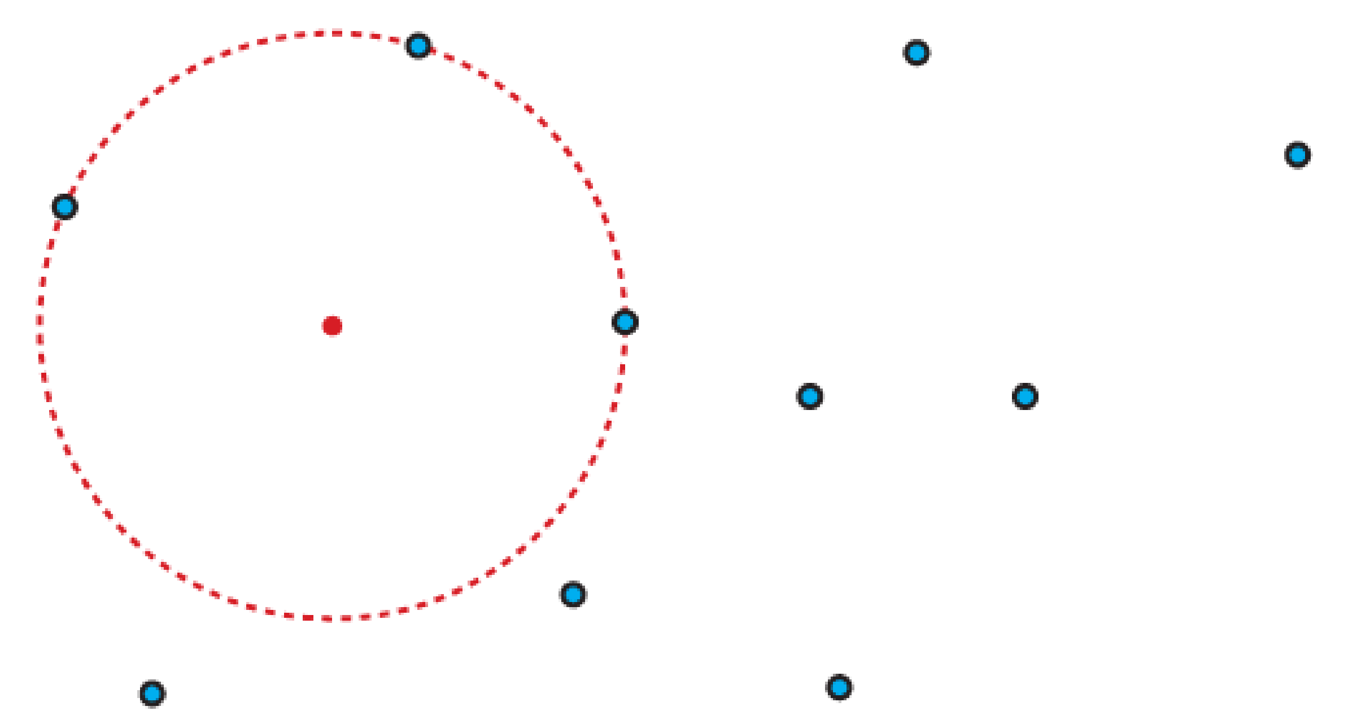
E THE LARGEST EMPTY CIRCLE PROBLEM

Given a set of sites in a plane, the **convex hull** of the sites is the smallest convex polygon which contains all of the sites.



The **Largest Empty Circle problem** is the problem of finding the largest circle, centred within the convex hull, whose interior does not contain any sites.

This problem can be formulated as finding the optimal position for a toxic waste dump, so as to maximise its distance from the nearest town. For this reason, this problem is also known as the **toxic waste dump problem**.



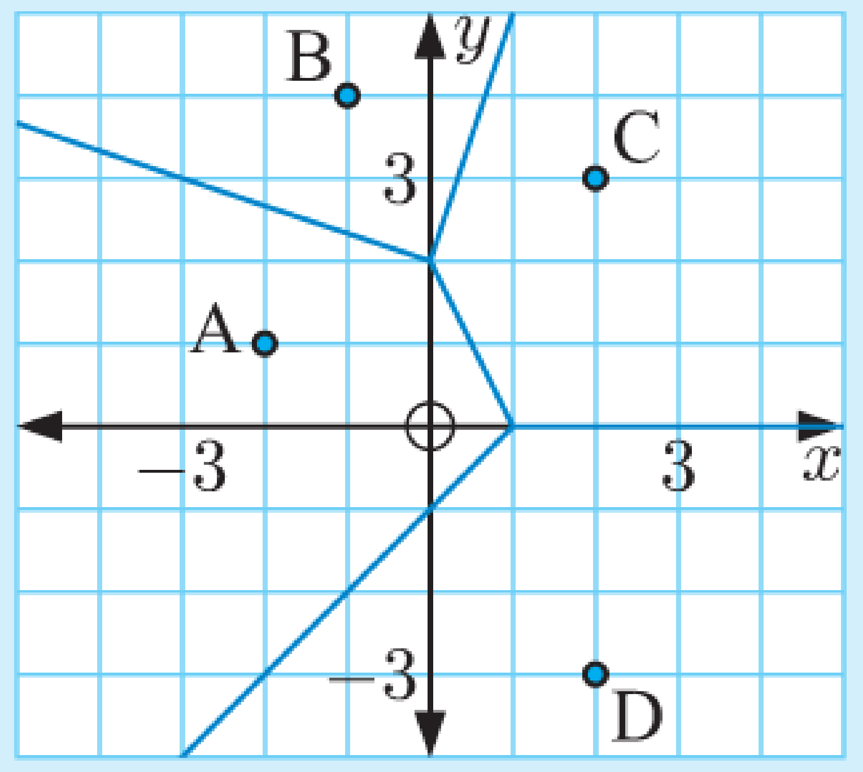
We can solve this problem by drawing the Voronoi diagram for the sites.

In the problems we will consider in this Section, all of the vertices of the Voronoi diagram lie within the convex hull. In this case, the optimal position for the circle's centre will occur at one of these vertices. The vertex with the greatest distance from its nearest site is the optimal position for the circle's centre.

Example 7 **Self Tutor**

The Voronoi diagram for the sites $A(-2, 1)$, $B(-1, 4)$, $C(2, 3)$, and $D(2, -3)$ is shown alongside.

Find the largest empty circle for these sites.



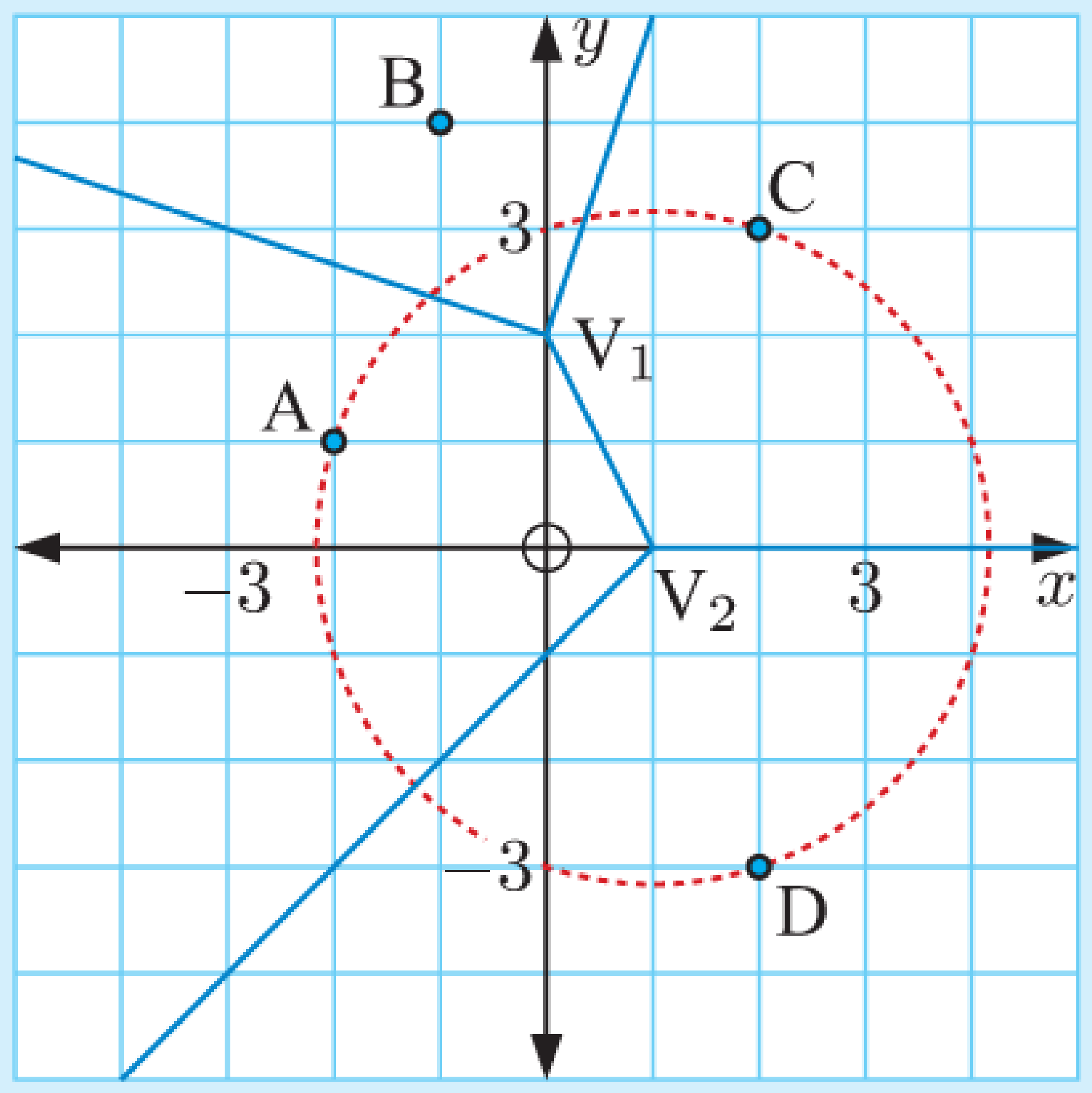
The Voronoi diagram has vertices $V_1(0, 2)$ and $V_2(1, 0)$.

V_1 is equidistant from A, B, and C.

$$V_1A = \sqrt{(-2-0)^2+(1-2)^2}$$
$$= \sqrt{(-2)^2+(-1)^2}$$
$$= \sqrt{5} \text{ units}$$

V_2 is equidistant from A, C, and D.

$$V_2A = \sqrt{(-2-1)^2+(1-0)^2}$$
$$= \sqrt{(-3)^2+1^2}$$
$$= \sqrt{10} \text{ units}$$

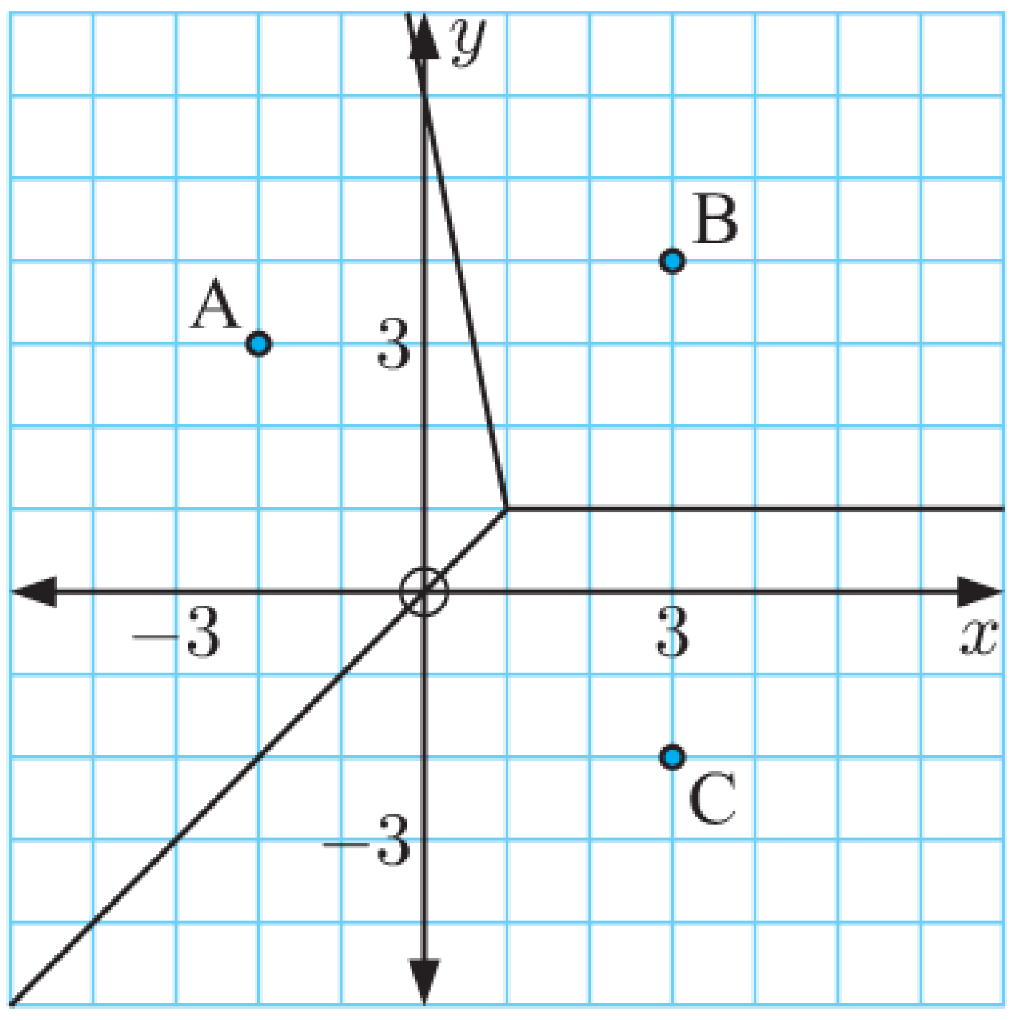


So, the largest empty circle has centre $V_2(1, 0)$ and radius $\sqrt{10}$ units.

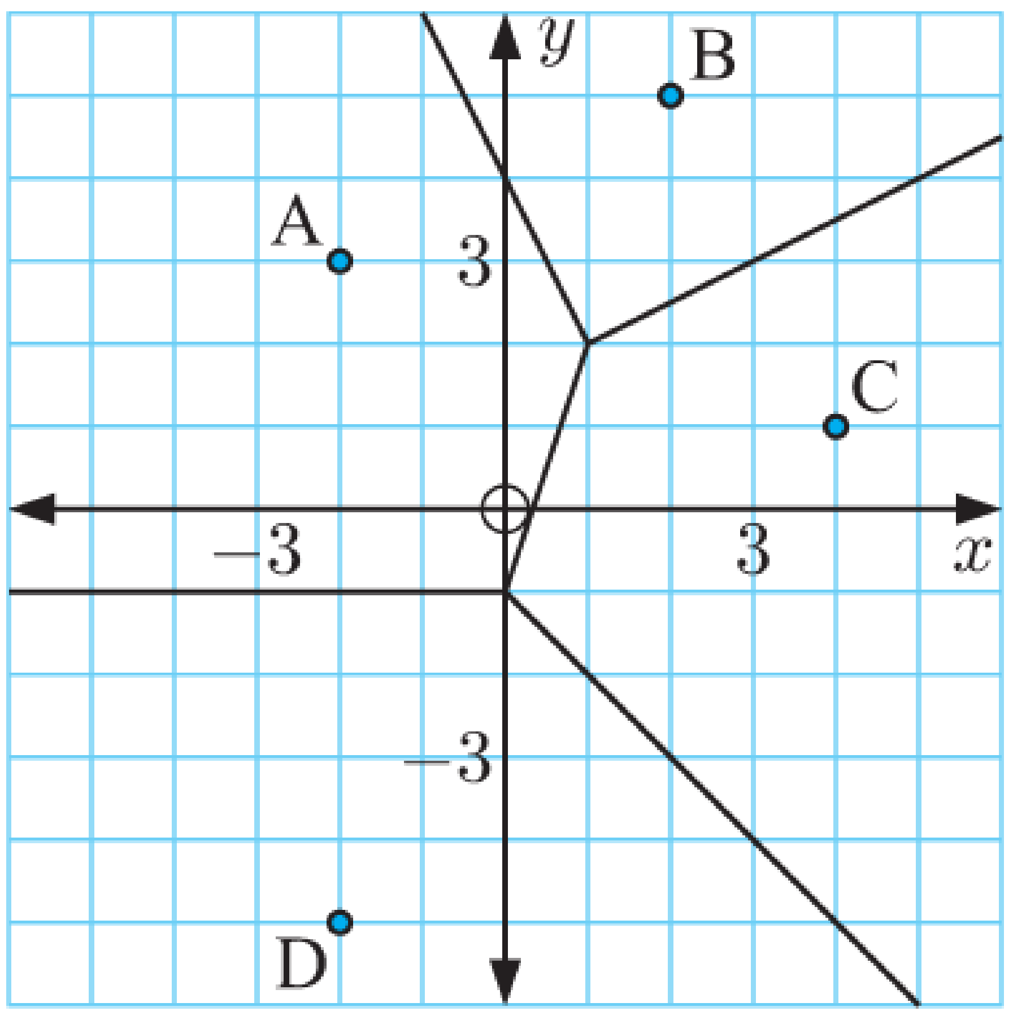
EXERCISE 17E

1 Find the centre and radius of the largest empty circle for these sets of sites:

a



b



2 This Voronoi diagram shows the locations of five towns in a rural area.

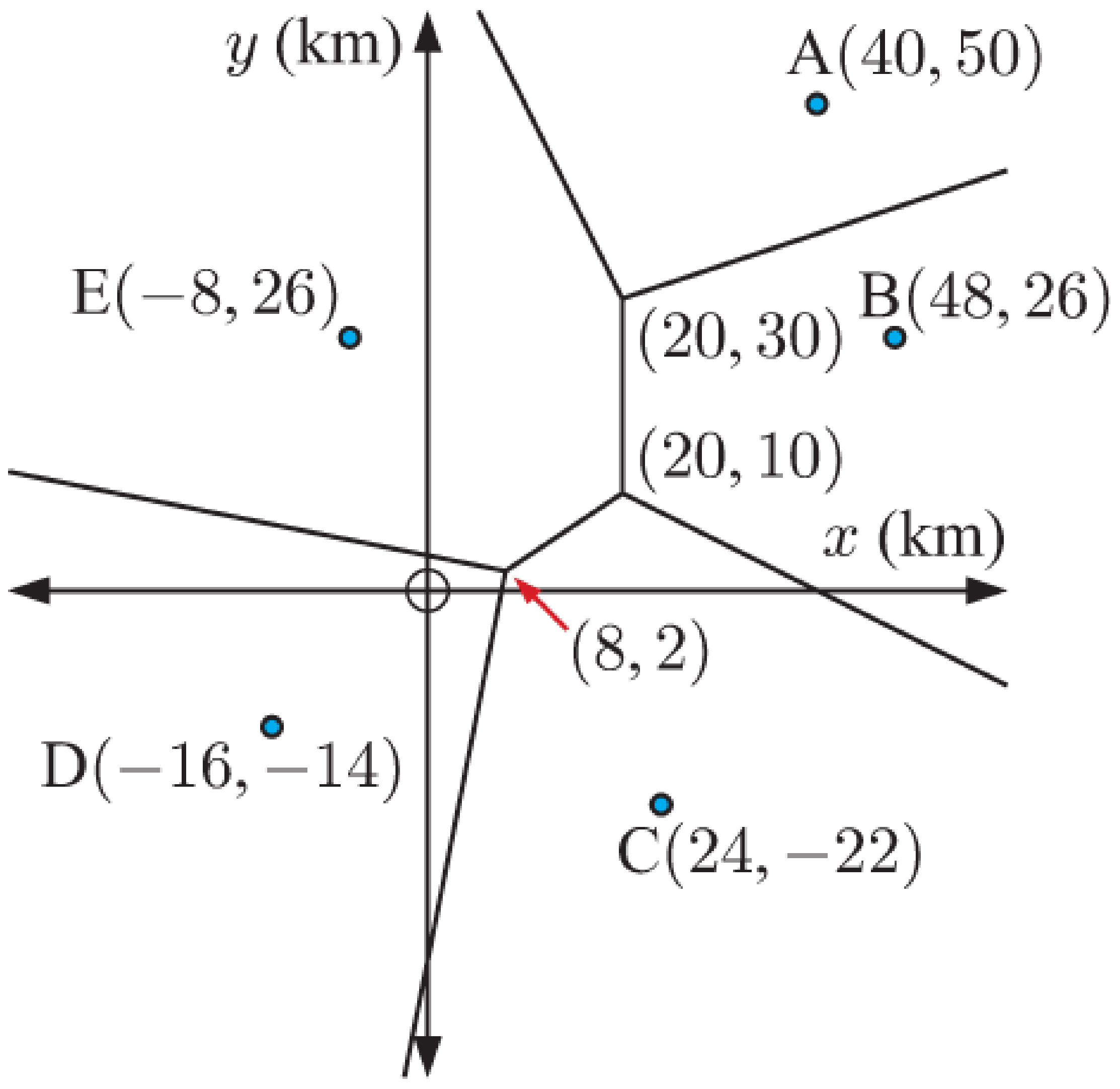
A rubbish dump is to be established somewhere in the region, so that it is as far as possible from the nearest town.

a Where should the rubbish dump be established?

b How far is the dump from the nearest town?

c Which towns are closest to the dump?

d A proposed location for the dump is $(25, 15)$. Show by direct calculation that the location you found in **a** is preferable.



3 Explain why it is impossible for the centre of the largest empty circle to lie inside a cell of the Voronoi diagram.

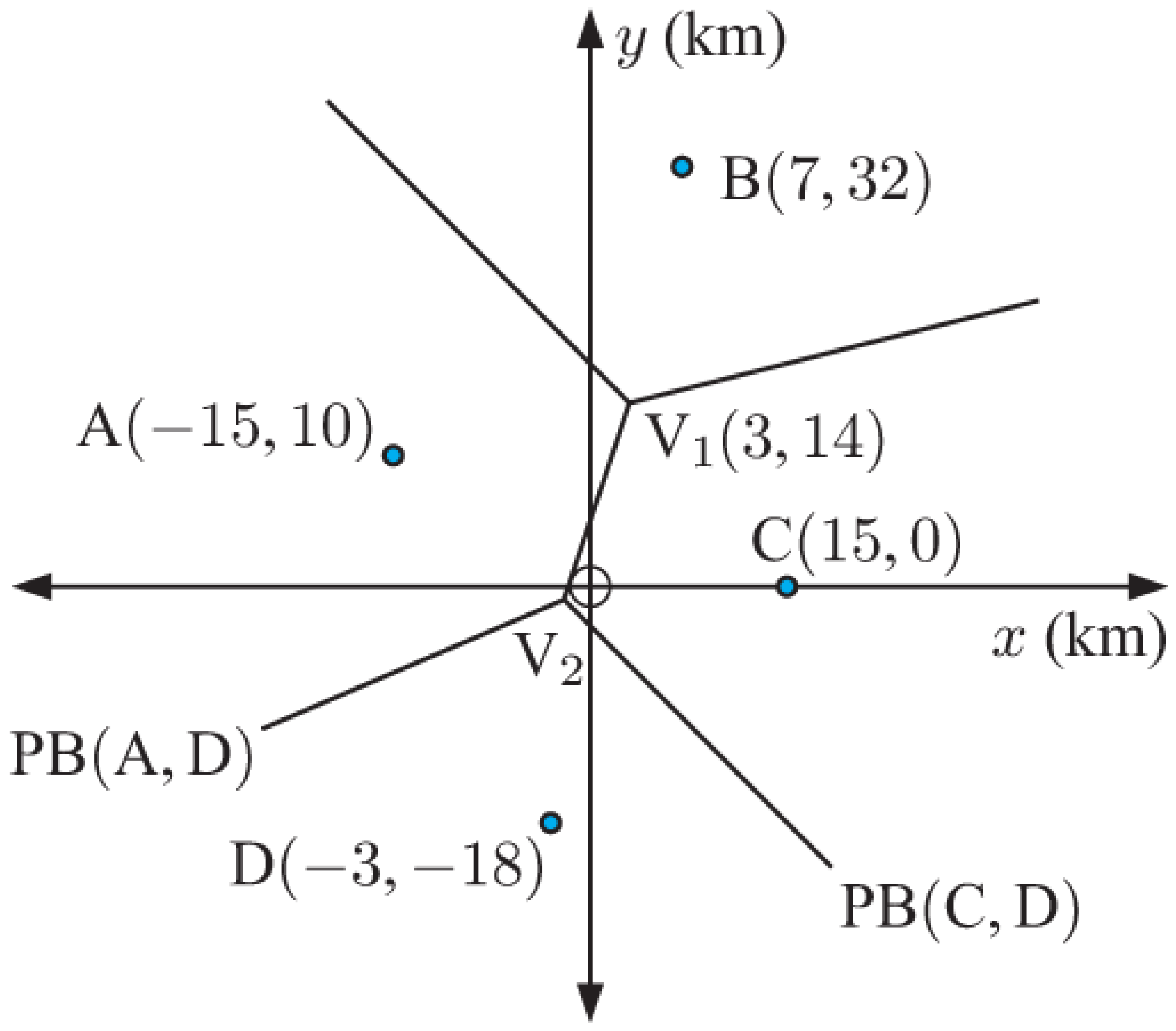
4 Consider this Voronoi diagram for the towns A, B, C, and D.

a Find the equation of:

i PB(A, D) **ii** PB(C, D)

b Hence find the coordinates of V_2.

c Find the optimal position for a toxic waste dump so that it is as far as possible from the nearest town.



5 Consider the sites $A(-5, -10)$, $B(11, 18)$, and $C(5, -12)$.

a Construct the Voronoi diagram for these sites, and find the equation of each edge.

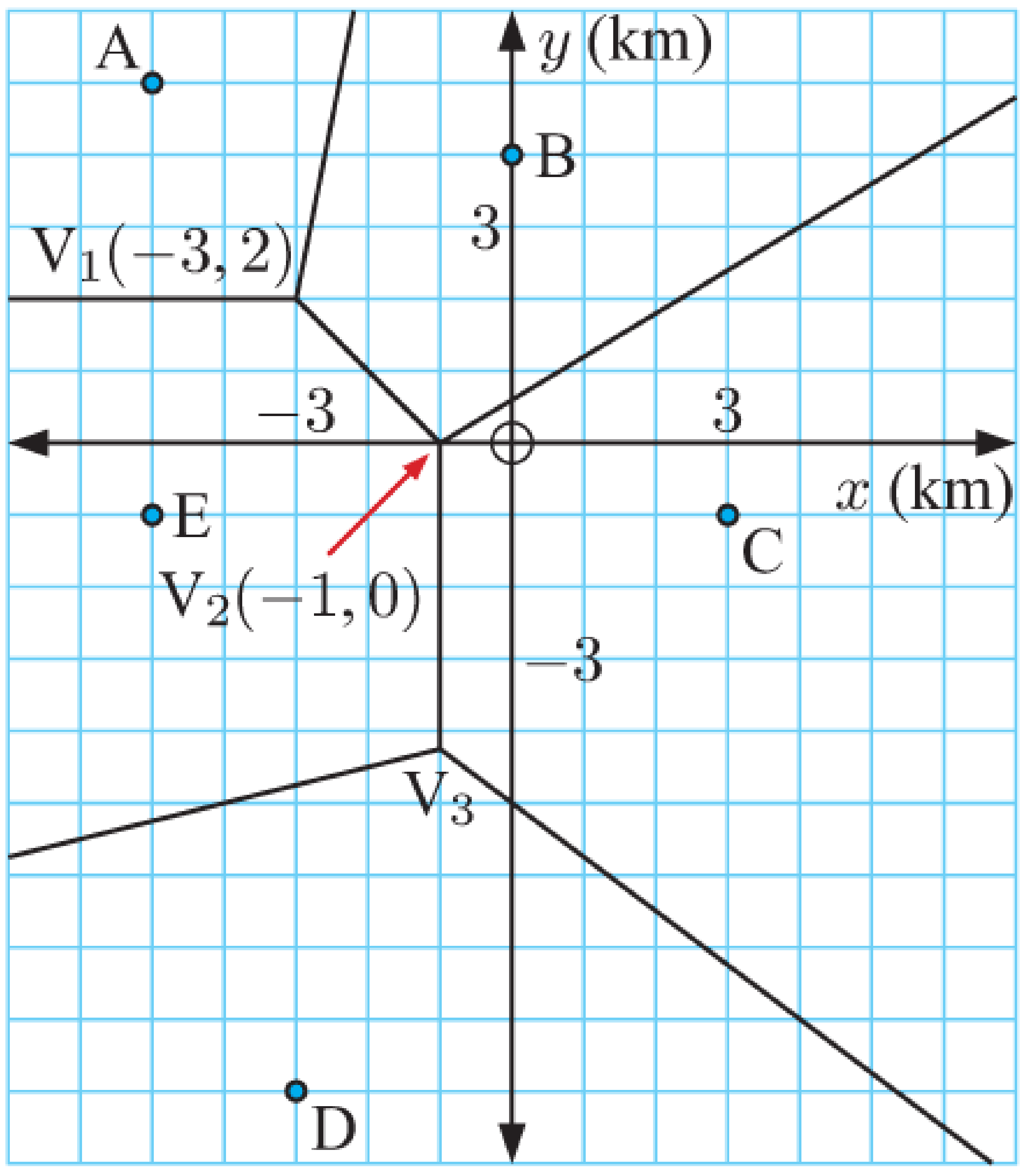
b Find the coordinates of the vertex of the diagram.

c Hence find the centre and radius of the largest empty circle.

6 Consider the sites $A(-2, 7)$, $B(4, 7)$, $C(6, 3)$, and $D(-2, -5)$.

- **a** Construct a Voronoi diagram for these sites.
- **b** Use your diagram to find the coordinates of the vertices.
- **c** Find the centre and radius of the largest empty circle.

7

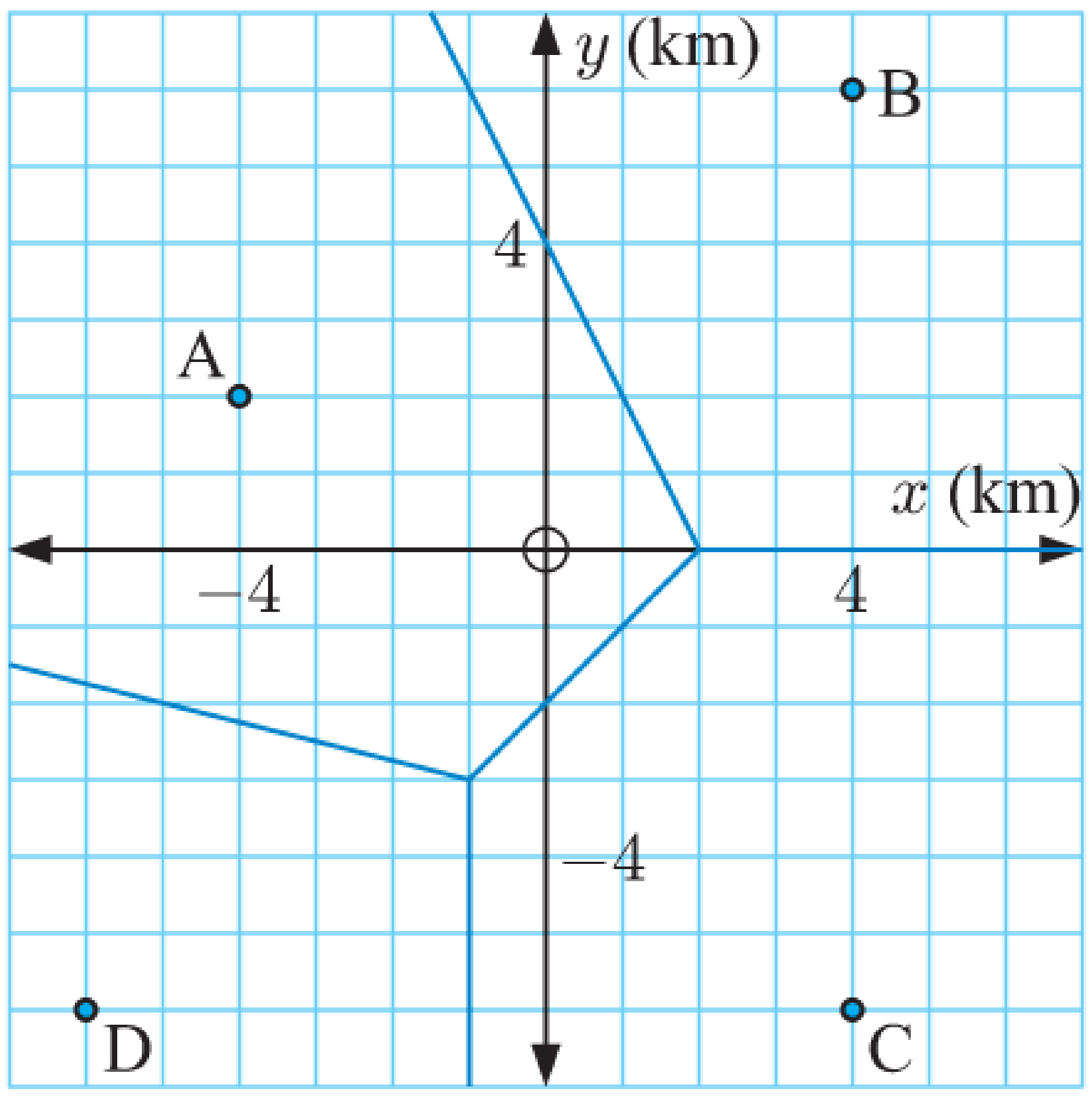


This Voronoi diagram shows the police stations in a city.

- **a** Which station is closest to:
  - **i** $(3, 4)$
  - **ii** $(-3, -3)$?
- **b** Find the coordinates of the vertex V_3.
- **c** A new police station is to be built. To maximise the efficiency of the stations, the new station will be built as far as possible from the existing stations.
  - **i** Where should the new station be built?
  - **ii** Show that this new station will now be the closest station to $(-3, -3)$.

8 Brigette would like to open a burger store. She has drawn a Voronoi diagram showing the existing burger stores in her area so she can locate her own store as far from the existing stores as possible.

- **a** Find the optimal position for Brigette's store, and state how far this location is from her nearest competitor.
- **b** Just before Brigette purchases the site, a new burger store opens at $(4, 2)$.
  - **i** Redraw the Voronoi diagram with the new store added.
  - **ii** Find the new optimal position for Brigette's store.

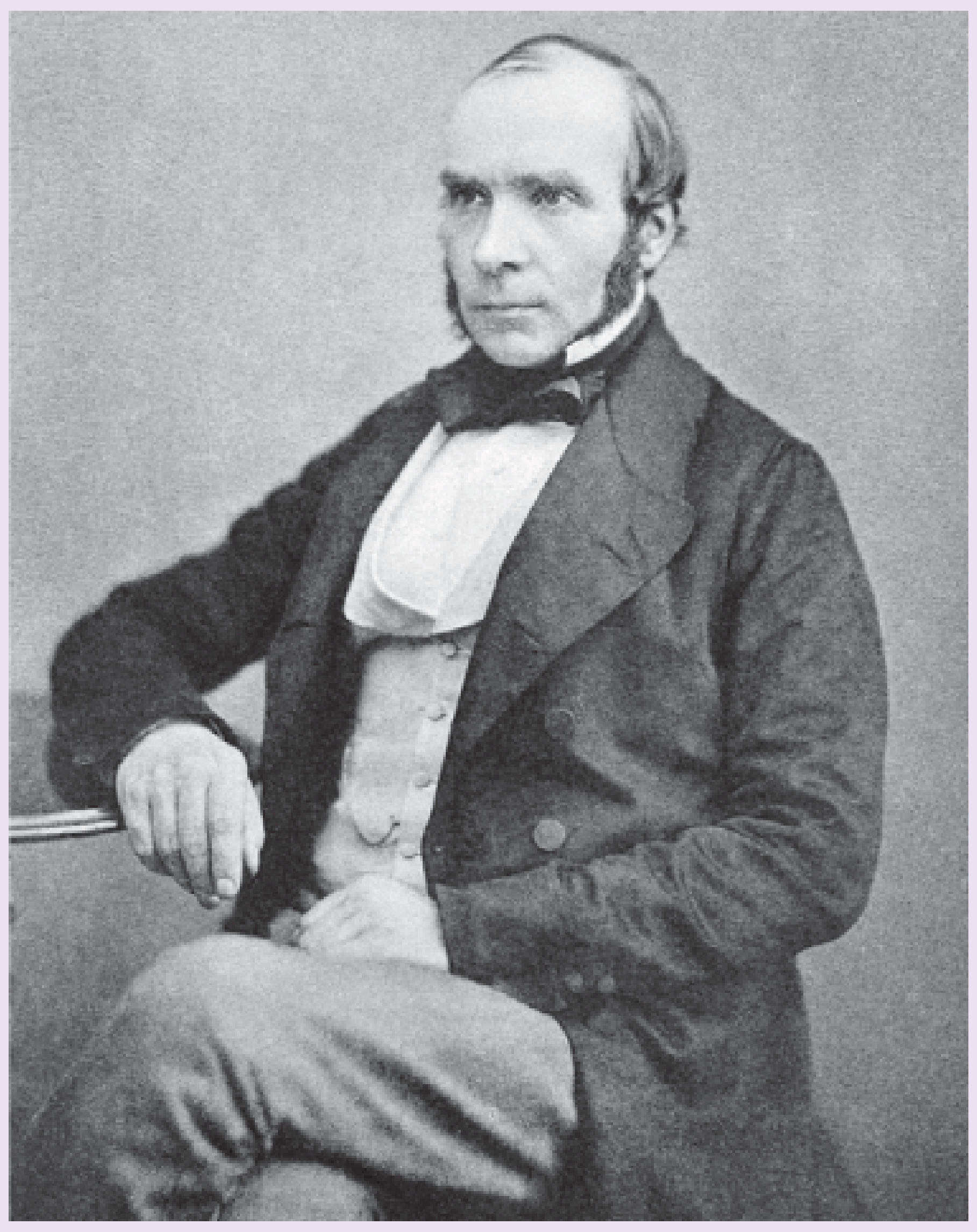


9 Answer the **Opening Problem** on page **418**.

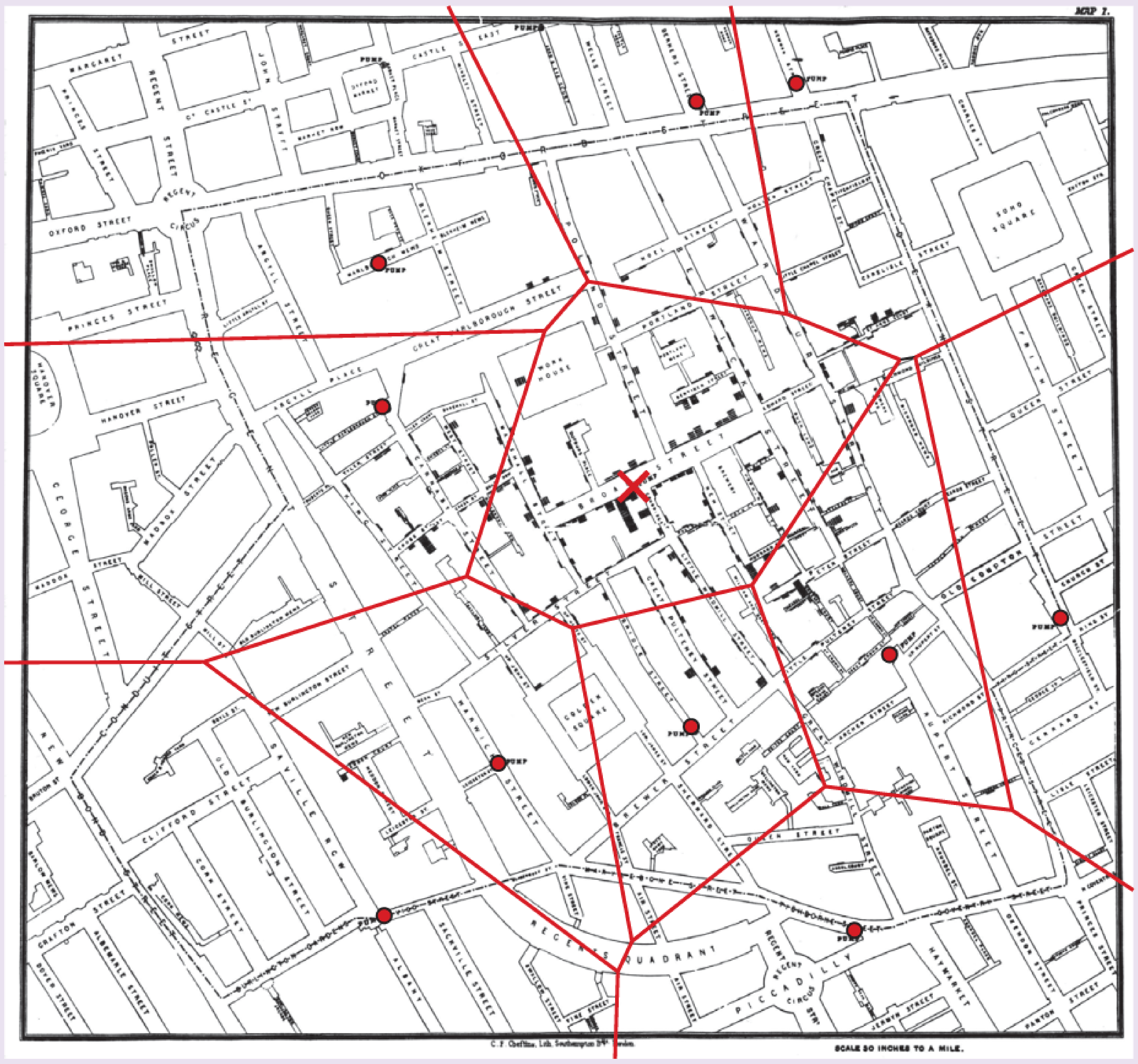
HISTORICAL NOTE THE BROAD STREET CHOLERA OUTBREAK

In 1854 in the Soho district of London, there was a serious cholera outbreak which killed 616 people. At the time, the cause of cholera was not fully understood. It was thought under the *miasma theory* that diseases such as cholera or the Black Death were caused by some kind of "bad air". The germ theory was not proposed until 1861 by **Louis Pasteur**.

In this outbreak, the English physician **John Snow** looked at the pattern of illness, and decided that the disease was spread by something in the water. At that time, water was drawn directly from the Thames River which was highly contaminated. Snow identified the source of the outbreak as the public water pump on Broad Street, and he persuaded the authorities to disable the pump by removing its handle.



John Snow


MAP 1.
OXFORD STREET
REGENT CIRCUS
MARGARET STREET
PRINCES STREET
HANOVER STREET
GREAT MARLBOROUGH STREET
SOHO SQUARE
WORK HOUSE
BROAD STREET
PUMP
REGENT QUADRANT
PICCADILLY
SCALE 30 INCHES TO A MILE.

The map above shows the locations of the cholera cases. The dark rectangles show where people died of cholera, the red dots show public water pumps, and the large red cross is the Broad Street pump. The Voronoi cells show the closest pump by straight line distance.

At the time of Snow's investigation, he did not have Voronoi diagrams to use. Voronoy had not yet been born! However, the overlaid Voronoi diagram clearly illustrates how the cholera cases were clustered around the Broad Street pump. This is what Snow recognised. He wrote "*it will be observed that the deaths either very much diminished, or ceased altogether, at every point where it becomes decidedly nearer to send to another pump than to the one in Broad Street.*"[1]

However, not all the cases occurred in the cell around the Broad Street pump. Many cases also occurred in the cell to the left. These people were much closer to the pump in Marlborough Street. However, when Snow investigated where people actually drank from, he found that "*the water of the pump in Marlborough Street, at the end of Carnaby Street, was so impure that many people avoided using it. And I found that the persons who died near this pump in September, had water from the Broad Street pump.*"

It was later discovered that the nappy of a baby who had contracted cholera from another source, had contaminated the water supply near the Broad Street pump.

[1] J. Snow. *On the Mode of Communication of Cholera* p. 47. John Churchill, London, 2nd edition, 1855.

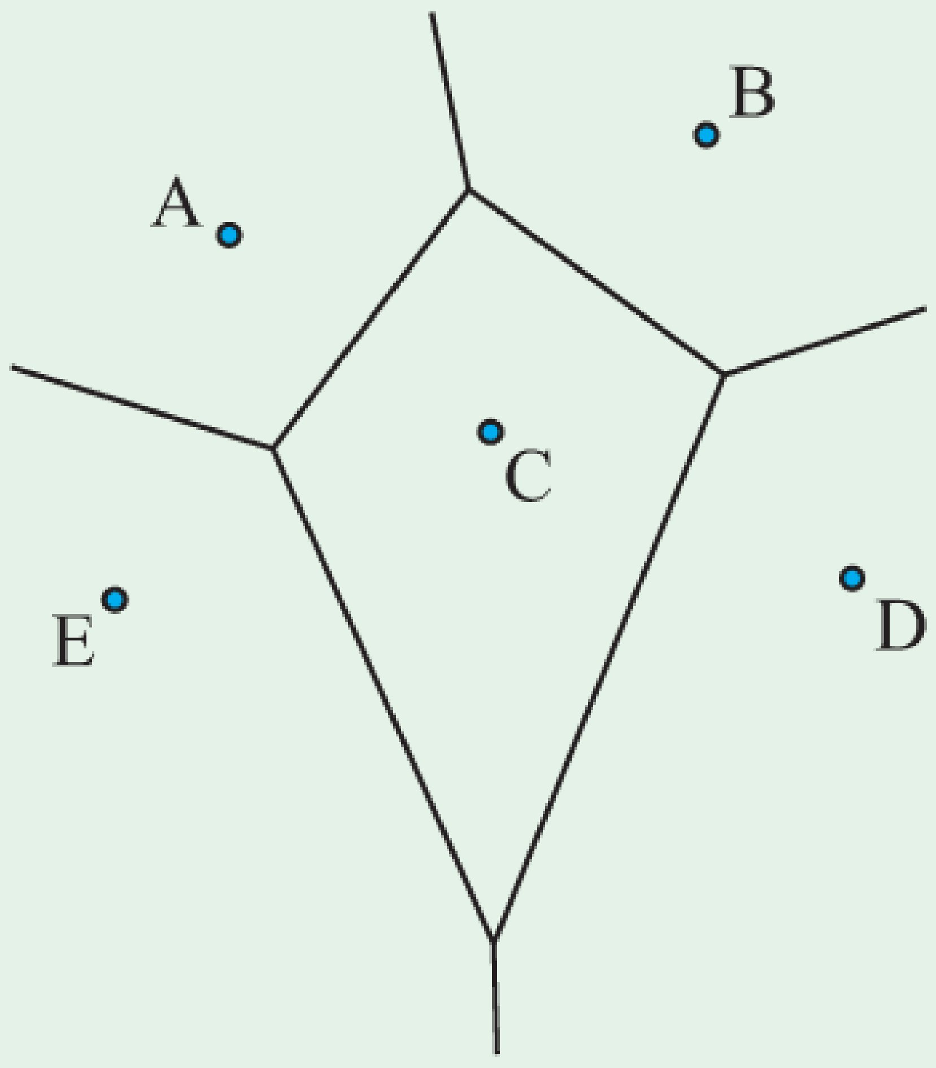
REVIEW SET 17A

1 Copy this Voronoi diagram, and indicate the parts of the diagram that are:

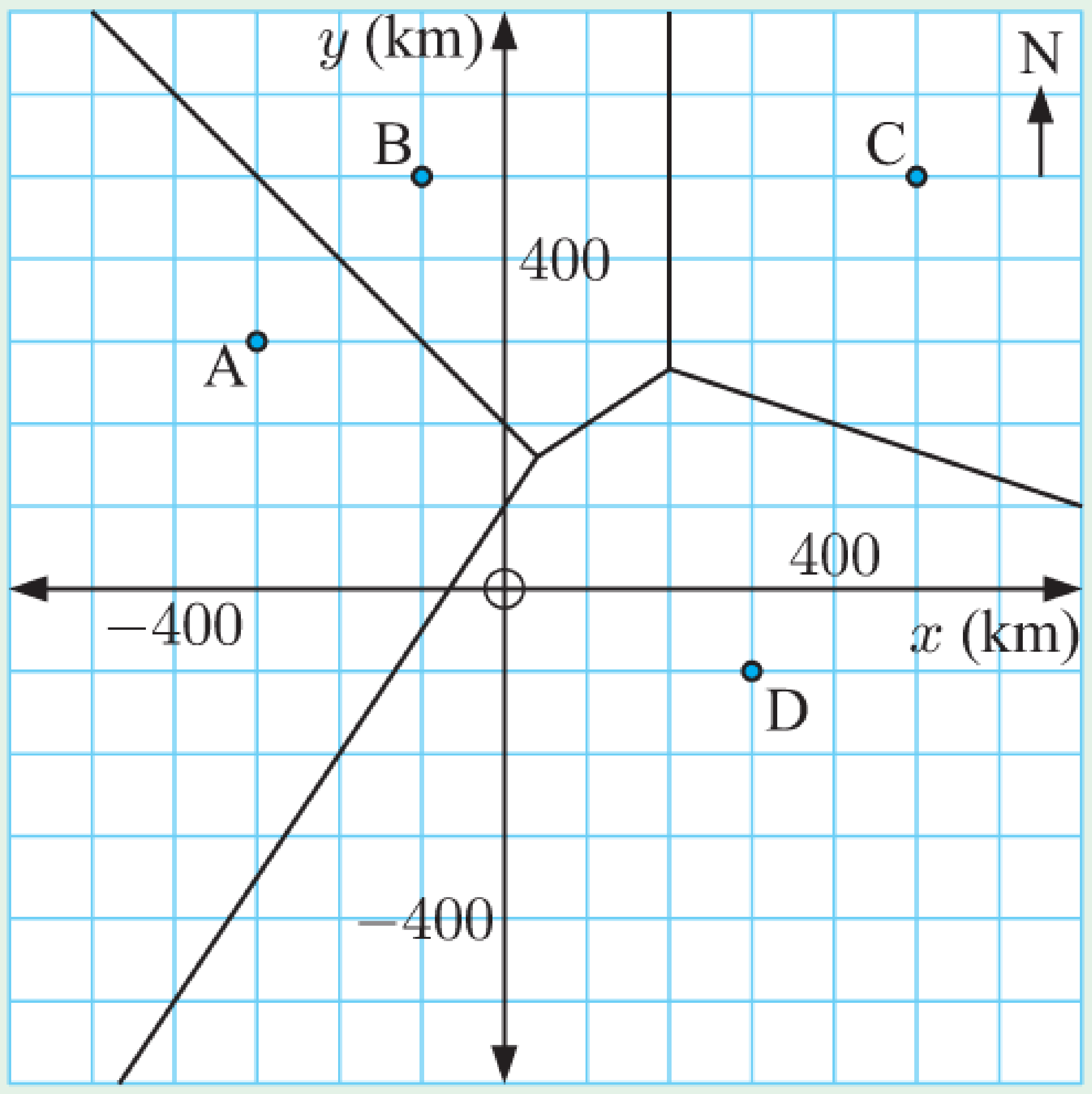
a closest to site B

b equally closest to C and D

c equally closest to A, C, and E.



2



This Voronoi diagram shows the airports in a country. If an aeroplane's passenger falls seriously ill during a flight, the aeroplane must land at the nearest airport.

a Identify the nearest airport to an aeroplane located at:

i (300, 400) **ii** (−100, 0)

b An aeroplane is currently at (100, 200).

i Which airports is the aeroplane closest to?

ii How far is the aeroplane from these airports? Give your answer to the nearest kilometre.

iii How far east must the aeroplane travel before it is closest to Airport C?

3 **a** Construct a Voronoi diagram for the sites A(−3, 2) and B(5, 6).

b Find the equation of the edge.

c Verify that the point (3, 0):

i lies on the edge **ii** is equidistant from A and B.

d Use your Voronoi diagram to identify the site closest to:

i (−1, 7) **ii** (2, −5)

4 This swimming pool has exits at A, B, and C.

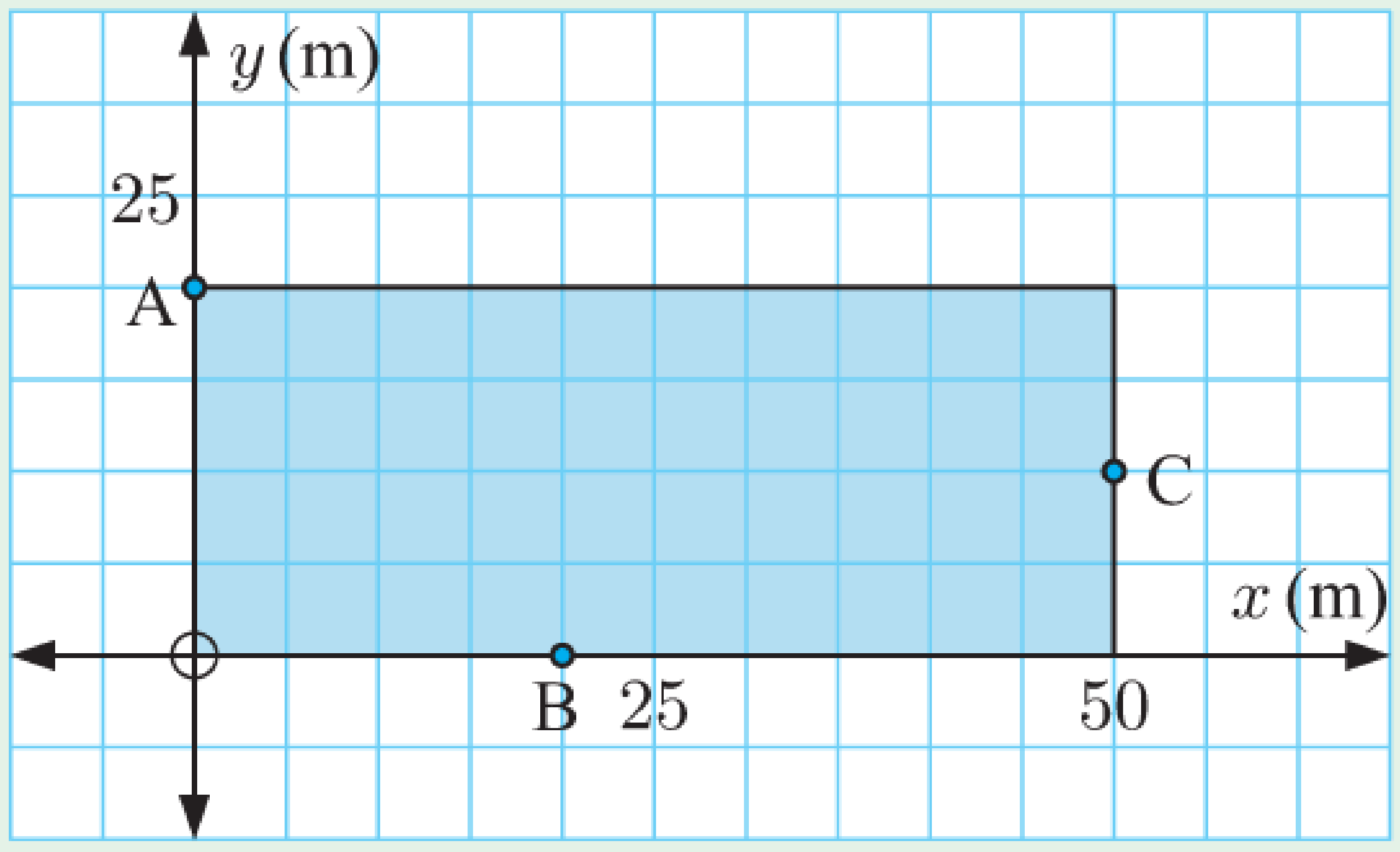
a Draw a Voronoi diagram for A, B, and C.

b Is there any point in the pool which is equidistant from all three exits? Explain your answer.

c Jenny is swimming at (35, 10).

i Which exit is Jenny closest to?

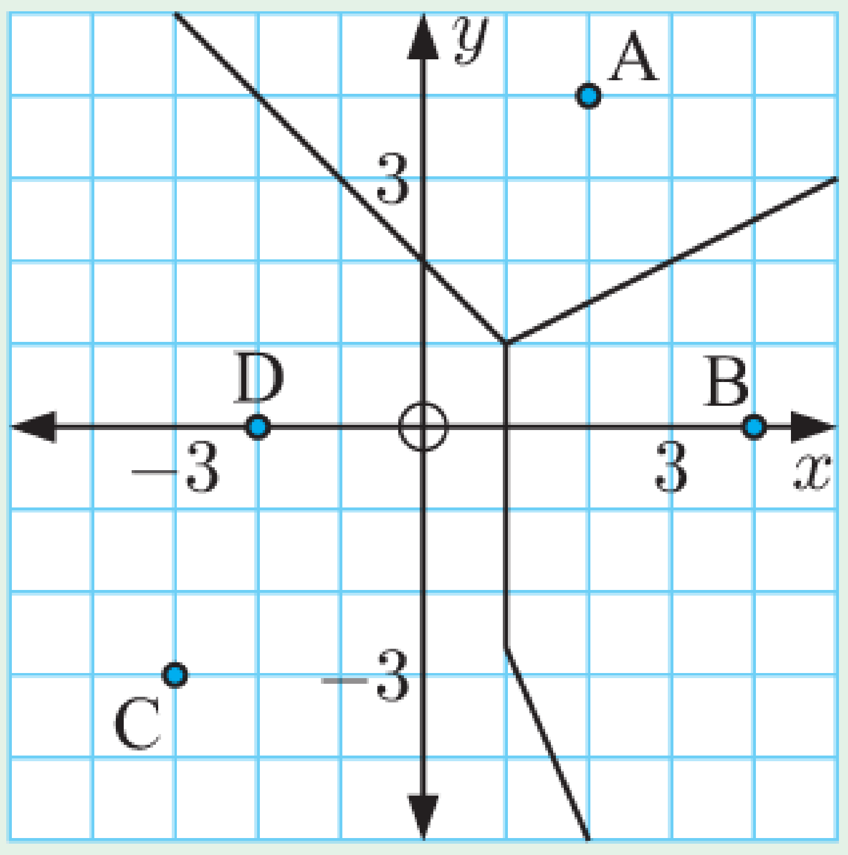
ii How far is Jenny from this exit?



d Find the proportion of the pool that is closest to exit:

i A **ii** B **iii** C.

5



a Explain why this Voronoi diagram must have an edge missing.

b Copy and complete the diagram, and find the equation of the missing edge. Write your equation in the form $ax + by + d = 0$.

PRINTABLE DIAGRAMS

6 Consider the Voronoi diagram alongside.

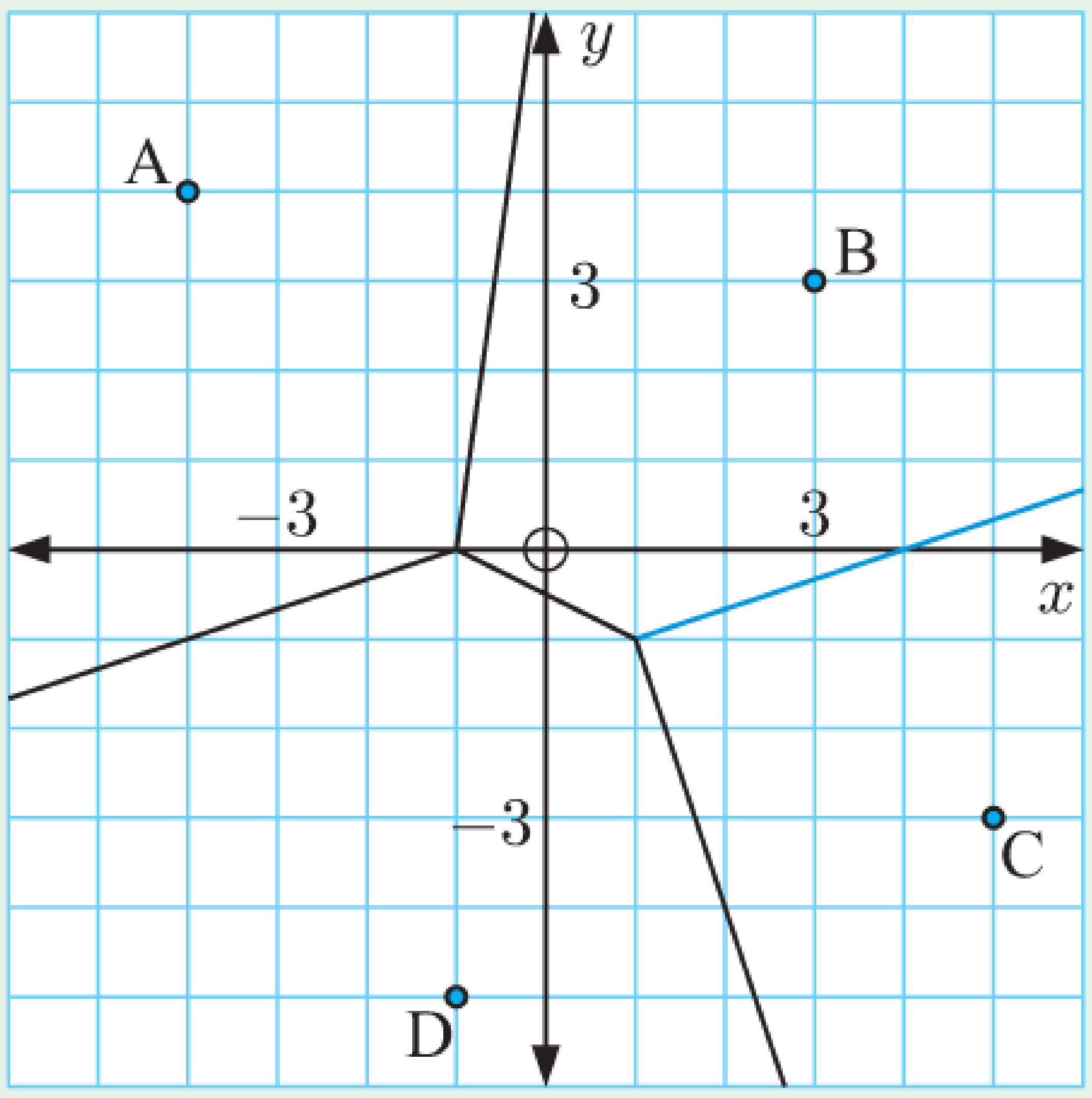
a Find the equation of the blue edge.

b Identify the site which is closest to:

i $(2,\ -1)$ **ii** $(-5,\ -2)$

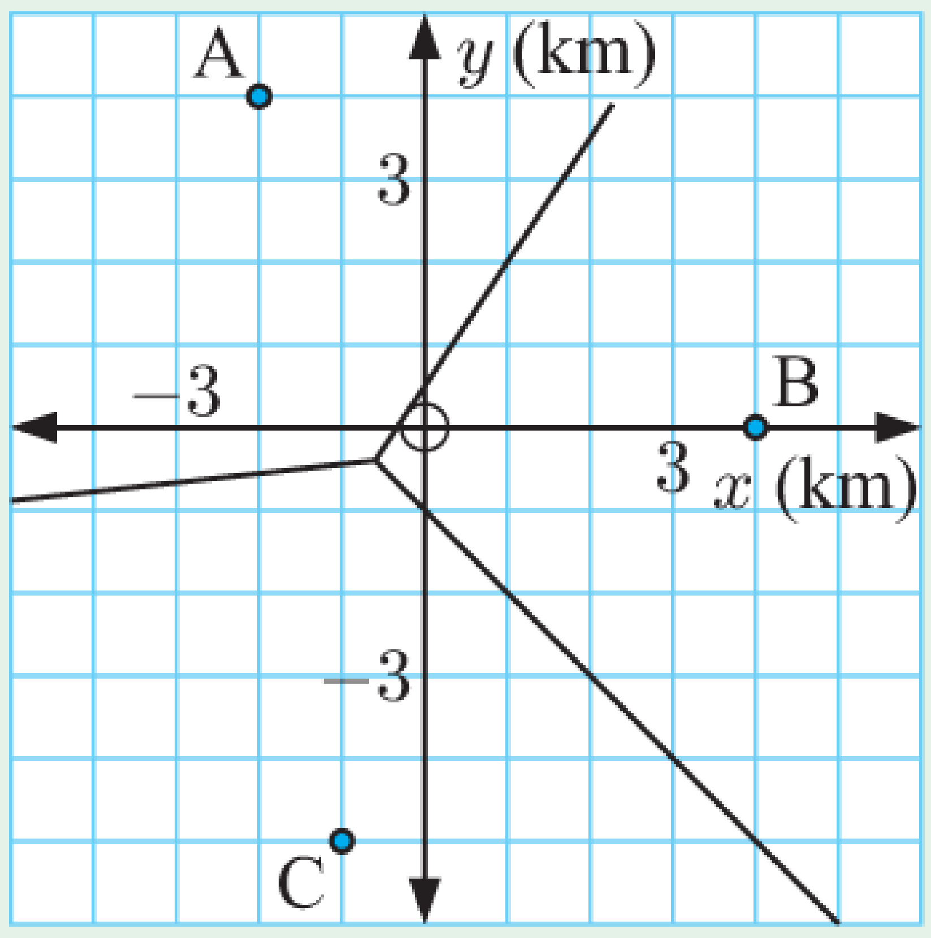
c Redraw the Voronoi diagram with a new site added at $\text{E}(-3,\ -6)$.

d Does the addition of site E affect your answers to **b**?



7 The wind speed is measured at three locations on an island.

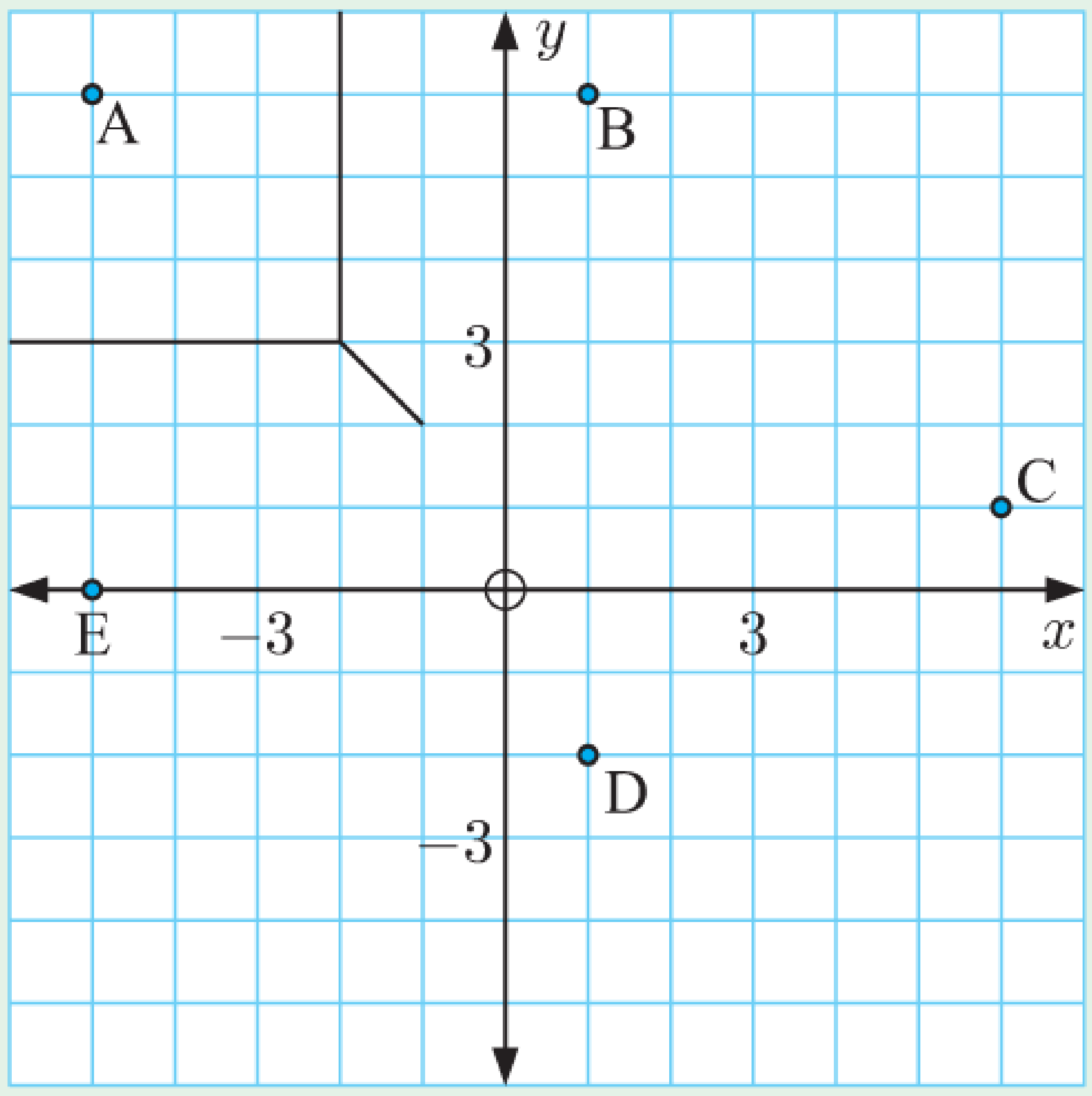
Location	*Wind speed* (km h^{-1})
A	14
B	11
C	19



Use nearest neighbour interpolation to estimate the wind speed at:

a $(0,\ 0)$ **b** $(-1,\ 3)$ **c** $(-4,\ -2)$

8

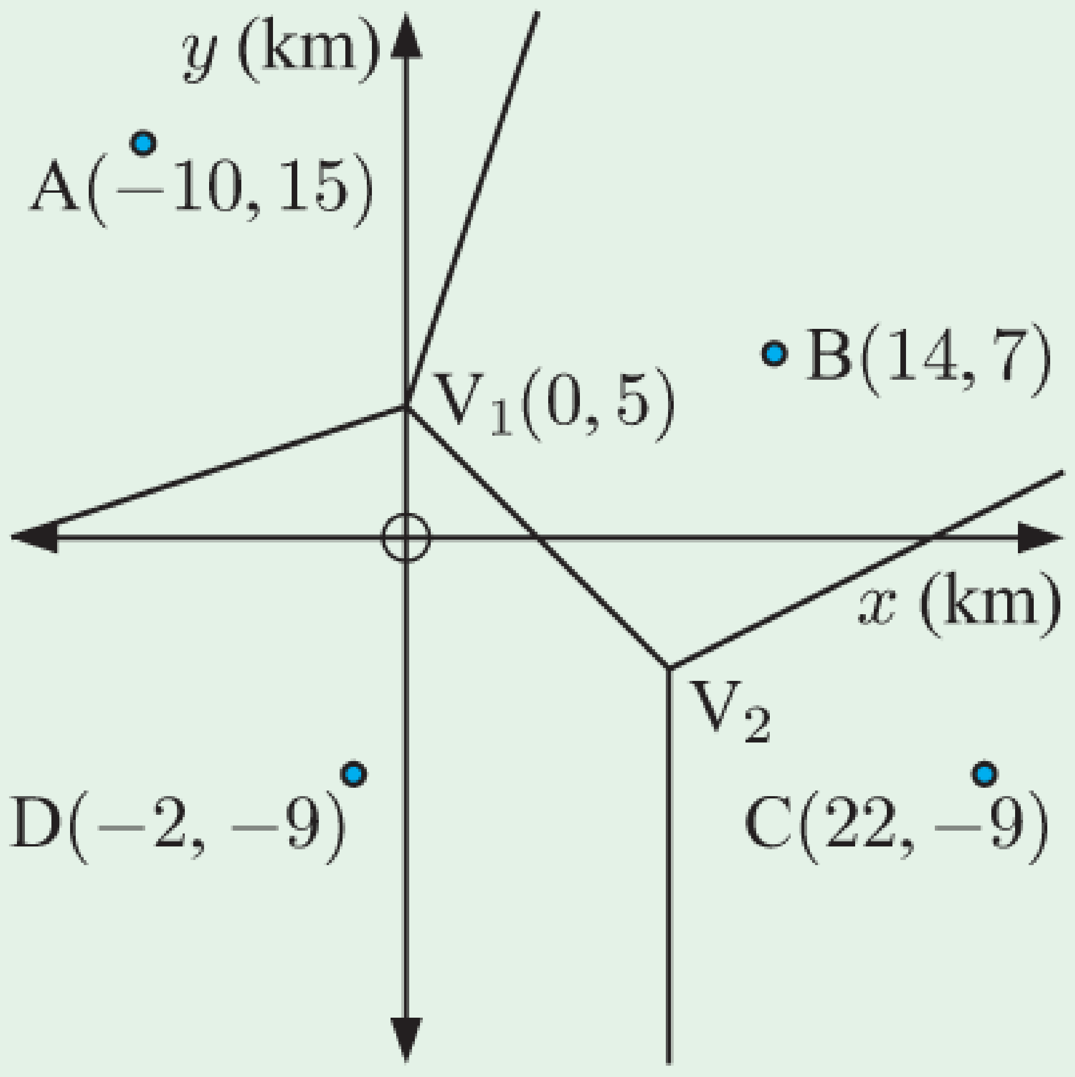


a Copy and complete this Voronoi diagram.

b Find the coordinates of the vertices of the Voronoi diagram.

c Find the centre and radius of the largest empty circle for these sites.

9 Consider this Voronoi diagram for the towns A, B, C, and D.



a Find the equation of:

i PB(B, C) **ii** PB(C, D)

b Hence find the coordinates of V_2.

c An observatory is to be built somewhere in the region, so that it is as far as possible from the town lights.

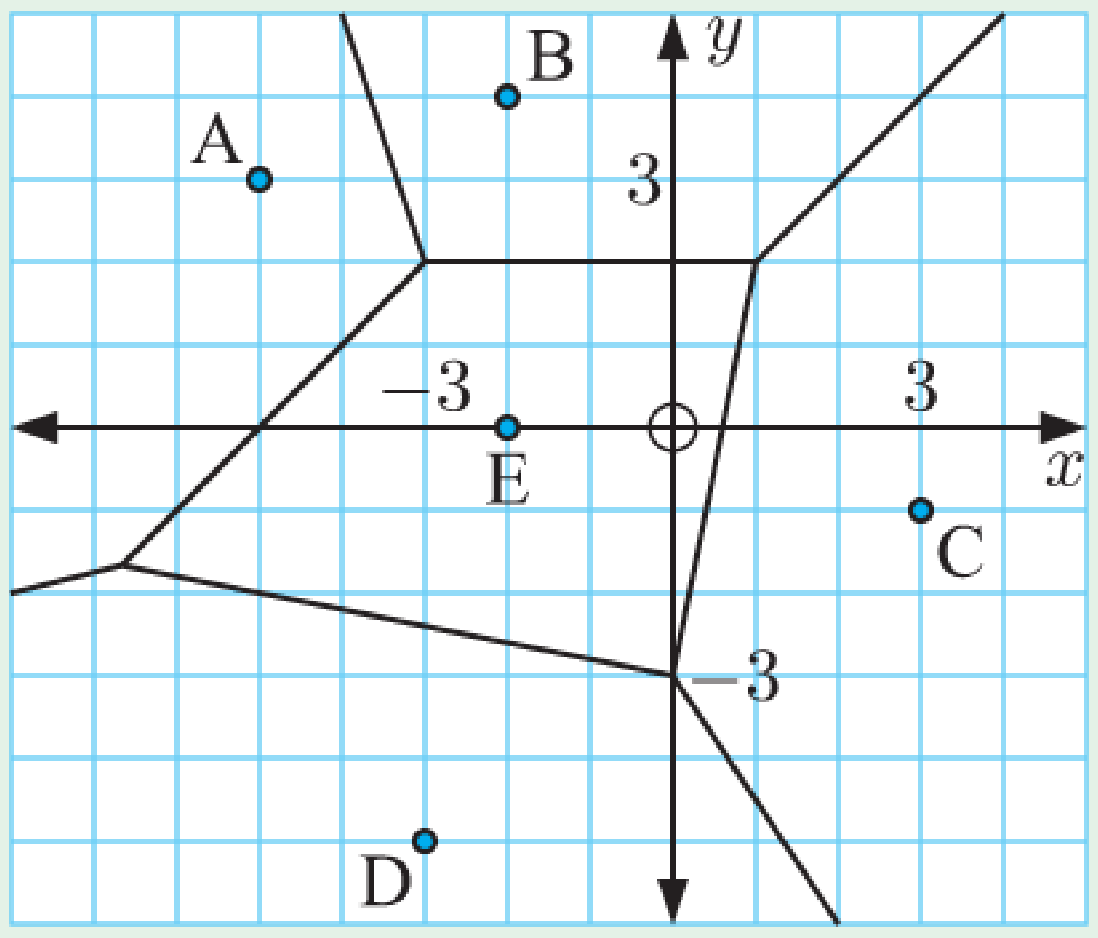
i Where should the observatory be built?

ii How far from the observatory will the nearest town be?

iii Which towns are closest to the observatory?

REVIEW SET 17B

1 Consider this Voronoi diagram for the sites A, B, C, D, and E.



a Identify the site(s) closest to:

i $(1, 0)$ **ii** $(-4, -2)$

iii $(3, -5)$ **iv** $(-3, 2)$

b Are there any points that are equally closest to sites B and D? Explain your answer.

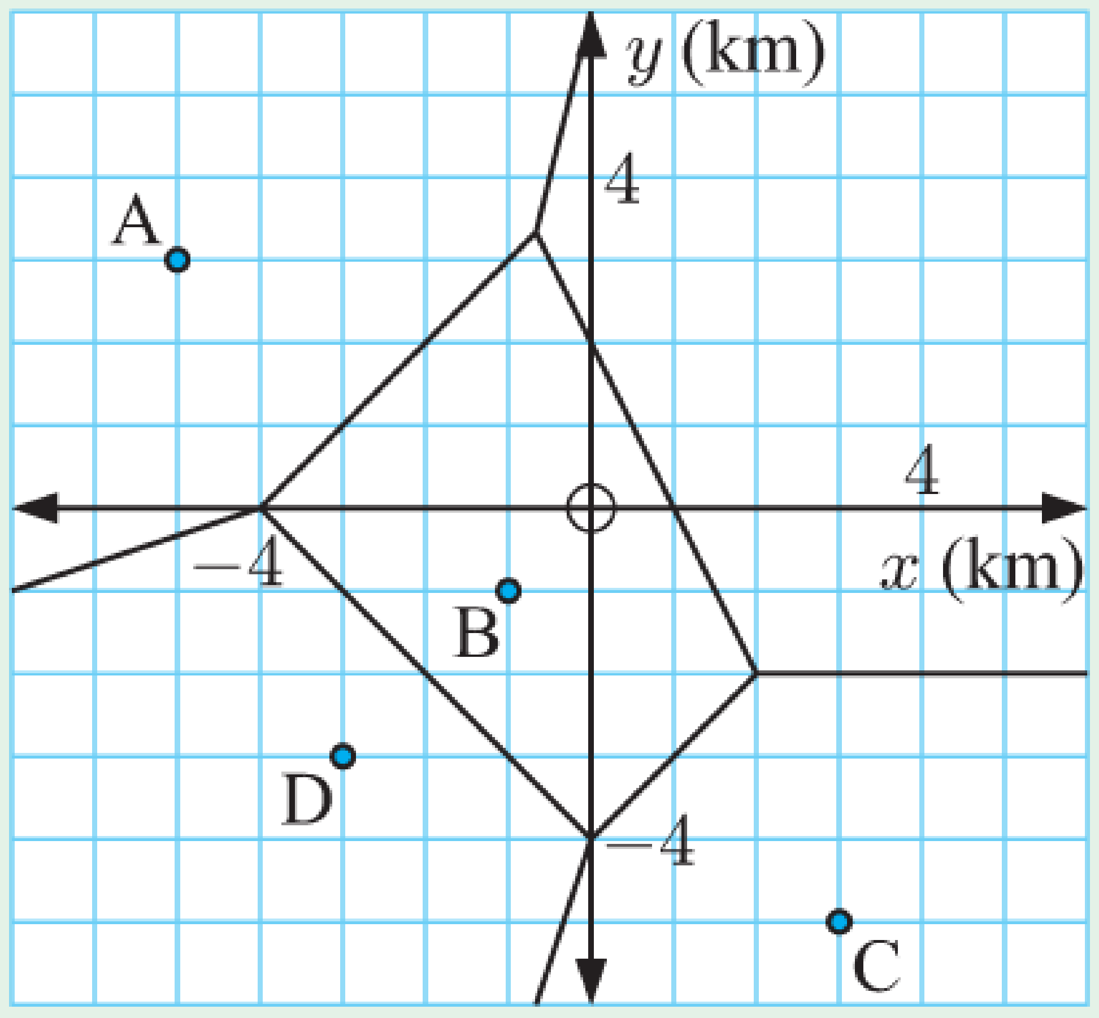
c Which point is equally closest to sites B, C, and E?

2 Let P be a point on an edge of a Voronoi diagram, and X be the site in a cell adjacent to that edge. Suppose a circle is drawn with centre P, passing through X. Explain why this circle must pass through one other site.

3 Construct a Voronoi diagram for the sites:

a A$(4, 7)$ and B$(2, -1)$ **b** A$(-5, 0)$, B$(1, 6)$, and C$(7, 4)$.

4 This Voronoi diagram shows the taxi ranks in a city.



a Explain why the diagram must have a site missing.

b Find the coordinates of the missing site E.

c Elizabeth is at $(-3, 2)$, and needs to catch a taxi.

i Which taxi rank is she closest to?

ii Assuming she can walk at 5 km h^{-1}, how long will she take to walk to this taxi rank? What assumption are you making in your answer?

d Albert is at $(-2, -2)$.

i Which taxi ranks is Albert closest to?

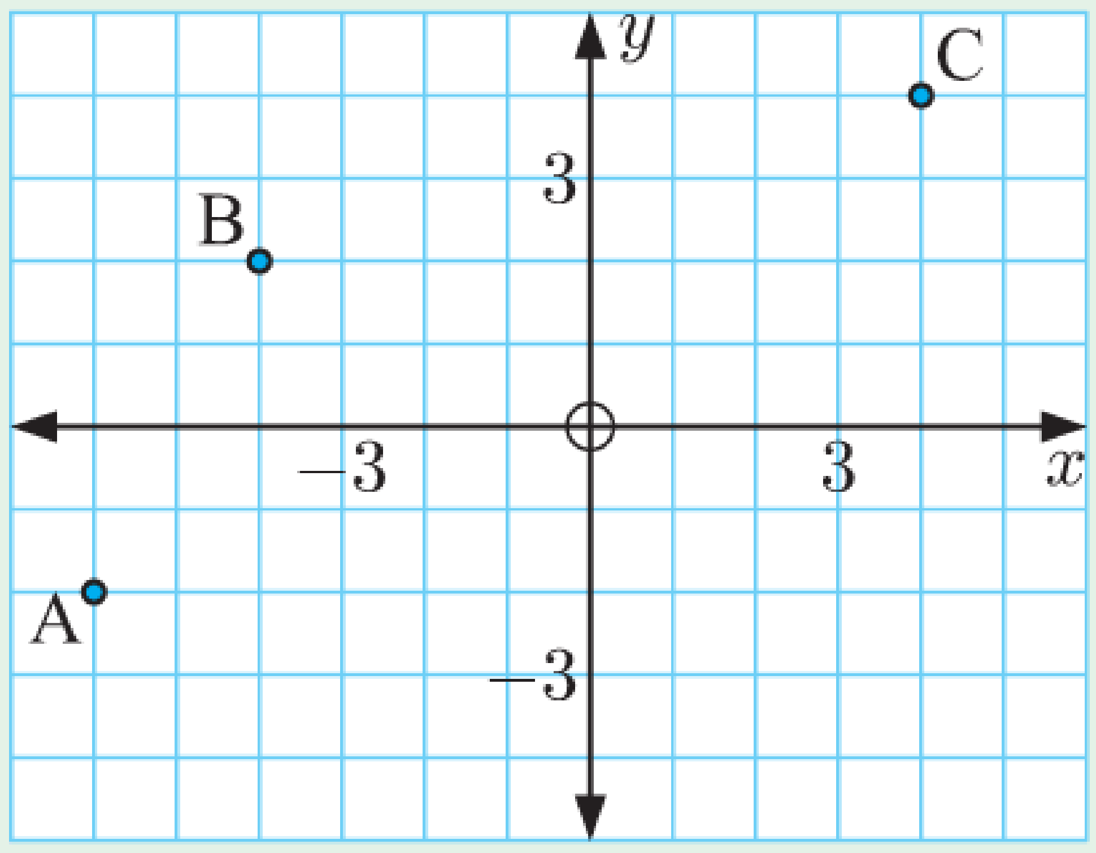
ii What other factors might Albert consider when deciding which taxi rank to use?

5 **a** Construct the Voronoi diagram for the sites shown.

b Use your Voronoi diagram to identify the site(s) closest to:

- **i** $(1, 2)$
- **ii** $(-1, -5)$
- **iii** $(-3, -1)$

PRINTABLE DIAGRAMS

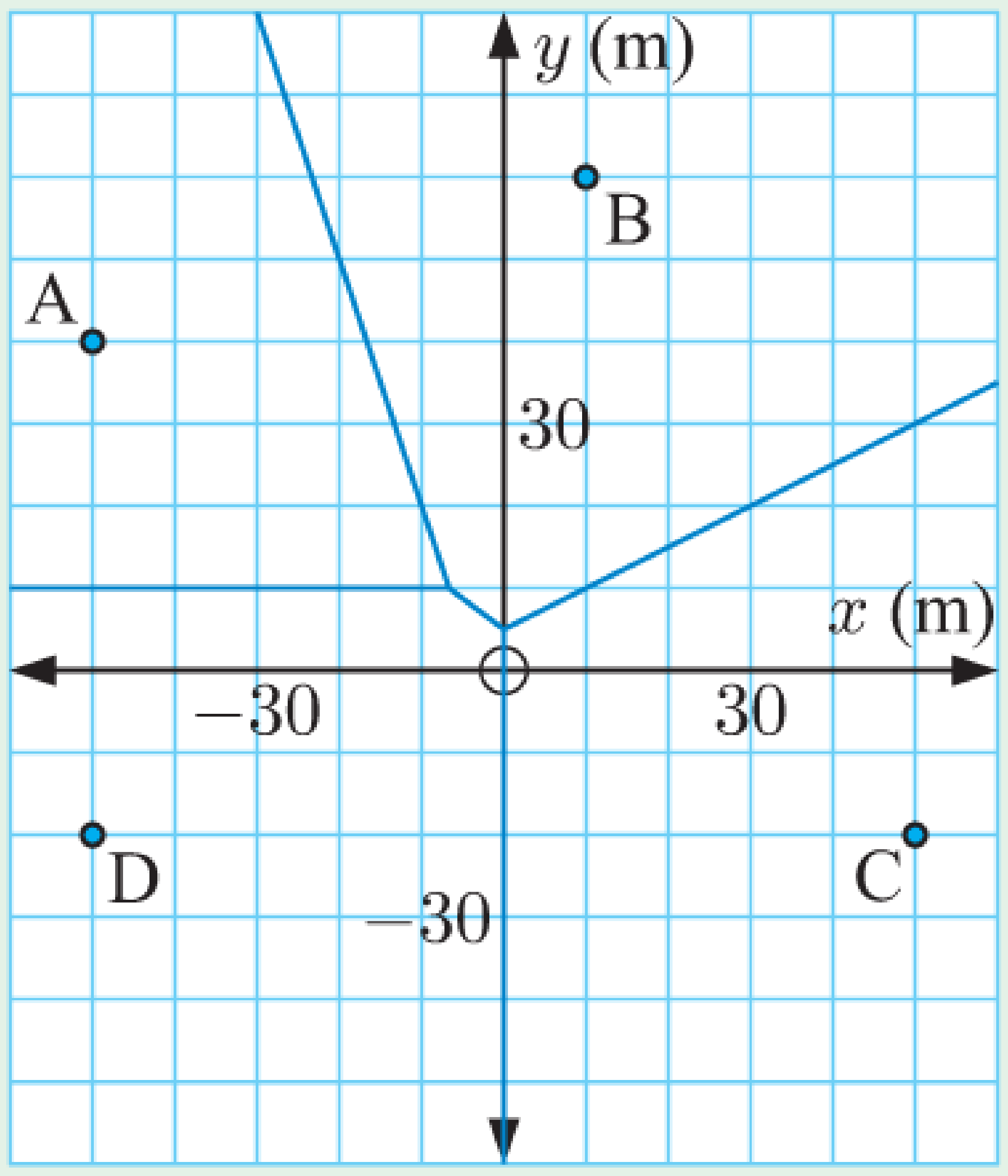


6 This Voronoi diagram shows the locations of rubbish bins in a park.

a Boris is at $(20, 20)$. How far is he from the closest bin?

b Suppose a new bin is placed at $E(50, 40)$.

- **i** Redraw the Voronoi diagram to include the new bin.
- **ii** Which cells have been affected by the introduction of the new bin?
- **iii** On your Voronoi diagram, shade the area of the park which was closest to Bin C, but is now closest to Bin E.



7 A diver measured the depth of a lake at four locations.

a Construct a Voronoi diagram for these locations.

b Use nearest neighbour interpolation to estimate the depth of the lake at:

- **i** $(-1, 3)$
- **ii** $(4, 0)$

Location	*Depth* (m)
$A(-2, -3)$	7.5
$B(-2, 5)$	9.2
$C(4, 3)$	6.1
$D(6, -1)$	6.9

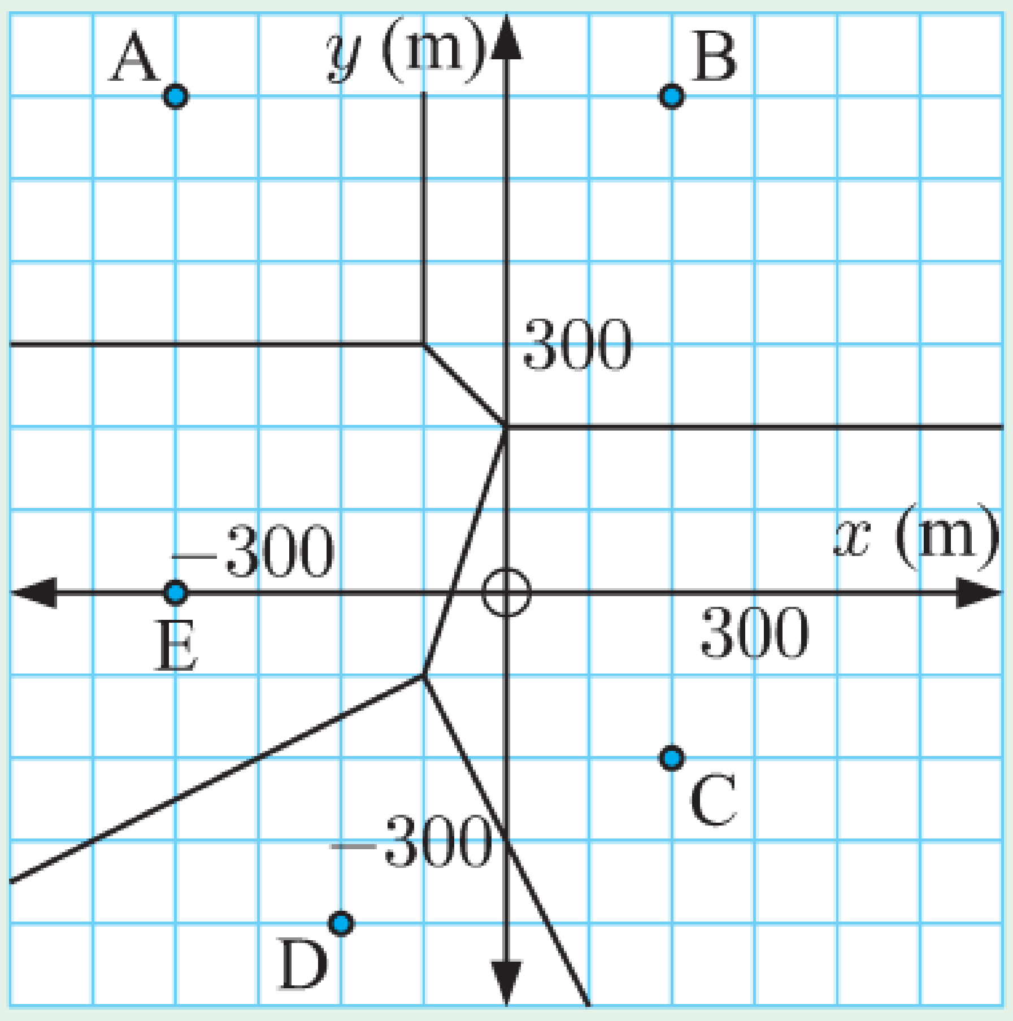
8 This Voronoi diagram shows the parking lots in a city.

a Identify the nearest parking lot to:

- **i** $(-100, 200)$
- **ii** $(0, -400)$

b A new parking lot is to be built in the city. It will be built as far as possible from the existing parking lots.

- **i** Find the optimal position for the new parking lot F.
- **ii** How far is this parking lot from the closest existing parking lots?
- **iii** Redraw the Voronoi diagram with the new parking lot added.
- **iv** Find the area of the region that is closest to the new parking lot.



c Joseph works at $(200, 200)$.

- **i** Show that Joseph is closest to the new parking lot.
- **ii** How much closer is Joseph to a parking lot, now that the new parking lot has been added?

ANSWERS

EXERCISE 1A.1

1 **a** 90 **b** 80 **c** 90 **d** 130 **e** 160
f 100 **g** 640 **h** 1820 **i** 700 **j** 3050
2 **a** 200 **b** 300 **c** 3800 **d** 4000 **e** 26 300
3 **a** 8000 **b** 4000 **c** 19 000 **d** 20 000 **e** 115 000
4 **a** 8850 m **b** 32 000 km^2 **c** 4 750 000 people
d 85 500 people **e** 7 700 000 km^2 **f** 5400 kg
g 17 000 km **h** 400 000 km **i** 428 000 000 people

EXERCISE 1A.2

1 **a** 6.2 **b** 6.18 **c** 3.3 **d** 17.40 **e** 2.132
f 0.2 **g** 0.10 **h** 102.38
2 9.6 s **3** 1.44 m **4** 0.01 cm
5 **a** 3.1 **b** 3.142 **c** 3.1416
6 **a** 0.3 **b** 0.26 **c** 0.263 158
7 **a** ≈ 1.414 **b** ≈ 2.236 **c** ≈ 4.796 **d** ≈ 1.587
e ≈ -2.466 **f** ≈ 7.663
8 **a** 499.32 **b** 228.84 **c** 9.11 **d** 31.75 **e** 26.67
f 0.88 **g** 7.41 **h** 5.93 **i** 0.48
9 1.74 goals per game
10 While 2.45 m $\approx$ 2.5 m is correct, Wang should have used the original value of 2.45 m to round to the nearest integer, so 2.45 m $\approx$ 2 m.

EXERCISE 1A.3

1 **a** 130 **b** 8300 **c** 2.6 **d** 0.013
e 160 000 **f** 1.1 **g** 4000 **h** 6.6
2 **a** 83 100 **b** 10 000 **c** 0.105 **d** 31.7
e 70.7 **f** 4.00 **g** 0.0367 **h** 20.0
3 **a** 16.38 **b** 438.2 **c** 6 874 000 **d** 0.028 89
4 **a** 100 000 people **b** 96 000 people **c** 96 300 people
5 **a** ≈ 2.65 **b** ≈ 6.28 **c** ≈ 2.12 **d** $\approx 1\,970\,000$
e ≈ 0.932 **f** ≈ 4.39 **g** ≈ 1.79 **h** ≈ 5.73
6 900 seats **7** 256 m^2 **8** **a** \$188 **b** \$188.06
9 1 200 000 m
10 **a** Eric has rounded each answer to 3 significant figures before using it in the next calculation, rather than using exact values.
b ≈ 56.5 cm^2

EXERCISE 1B

1 **a** 180 **b** 420 **c** 2400 **d** 600
e 25 000 **f** 24 000 **g** 1000 **h** 2400
i 5000 **j** 60 000 **k** 120 000 **l** 120 000
m 3 **n** 3.5 **o** 180
2 **a** 30 **b** 12 **c** 25 **d** 15
e 200 **f** 1000 **g** 5 **h** 250
i 0.2 **j** 10 **k** 2000 **l** 5
3 **a** €24 **b** \$600 **c** \$630 **d** £60
4 **a** 240 km **b** 4000 days **c** 20 tonnes **d** €4000

EXERCISE 1C

1 **a** $\pm\frac{1}{2}$ cm **b** $\pm\frac{1}{2}$ mL **c** ± 50 mL **d** ± 250 g
e ± 0.05°C
2 67.5 kg $< w <$ 68.5 kg
3 **a** 26.5 mm to 27.5 mm **b** 38.25 cm to 38.35 cm
c 4.75 m to 4.85 m **d** 1.45 kg to 1.55 kg
e 24.5 g to 25.5 g **f** 3.745 kg to 3.755 kg
4 36.35°C $< T <$ 36.45°C
5 **a** 1.055 km **b** 9.715 km **c** 10.05 km
The watch displays the correct distance accurate to ± 0.005 km.
6 **a** 6.4 m, as it is the most different from the other measurements.
b 6.05 m, as it is in the range of values for a measurement of 6.0 m or 6.1 m.
c 10 cm
7 **a** 2.35 m $< l <$ 2.45 m **b** $2.35n$ m $< L < 2.45n$ m
8 4 s $< t <$ 6 s **9** 248 ± 2 cm **10** 788 cm $< l <$ 792 cm
11 **a** 55.25 cm^2 **b** 41.25 cm^2
12 1092.25 ± 34 cm^2 **13** 36.125 ± 4.25 cm^2
14 196.5 ± 52.125 cm^3 **15** $\approx 1523.90 \pm 21.79$ cm^3
16 $\approx 1197.73 \pm 275.28$ cm^3 **17** 339.95 ± 7.74 cm^3
18 volume $= \frac{4}{3}\pi r^3$, surface area $= 4\pi r^2$
The rounding will have more effect on the volume, as the error is multiplied through 3 times rather than twice.
19 **a** 36.69 ± 1.15 cm^3 **b** 73.48 ± 1.54 cm^2

EXERCISE 1D

1 **a** €2460, $\approx 0.180\%$ **b** 467 people, $\approx 1.48\%$
c \$1890, $\approx 0.413\%$ **d** 189 cars, $\approx 6.72\%$
2 **a** 1.238 kg, $\approx 19.8\%$ **b** 2.4 m, $\approx 2.46\%$
c 3.8 L, $\approx 16.0\%$ **d** 22 hours, $\approx 30.6\%$
3 **a** 99.91 m^2 **b** 100 m^2 **c** 0.09 m^2, $\approx 0.0901\%$
4 **a** 3254.224 cm^3 **b** 3240 cm^3
c 14.224 cm^3, $\approx 0.437\%$
5 **a** 72 m^2 **b** \$6120 **c** 77.08 m^2 **d** $\approx 6.59\%$
e no **f** \$7650
6 **a** 5 m^2 **b** 1.7 m **c** 5.1 m^2 **d** 2%
7 **a** 65.25 km h^{-1} **b** 4.75 km h^{-1}, $\approx 7.28\%$
8 **b** **Hint:** Let the measure have the value 1.
c $\approx 1.50 \times 10^{-4}\,\%$
9 **a** ≈ 3.22 m^2 **b** $3.0375 < A < 3.4075$ **c** $\approx 6.01\%$
10 **a** ≈ 251 cm^3 **b** $173 < V < 350$ **c** $\approx 45.1\%$
11 **a** ≈ 4.40 hours ($\approx$ 4 h 24 min 5 s)
b **i** ≈ 45.7 s **ii** $\approx 0.287\%$

REVIEW SET 1A

1 **a** 7400 **b** 32 200 **c** 10 500 **d** 409 000
2 **a** 2.72 **b** 2.718 28 **c** 2.718 281 83
3 **a** ≈ 5.20 **b** ≈ 16.6 **c** ≈ 0.0289
4 **a** 350 **b** 1800 **c** 6
5 **a** 200
b Less, as in our one figure approximation we increased the numerator and decreased the denominator.
6 32.5 m $< P <$ 35.5 m
7 **a** accurate to $\pm\frac{1}{2}$ cm **b** 35.5 cm to 36.5 cm
c 1260.25 cm^2 $< A <$ 1332.25 cm^2
8 **a** \$590, $\approx 22.8\%$ **b** 0.109 cm, $\approx 0.417\%$
c 386 people, $\approx 8.80\%$
9 $\approx 0.0811\%$
10 **a** $\pi \times 1.4^2 \approx 6.16$ m^2 **b** $\pi \times 1.5^2 \approx 7.07$ m^2
c ≈ 0.911 m^2, $\approx 14.8\%$

REVIEW SET 1B

1 **a** 74 820 **b** 74 800 **c** 75 000
2 19.1 customers per day **3** **a** 6.41 **b** 6.406
4 **a** 2100 **b** 48 **c** 3

5 **a** £431.20 **b** £450 **c** $\approx 4.36\%$

6 **a** $14.85 \text{ s} < t < 14.95 \text{ s}$ **b** $6.66 \text{ m s}^{-1} < s < 6.77 \text{ m s}^{-1}$

7 $267.5 \text{ cm}^2 < A < 355.5 \text{ cm}^2$

8 **a** ≈ 175 cm **b** ≈ 2.87 cm, $\approx 1.61\%$

9 **a** **i** 2 m **ii** 2.24 m **iii** 2.236 m

b **i** 1 **ii** 3 **iii** 4 **c** 2.24 m and 2.236 m

10 **a** $\approx 4.51\%$ **b** $\approx 1.32\%$ **c** $\approx 0.0507\%$

d $\approx 0.0402\%$ **e** $\approx 8.49 \times 10^{-6}\,\%$

11 **a** $\approx 38.5 \text{ cm}^2$ **b** $37.4 < A < 39.6 \text{ cm}^2$ **c** $\approx 2.92\%$

EXERCISE 2A

1 **a** \$232 **b** \$13 920 **c** \$1920

2 **a** £282.60 **b** £10 173.60 **c** £673.60

3 **a** \$1036.80 **b** \$2610 **4** €1903.28

5 **a** Balance Bank **b** Cash Credit Union

c If Becky is able to afford the larger repayments, she should choose Cash Credit Union as she will pay less interest.

6 **b** \$27 509.01

c **i** \$200 **ii** \$186.21

The balance of the loan is less in month 6 which means the interest paid will also be less.

d **i** \$4.02 **ii** \$6497.40 (\$6498 using technology)

e The monthly repayment was rounded up, so every month the payments have reduced the balance by a little extra.

7 **a** \$148.64 **b** \$729.28

8 **a** £490.61 **b** £15 598.73

9 **a** \$8500 **b** **i** \$518.58 **ii** \$1871.60 **iii** \$5492.57

10 **a** 6.30% p.a. **b** \$395.65

11 **a** **i** \$789.19 **ii** \$512.92 **iii** \$395.92

b The 3 year loan charges the least interest of \$3410.84 as more is paid off each month and therefore less interest is charged.

12 **a** €386.90 **b** €5214 **c** €10 169.13

d Ally pays more interest in the first $2\frac{1}{2}$ years than in the second $2\frac{1}{2}$ years.

13 **a** \$1827.33 **b** \$188 559.20 **c** \$162 745.03

d **i** \$3165.28 **ii** \$159 196.40 **iii** \$29 362.80

EXERCISE 2B

1 **a** 25 years 4 months **b** \$3693.84

2 **a** €3163.24 **b** €413 500.41

3 **a** 12 years 7 months **b** 3 years 1 month longer

4 **a** £5614.06

b No, he can only afford to spend £5614.06 per month. Otherwise his money will run out before he turns 84.

5 **a** \$1 094 748.09

b **i** \$8600.27 **ii** 11 years 11 months

6 **a** £11 512.29 **b** £394 007.62 **c** £1312.64

7 **a** \$4500 × 12 × 20 = \$1 080 000

b Maggie will earn interest on the money in the annuity account as she makes her regular withdrawals.

c \$618 117.53

8 The money will last forever.

9 **a** 7.19% **b** **i** 2 years 10 months **ii** €679.24

10 **a** \$5121.03 **b** \$322 605.07 **c** \$6708.44

REVIEW SET 2A

1 **a** \$455.43 **b** \$27 325.80 **c** \$4325.80

2 **a** €157.24 **b** €1086.93

3 **a** \$2884.74

b Total repayments = \$2884.74 × 12 × 25 = \$865 422

Total interest charged = \$865 422 − \$410 000 = \$455 422

4 **a** 8 years 7 months **b** \$2996.23

5 **a** €7861.43 **b** 14 years 3 months **c** €727 698.90

6 **a** \$799 813.28 **b** \$314 877.35

REVIEW SET 2B

1 **a** \$279.08 **b** \$1395.84

2 **a** \$17 500 **b** \$1260.97 **c** \$2675.52 **d** \$9347.67

3 **a** **i** 11 742.52 pesos **ii** 8286.45 pesos

b The 4 year loan charges the least interest of 63 640.96 pesos as more is paid off each month and therefore less interest is charged.

4 **a** An annuity fund is an investment where an individual makes a lump-sum deposit, and then makes regular *withdrawals* from the account. We have previously considered compound interest investments that make regular *deposits* into an account.

b Diane is technically correct, but she will be able to withdraw more than £2000 per month since the money in the fund will earn interest.

c £3167.02

5 **a** €2467.29 **b** €448.52

6 **a** 4.90% p.a. **b** 6 years 7 months

EXERCISE 3A

1 **a**, **d**, and **e** are functions, since in each case, no two different ordered pairs have the same x-coordinate.

2 **a** Is a function, since for any value of x there is at most one value of y.

b Is a function, since for any value of x there is at most one value of y.

c Is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2$, $y = \pm\sqrt{5}$.

3 **a** function **b** function **c** function

d not a function **e** not a function **f** function

g function **h** not a function

4 Not a function as a 2 year old child could pay \$0 or \$20.

5 No, because a vertical line (the y-axis) would cut the relation more than once.

6 No. A vertical line is not a function. It will not pass the "vertical line" test.

7 **a** $y^2 = x$ is a relation but not a function.

$y = x^2$ is a function (and a relation).

$y^2 = x$ has a horizontal axis of symmetry (the x-axis).

$y = x^2$ has a vertical axis of symmetry (the y-axis).

Both $y^2 = x$ and $y = x^2$ have vertex $(0, 0)$.

$y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin *or* $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.

b **i** The part of $y^2 = x$ in the first quadrant.

ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.

8 **a** Both curves are functions since any vertical line will cut each curve at most once.

b $y = \sqrt[3]{x}$

EXERCISE 3B

1 **a** 2 **b** 8 **c** -1 **d** -13 **e** 1

2 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$

3 **a** -3 **b** 3 **c** 3 **d** -3 **e** $\frac{15}{2}$

4 **a** **i** 1 **ii** -1 **b** $x = -4$

5 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$ **b** $x = 4$ **c** $x = \frac{9}{5}$

6 **a** $f(-1) = 1 - 3(-1) = 4$ **b** $x = -\frac{2}{3}$
$g(11) = \sqrt{11+5} = 4 = f(-1)$

7 **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $7 - 6a$
e $1 - 3x$ **f** $7 - 3x - 3h$

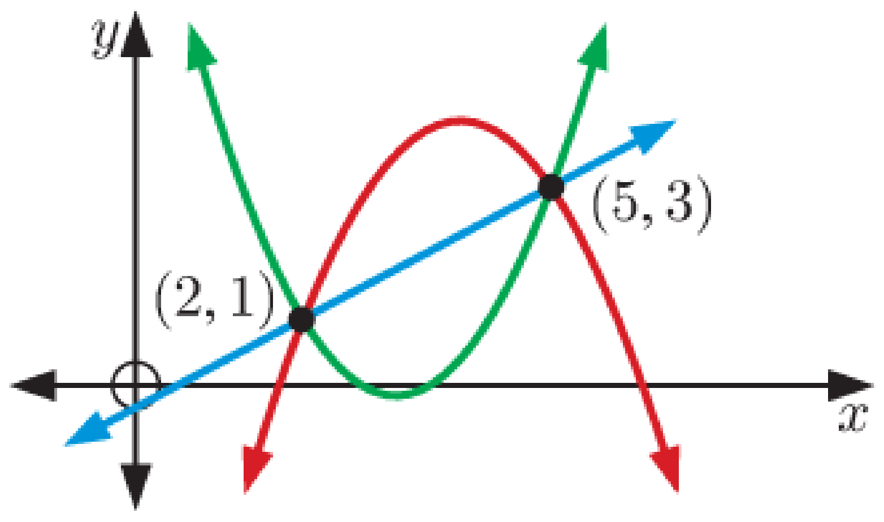
8 **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
c $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
e $18x^2 + 9x - 1$ **f** $2x^2 + (4h+3)x + 2h^2 + 3h - 1$

9 **a** $9x^2$ **b** $\dfrac{x^2}{4}$ **c** $3x^2$ **d** $2x^2 - 4x + 7$

10 **a** $-\dfrac{1}{x}$ **b** $\dfrac{2}{x}$ **c** $\dfrac{2+3x}{x}$ **d** $\dfrac{2x+1}{x-1}$

11 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

12 **Note:** Other answers are possible.



13 $f(x) = -2x + 5$

14 **a** $P(3) = 35$; there are 35 L of petrol in the tank after 3 minutes.
b $t = 4.5$; after $4\frac{1}{2}$ minutes there are 50 L of petrol in the tank.
c 5 L

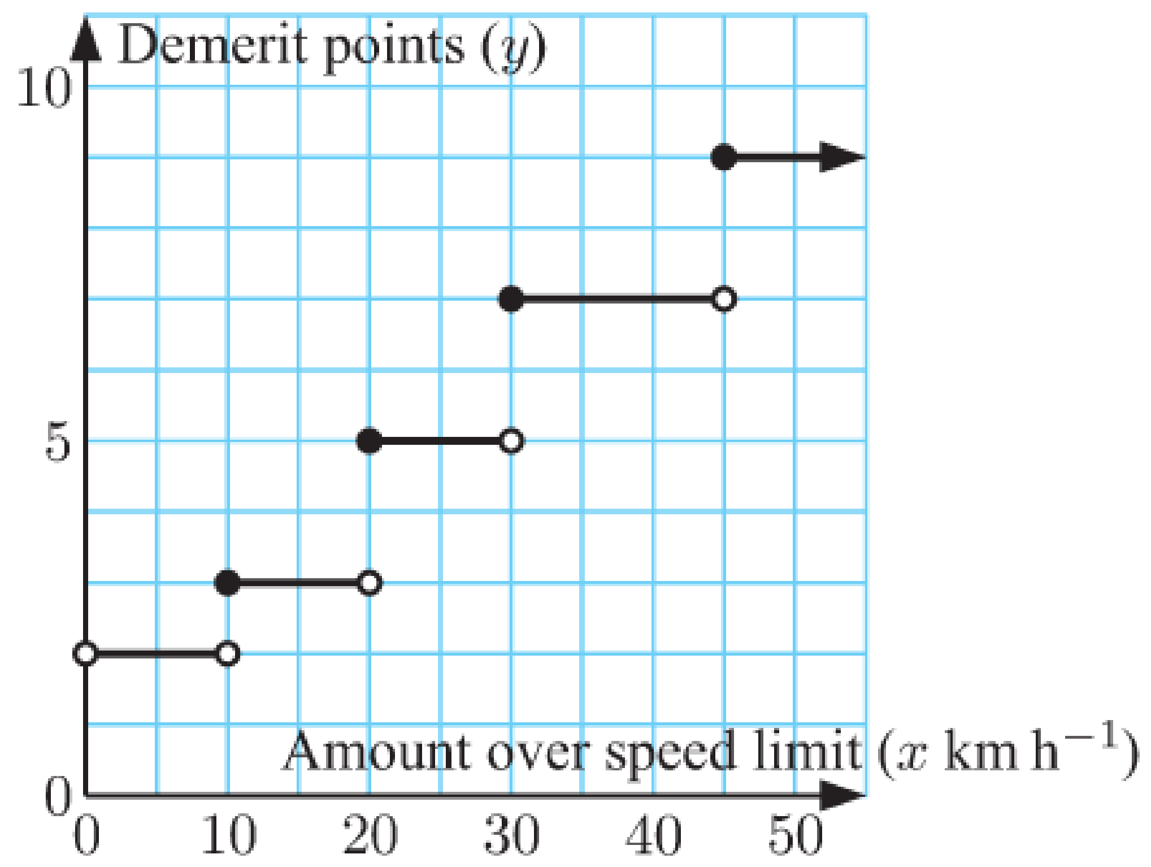
15 **a** $H(30) = 800$; after 30 minutes the balloon is 800 m high.
b $t = 20$ or 70; after 20 minutes and after 70 minutes the balloon is 600 m high.
c $0 \leqslant t \leqslant 80$ **d** 0 m to 900 m

16 $a = 3$, $b = -2$ **17** $a = 3$, $b = -1$, $c = -4$

18 **a** $V(4) = 5400$; $V(4)$ is the value of the photocopier in pounds after 4 years.
b $t = 6$; after 6 years the value of the photocopier is £3600.
c £9000 **d** $0 \leqslant t \leqslant 10$

EXERCISE 3C

1 **a**



b Domain is $\{x \mid x > 0\}$, Range is $\{2, 3, 5, 7, 9\}$

2 **a** At any moment in time there can be only one temperature, so the graph is a function.
b Domain is $\{t \mid 0 \leqslant t \leqslant 30\}$, Range is $\{T \mid 15 \leqslant T \leqslant 25\}$

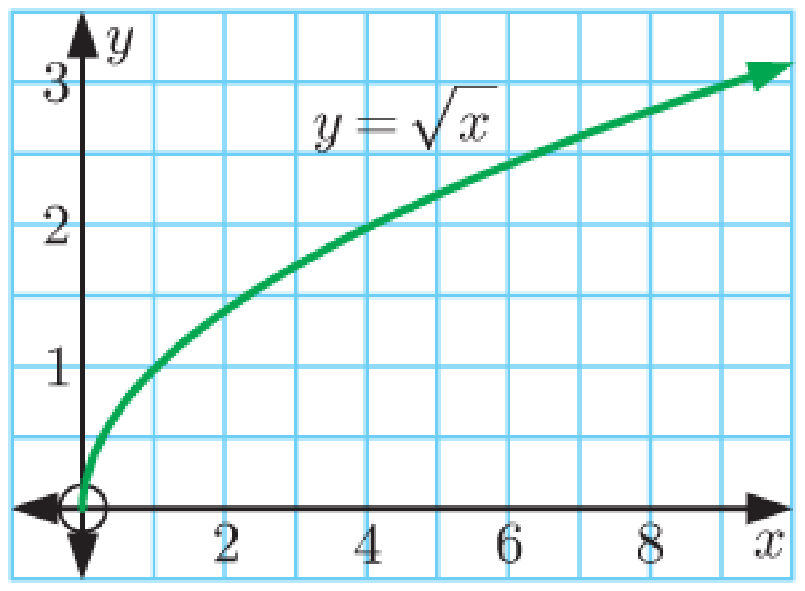
3 **a** Domain is $\{x \mid x \geqslant -1\}$, Range is $\{y \mid y \leqslant 3\}$
b Domain is $\{x \mid -1 < x \leqslant 5\}$, Range is $\{y \mid 1 < y \leqslant 3\}$
c Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq -1\}$
d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 2\}$
e Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -1\}$
f Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant \frac{25}{4}\}$
g Domain is $\{x \mid x \geqslant -4\}$, Range is $\{y \mid y \geqslant -3\}$
h Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -2\}$
i Domain is $\{x \mid x \neq \pm 2\}$,
Range is $\{y \mid y \leqslant -1$ or $y > 0\}$

4 **a** true **b** false **c** true **d** true

5 **a** $\{x \mid x \geqslant 0\}$ **b**

x	0	1	4	9	16
$f(x)$	0	1	2	3	4

c



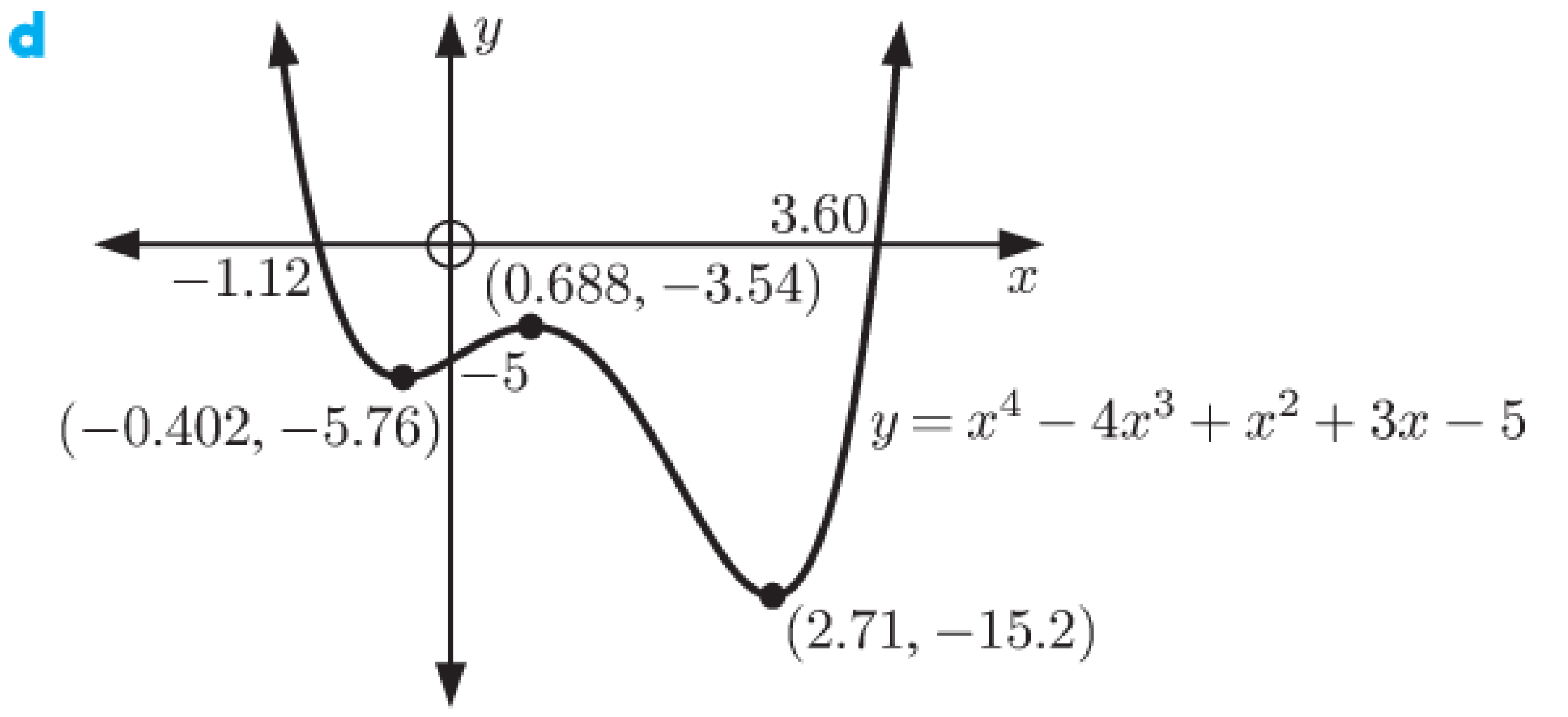
d $\{y \mid y \geqslant 0\}$

6 **a** Domain is $\{x \mid x \geqslant -6\}$, Range is $\{y \mid y \geqslant 0\}$
b Domain is $\{x \mid x \neq 0\}$, Range is $\{y \mid y > 0\}$
c Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 0\}$
d Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y < 0\}$
e Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq 0\}$
f Domain is $\{x \mid x \leqslant 4\}$, Range is $\{y \mid y \geqslant 0\}$

EXERCISE 3D

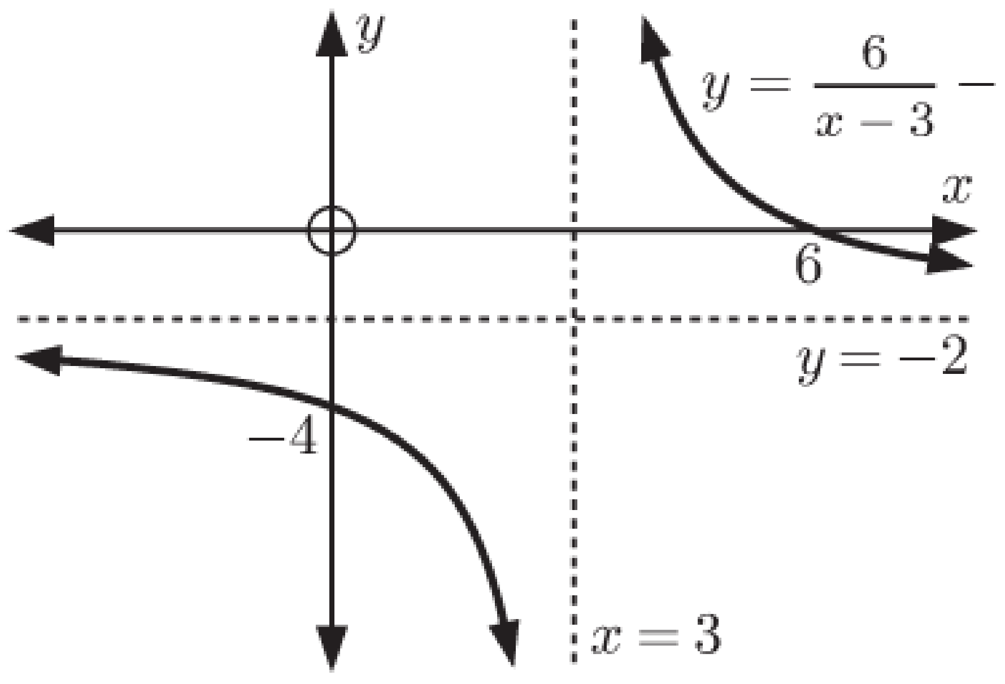
1 **a** -12
b **i** ≈ -1.05, 1.84, and 6.20
ii local minimum $(0.225, -12.3)$,
local maximum $(4.44, 25.1)$

2 **a** x-intercepts are ≈ -1.12 and ≈ 3.60, y-intercept is -5
b local maximum $(0.688, -3.54)$,
local minimum $(-0.402, -5.76)$ and $(2.71, -15.2)$
c as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to \infty$
d



e Range is $\{y \mid y \geqslant -15.2\}$

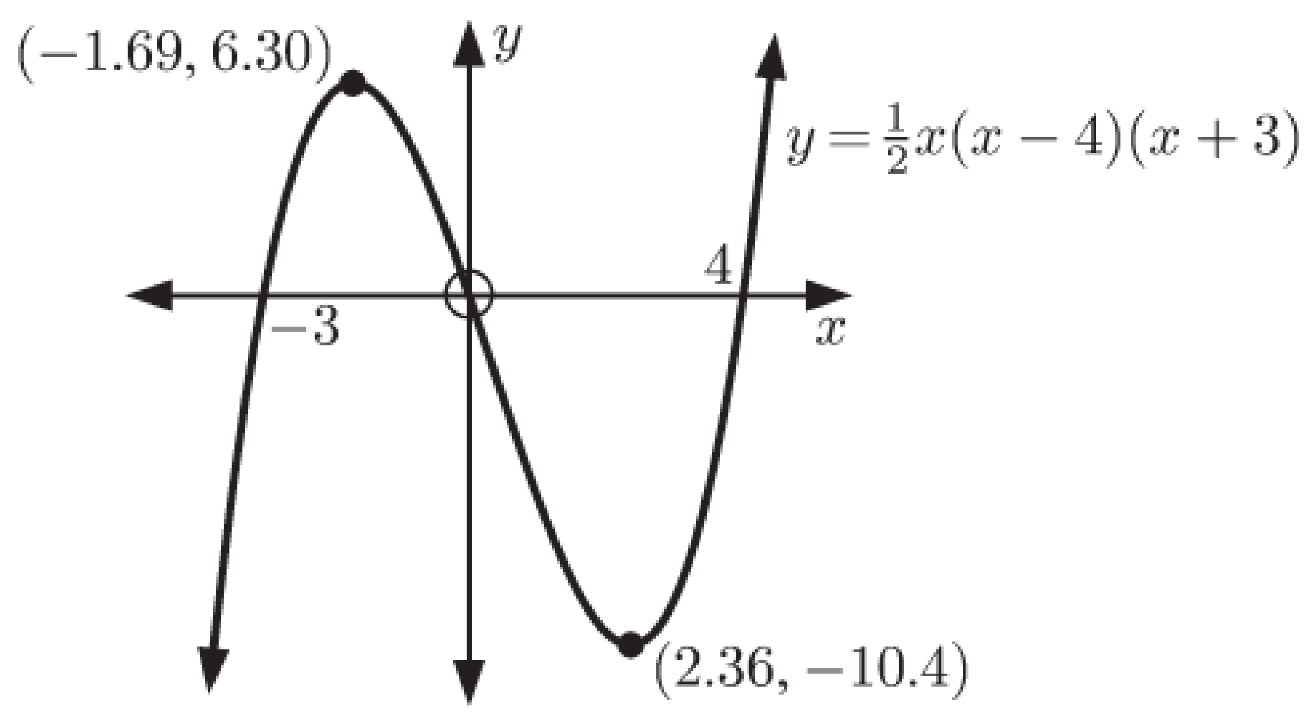
3 **a** x-intercept is 6, y-intercept is -4
b horizontal asymptote $y = -2$, vertical asymptote $x = 3$
c



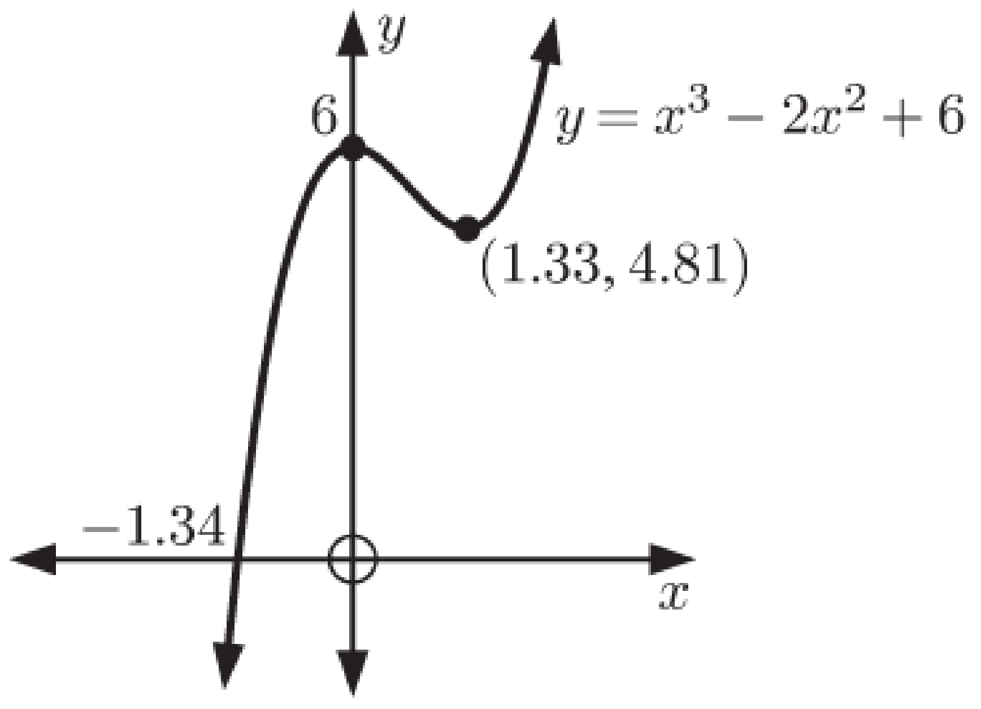
d Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq -2\}$

4 **a** **i** x-intercepts -3, 0, and 4, y-intercept 0

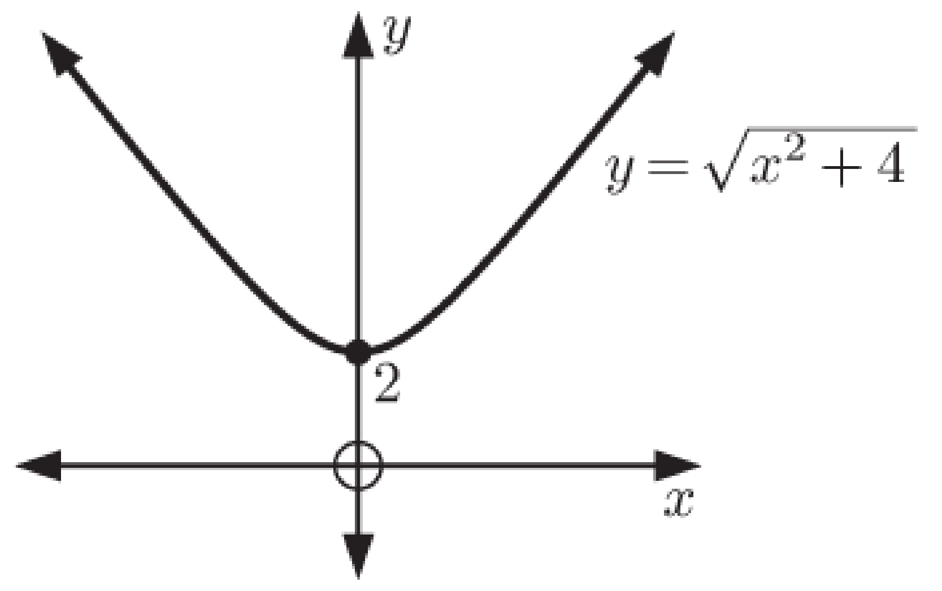
ii max. turning point $(-1.69, 6.30)$
min. turning point $(2.36, -10.4)$
iii no asymptotes
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$
v



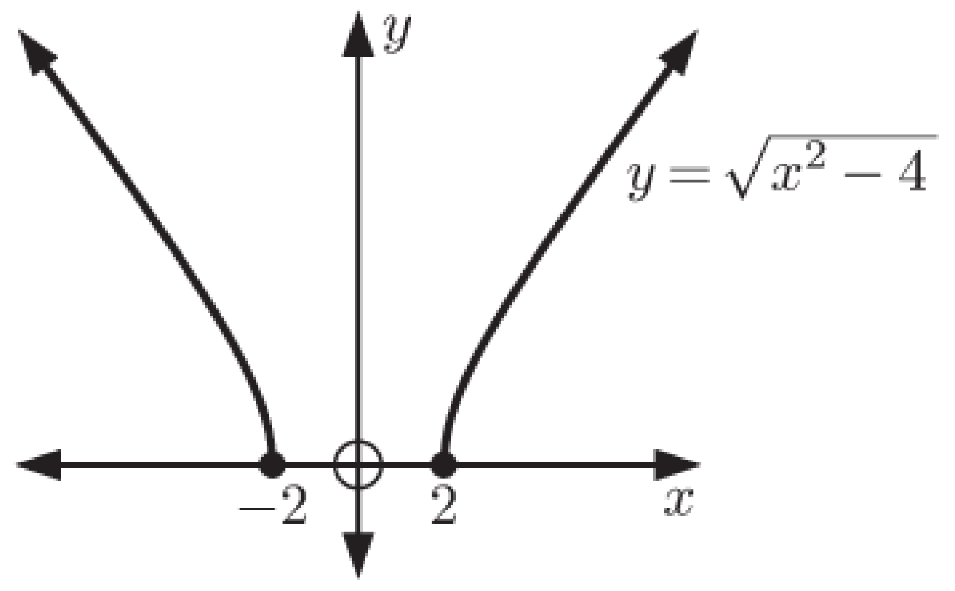
b **i** x-intercept ≈ -1.34, y-intercept 6
ii max. turning point $(0, 6)$
min. turning point $(1.33, 4.81)$
iii no asymptotes
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$
v



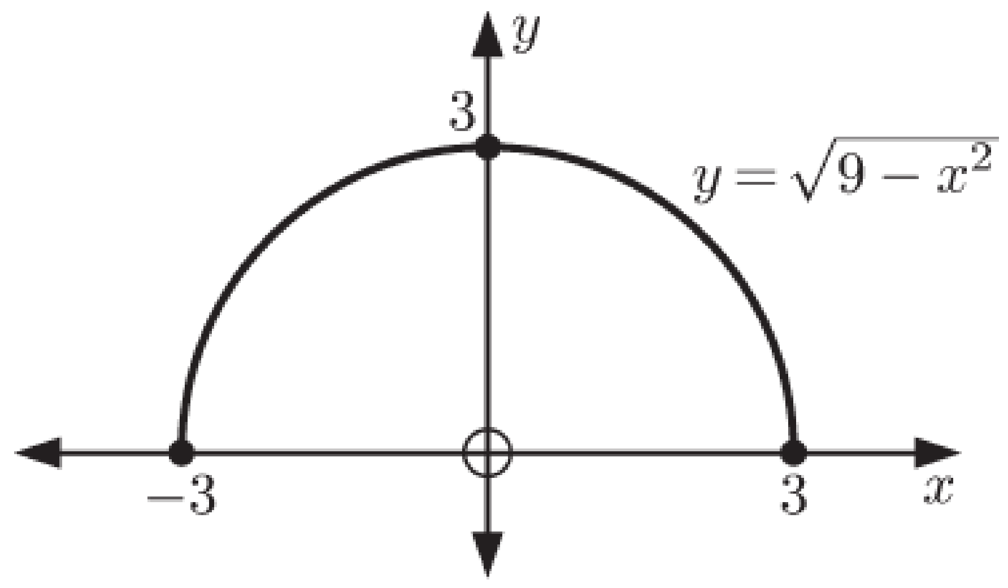
c **i** y-intercept 2, no x-intercepts
ii min. turning point $(0, 2)$ **iii** no asymptotes
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 2\}$
v



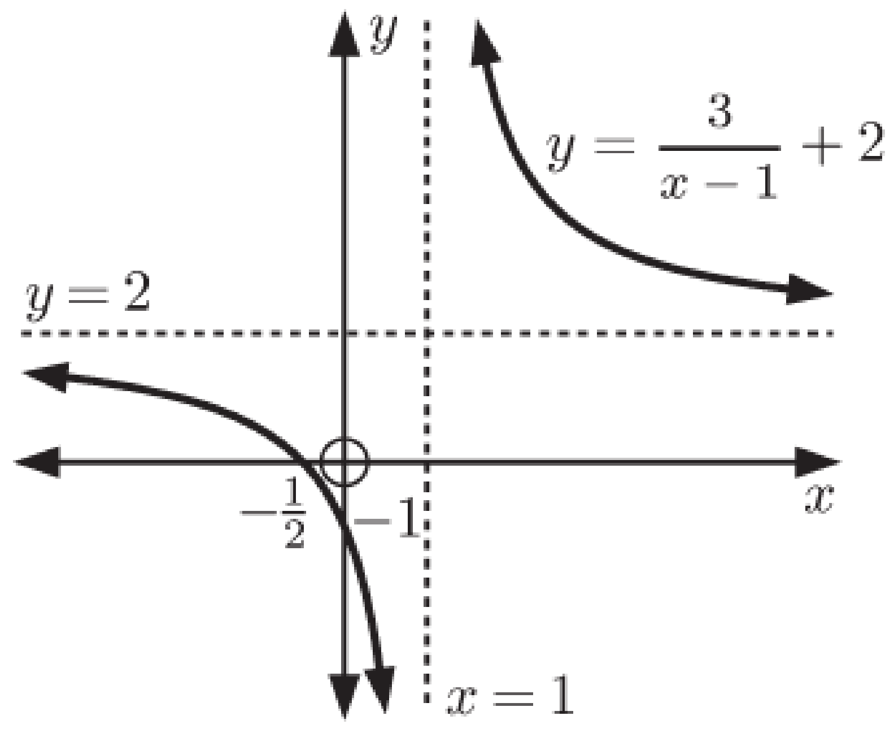
d **i** x-intercepts -2 and 2, no y-intercept
ii no turning points **iii** no asymptotes
iv Domain is $\{x \mid x < -2 \text{ or } x > 2\}$,
Range is $\{y \mid y > 0\}$
v



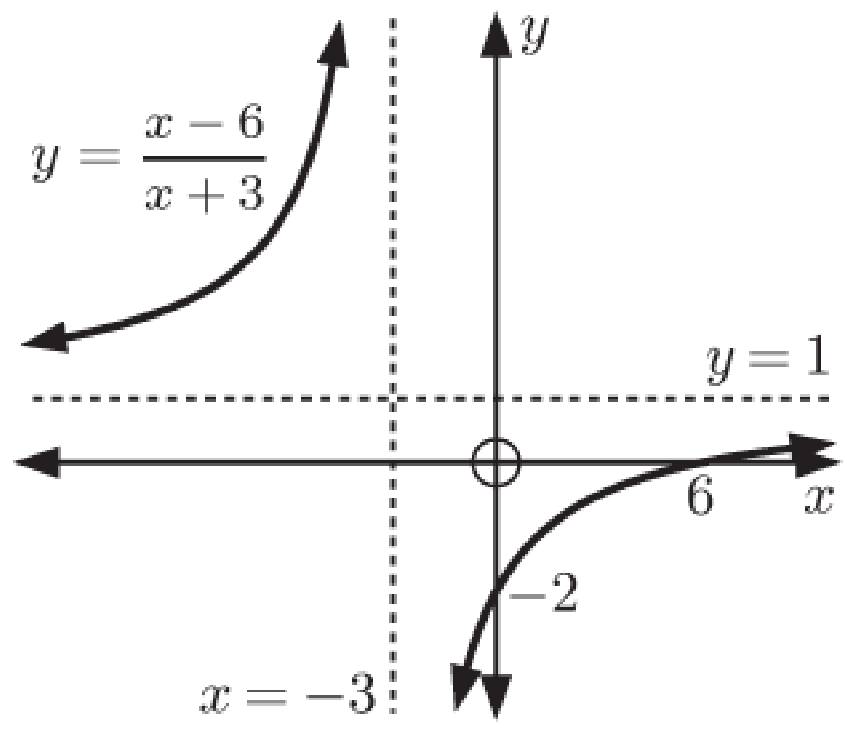
e **i** x-intercepts -3 and 3, y-intercept 3
ii max. turning point $(0, 3)$ **iii** no asymptotes
iv Domain is $\{x \mid -3 \leqslant x \leqslant 3\}$,
Range is $\{y \mid 0 \leqslant y \leqslant 3\}$
v



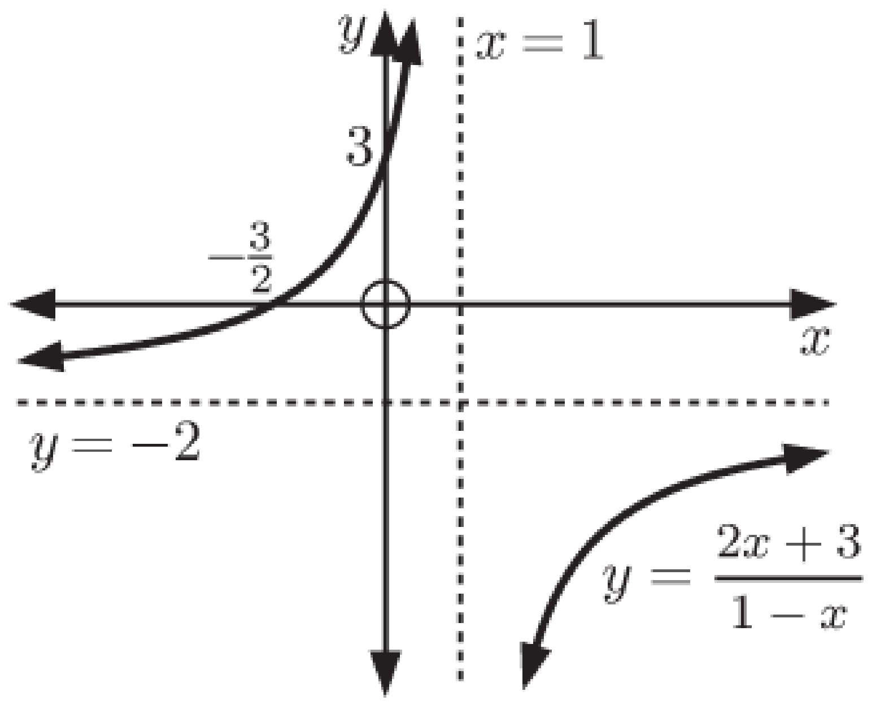
f **i** x-intercept $-\frac{1}{2}$, y-intercept -1
ii no turning points
iii vertical asymptote $x = 1$, horizontal asymptote $y = 2$
iv Domain is $\{x \mid x \neq 1\}$, Range is $\{y \mid y \neq 2\}$
v



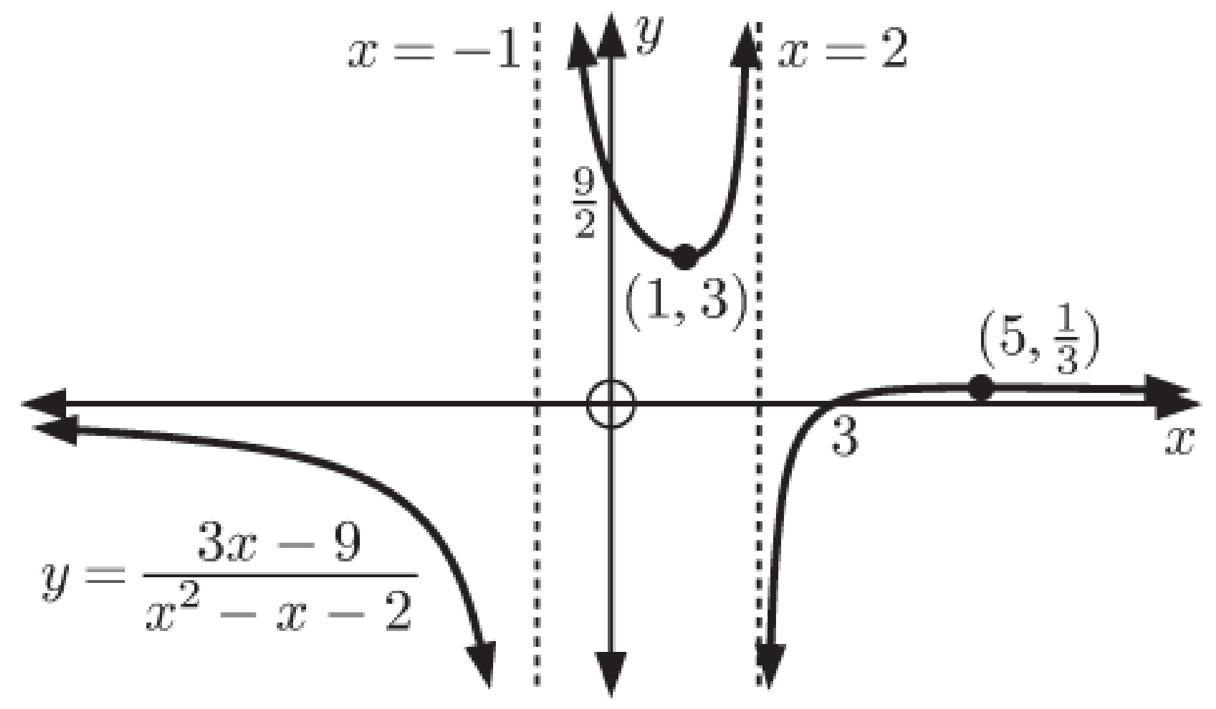
g **i** x-intercept 6, y-intercept -2 **ii** no turning points
iii vertical asymptote $x = -3$,
horizontal asymptote $y = 1$
iv Domain is $\{x \mid x \neq -3\}$, Range is $\{y \mid y \neq 1\}$
v



h **i** x-intercept $-\frac{3}{2}$, y-intercept 3
ii no turning points
iii vertical asymptote $x = 1$,
horizontal asymptote $y = -2$
iv Domain is $\{x \mid x \neq 1\}$, Range is $\{y \mid y \neq -2\}$
v

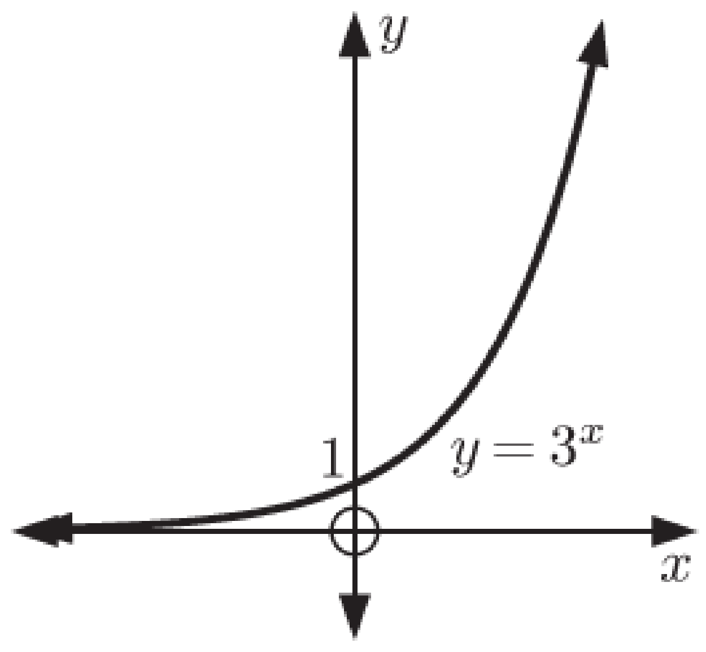


i **i** x-intercept 3, y-intercept $\frac{9}{2}$
ii min. turning point $(1, 3)$, max. turning point $(5, \frac{1}{3})$
iii vertical asymptotes $x = -1$ and $x = 2$,
horizontal asymptote $y = 0$
iv Domain is $\{x \mid x \neq -1 \text{ or } 2\}$,
Range is $\{y \mid y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$
v

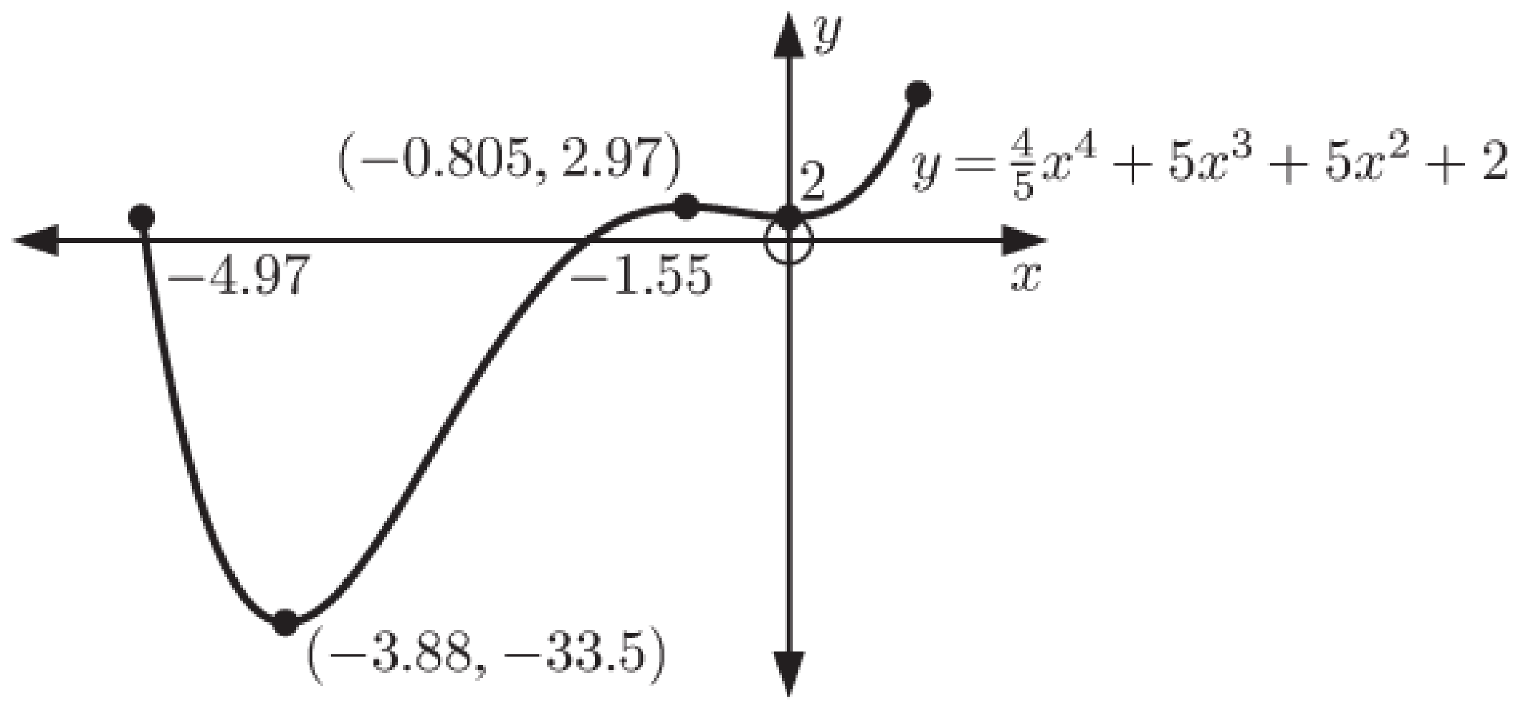


j **i** y-intercept 1, no x-intercepts
ii no turning points
iii horizontal asymptote $y = 0$
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 0\}$

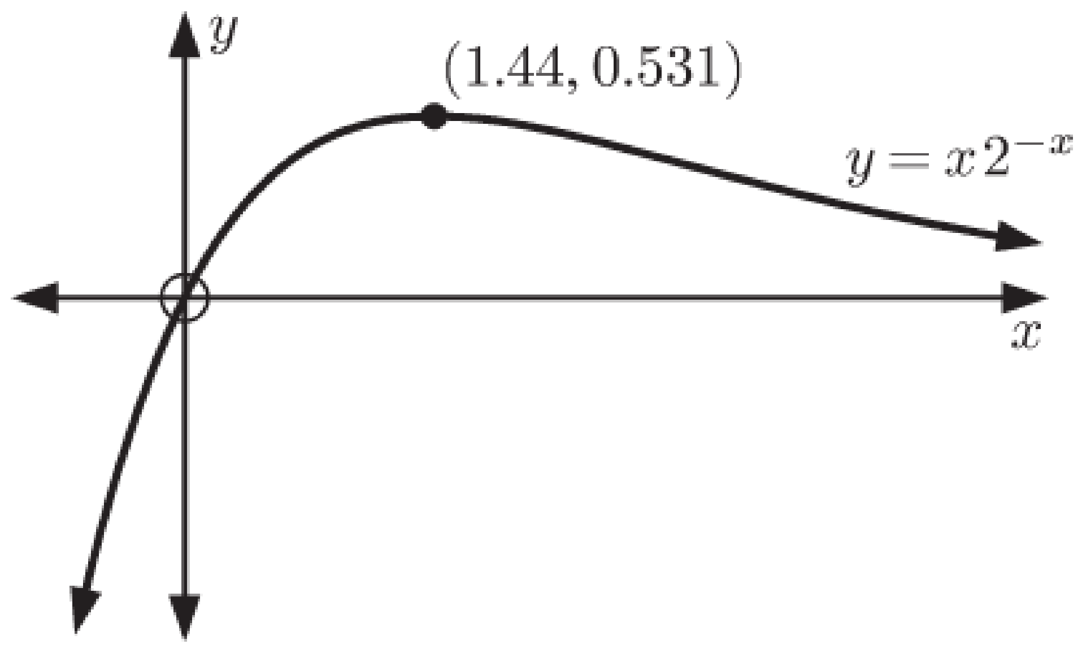
v



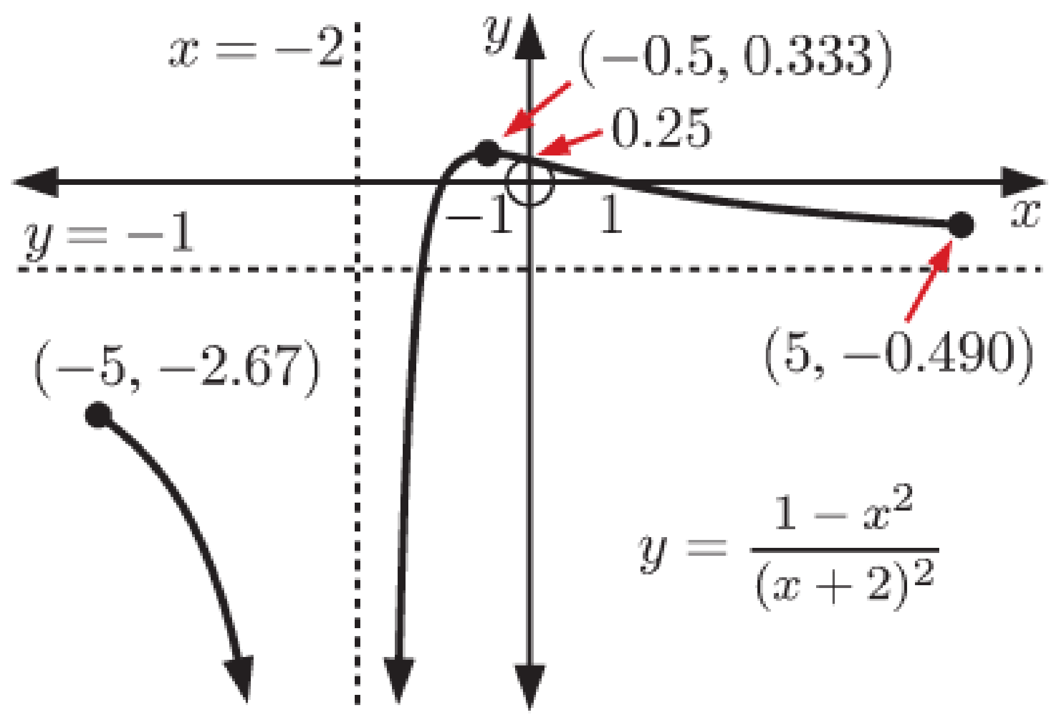
k **i** x-intercepts ≈ -4.97 and ≈ -1.55, y-intercept 2
ii min. turning points $(-3.88, -33.5)$, $(0, 2)$
max. turning point $(-0.805, 2.97)$
iii no asymptotes
iv Domain is $\{x \mid -5 \leqslant x \leqslant 1\}$,
Range is $\{y \mid -33.5 \leqslant y \leqslant 12.8\}$
v



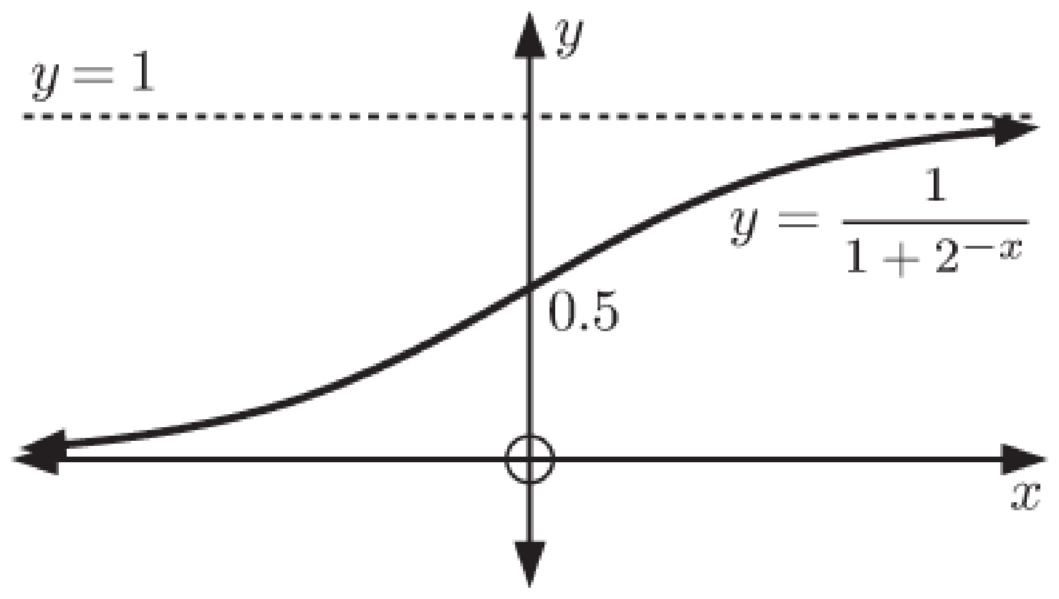
l **i** x-intercept 0, y-intercept 0
ii local maximum at $(1.44, 0.531)$
iii horizontal asymptote $y = 0$
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant 0.531\}$
v



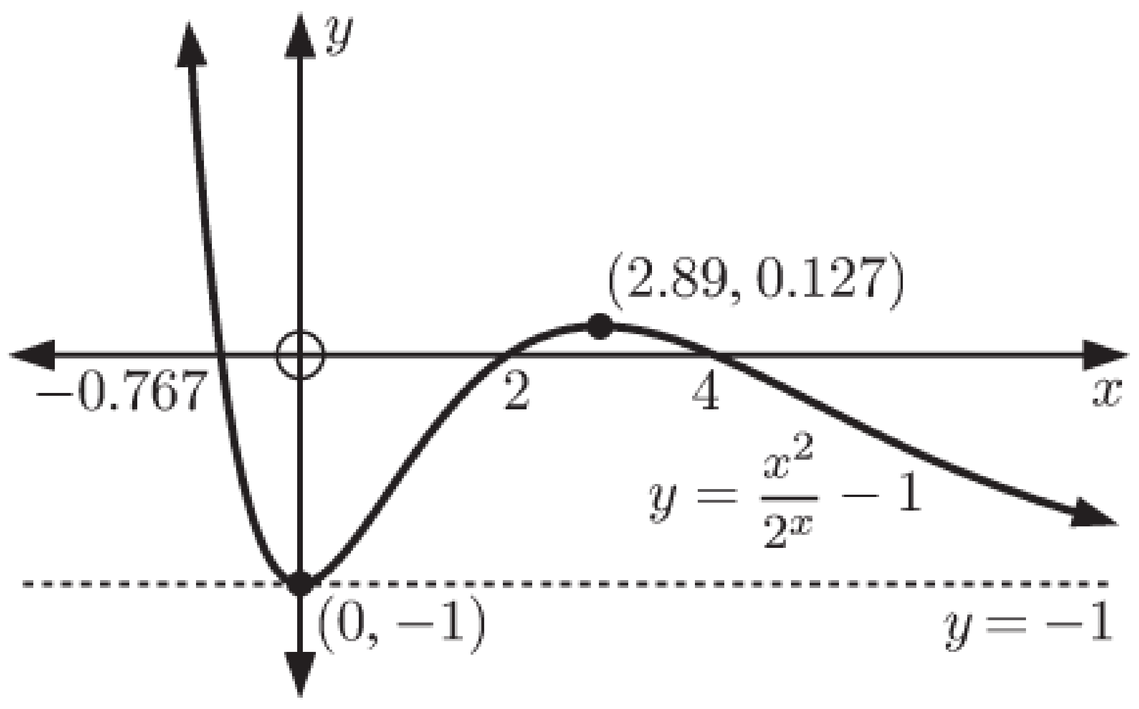
m **i** x-intercepts -1 and 1, y-intercept 0.25
ii local maximum at $(-0.5, 0.333)$
iii vertical asymptote $x = -2$,
horizontal asymptote $y = -1$
iv Domain is $\{x \mid -5 \leqslant x \leqslant 5, x \neq -2\}$,
Range is $\{y \mid y < -1, -1 < y \leqslant 0.333\}$
v



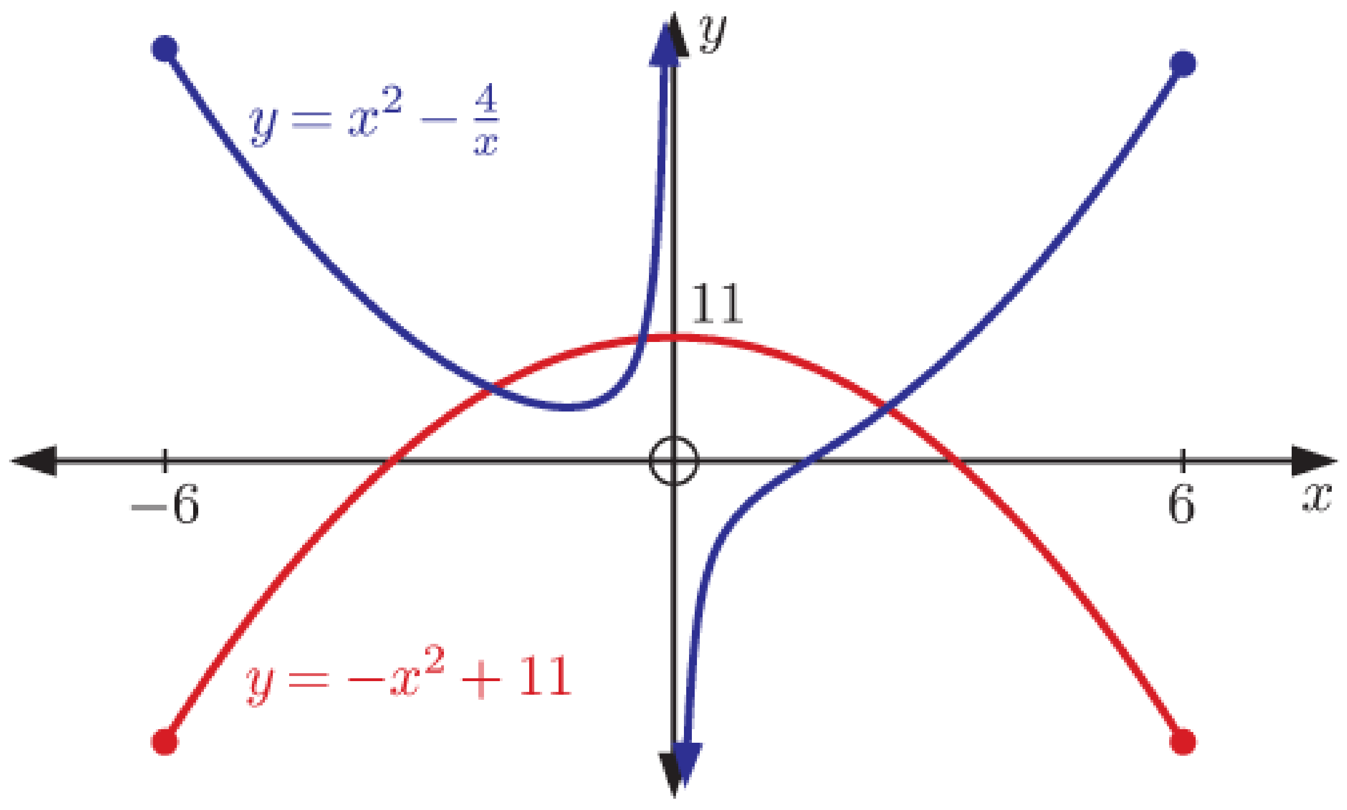
n **i** y-intercept 0.5, no x-intercepts **ii** no turning points
iii horizontal asymptotes $y = 0$ and $y = 1$
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y < 1\}$
v



o **i** x-intercepts ≈ -0.767, 2, and 4, y-intercept -1
ii min. turning point $(0, -1)$
max. turning point $(2.89, 0.127)$
iii horizontal asymptote $y = -1$
iv Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -1\}$
v

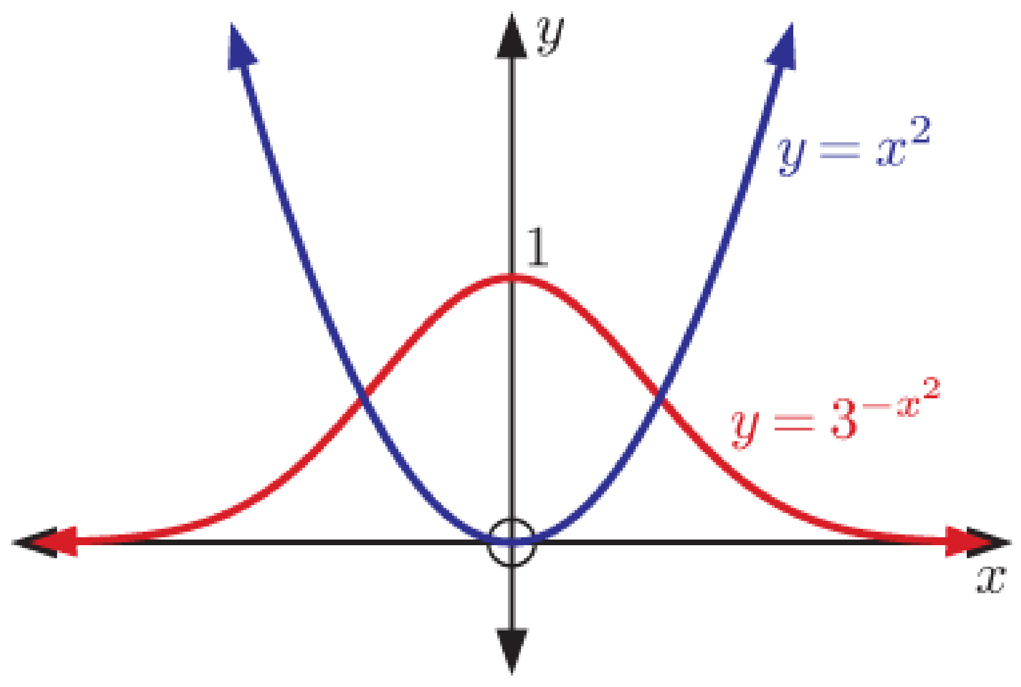


5 **a**



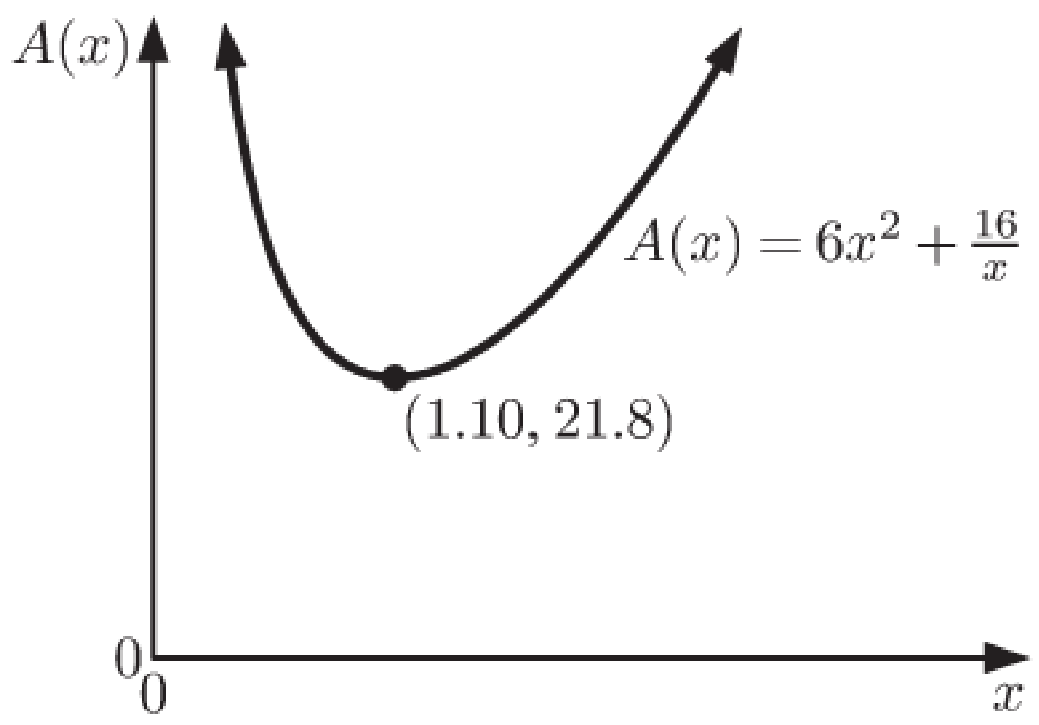
b $x \approx -0.373$ and ≈ -2.14

6 **a**



b no x-intercept, y-intercept 1
max. turning point $(0, 1)$
horizontal asymptote $y = 0$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 1\}$
c $x \approx -0.740$ or $x \approx 0.740$

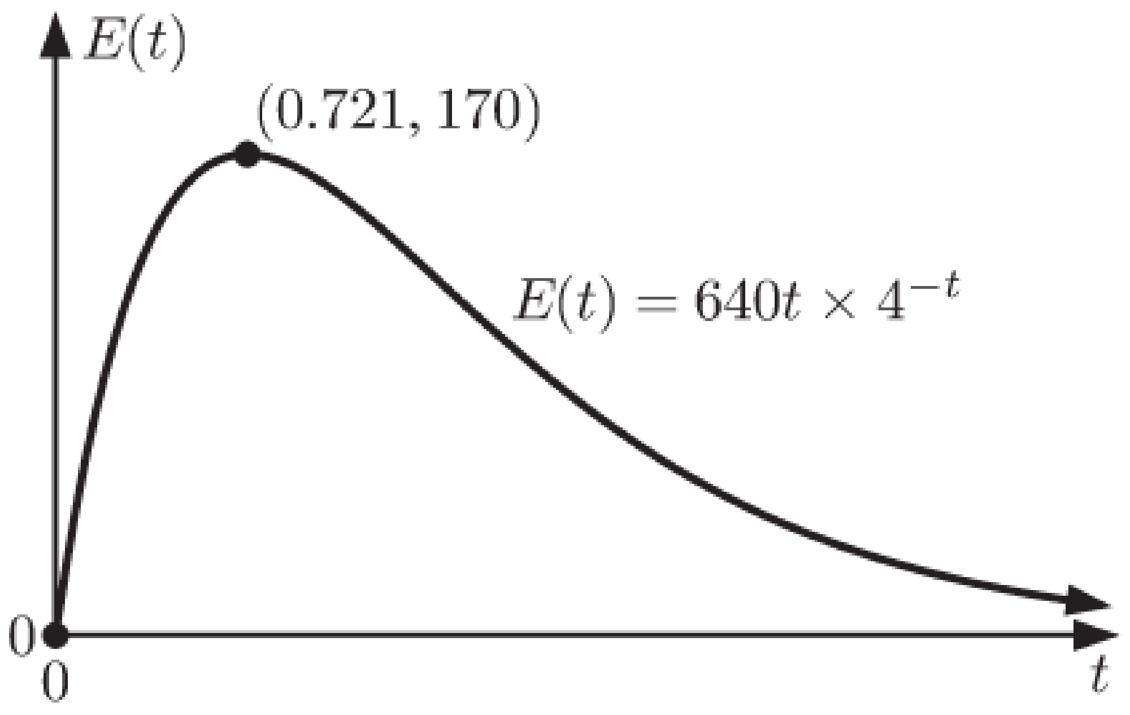
7 **c** vertical asymptote $x = 0$
d



e 32 m^2

f local minimum $(1.10, 21.8)$. The box will have minimum surface area of ≈ 21.8 m^2 when $x \approx 1.10$.

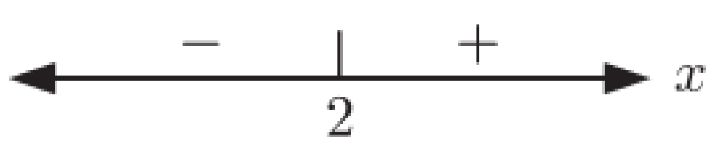
8 **a** $E(1) = 160$, $E(4) = 10$. These are the effects of the injection 1 and 4 hours after it was administered.
b

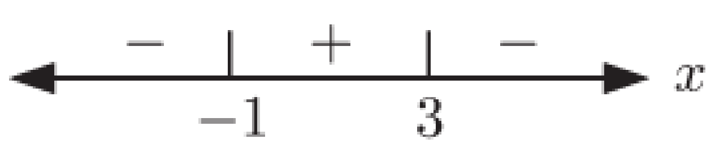


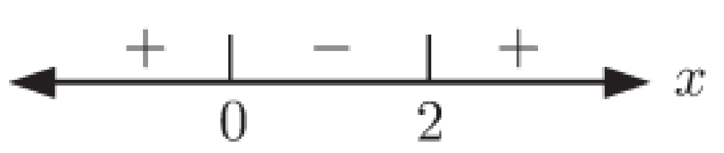
c As $t \to \infty$, $E(t) \to 0^+$; the effectiveness of the drug approaches zero.

d local maximum $(0.721,\ 170)$; at ≈ 0.721 hours after the injection, the drug has reached maximum effect of ≈ 170 units and the effect will now begin to decrease.

EXERCISE 3E

1 **a**


b


c


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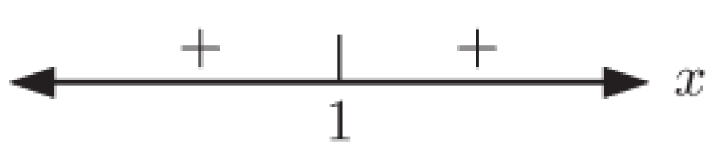

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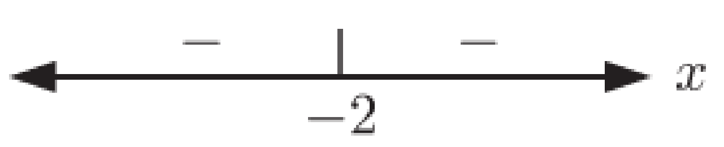

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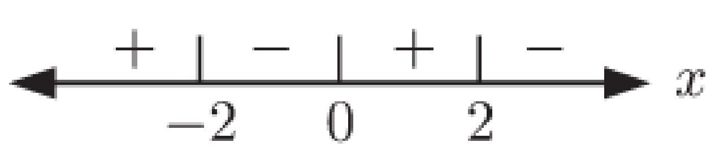

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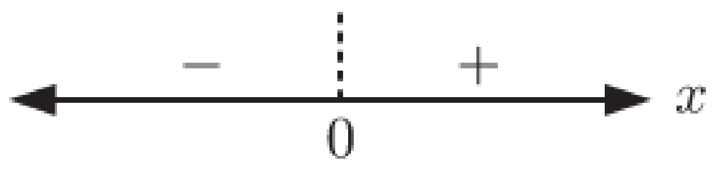

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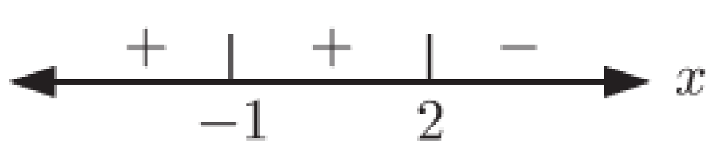

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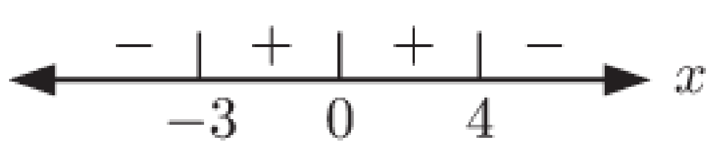

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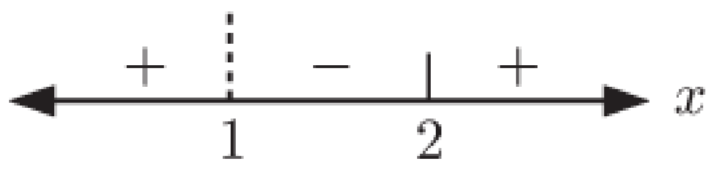

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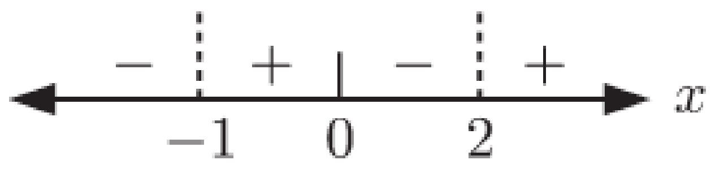

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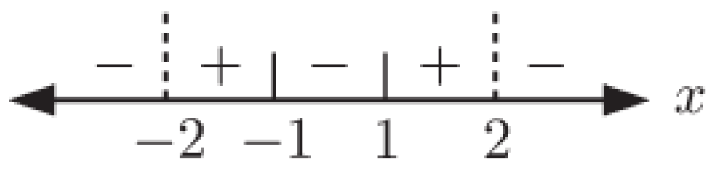

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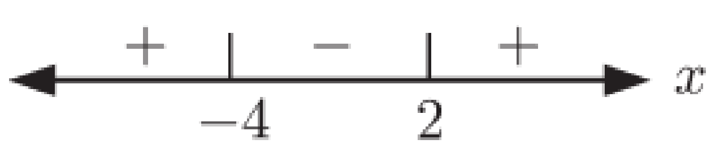

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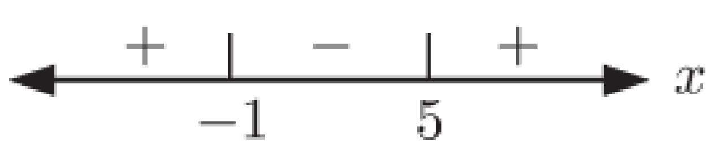

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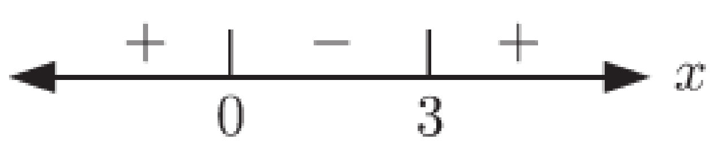

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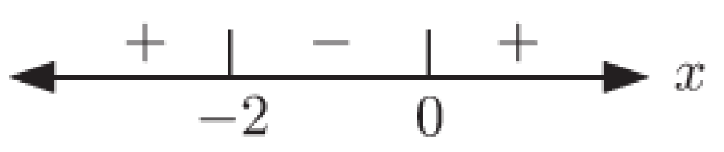

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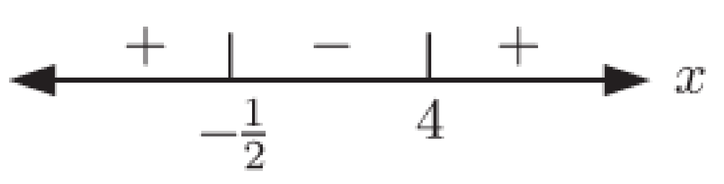

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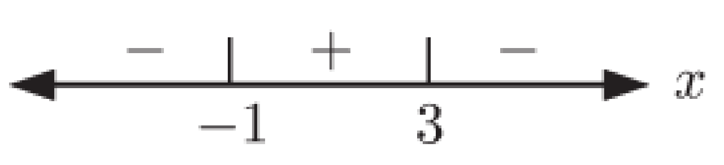

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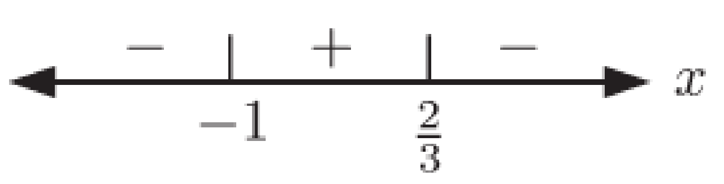

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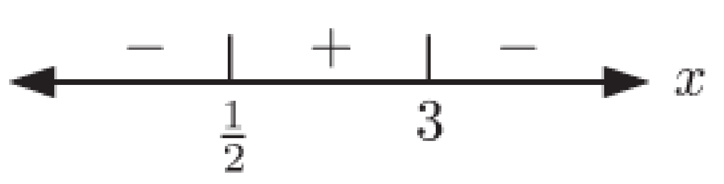

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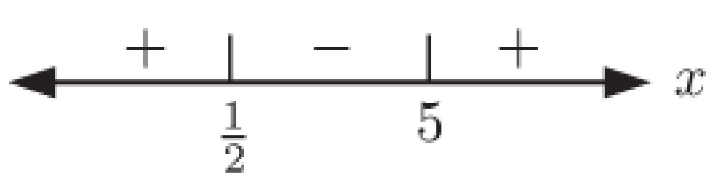

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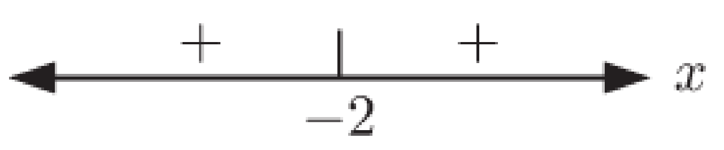

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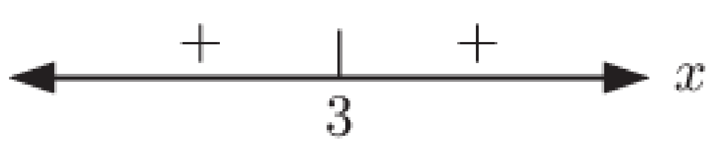

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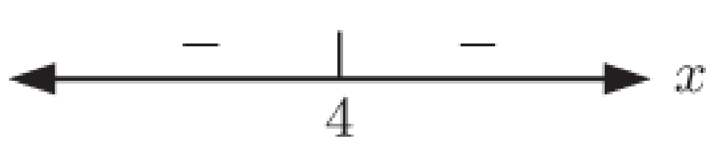

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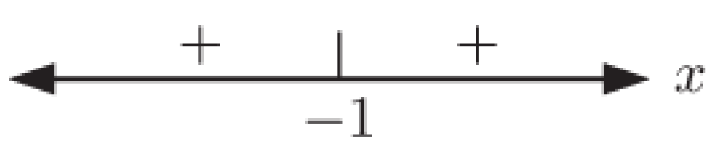

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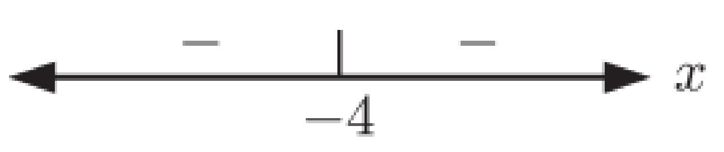

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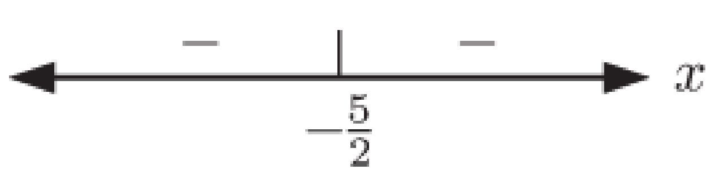

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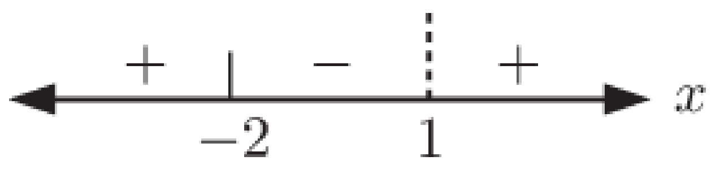

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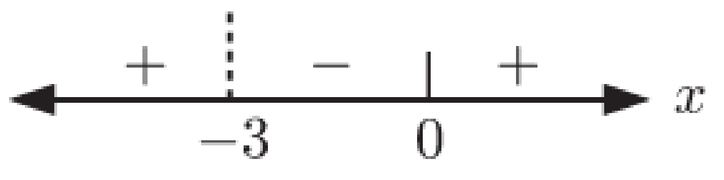

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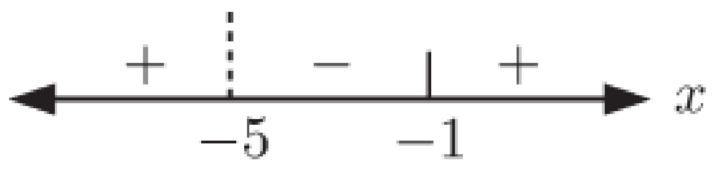

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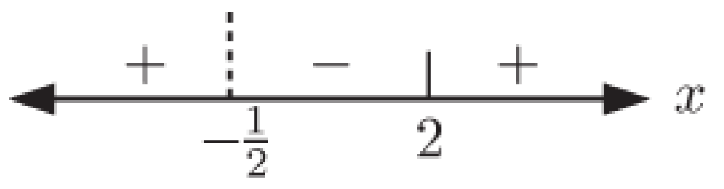

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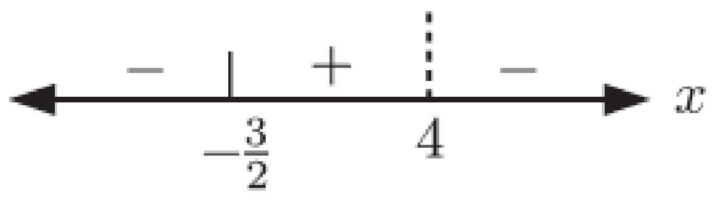

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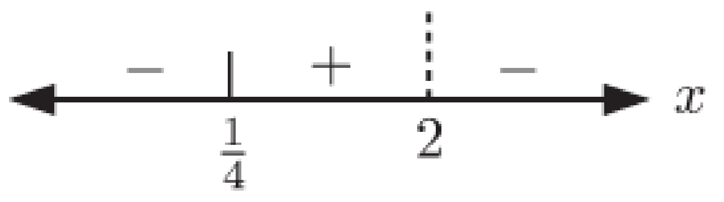

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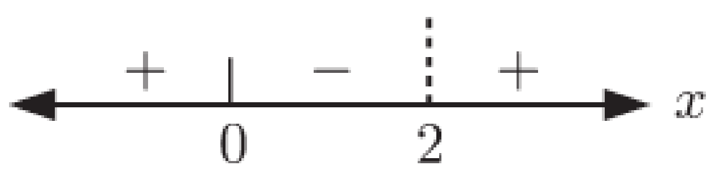

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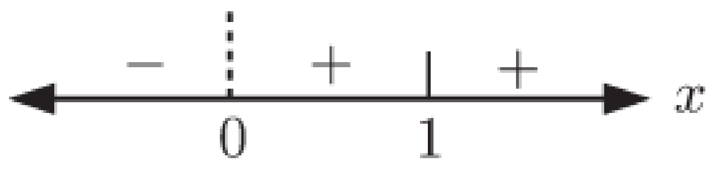

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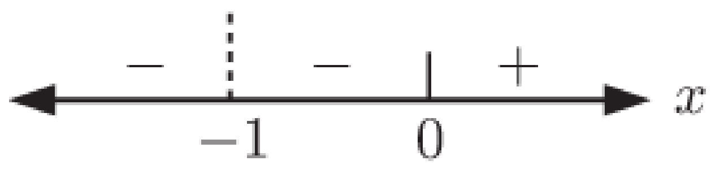

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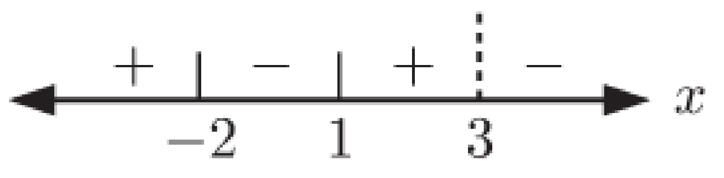

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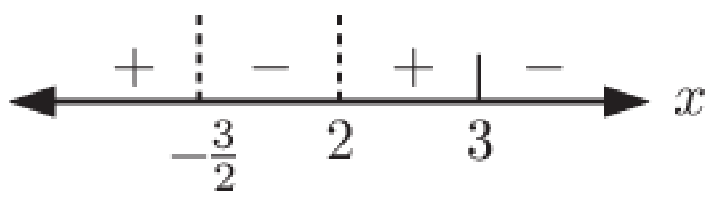

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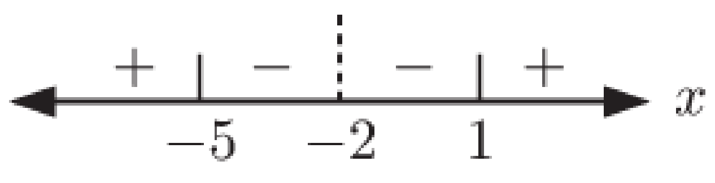
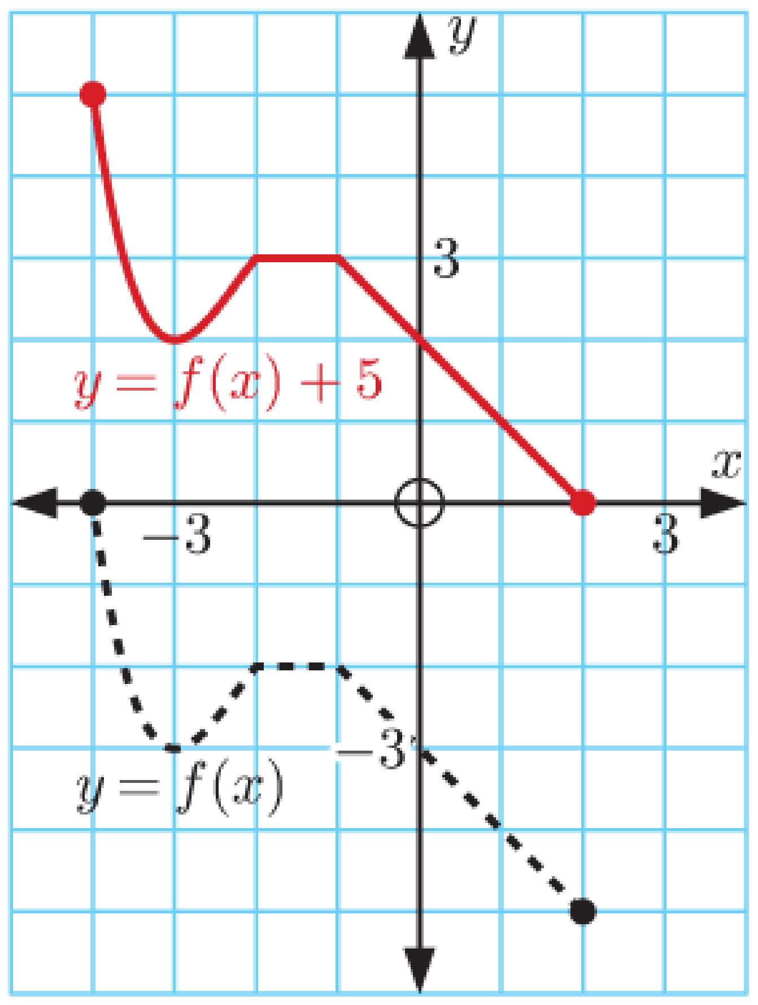


EXERCISE 3F

1 **a**


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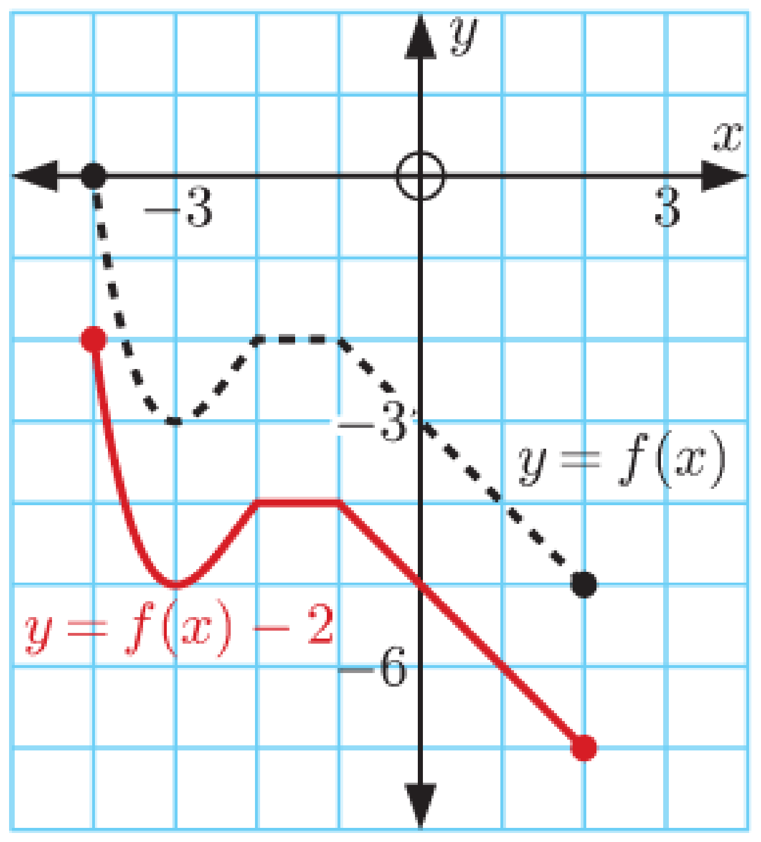

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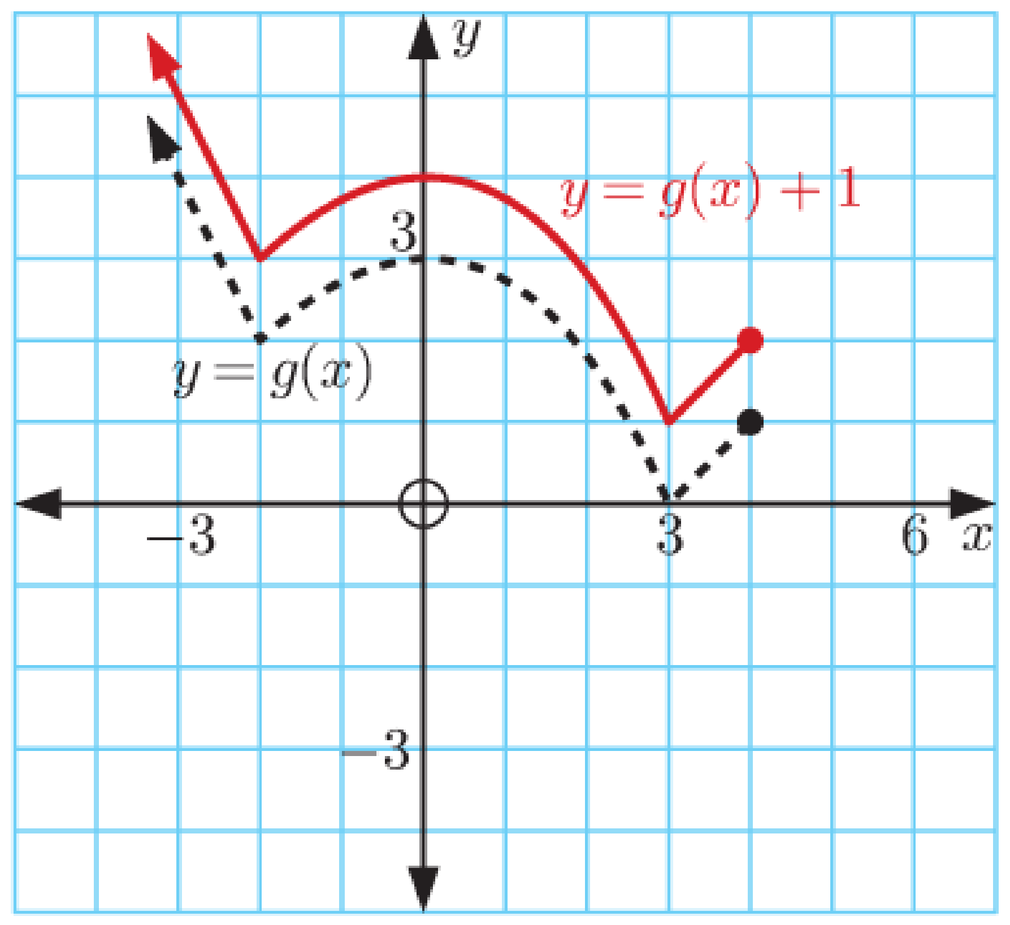

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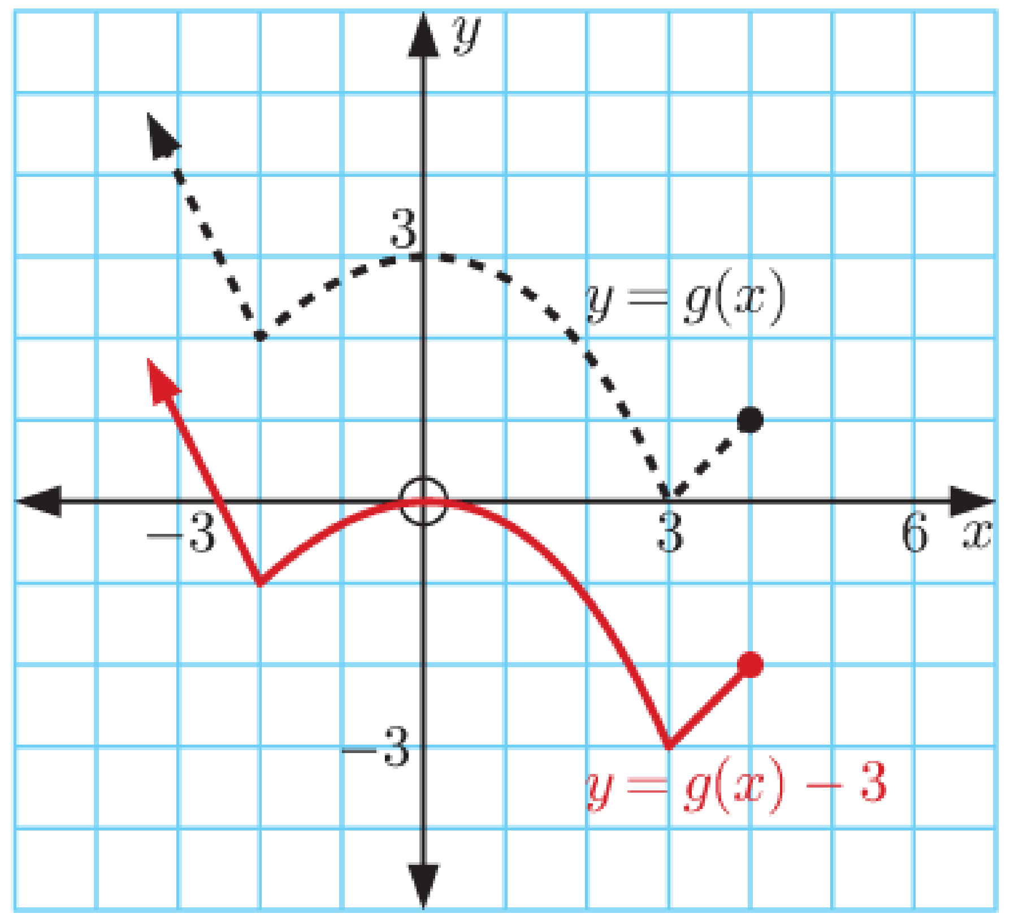
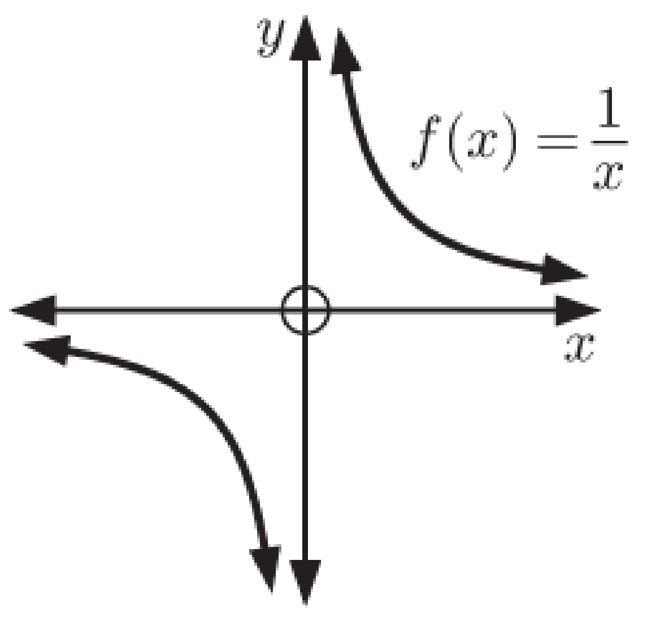


3 $g(x) = f(x) + 3$

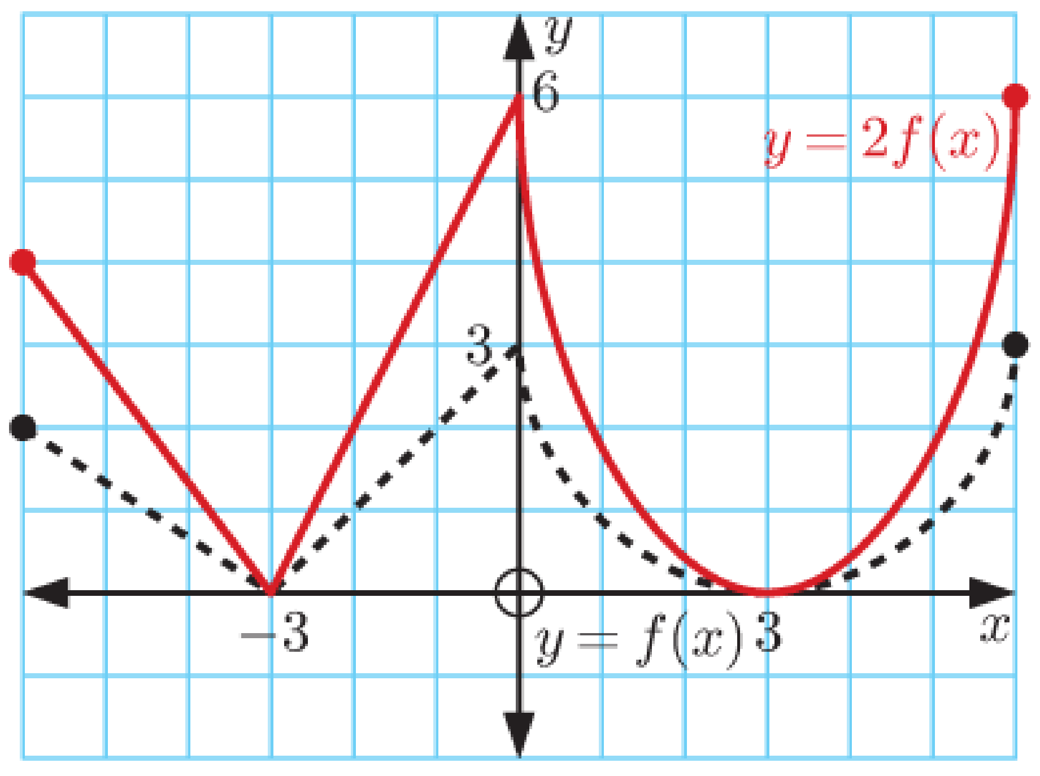
4 **a** $g(x) = 2x - 1$ **b** $g(x) = -x^2 + 5x - 4$

c $g(x) = x^3 - 4 - \dfrac{1}{x}$

5 **a**


b **i** $g(x) = \dfrac{1}{x} + k$

ii $y = k$

6 **a**


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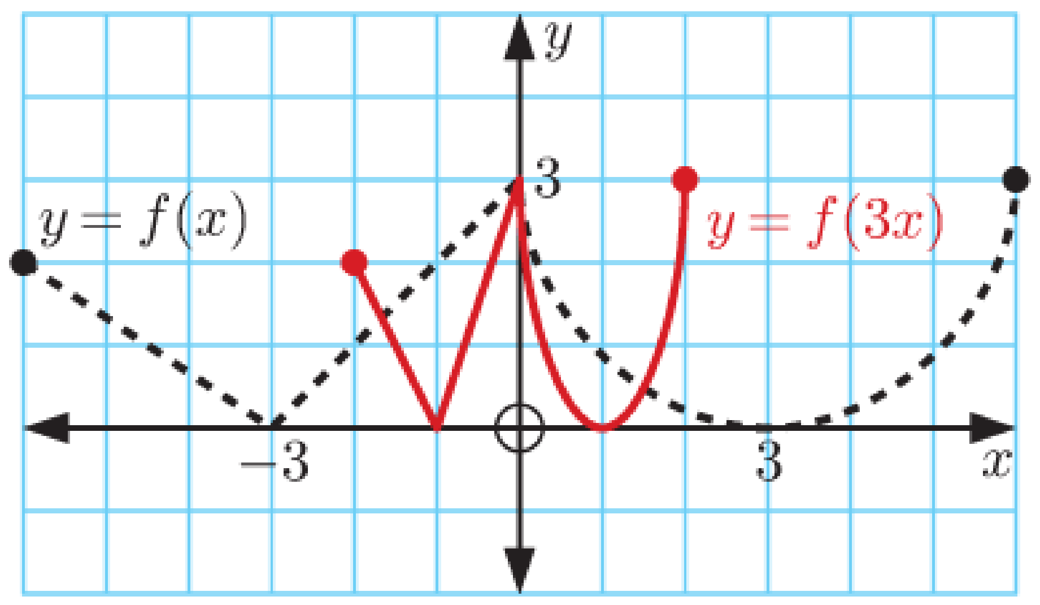
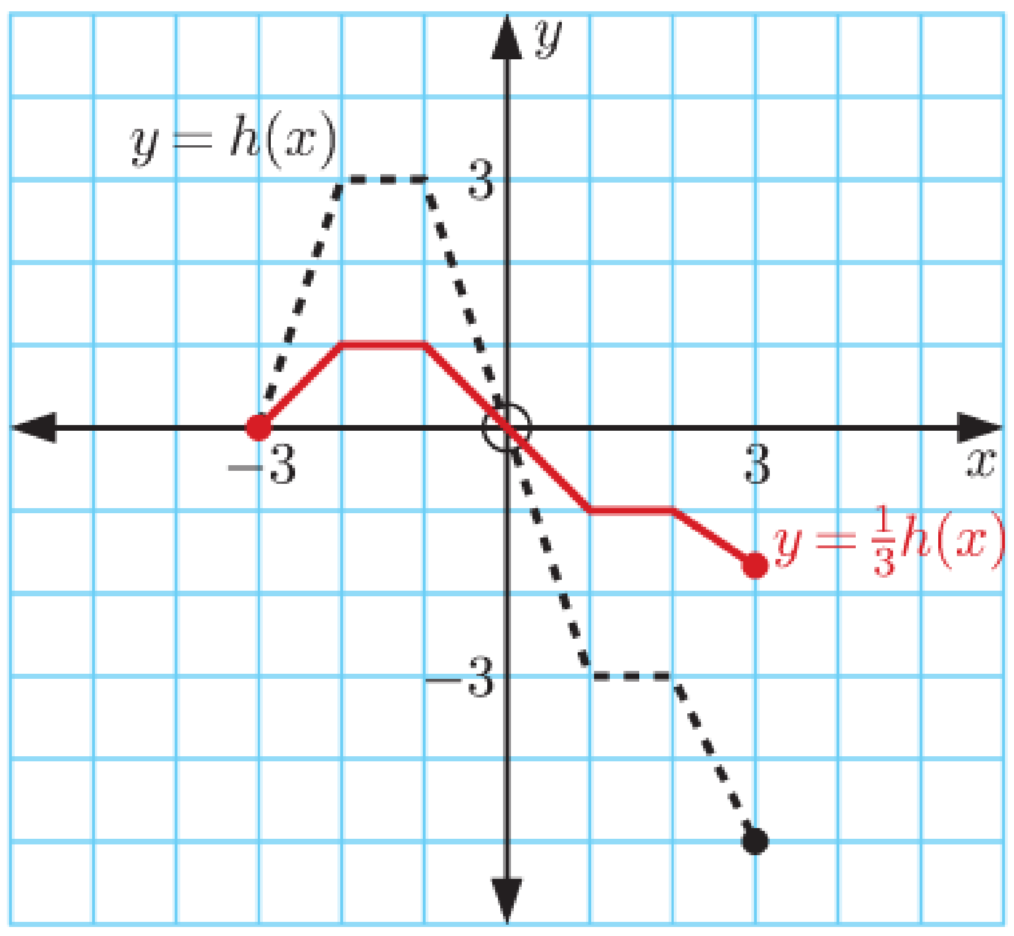


7 **a**



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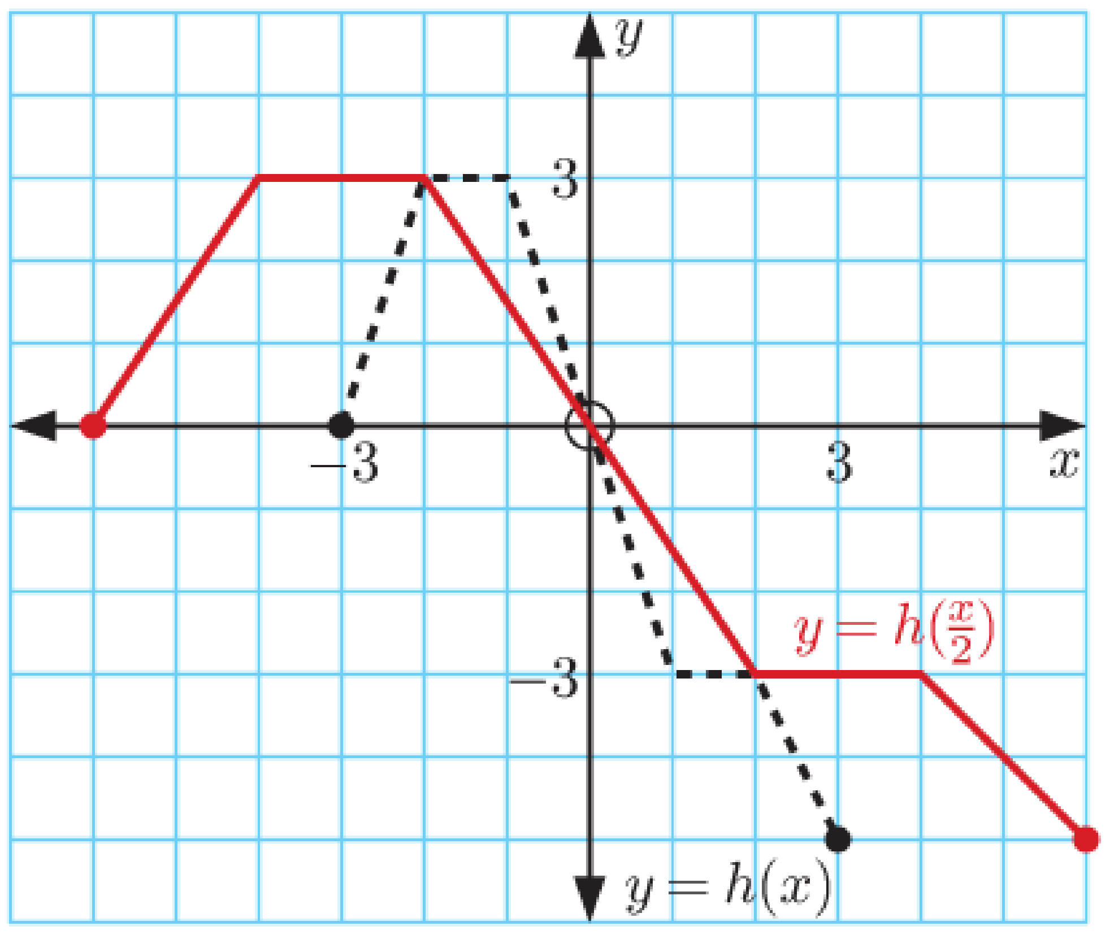
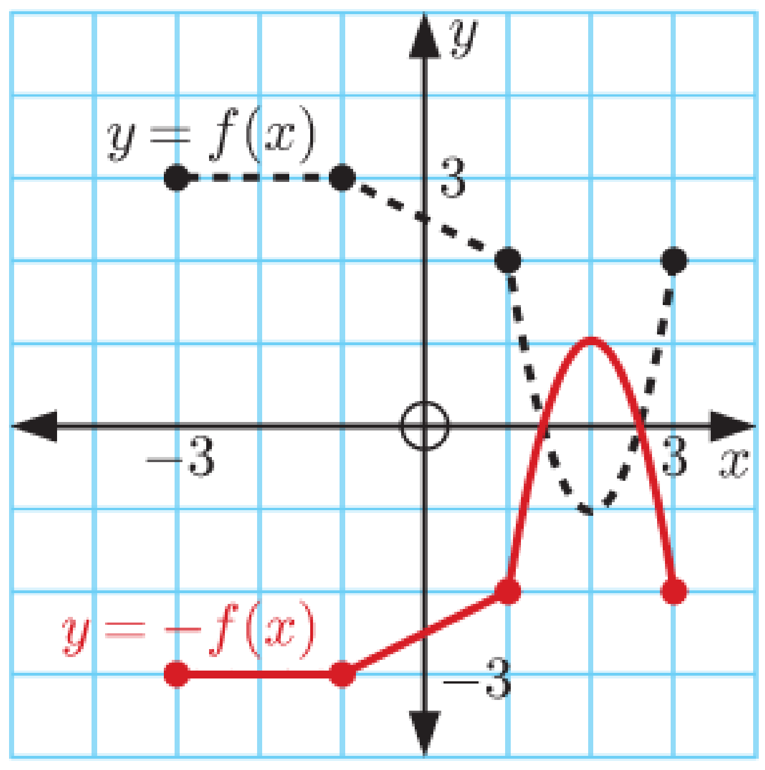


8 **a** $g(x) = 2f(x)$ **b** $g(x) = f\left(\dfrac{x}{3}\right)$ **9** cm

10 **a** $g(x) = 2x^2 + 4$ **b** $g(x) = 5 - x$

c $g(x) = \frac{1}{4}x^3 + 2x^2 - \frac{1}{2}$ **d** $g(x) = 8x^2 + 2x - 3$

11 **a**



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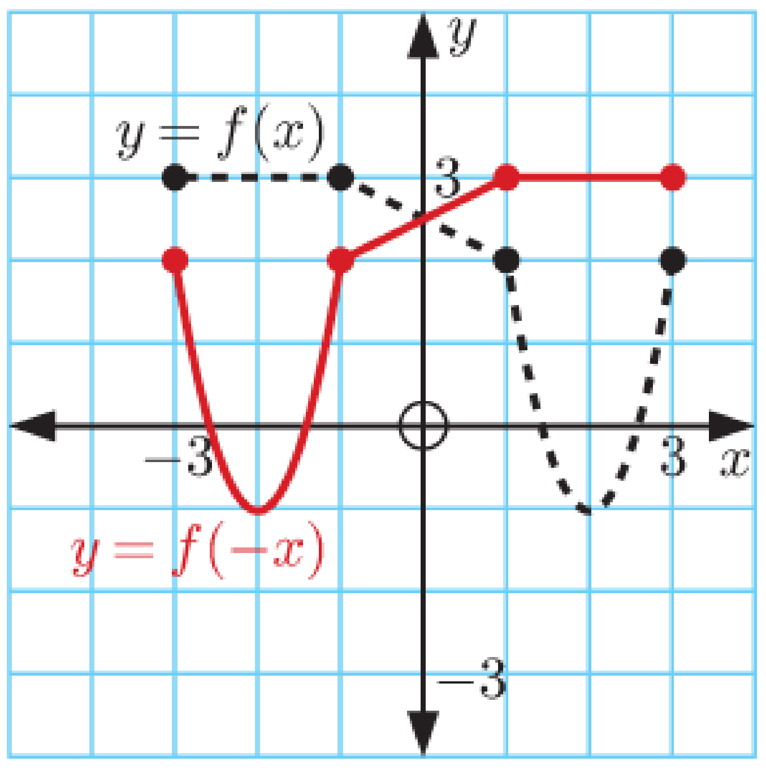
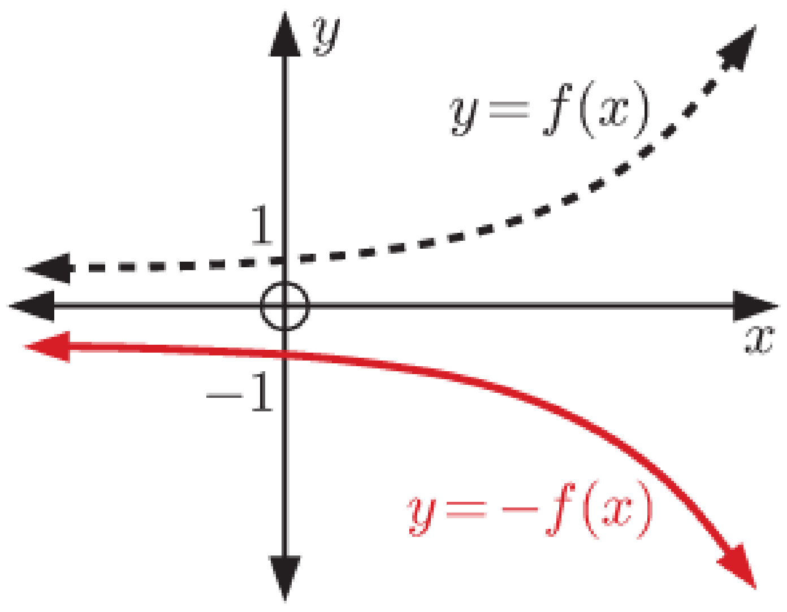


12 **a**



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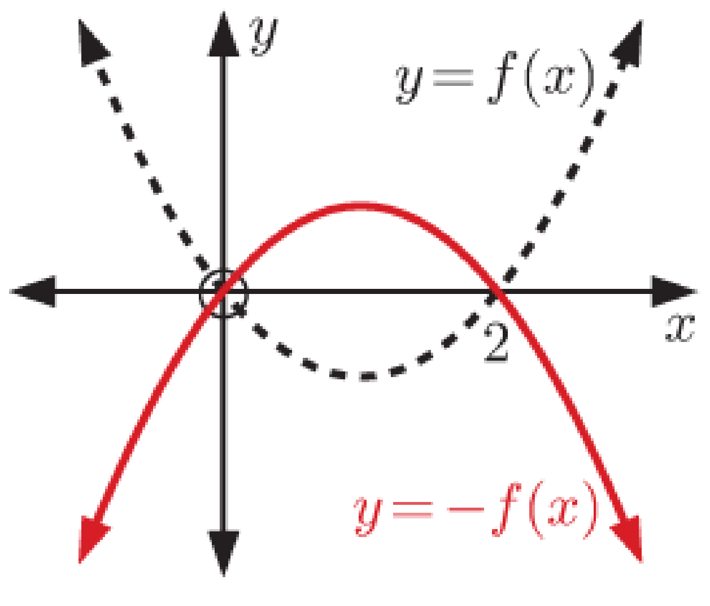

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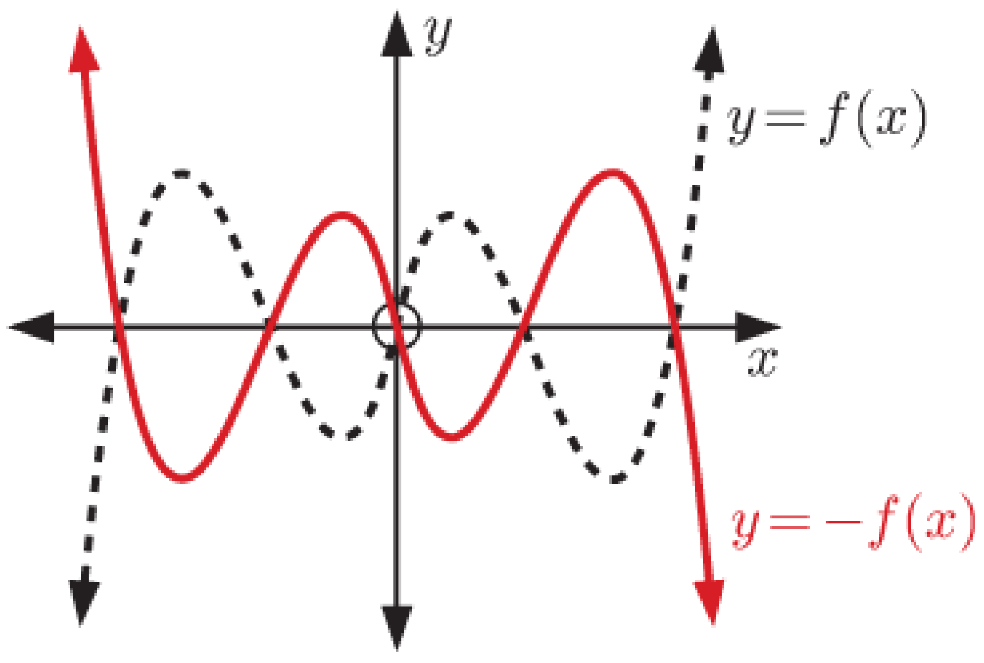
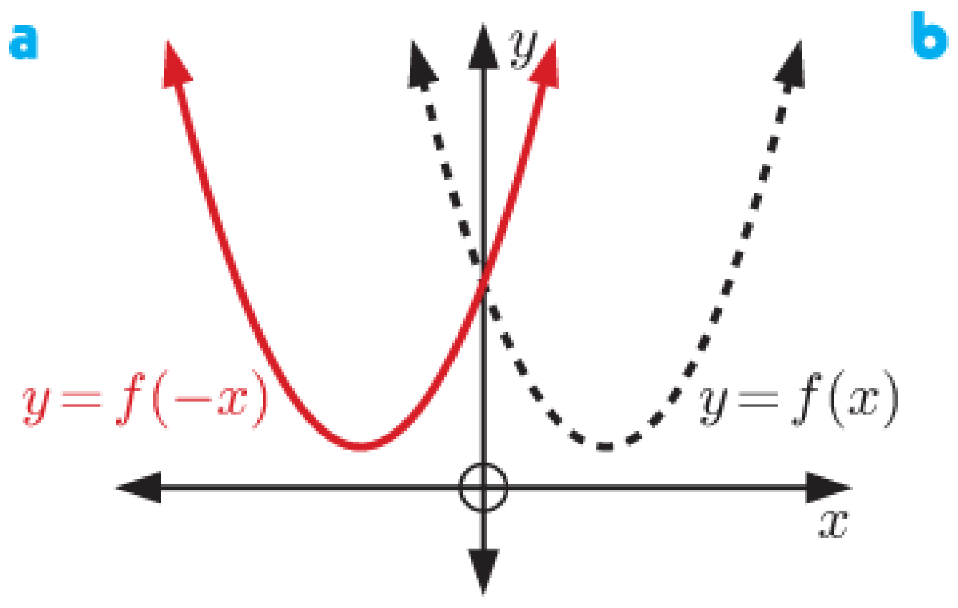


13 **a**



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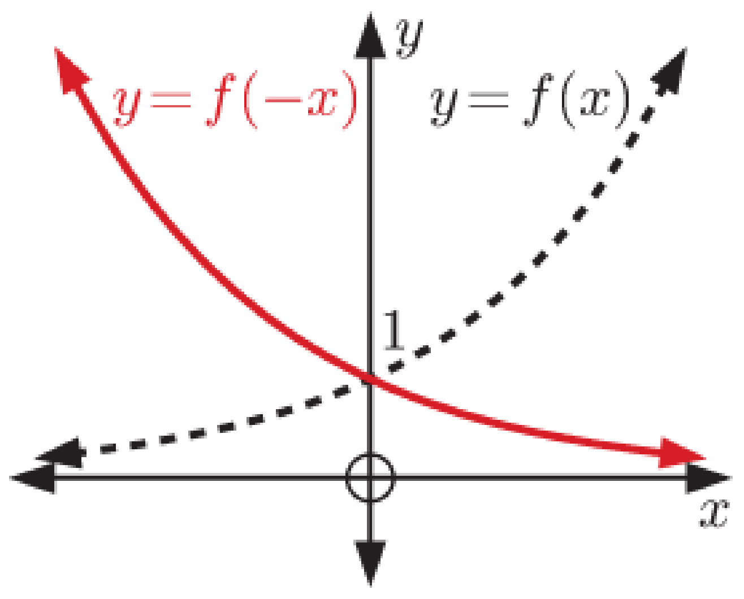

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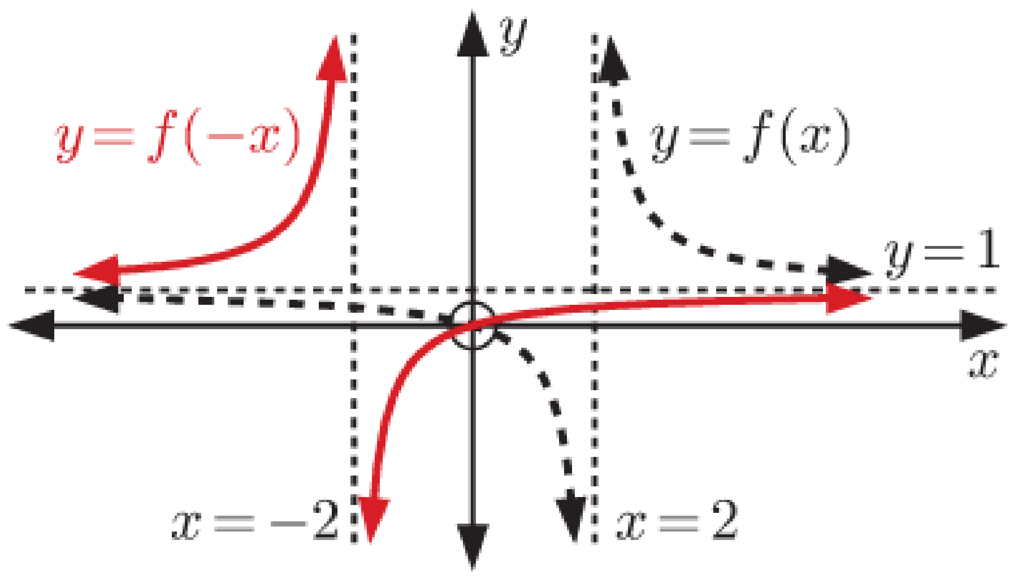


14 **a** $g(x) = -5x - 7$ **b** $g(x) = 2^{-x}$

c $g(x) = -2x^2 - 1$

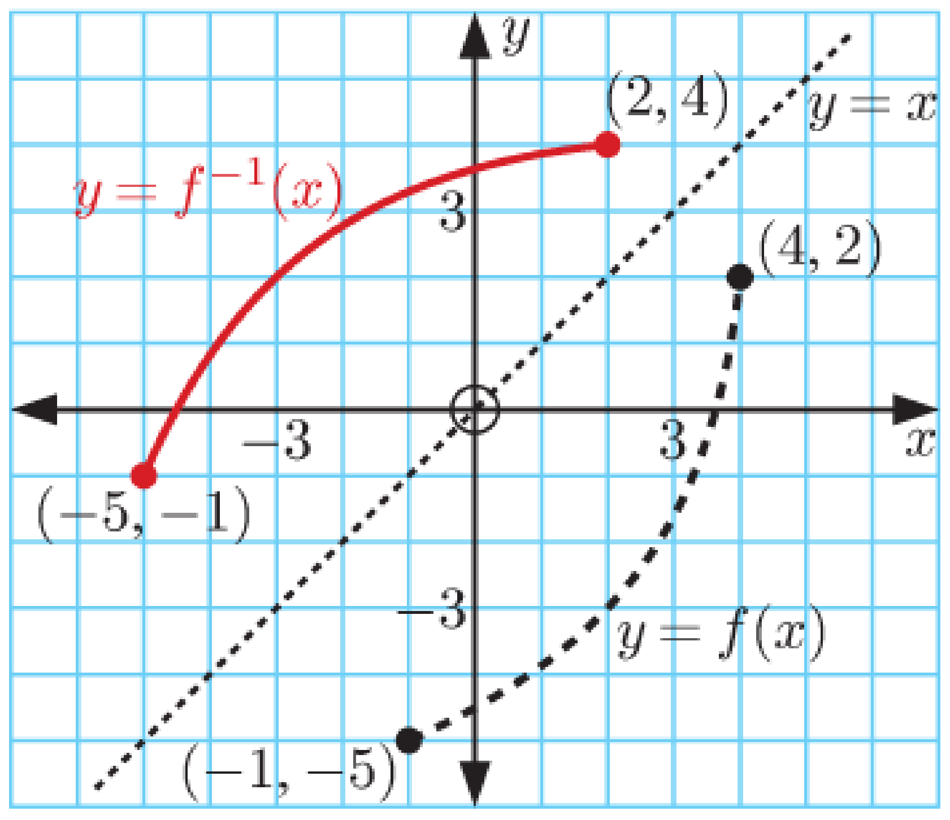
d $g(x) = x^4 + 2x^3 - 3x^2 - 5x - 7$

EXERCISE 3G

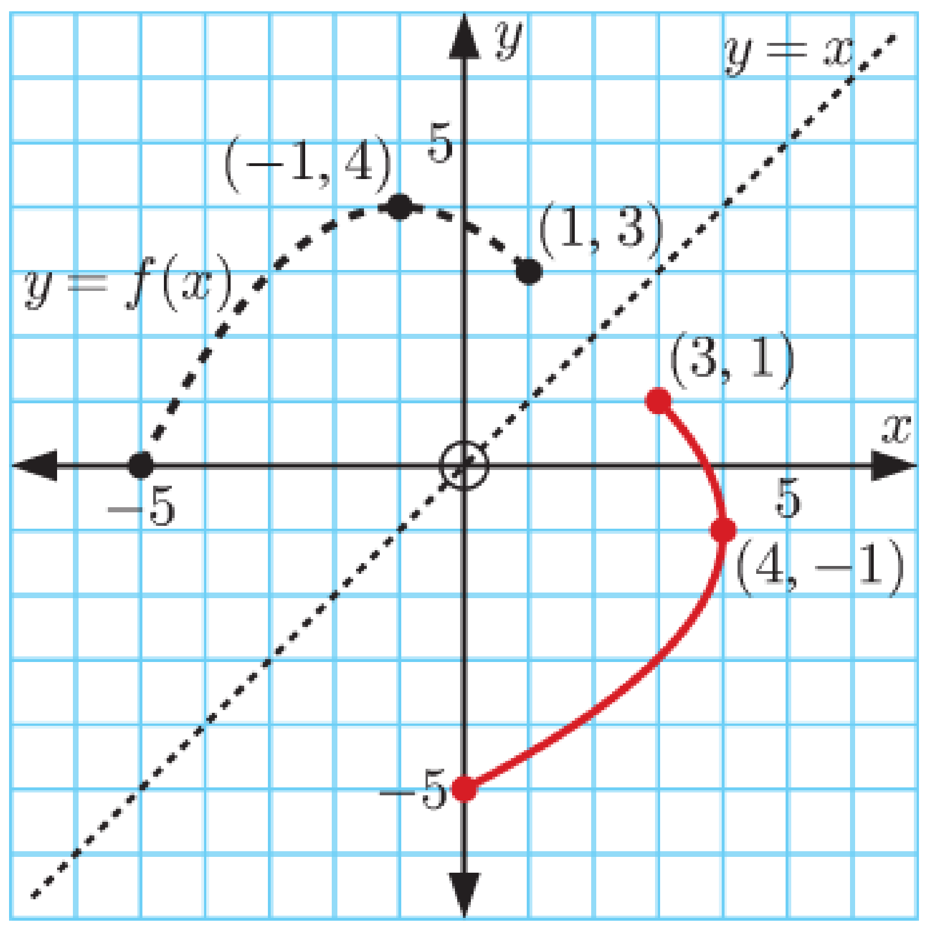
1 (7, −3), (4, 0), and (−6, 2)

2 **a** $f(x)$ passes both the *vertical* line test and *horizontal* line test.

b



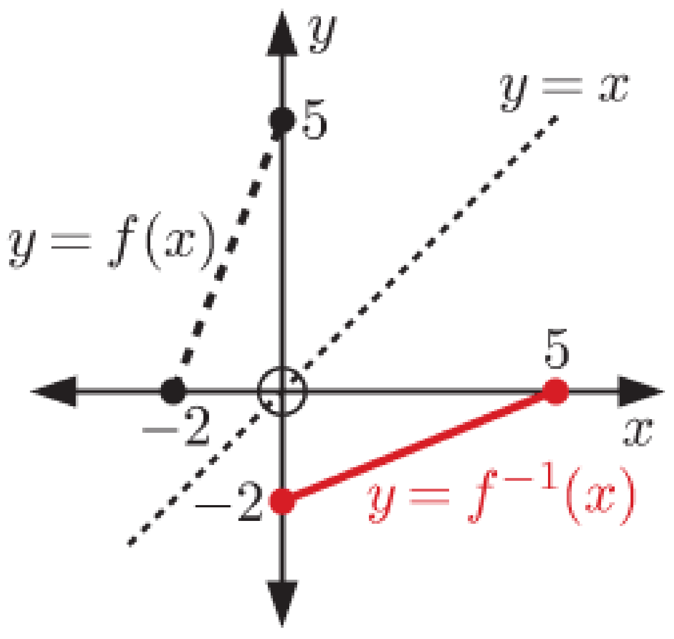
3 **a**



b No, $f(x)$ does not have an inverse since it does not pass the horizontal line test.

4 Range is $\{y \mid -2 \leqslant y < 3\}$

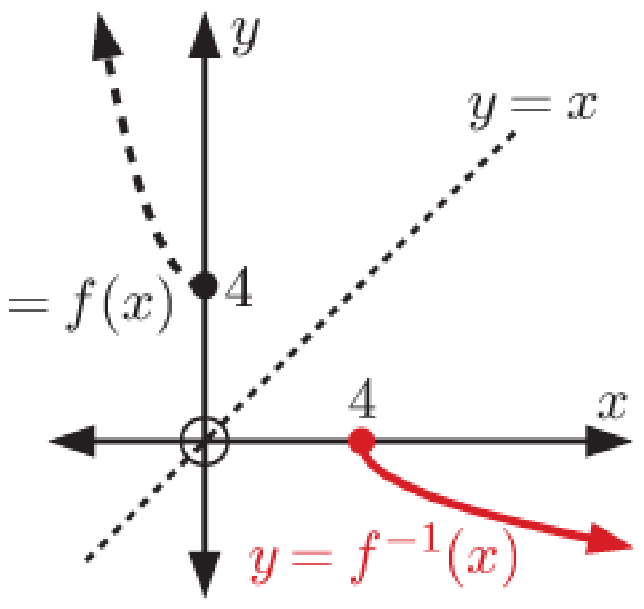
5 **a**



f:
Domain is $\{x \mid -2 \leqslant x \leqslant 0\}$
Range is $\{y \mid 0 \leqslant y \leqslant 5\}$

f^{-1}:
Domain is $\{x \mid 0 \leqslant x \leqslant 5\}$
Range is $\{y \mid -2 \leqslant y \leqslant 0\}$

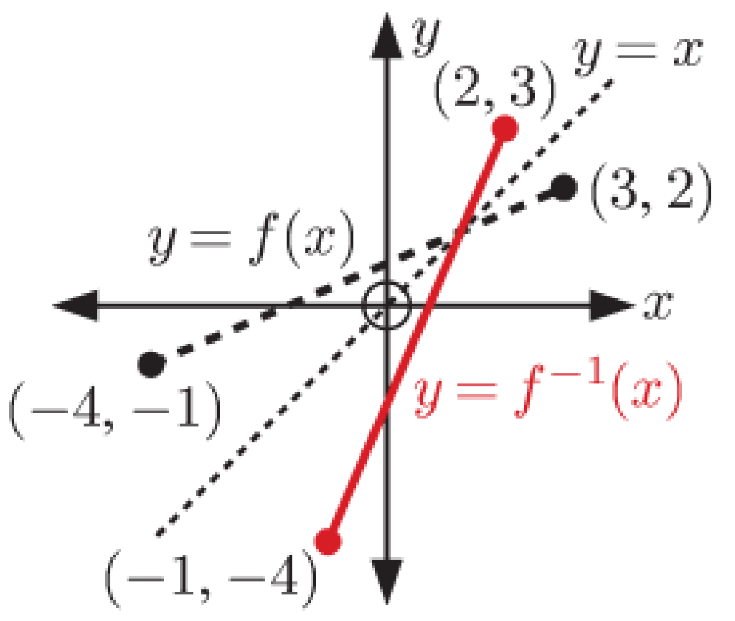
b



f:
Domain is $\{x \mid x \leqslant 0\}$
Range is $\{y \mid y \geqslant 4\}$

f^{-1}:
Domain is $\{x \mid x \geqslant 4\}$
Range is $\{y \mid y \leqslant 0\}$

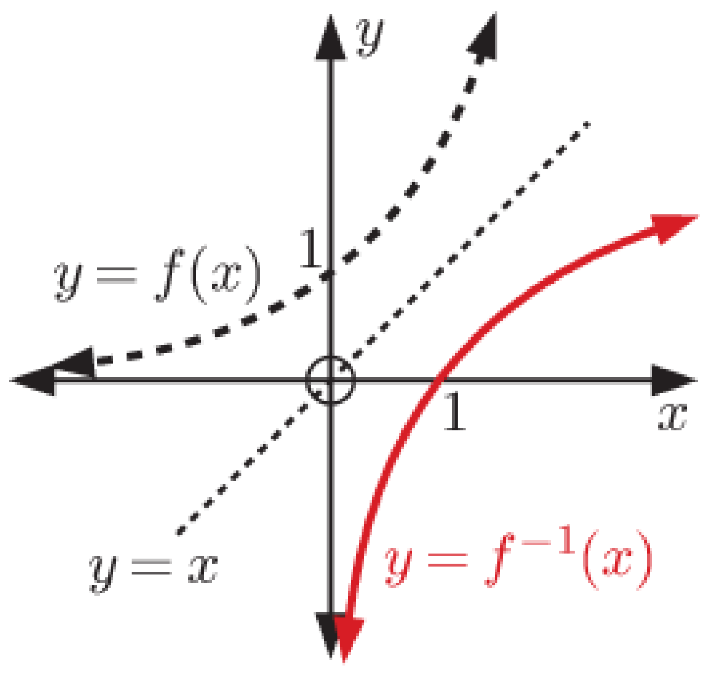
c



f:
Domain is $\{x \mid -4 \leqslant x \leqslant 3\}$
Range is $\{y \mid -1 \leqslant y \leqslant 2\}$

f^{-1}:
Domain is $\{x \mid -1 \leqslant x \leqslant 2\}$
Range is $\{y \mid -4 \leqslant y \leqslant 3\}$

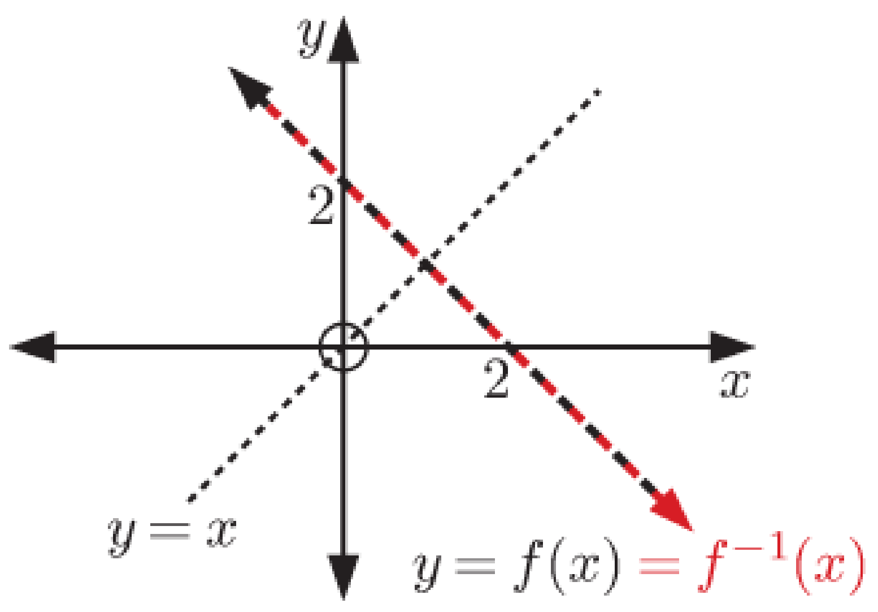
d



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 0\}$

f^{-1}:
Domain is $\{x \mid x > 0\}$
Range is $\{y \mid y \in \mathbb{R}\}$

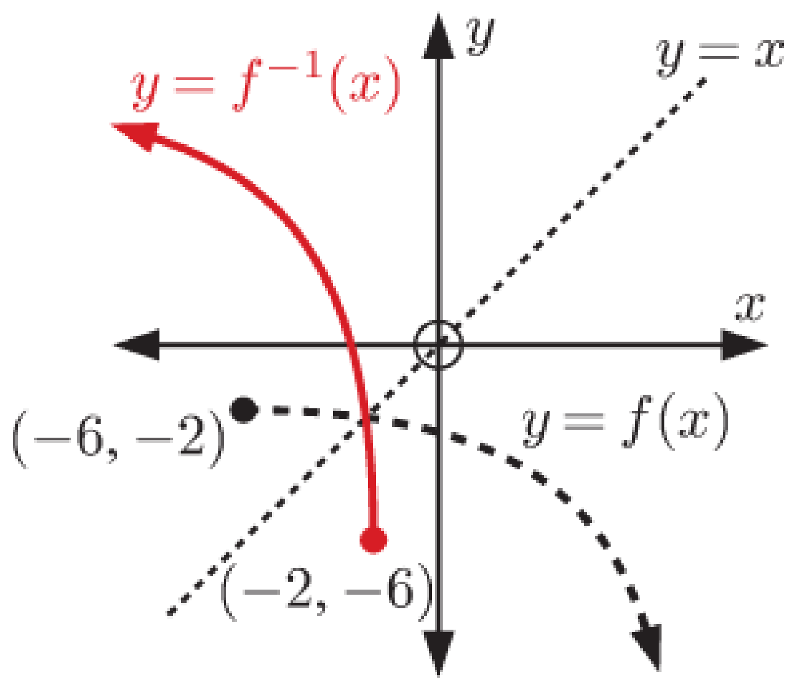
e



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f



f:
Domain is $\{x \mid x \geqslant -6\}$
Range is $\{y \mid y \leqslant -2\}$

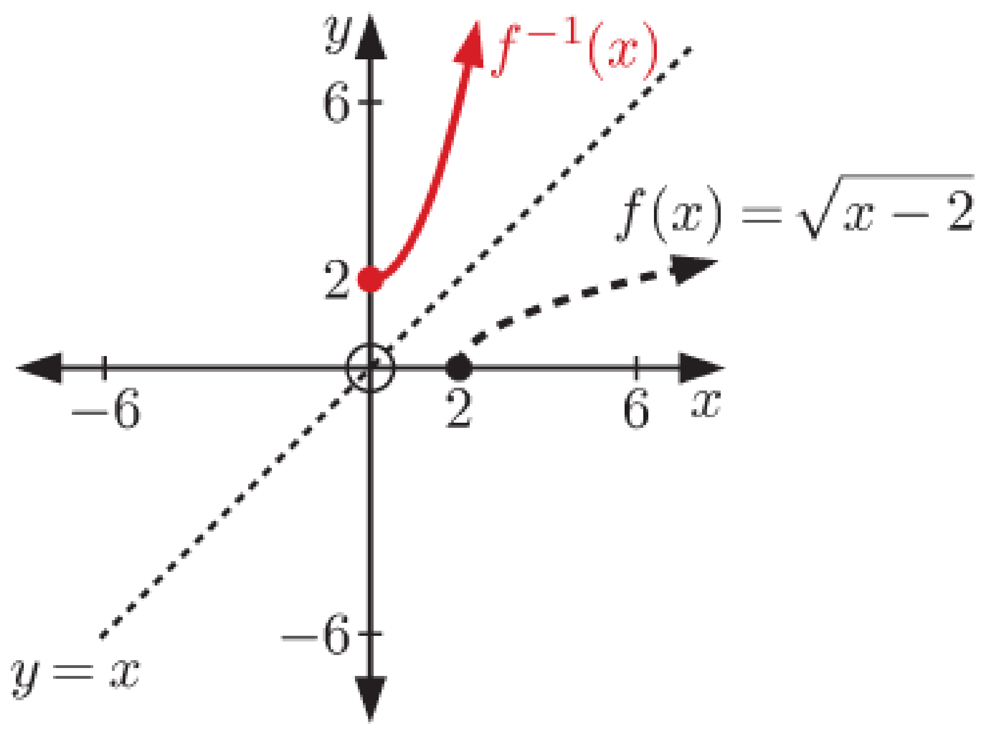
f^{-1}:
Domain is $\{x \mid x \leqslant -2\}$
Range is $\{y \mid y \geqslant -6\}$

6 x-intercept -3, y-intercept 5

7 $f(x)$ does not pass the horizontal line test ($y = 0$ passes through the two x-intercepts).
$\therefore$ $f(x)$ is not invertible.

8 **a** Domain is $\{x \mid x \geqslant 2\}$, Range is $\{y \mid y \geqslant 0\}$

b



c Domain is $\{x \mid x \geqslant 0\}$, Range is $\{y \mid y \geqslant 2\}$

d $x = 1$

9 **a** Domain is $\{x \mid x \geqslant -1\}$, Range is $\{y \mid y \geqslant -3\}$

b $x = 3$

10 **a** is the only one which has an inverse function.

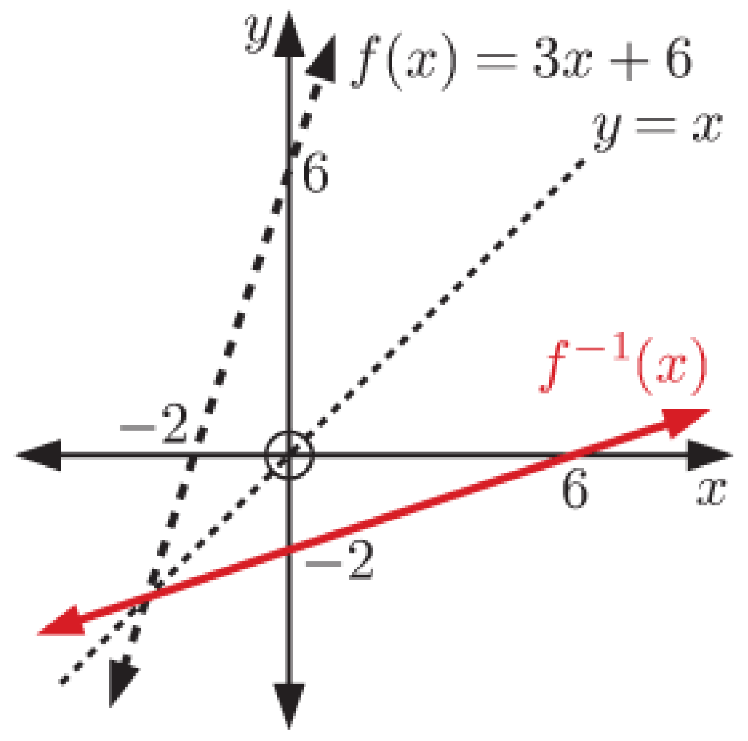
11 **a**

x	-2	-1	0	1	2
$f(x)$	13	8	5	4	5

b There are two points with the same y-coordinate. $f(x)$ does not pass the horizontal line test and hence is not invertible.

12 **a** x-intercept -2, y-intercept 6

b



c $f^{-1}(x) = \frac{1}{3}x - 2$

13 $f^{-1}(x) = \dfrac{x - c}{m}$, $m \neq 0$

REVIEW SET 3A

1 **a** Function, no two ordered pairs have the same x-coordinate.
b Not a function, $(3, 2)$ and $(3, 5)$ have the same x-coordinate.

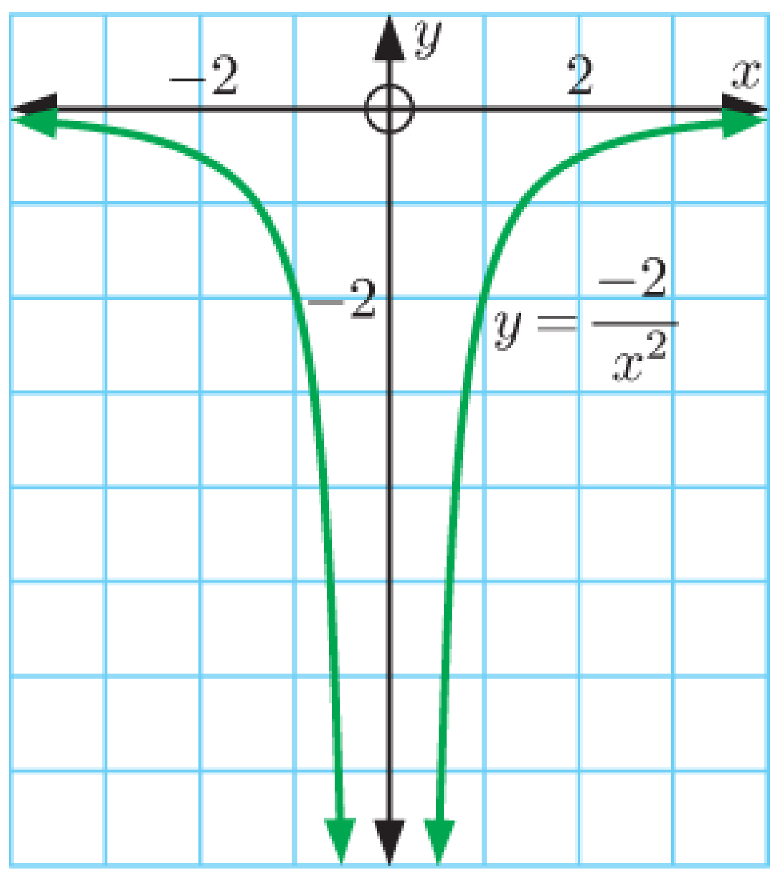
2 **a** 0 **b** -15 **c** $-\frac{5}{4}$

3 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$ **ii** Range is $\{y \mid y > -4\}$
iii yes, it is a function
b **i** Domain is $\{x \mid x \in \mathbb{R}\}$ **ii** Range is $\{y \mid y = 2\}$
iii yes, it is a function
c **i** Domain is $\{x \mid x \in \mathbb{R}\}$
ii Range is $\{y \mid y \leqslant -1$ or $y \geqslant 1\}$
iii no, not a function
d **i** Domain is $\{x \mid x \in \mathbb{R}\}$
ii Range is $\{y \mid -5 \leqslant y \leqslant 5\}$ **iii** yes, it is a function

4 **a** **i** 2 **ii** 0 **b** $x = -4$ **5** $a = -6$, $b = 13$

6 Domain is $\{t \mid 0 \leqslant t \leqslant 140\}$,
Range is $\{N \mid 70 \leqslant N \leqslant 110\}$

7 **a** $x = 0$ **b**

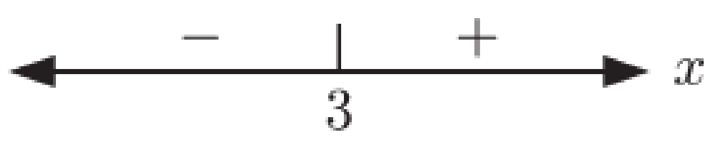


c Domain is $\{x \mid x \neq 0\}$
Range is $\{y \mid y < 0\}$

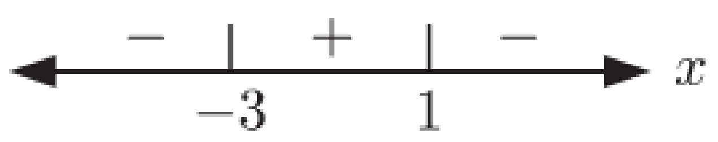
8 **a** $f(-3) = (-3)^2 = 9$, $g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3}) = 9$
b $x = -4$

9 **a** $x^2 - 2x + 5$ **b** $3x^2 - 6x$ **c** $4x^2 - 4x$ **d** $x^2 + 2x$

10 **a**



b



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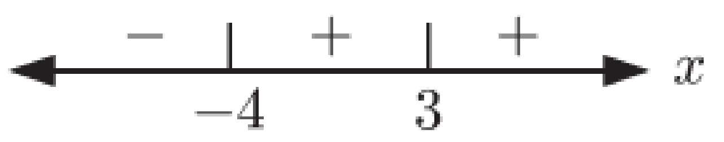
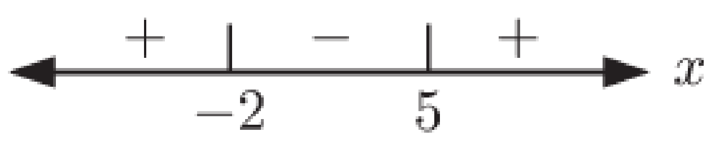


11 **a**



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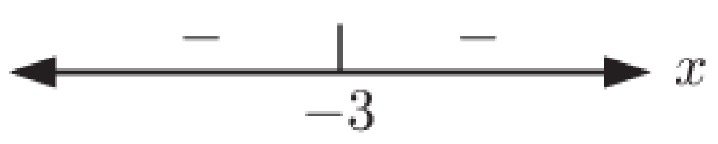

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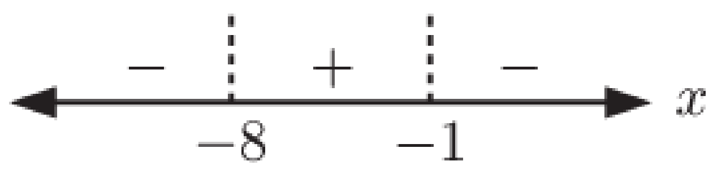
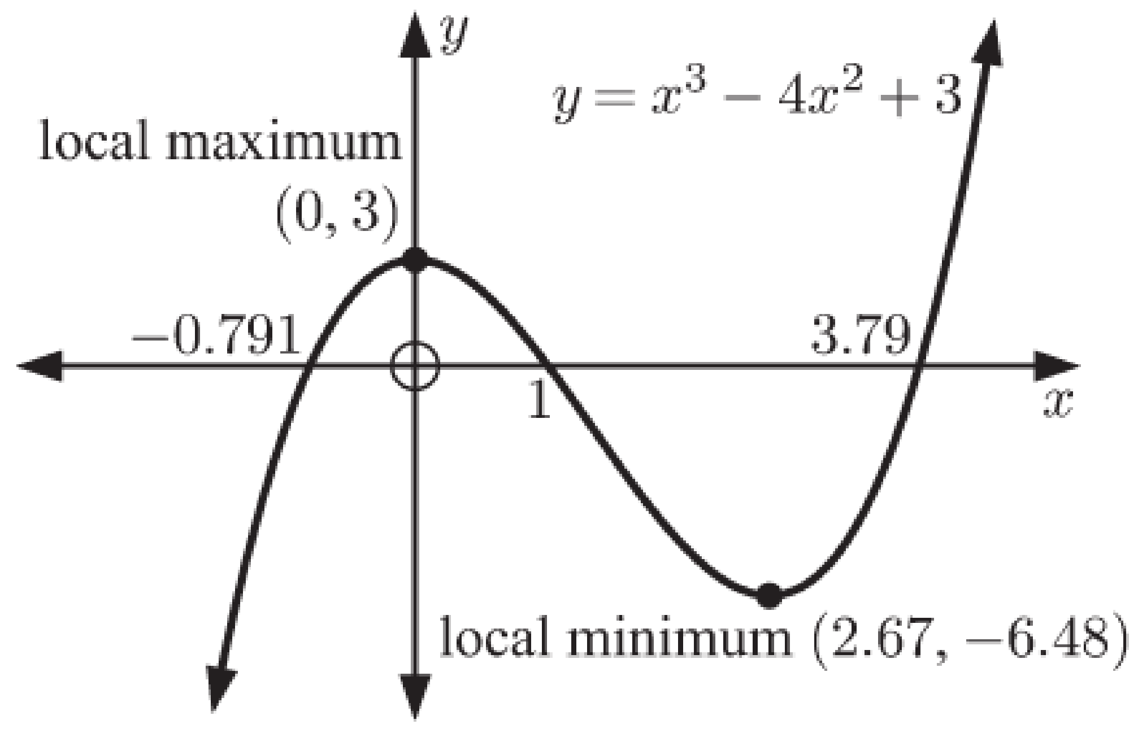


12 **a** **i** x-intercepts ≈ -0.791, 1, and ≈ 3.79, y-intercept 3
ii local maximum $(0, 3)$,
local minimum $(2.67, -6.48)$
iii no asymptotes

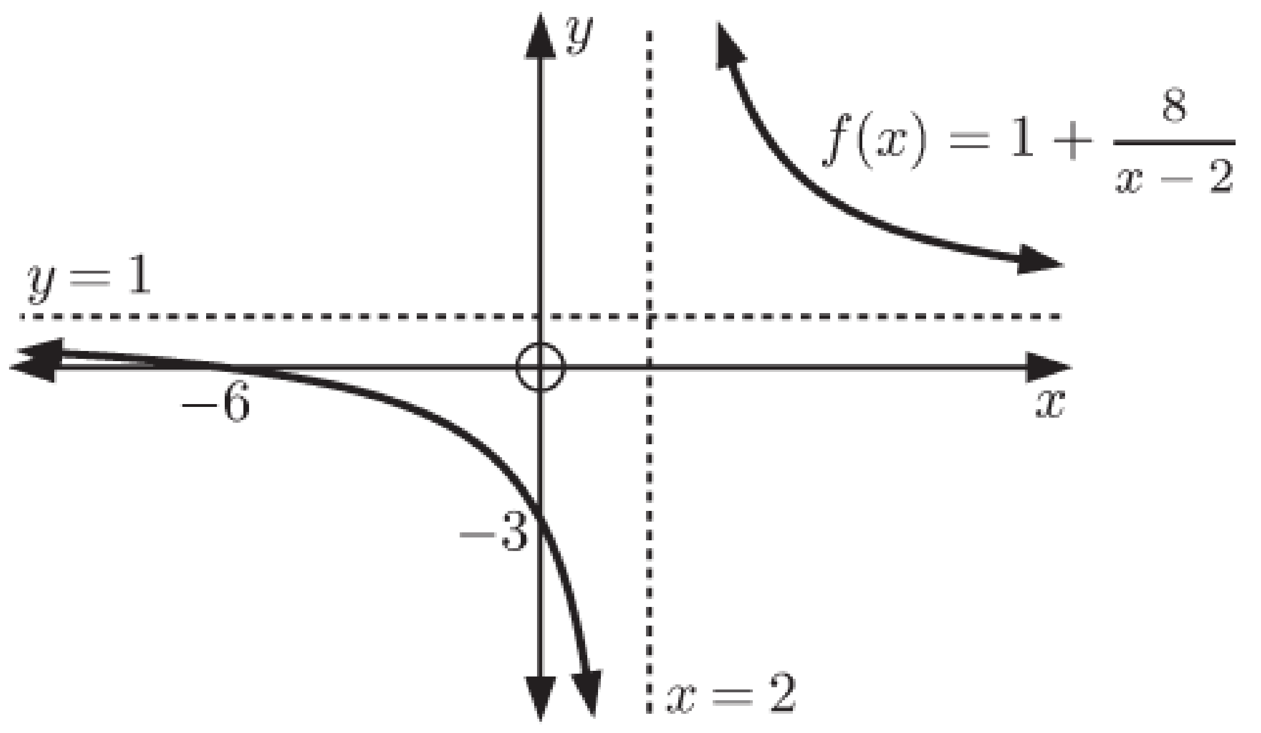
iv



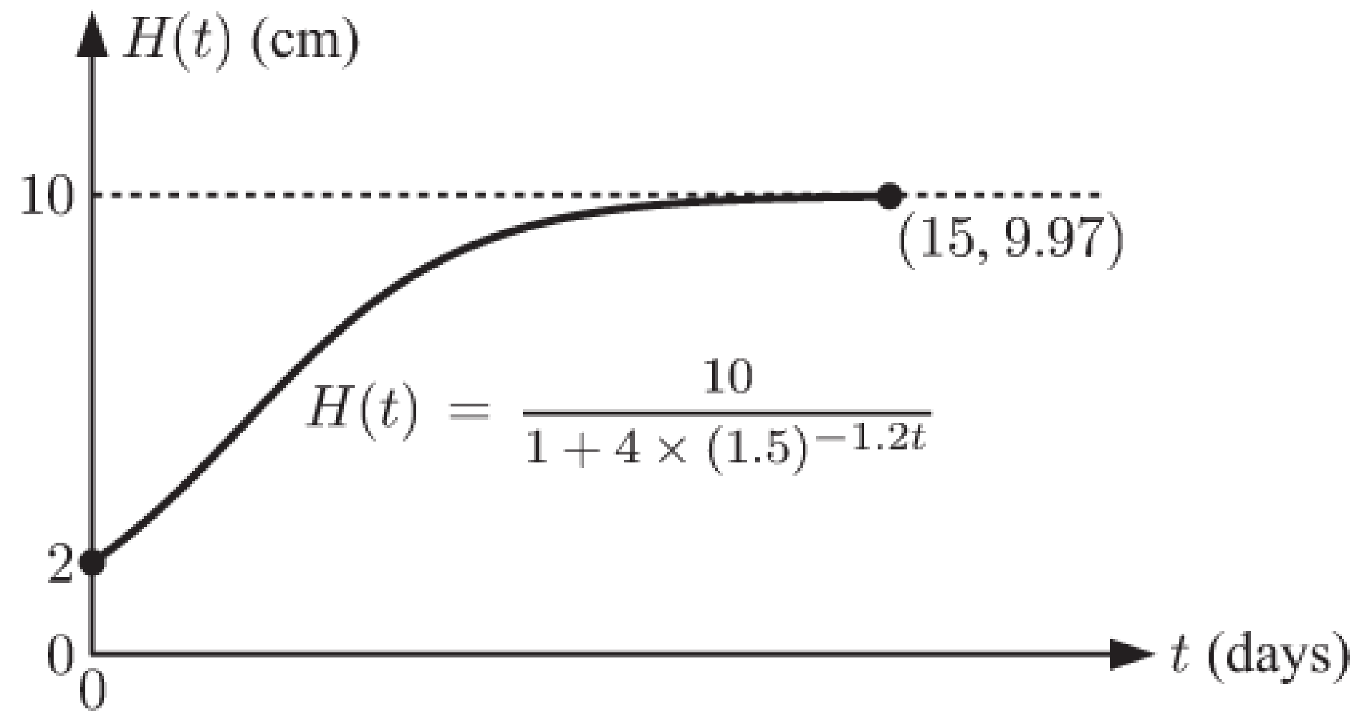
b **i** x-intercept -6, y-intercept -3 **ii** no turning points

iii vertical asymptote $x = 2$,
horizontal asymptote $y = 1$

iv

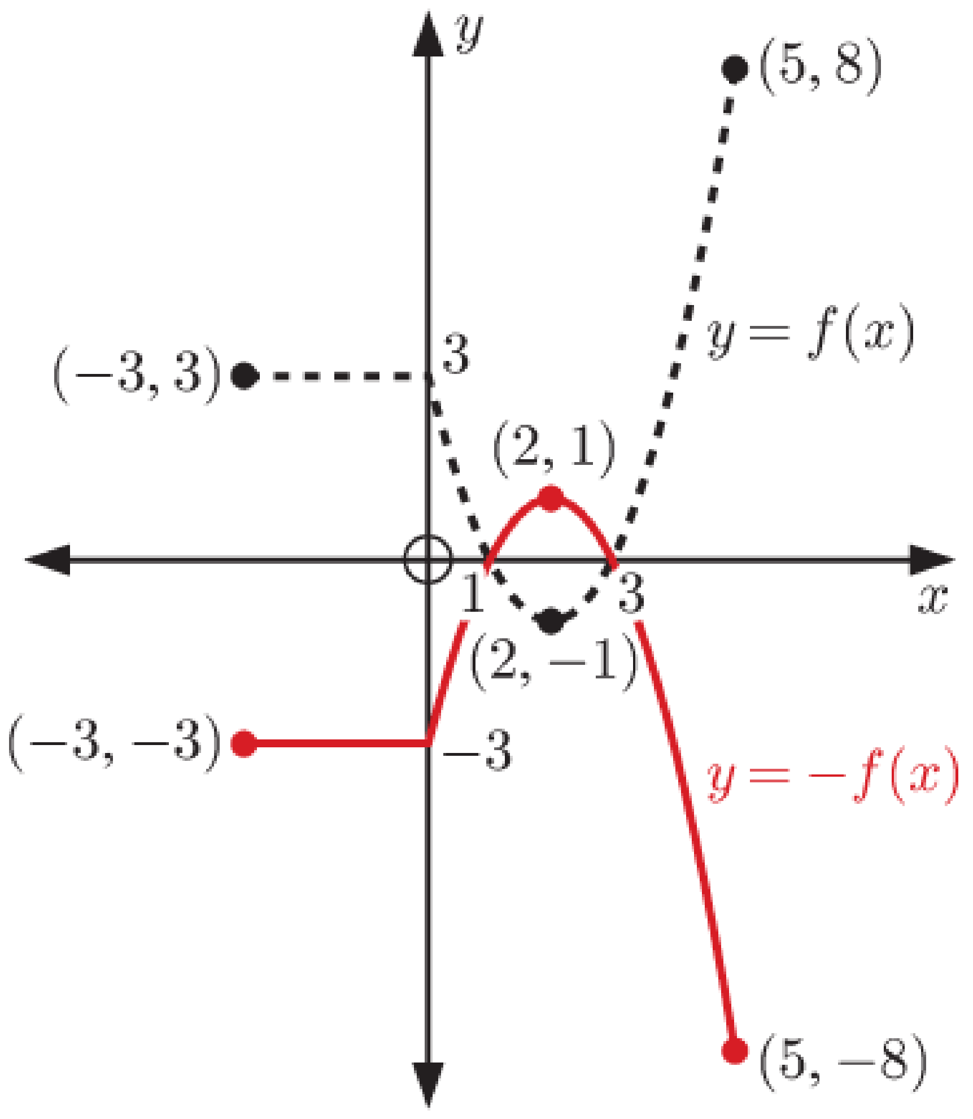


13 **a**

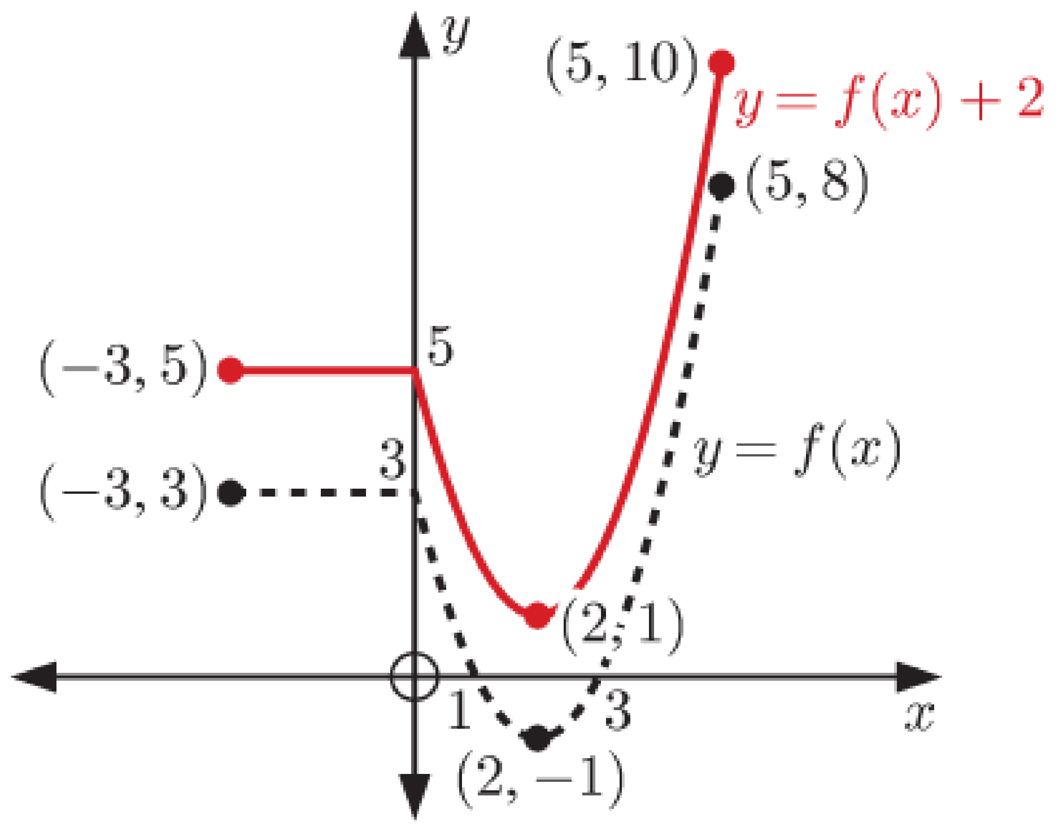


b 2 cm **c** ≈ 8.83 cm **d** ≈ 2.85 days **e** yes, 10 cm

14 **a**



b



15 **a** $g(x) = 4x - 10$ **b** $g(x) = 5x^2 + 30$

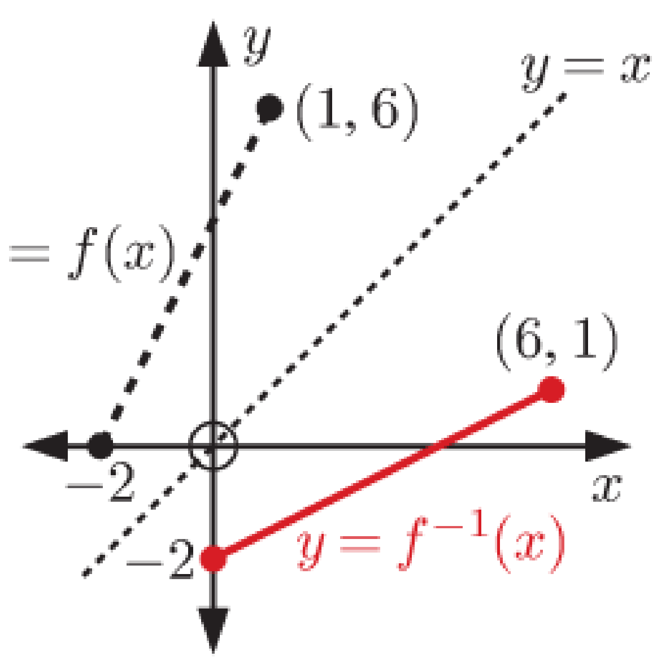
c $g(x) = \frac{2}{9}x^2 - \frac{1}{3}x + 4$ **d** $g(x) = -x^3$

16 Domain is $\{x \mid -2 \leqslant x \leqslant 3\}$, Range is $\{y \mid -5 \leqslant y \leqslant 3\}$
$g(x)$ is the graph of $f(x)$ translated down 4 units.

17 $(4, -1)$ and $(-2, 6)$

18 **a** and **c** both have inverse functions.

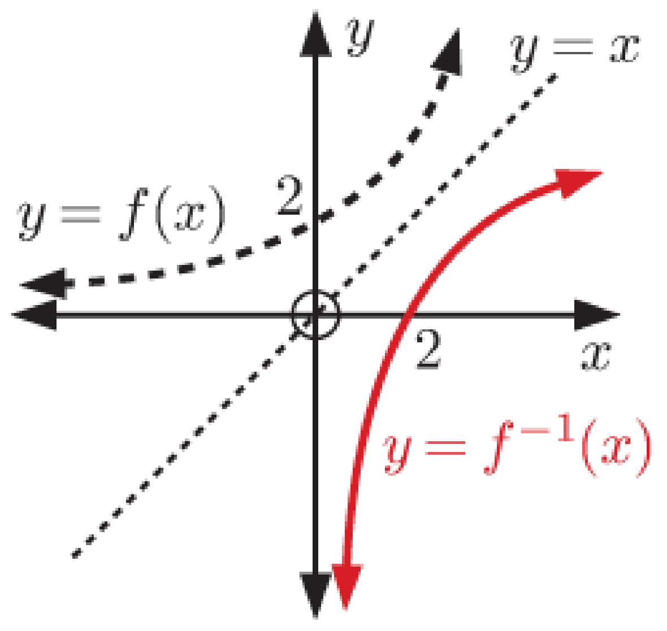
19 **a**



f:
Domain is $\{x \mid -2 \leqslant x \leqslant 1\}$
Range is $\{y \mid 0 \leqslant y \leqslant 6\}$

f^{-1}:
Domain is $\{x \mid 0 \leqslant x \leqslant 6\}$
Range is $\{y \mid -2 \leqslant y \leqslant 1\}$

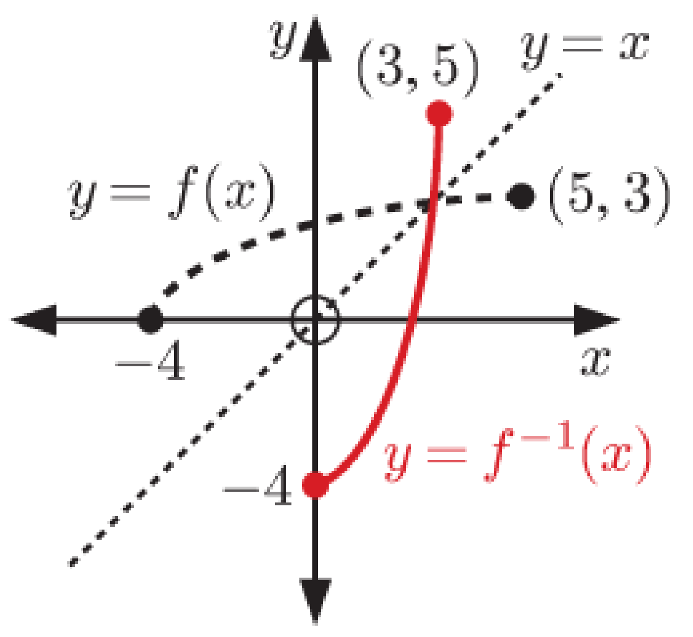
b



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 0\}$

f^{-1}:
Domain is $\{x \mid x > 0\}$
Range is $\{y \mid y \in \mathbb{R}\}$

c



f:
Domain is $\{x \mid -4 \leqslant x \leqslant 5\}$
Range is $\{y \mid 0 \leqslant y \leqslant 3\}$

f^{-1}:
Domain is $\{x \mid 0 \leqslant x \leqslant 3\}$
Range is $\{y \mid -4 \leqslant y \leqslant 5\}$

REVIEW SET 3B

1 **a** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -4\}$

b Domain is $\{x \mid x \geqslant -2\}$, Range is $\{y \mid 1 \leqslant y < 3\}$

c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y = -1, 1, \text{ or } 2\}$

2 **a** -2 **b** $x^2 - x - 2$ **c** $16x^2 - 12x$

3 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -5\}$

ii x-intercepts -1 and 5, y-intercept $-\frac{25}{9}$

iii is a function

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y = 1 \text{ or } -3\}$

ii no x-intercepts, y-intercept 1 **iii** is a function

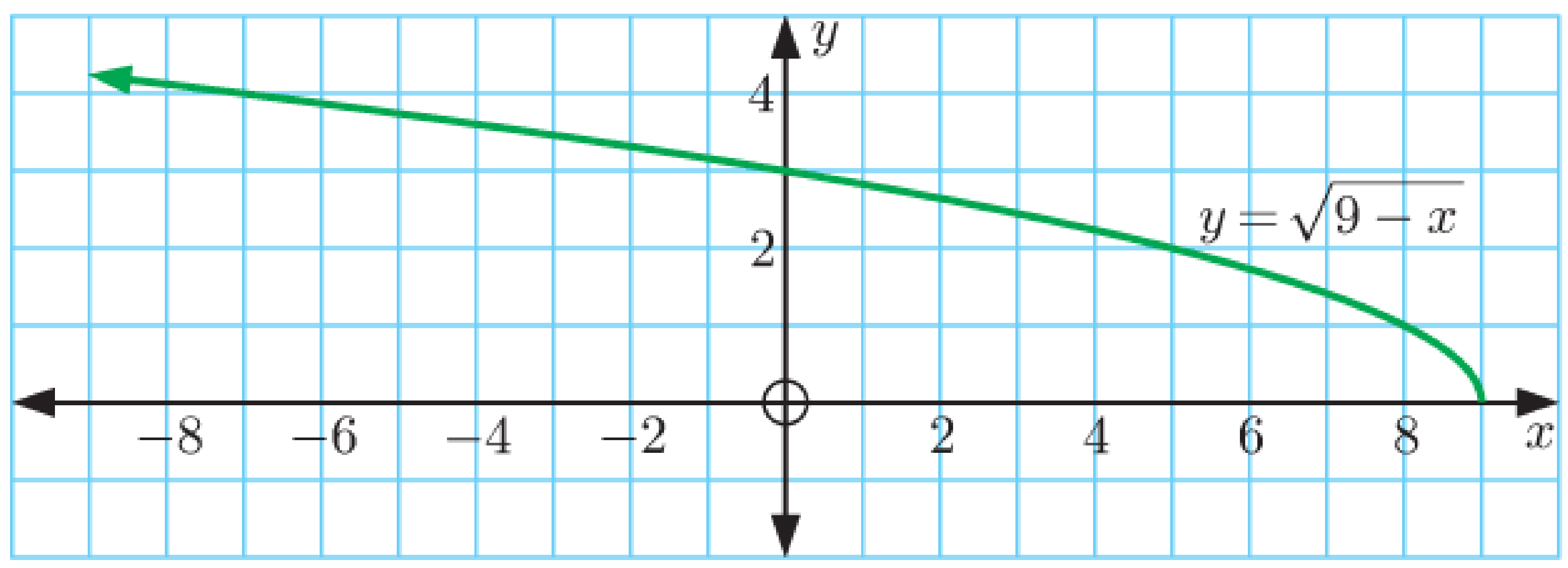
4 **a** is a function **b** is not a function

5 **a** **i** -4 **ii** $-\frac{1}{2}$ **iii** 2

b $x = -2$ **c** $\dfrac{3x - 4}{x + 1}$ **d** $x = -9$

6 **a** 12 **b** $x = \pm 1$

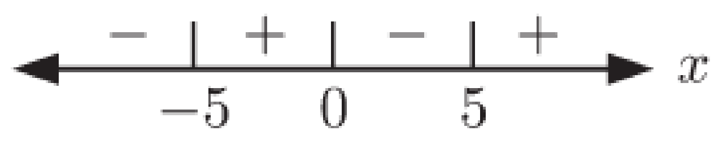
7 **a**



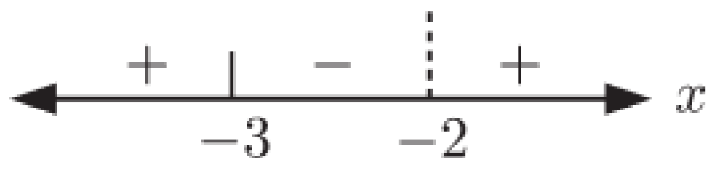
b It is a function.

c Domain is $\{x \mid x \leqslant 9\}$, Range is $\{y \mid y \geqslant 0\}$

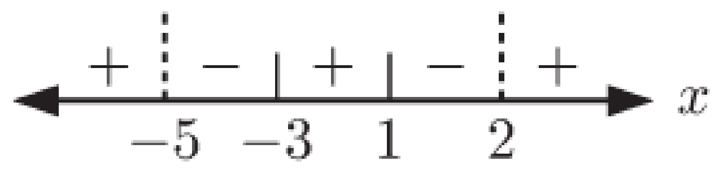
8 **a**



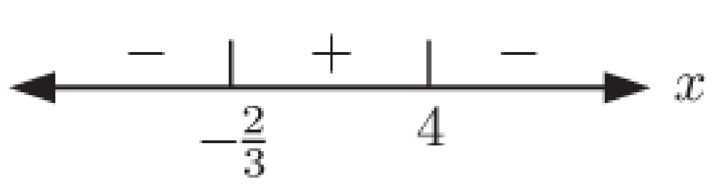
b



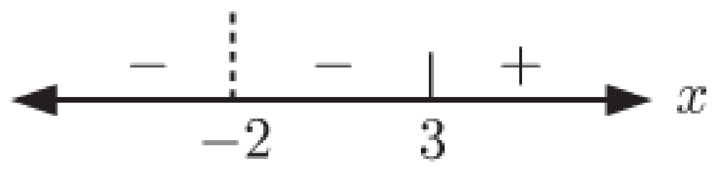
c



9 **a**



b

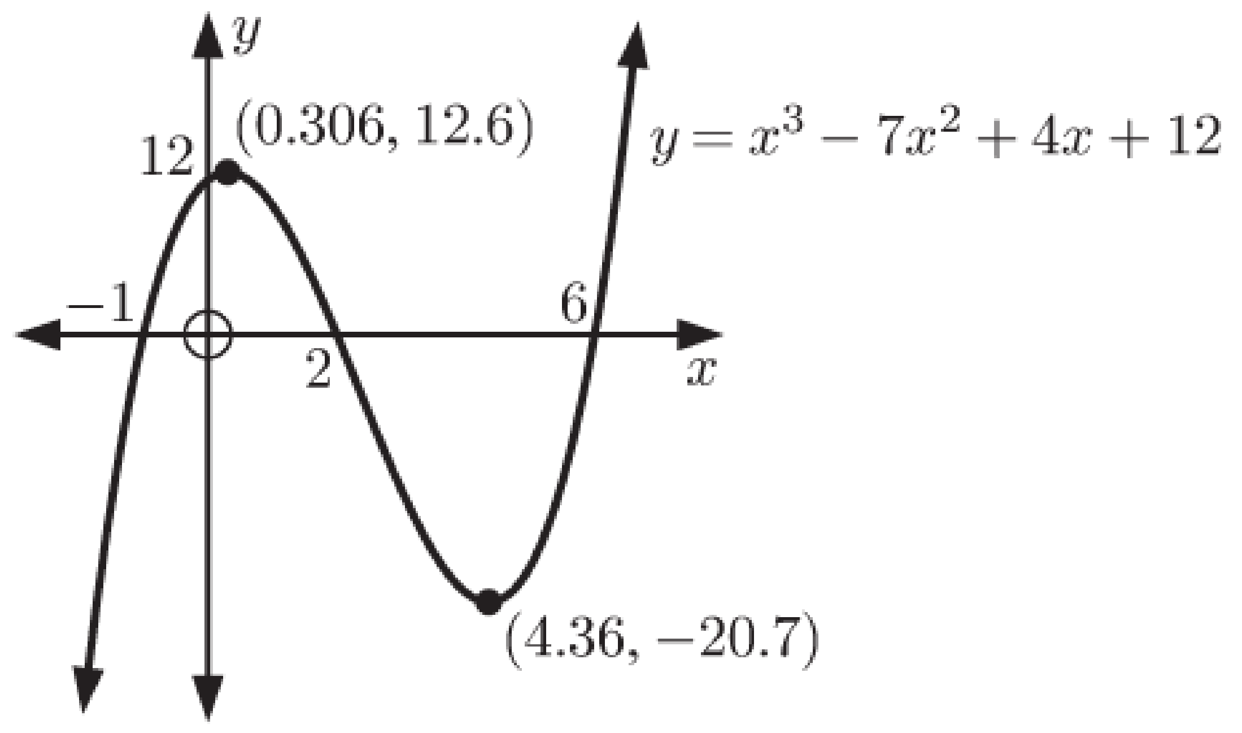


10 **a** **i** x-intercepts -1, 2, and 6, y-intercept 12

ii max. turning point $(0.306,\ 12.6)$
min. turning point $(4.36,\ -20.7)$

iii Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iv

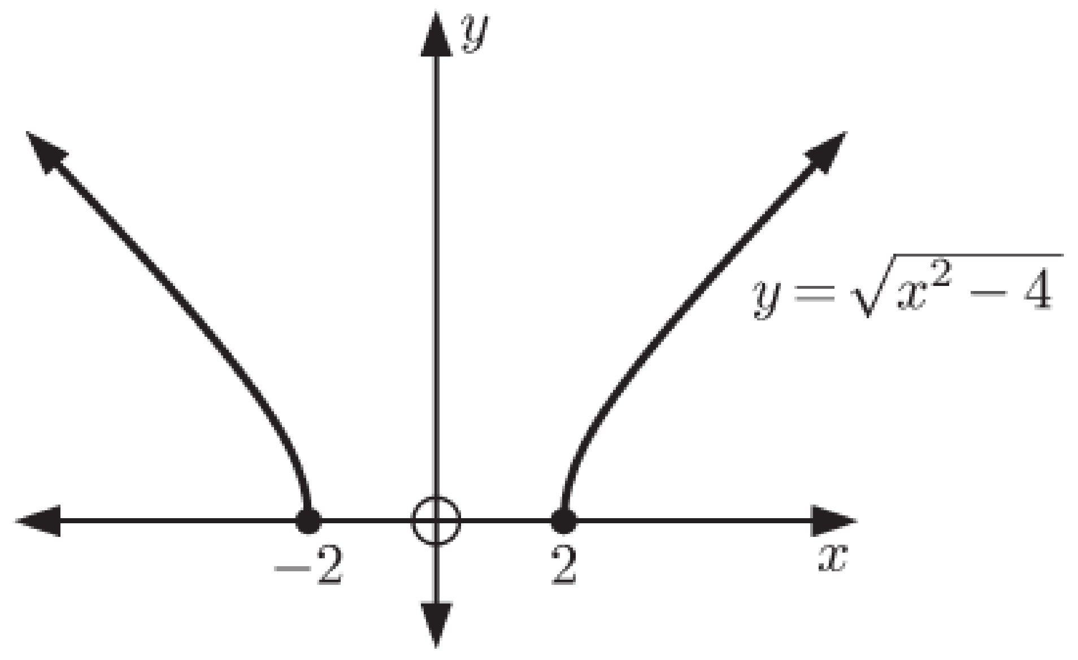


b **i** x-intercepts -2 and 2, no y-intercept

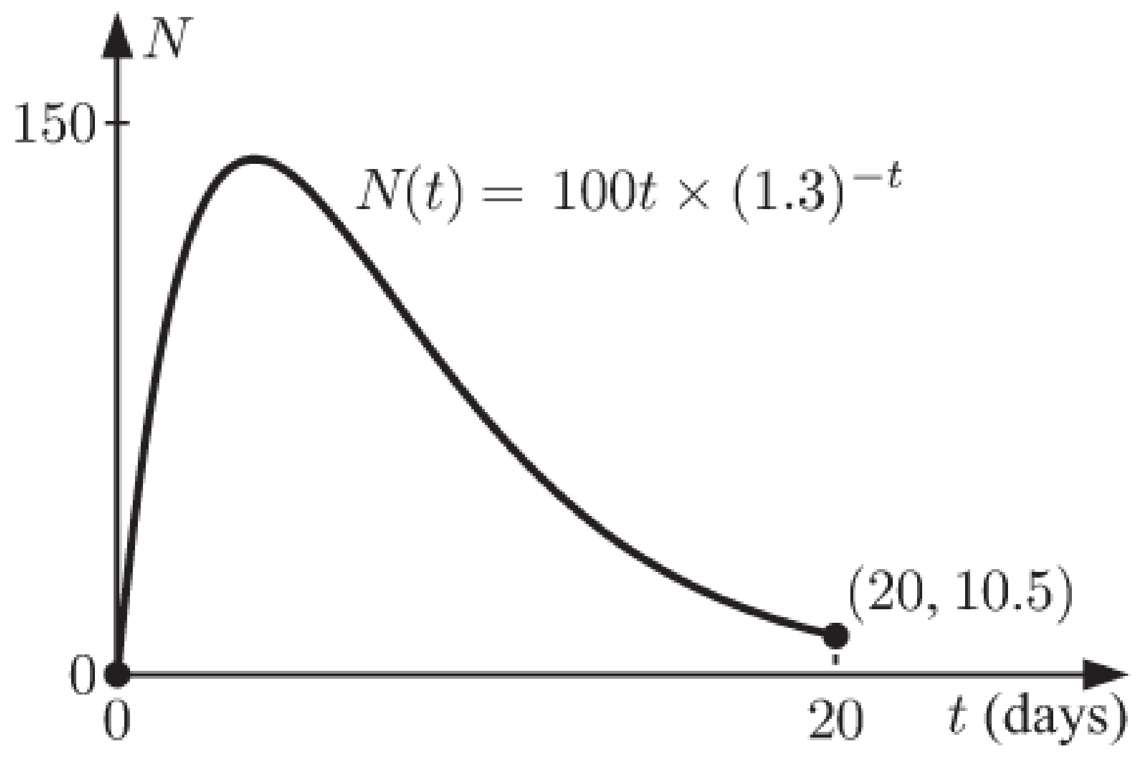
ii no turning points

iii Domain is $\{x \mid x \leqslant -2 \text{ or } x \geqslant 2\}$
Range is $\{y \mid y \geqslant 0\}$

iv



11 **a**



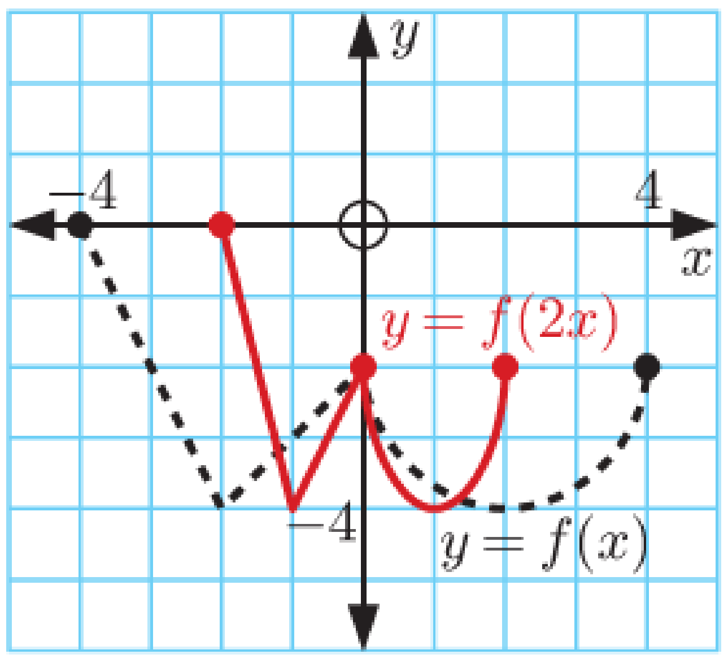
b ≈ 77 people **c** $t \approx 3.81$ days, ≈ 140 people

d ≈ 18.3 days

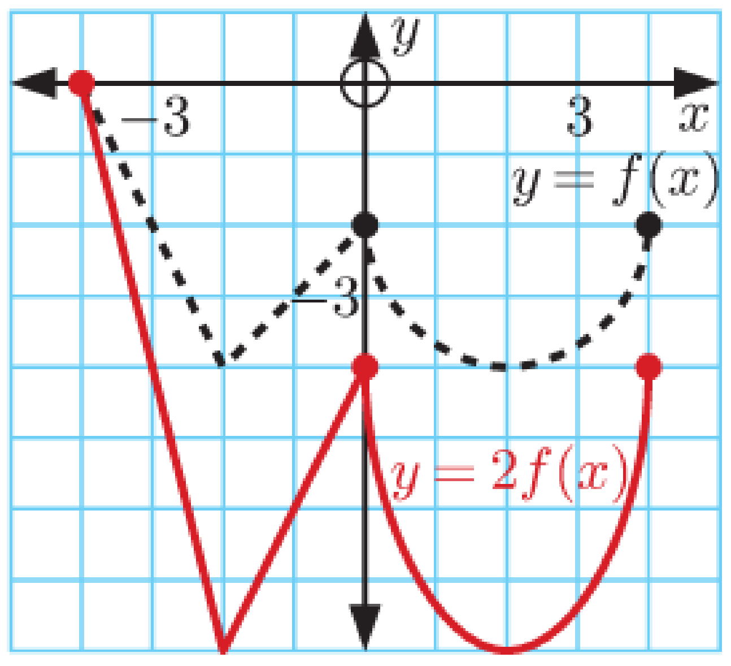
12 **a** $g(x) = 3x - x^2$ **b** $g(x) = 16 - x$

c $g(x) = \frac{1}{12}x + 2$

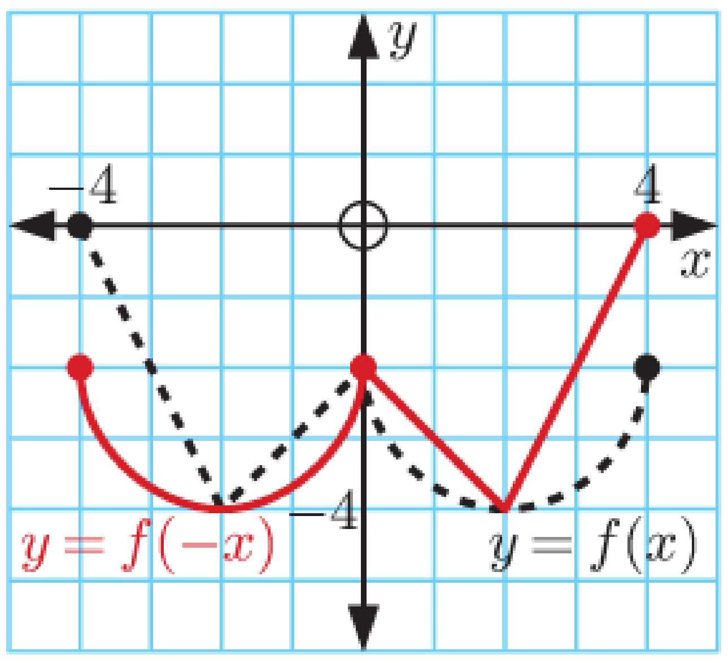
13 **a**



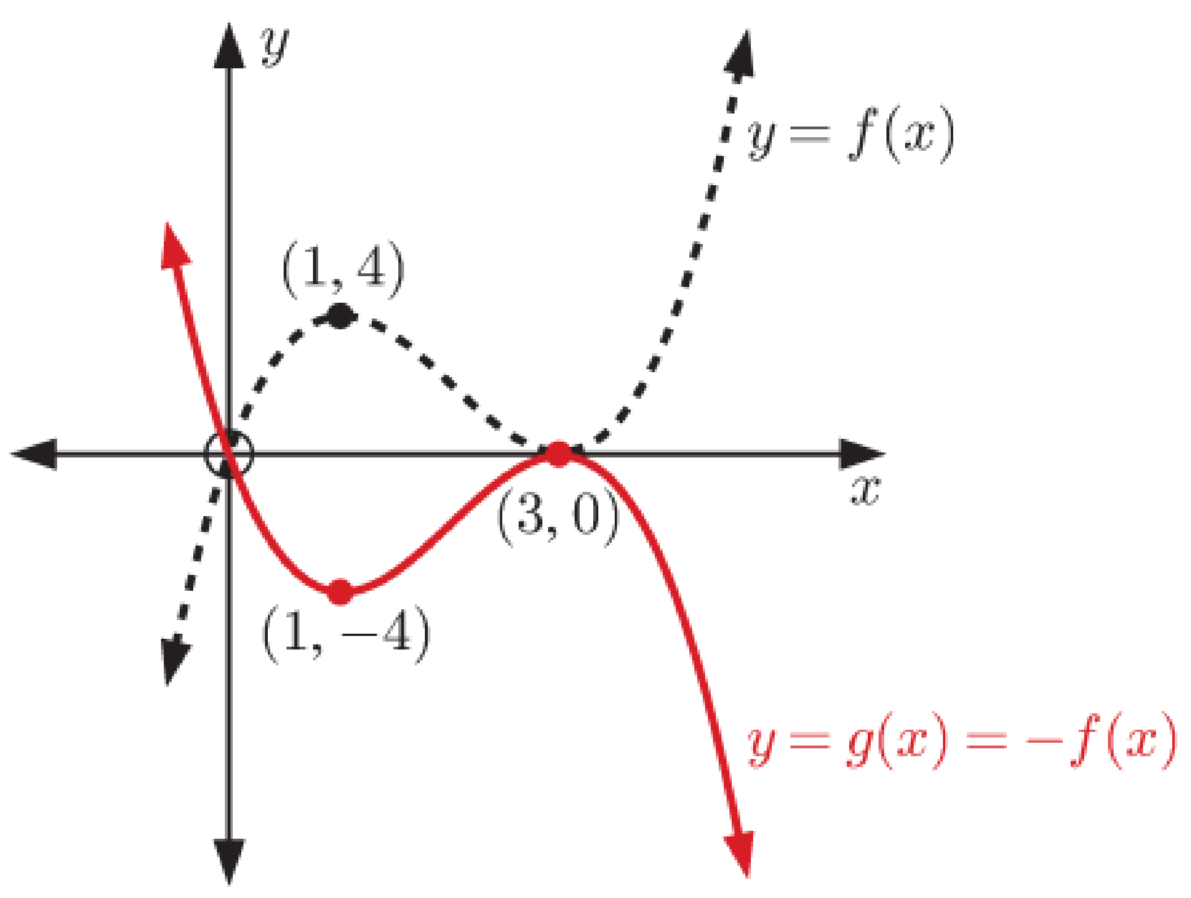
b



c



14 **a**

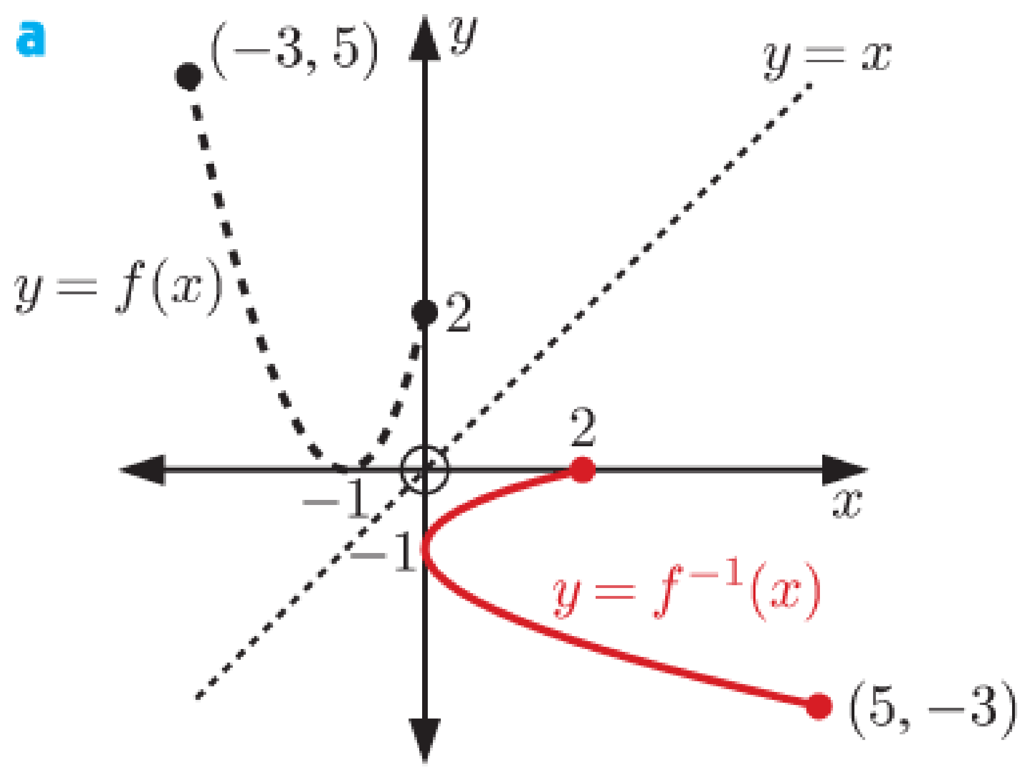


b max. turning point $(3,\ 0)$
min. turning point $(1,\ -4)$

15 **a** x-intercept -4, y-intercept -1

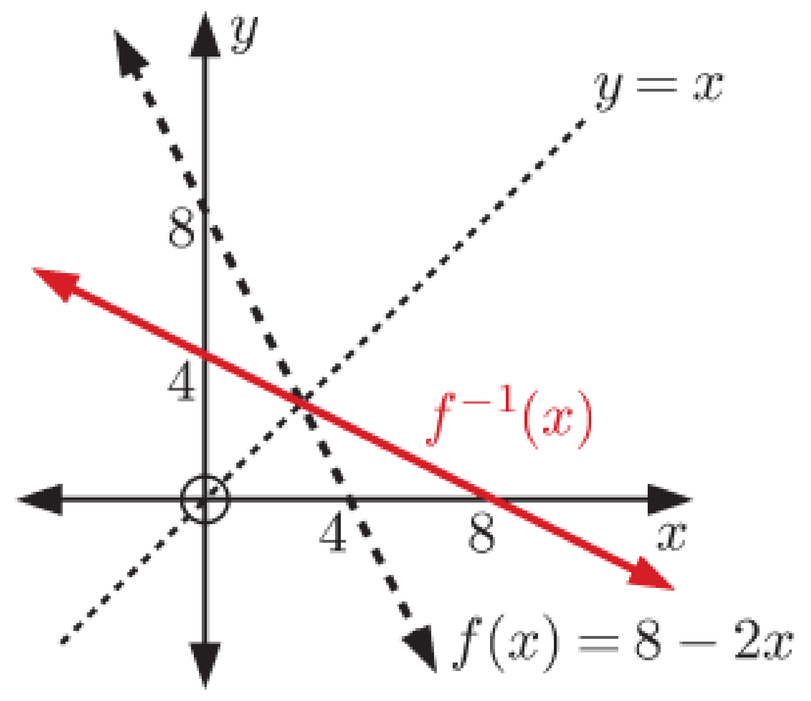
b x-intercept -1, y-intercept 4

16 **a**



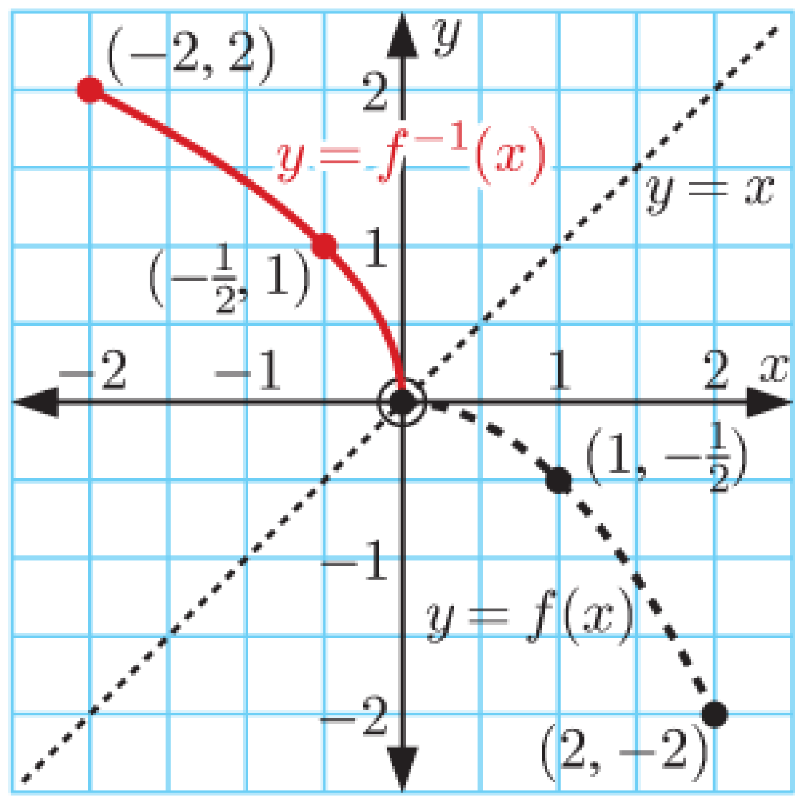
b No, $f(x)$ does not have an inverse since it does not pass the horizontal line test.

17 **a**, **b** x-intercept 4, y-intercept 8



c $f^{-1}(x) = -\frac{1}{2}x + 4$

18 **a**



b Range is
$\{y \mid 0 \leqslant y \leqslant 2\}$

c **i** $x = \sqrt{3}$

ii $x = -\frac{1}{2}$

EXERCISE 4A

1 **a** We have assumed that the cyclist travelled at a constant speed of 30 km h^{-1} the entire time. This is not realistic, the cyclist will travel at different speeds uphill, downhill, and on flat ground.

b 60 km

2 **a** Briony constructed her model by finding the equation of the line through $(0,\ 8)$ and $(12,\ 23)$. Briony has assumed that the laptop will charge at a constant rate, and indefinitely. These assumptions are not realistic, but they may be satisfactory for this problem.

b $0 \leqslant C \leqslant 100, \quad t \geqslant 0$
Briony's model suggests that it is possible to have charge greater than 100%, which is not possible.

c 73.6 minutes

d **i** The laptop likely charges at a faster rate earlier on, then at a slower rate as it approaches a full charge.
ii Yes, 73.6 minutes was a useful estimate.

3 **a** $t = \frac{3}{20}d$ seconds **b** 75 seconds

c Rick will take a longer time to run 500 m than our prediction. He will not be able to run 500 m at the same pace that he runs 100 m.

4 **a** **C**, it is usually coldest at dawn, and warmest in the afternoon. Daily temperature should be roughly periodic.

b $\approx 28°$C

5 **a** **B**, most light globes last for 200 hours, then the number of working globes quickly decreases.

b ≈ 95 light globes

6 **a** Darren has assumed that the Earth is perfectly spherical, and that the lighthouse is perpendicular to the Earth's surface. These are reasonable assumptions.

b $D \approx 22.6$ km

7 **a** $h = (\frac{2}{15}x + 3)$ m; we have assumed that the power line is hanging perfectly straight and that the poles are directly opposite.

b $0 \leqslant x \leqslant 15$ **c** 4 m

d Yes, if the poles are 3 m and 5 m high, we expect the height of the power line above the middle of the road to be 4 m.

e Yes, provided he knows the width and height of the truck, he should be able to reasonably decide whether or not the truck can fit.

8 **a** **i** ≈ 71.8 seconds **ii** ≈ 51.7 seconds
We have assumed that there is no current, and that Antonio can swim/jog at a constant speed, in all situations.

b Swimming to C then jogging to B appears to be quicker.

c Swim to the point 16.2 m from C and jog 83.8 m to B, total time ≈ 49.7 seconds.

9 $4\frac{8}{19}$ hours (≈ 4 h 25 min)

10 $9\frac{3}{8}$ minutes (≈ 9 min 23 s) **11** 15 hours **12** 6 hours

EXERCISE 4B

1 **a** $C = 2.5x$

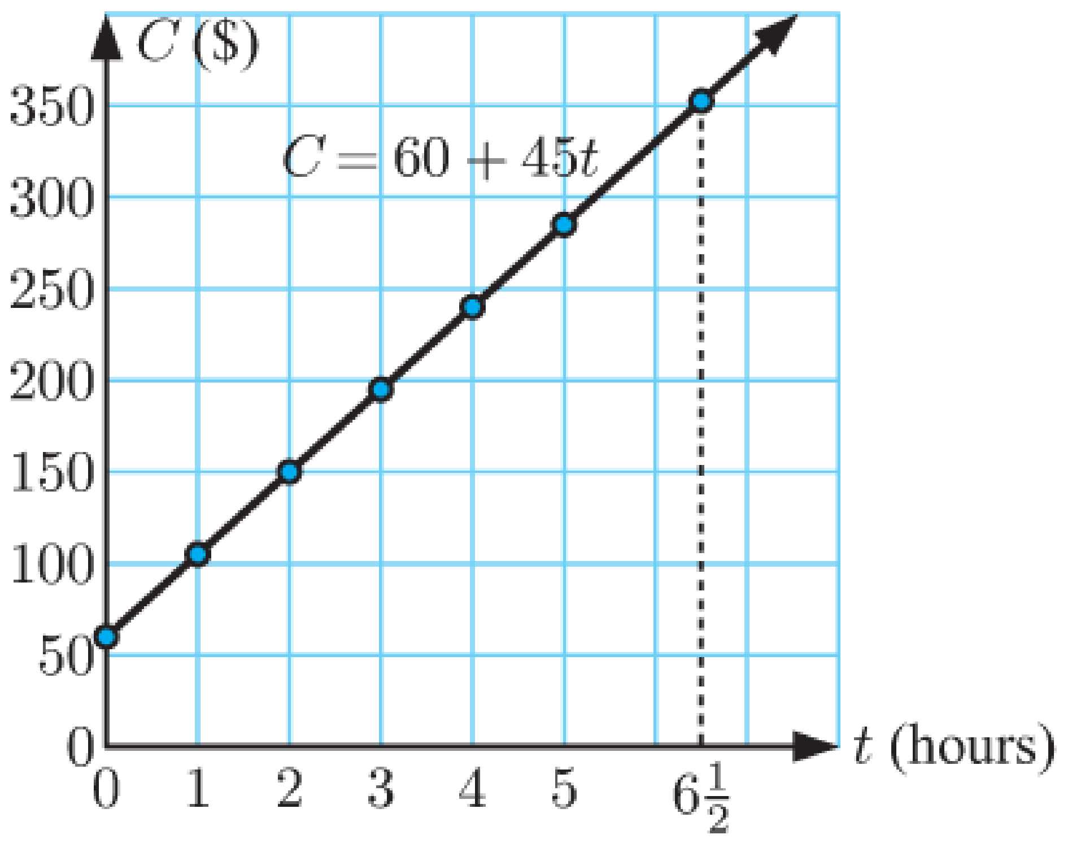
b The model fits the data points exactly.
$\therefore$ the model is exact.

c yes, \$30

2 **a**

Time (t hours)	0	1	2	3	4	5
Cost (\$$C$)	60	105	150	195	240	285

b



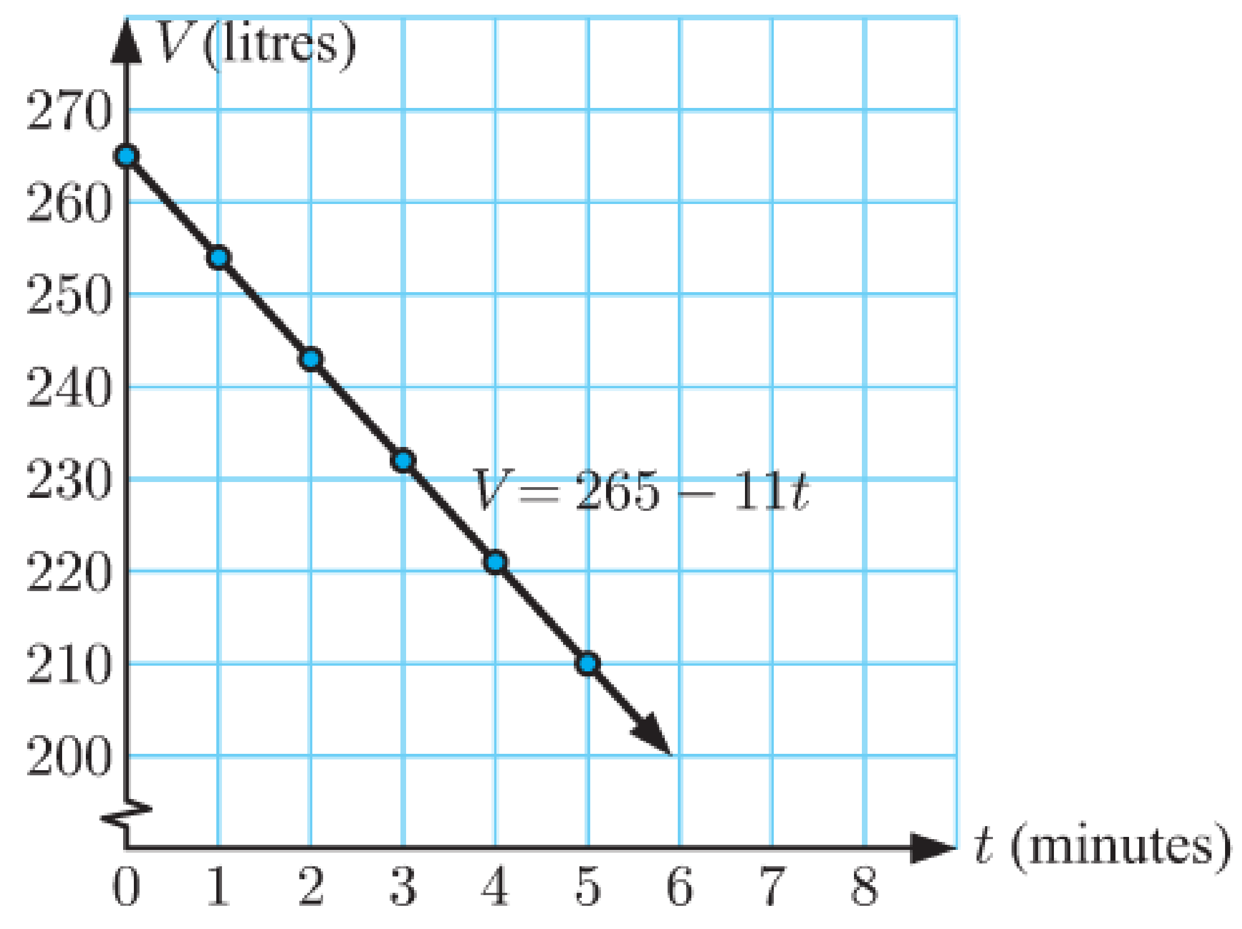
c $C = 45t + 60$

d \$352.50

3 **a**

Time (t minutes)	0	1	2	3	4	5
Volume (V litres)	265	254	243	232	221	210

b



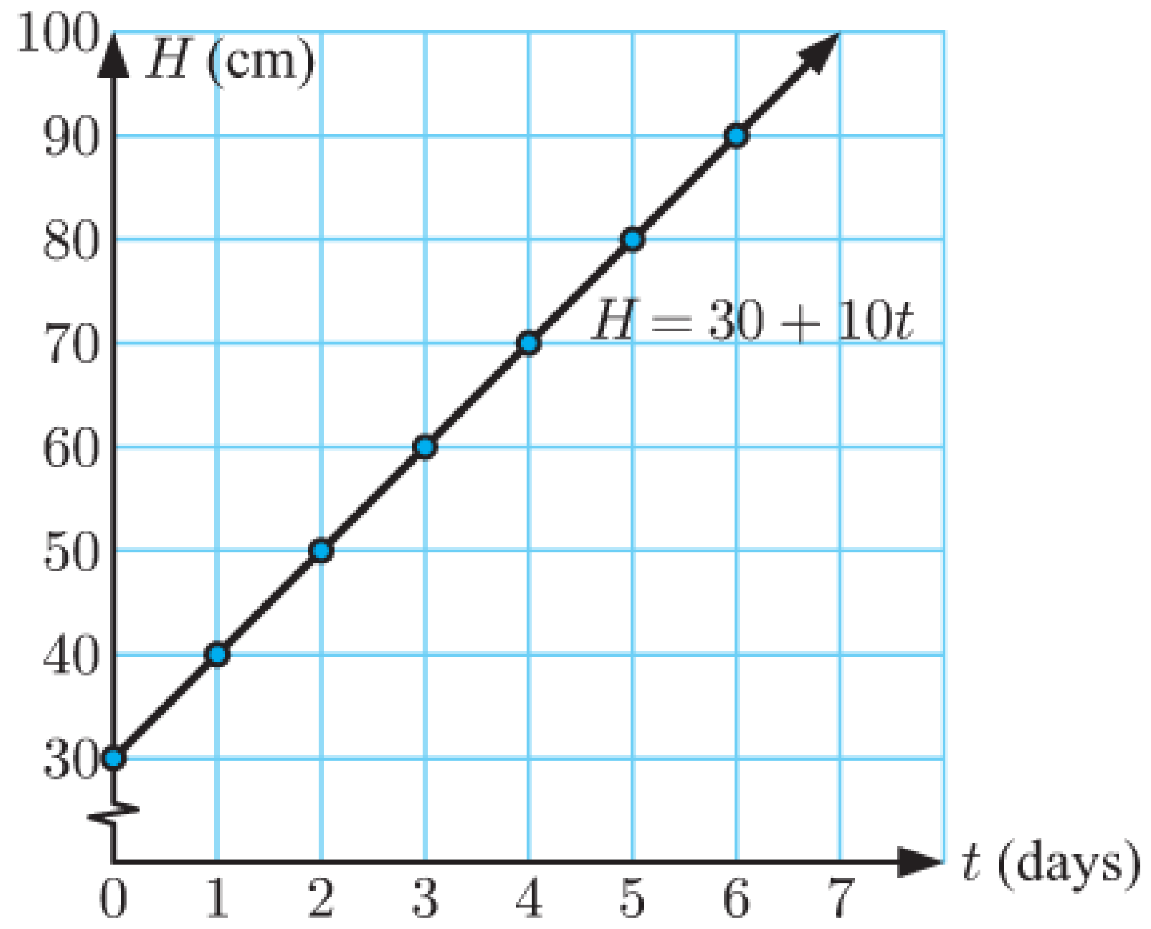
c $V = 265 - 11t$

d **i** 100 litres **ii** $\frac{265}{11} \approx 24.1$ minutes

4 **a**

t (days)	0	1	2	3	4	5	6
H (cm)	30	40	50	60	70	80	90

b



c It does not make sense to extend the model for $t < 0$ since the number of days after planting cannot be negative. It does not make sense to extend the model indefinitely for $t > 6$ since the plant will not continue to grow at this rate forever. However, we will assume that it does for the purposes of this model. $H(t)$ has domain $\{t \mid t \geqslant 0\}$.

d $H = 10t + 30$ **e** 7 days

5 **a** The points do not lie exactly on the line.
$\therefore$ the model is approximate.

b ≈ 160 pieces of litter; this estimate is likely to be inaccurate as it is an extrapolation. There would probably not be this much litter for Jack to pick up.

6 **a** 54.85 s, while the actual time was 57.0 s.
The values differ since the model is not exact. Some times will be higher or lower than what is predicted.

b The model would not be accurate for years much earlier than 1957, as the model would begin to predict unrealistically large times. The model should still be useful in estimating times a few years prior to 1957.

c 2005

d It is not suitable to extrapolate future times. Improvements in time no longer follow a linear trend.

EXERCISE 4C

1 **a** base rate: gradient $= 2$; the first 25 kL of water costs \$2 per kL.
higher rate: gradient $= 4$; each kL used above 25 kL costs \$4 per kL.

b \$160 **c** 15 kL

2 **a** 10 GB **b** \$5 per GB **c** \$45 **d** 18 GB

3 **a** 45 min **b** 16 km

c gradient $= 0.5$; the cyclist was travelling at 0.5 km per min for the first 10 km.

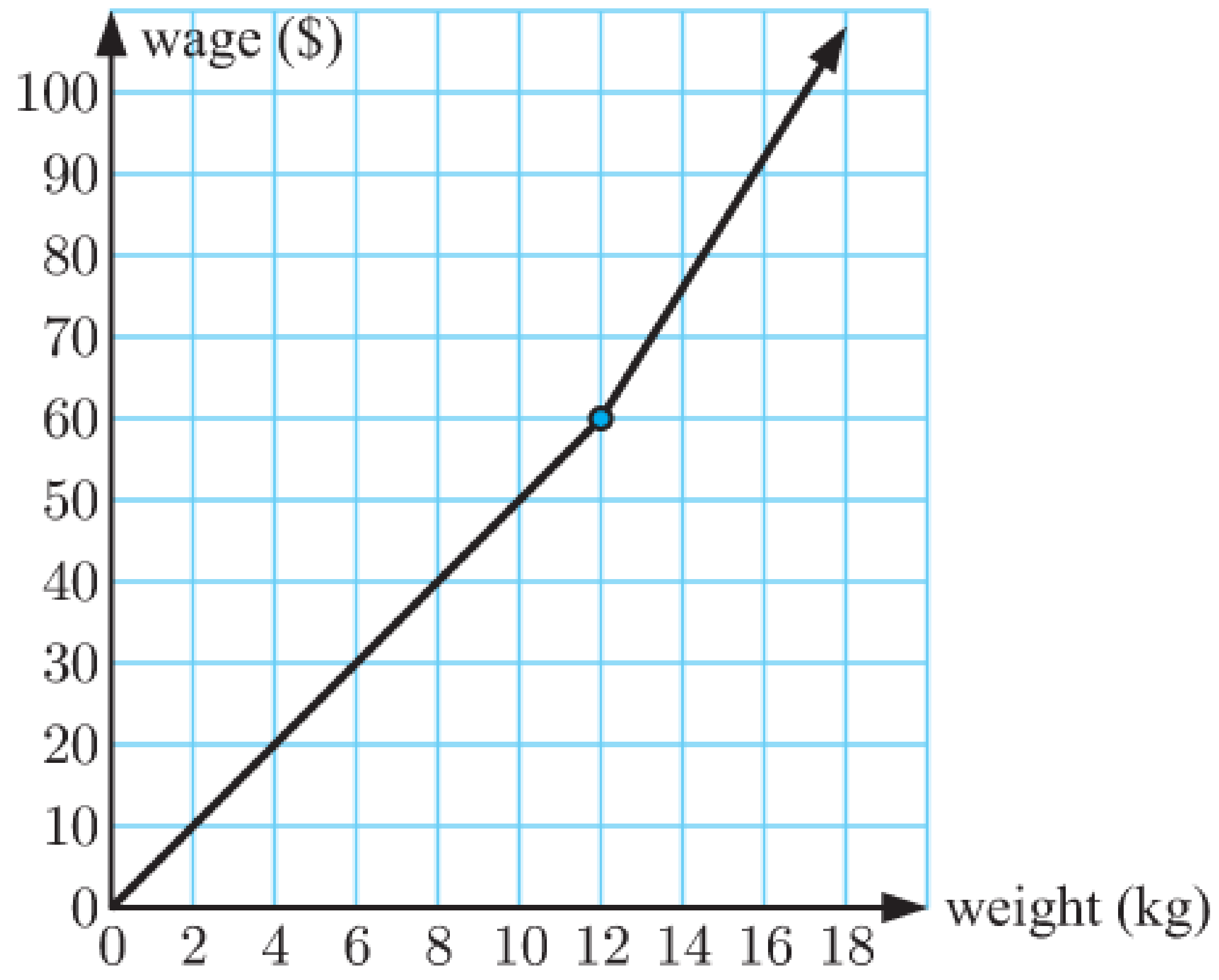
d 10 min **e** 12 km

4 **a** The car was initially travelling at 10 m s^{-1}, then braked for 6 seconds to a complete stop. The car stopped for 5 seconds before accelerating for 4 seconds to a speed of 7 m s^{-1} which it maintained for the remaining 5 seconds.

b We have assumed that the car accelerates and decelerates at a constant rate. These assumptions are reasonable.

c **i** 5 m s^{-1} **ii** 30 m **d** 79 m

5 **a**



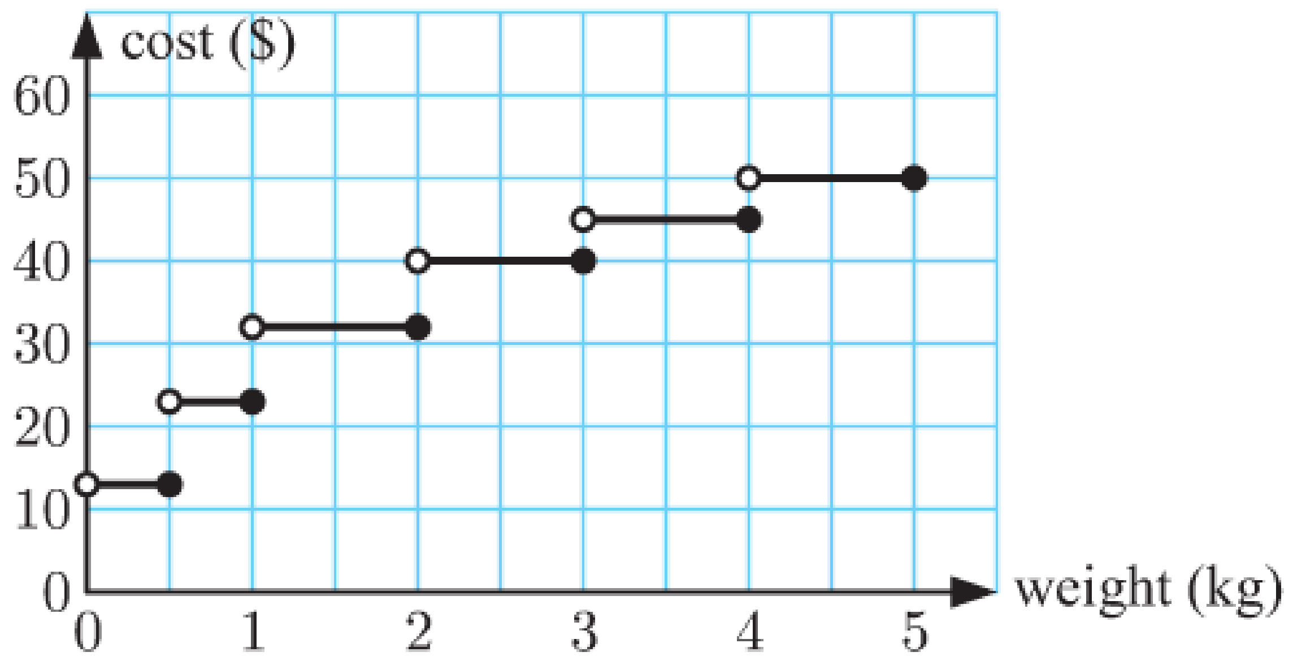
b \$45

c 17 kg

6 **a** **i** \$75 **ii** \$95 **b** **i** 15 min **ii** 60 min

c \$155, assuming that the pattern shown in the graph continues.

7 **a**



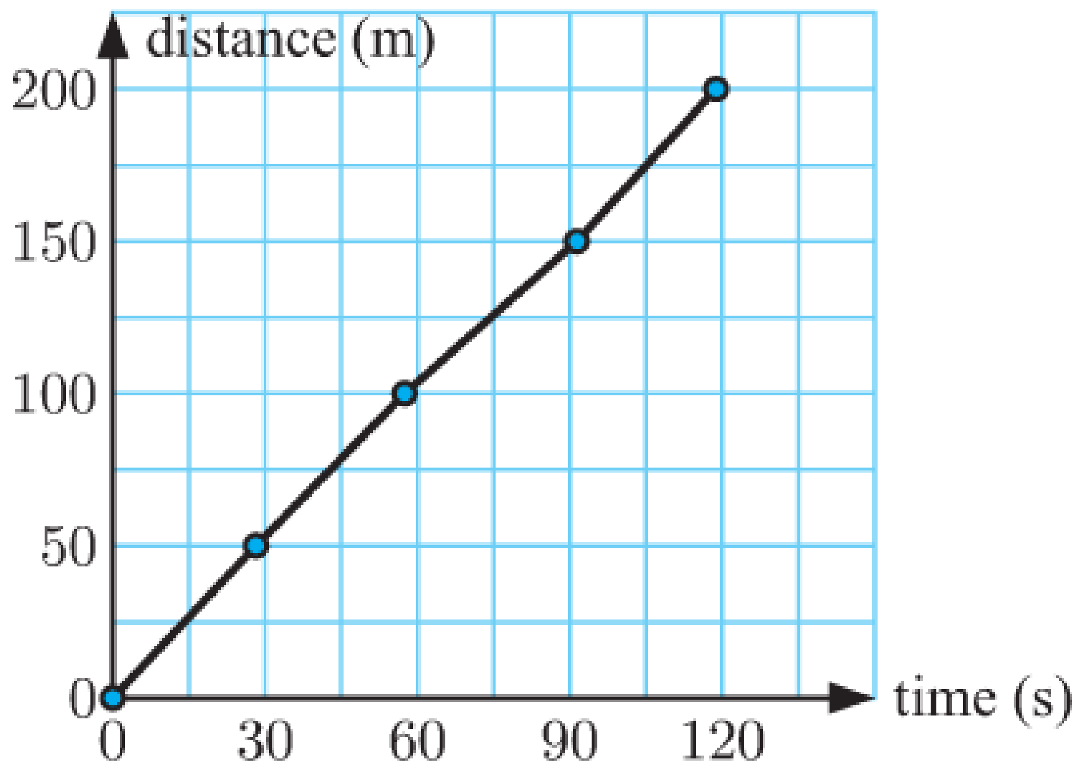
b **i** \$23

ii \$32

iii \$45

c \$22

8 **a**



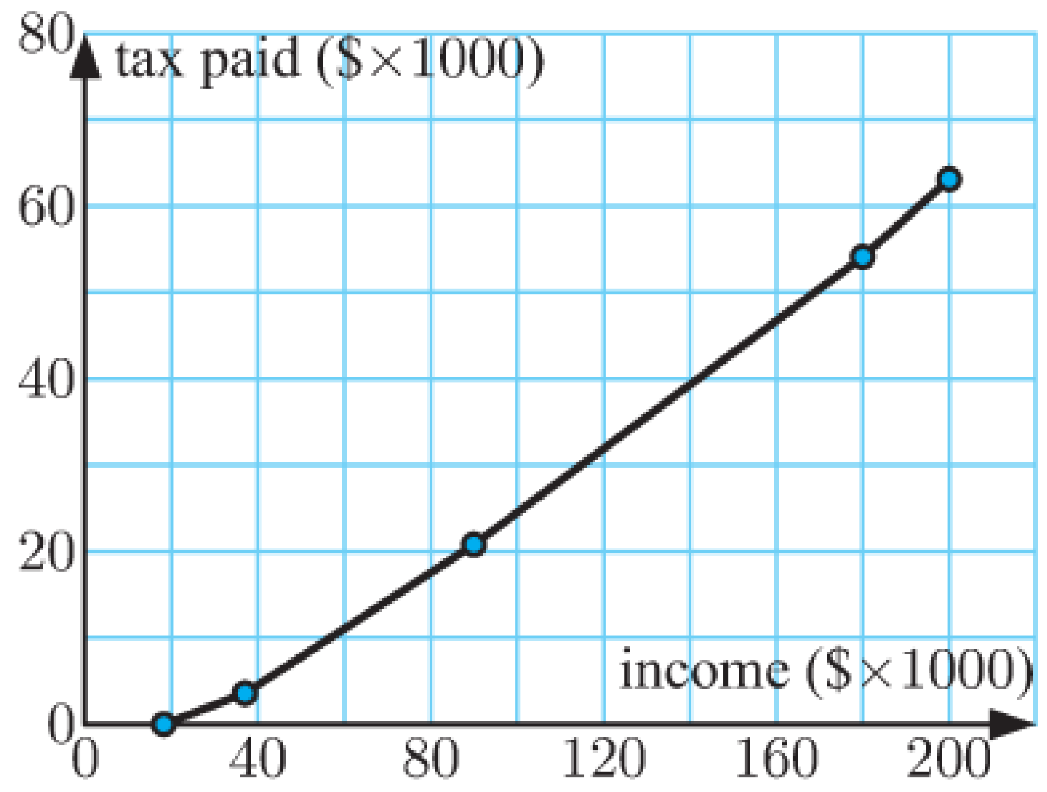
b We assume Laura swims at a constant rate in each 50 m leg.

c ≈ 104 m

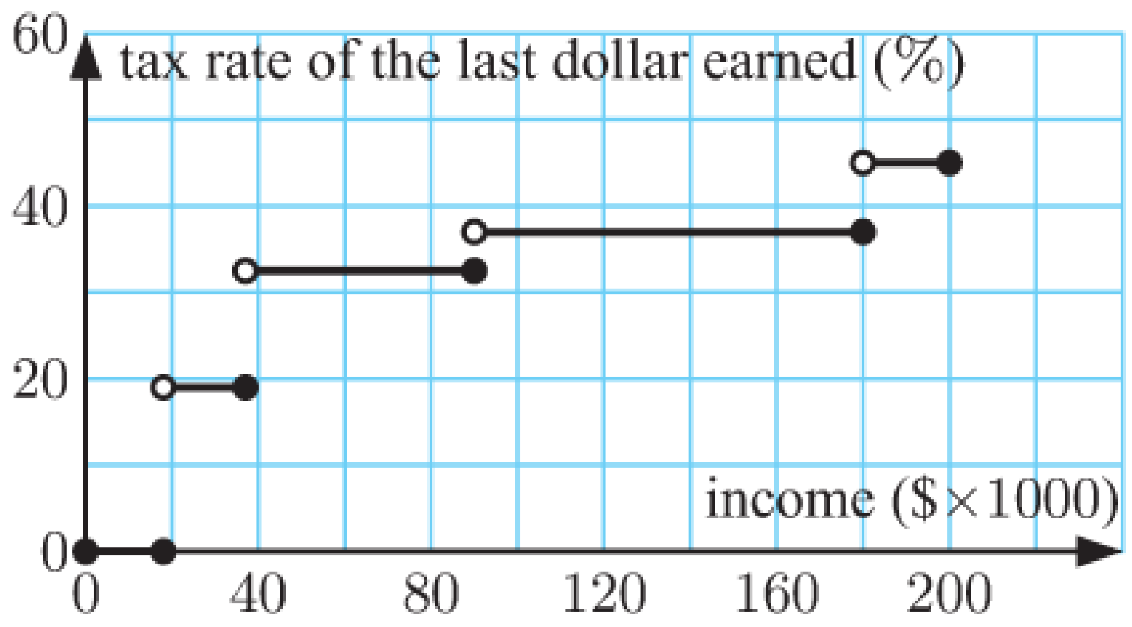
9 **a** £448.80 **b** **i** £9628.80 **ii** £43 543.20

c **i** £50 118.15 **ii** ≈ 32.6%

10 **a**



b



c **i** \$1862 **ii** \$7244.50 **iii** \$23 072.50

EXERCISE 4D

1 **a** $y = \frac{1}{2}x + 1$

2 **a** $a + b + c = 7$
$4a + 2b + c = 10$
$9a + 3b + c = 11$

b $a = -1, \ b = 6, \ c = 2$

c $y = -x^2 + 6x + 2$

3 **a** $y = -3x^2 + 10x$ **b** $y = x^3 - 4x^2 + 5x$

c $y = 2x + \dfrac{6}{x} - 3$

4 $a = 0.2, \ b = 0.5, \ c = 1$

5 **a** $h = \dfrac{36}{r} - r$

b **Hint:** Make h the subject in the equation of the surface area of a cylinder.

c 72π cm^2

d $h = -5$ cm, which is not reasonable, as height cannot be negative.

e 0 cm $< r <$ 6 cm

6 **a** $d = 5.209$ **b** $a = 0.003, \ b = -0.117, \ c = 0.921$

c 0.829 m

REVIEW SET 4A

1 **a** $D = \frac{5}{2}t$ m **b** 1500 m

c The actual distance will be less than our prediction. Ben will not be able to kayak at the same speed for 10 minutes as he can for 40 seconds.

2 **a** **A** **b** ≈ 40°C

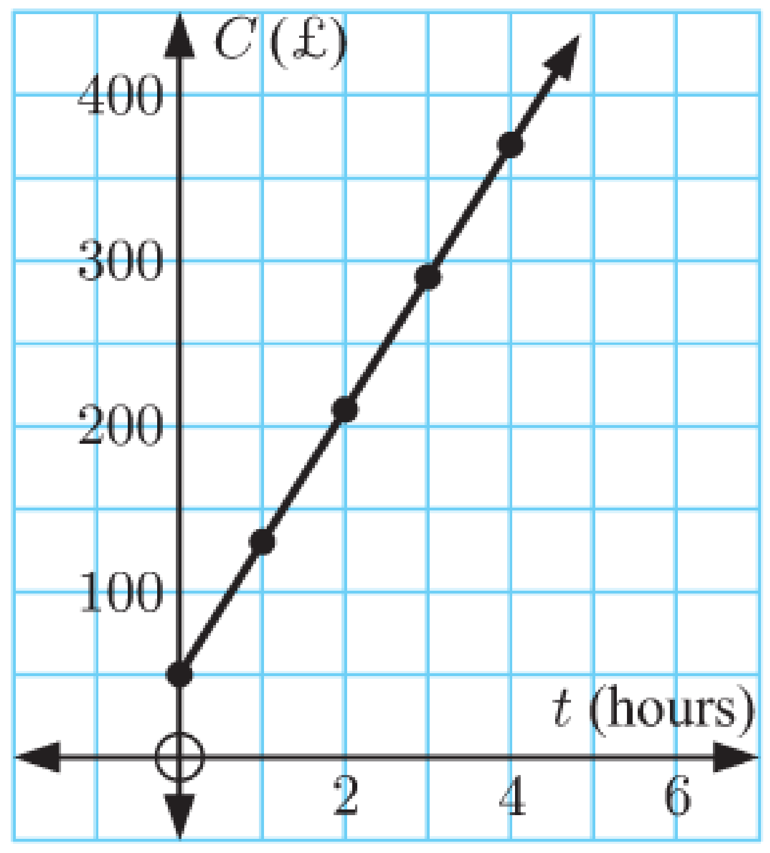
3 **a** gradient = −5, A-intercept = 160
The barrel initially contained 160 L of oil and is leaking at a rate of 5 L per minute.

b $A = 160 - 5t$ **c** 85 litres **d** $0 \leqslant t \leqslant 32$

4 **a**

Time (t hours)	0	1	2	3	4
Cost (£C)	50	130	210	290	370

b



c $C = 80t + 50$

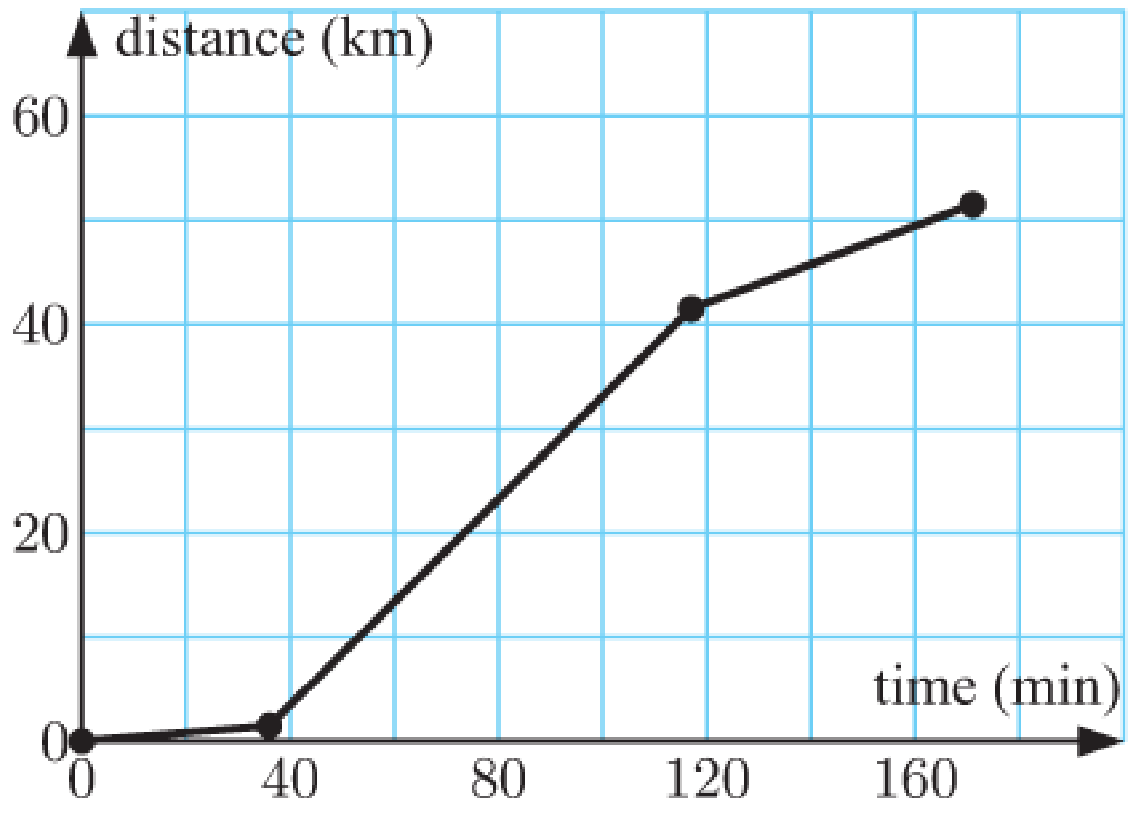
d £530

5 **a** $1\frac{1}{2}$ days

b The assumption is not reasonable, the job will take more time than our prediction in **a**.

6 **a** **i** \$25 **ii** \$35 **b** 1 hour

7 **a**



b We have assumed that Alana travels at a constant speed during each leg of the race.

c ≈ 13 km

8 **a** $y = 3x + \dfrac{4}{x}$ **b** $y = 2x^3 - 4x^2 + 3x$

9 **a** $a = \frac{7}{100}$, $b = \frac{6}{5}$ **b** 99 m

c No, braking distance is dependent on the car and the reaction time of the driver.

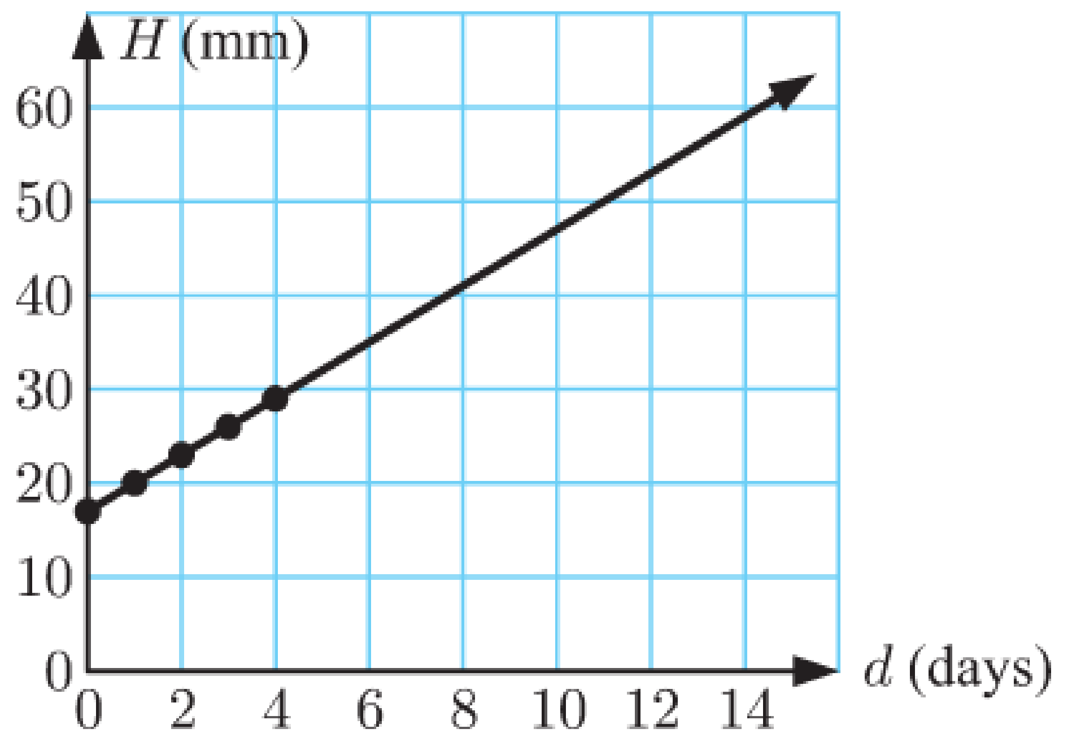
REVIEW SET 4B

1 $1\frac{8}{13}$ hours ($\approx$ 1 h 37 min), assuming that Todd and Sophie can work together at the same rate that they could individually.

2 **a**

d (days)	0	1	2	3	4
H (mm)	17	20	23	26	29

b



c $H = 3d + 17$ **d** 53 mm **e** every 21 days (3 weeks)

3 **a** Yes, these are reasonable assumptions. Not all of the wood in a tree is usable, and assuming that the trees are cylindrical makes it easier to perform calculations.

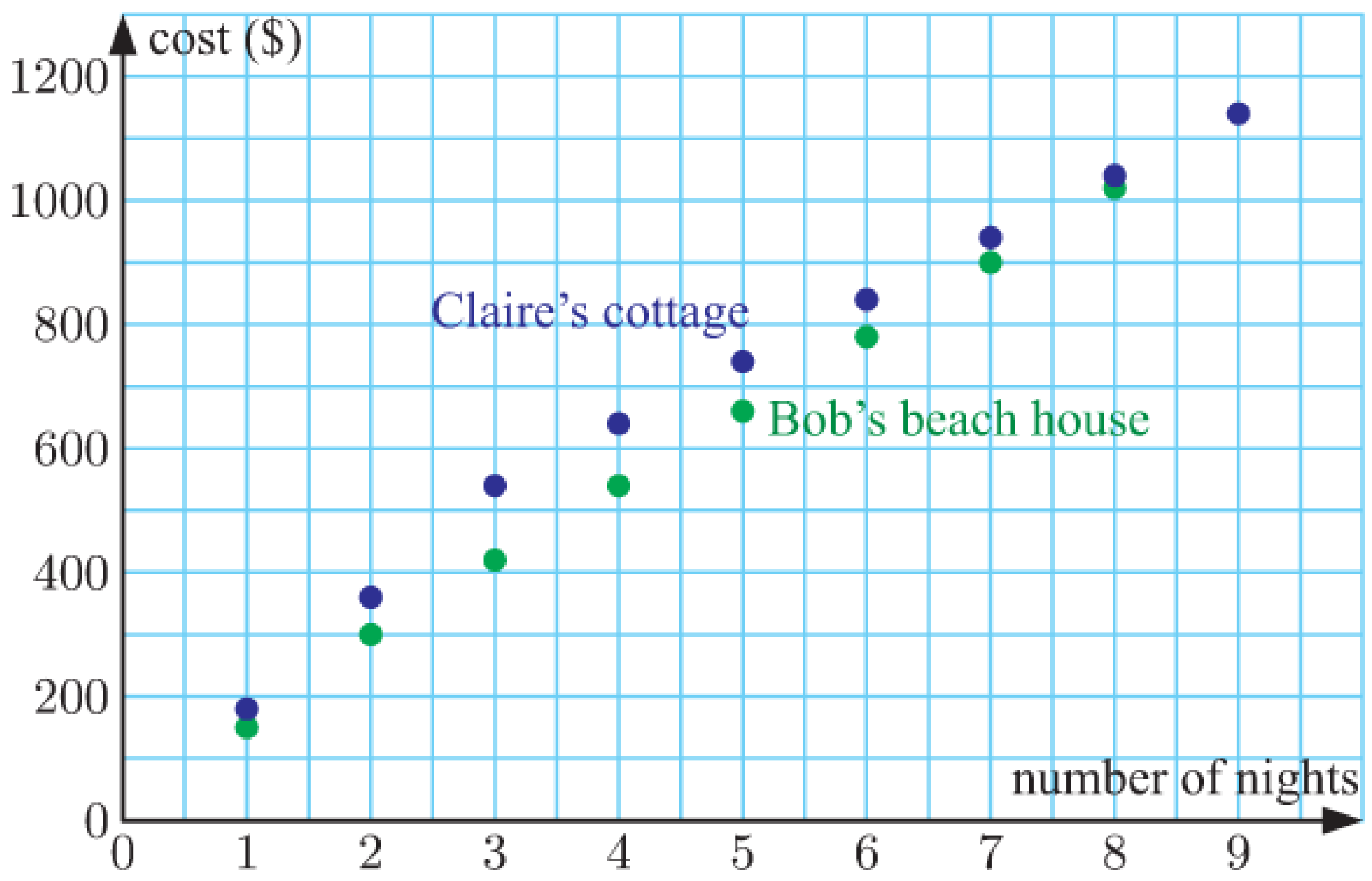
b $V = 0.8 \times \pi(0.225)^2 h$ m^3 or $V = \dfrac{81\pi h}{2000}$ m^3

c ≈ 1.91 m^3

4 **a** The points do not lie exactly on the line.
$\therefore$ the model is approximate.

b 2040. This is an extrapolation, so this prediction is not likely to be reasonable.

5 **a**

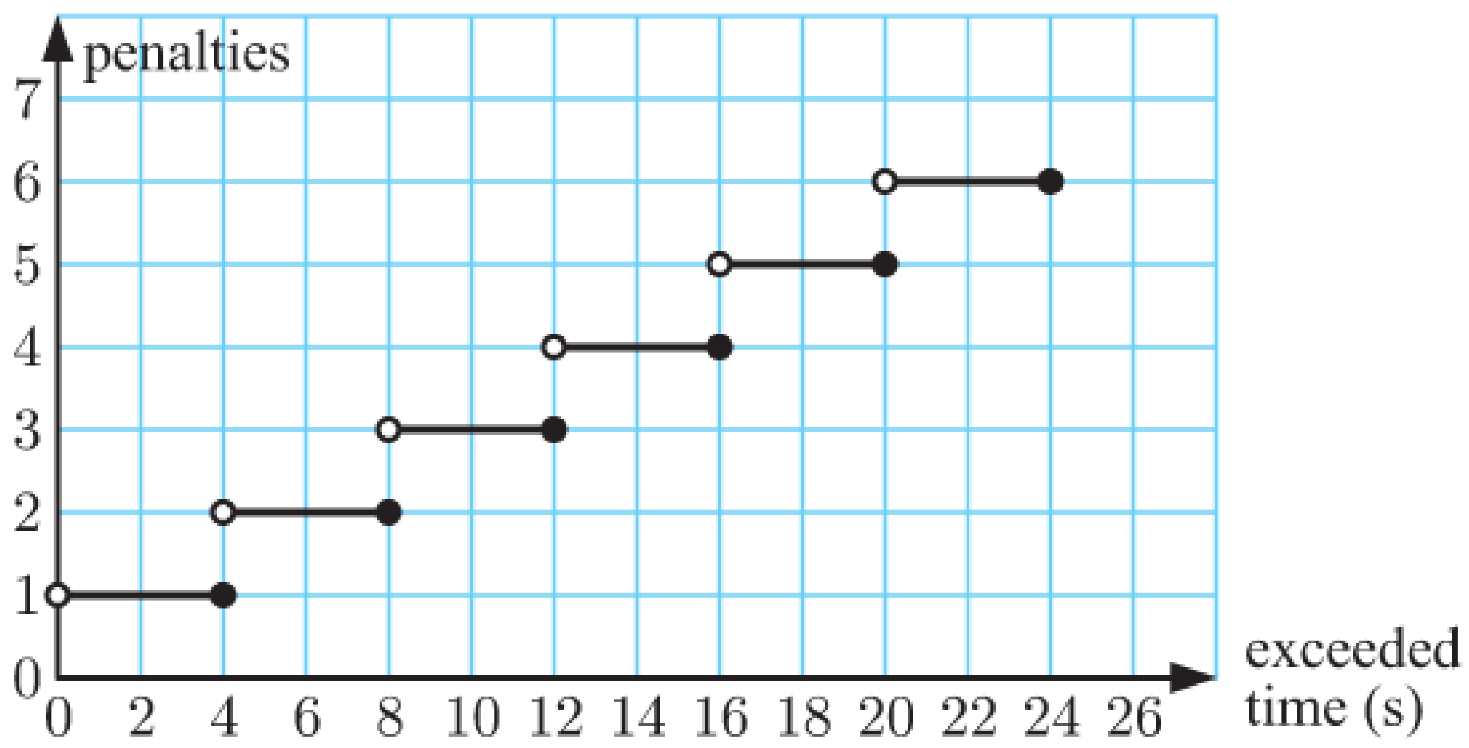


b **i** \$540 **ii** \$640 **c** 9 nights

d Bob's Beach House will be cheaper by \$20.

6 **a** piecewise linear model **b** $\approx 32.7\%$ **c** €36 841.48

7 **a**



b **i** 1 time penalty **ii** 2 time penalties
iii 0 time penalties **iv** 4 time penalties

8 **a** $y = x + 2\sqrt{x}$ **b** $y = -x^2 + 5x + \dfrac{8}{x}$

9 **a** If $x = 0$, the metal cannot be folded into a tray, and hence $V = 0$. $\therefore$ $d = 0$

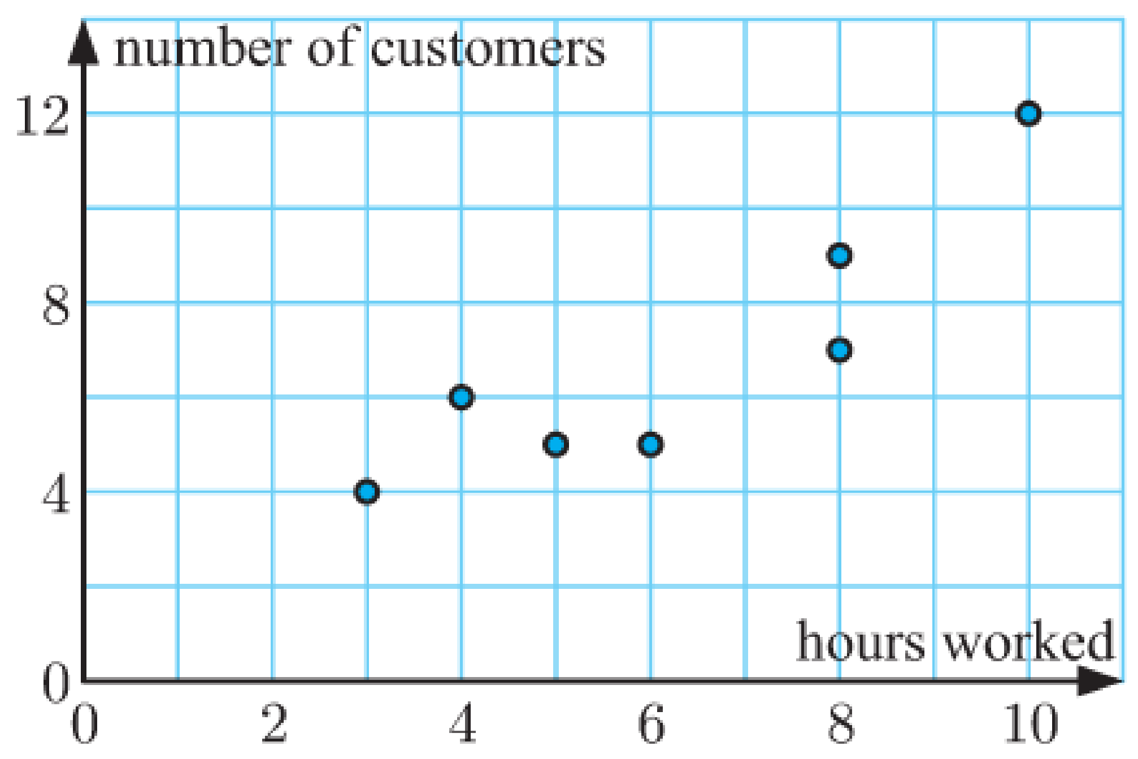
b $a = 4$, $b = -120$, $c = 900$ **d** $0 < x < 15$

EXERCISE 5A

1 **a** weak, positive, linear correlation, with no outliers
b strong, negative, linear correlation, with one outlier
c no correlation
d strong, negative, non-linear correlation, with one outlier
e moderate, positive, linear correlation, with no outliers
f weak, positive, non-linear correlation, with no outliers

2 **a** *Hours worked* is the explanatory variable.
Number of customers is the response variable.

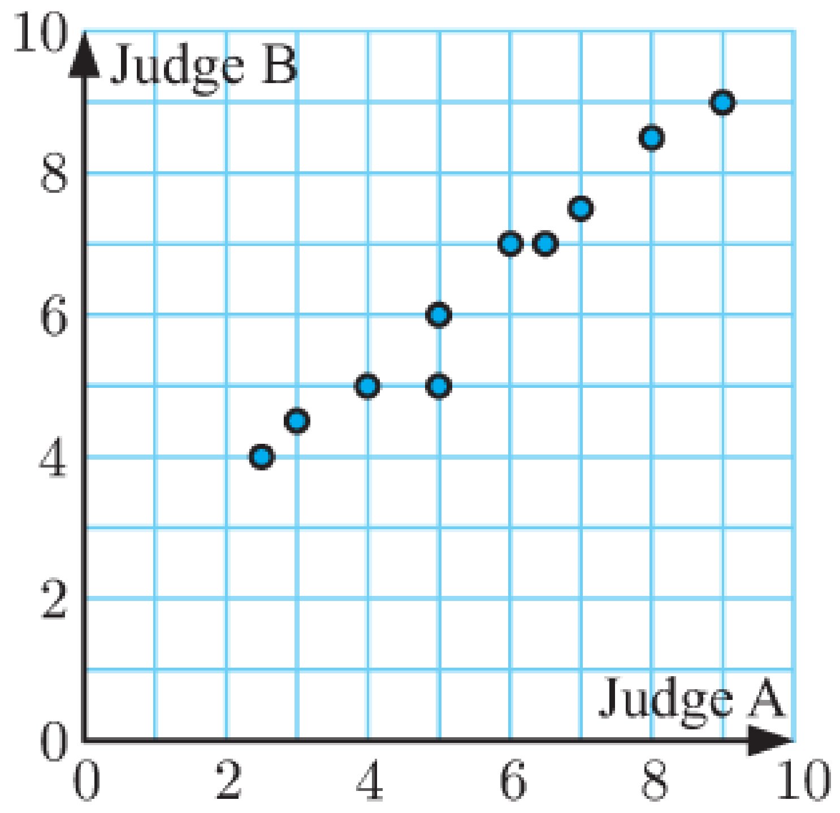
b



c **i** Monday and Friday **ii** Wednesday and Sunday

d The more hours that Tiffany works, the more customers she is likely to have.

3 **a**

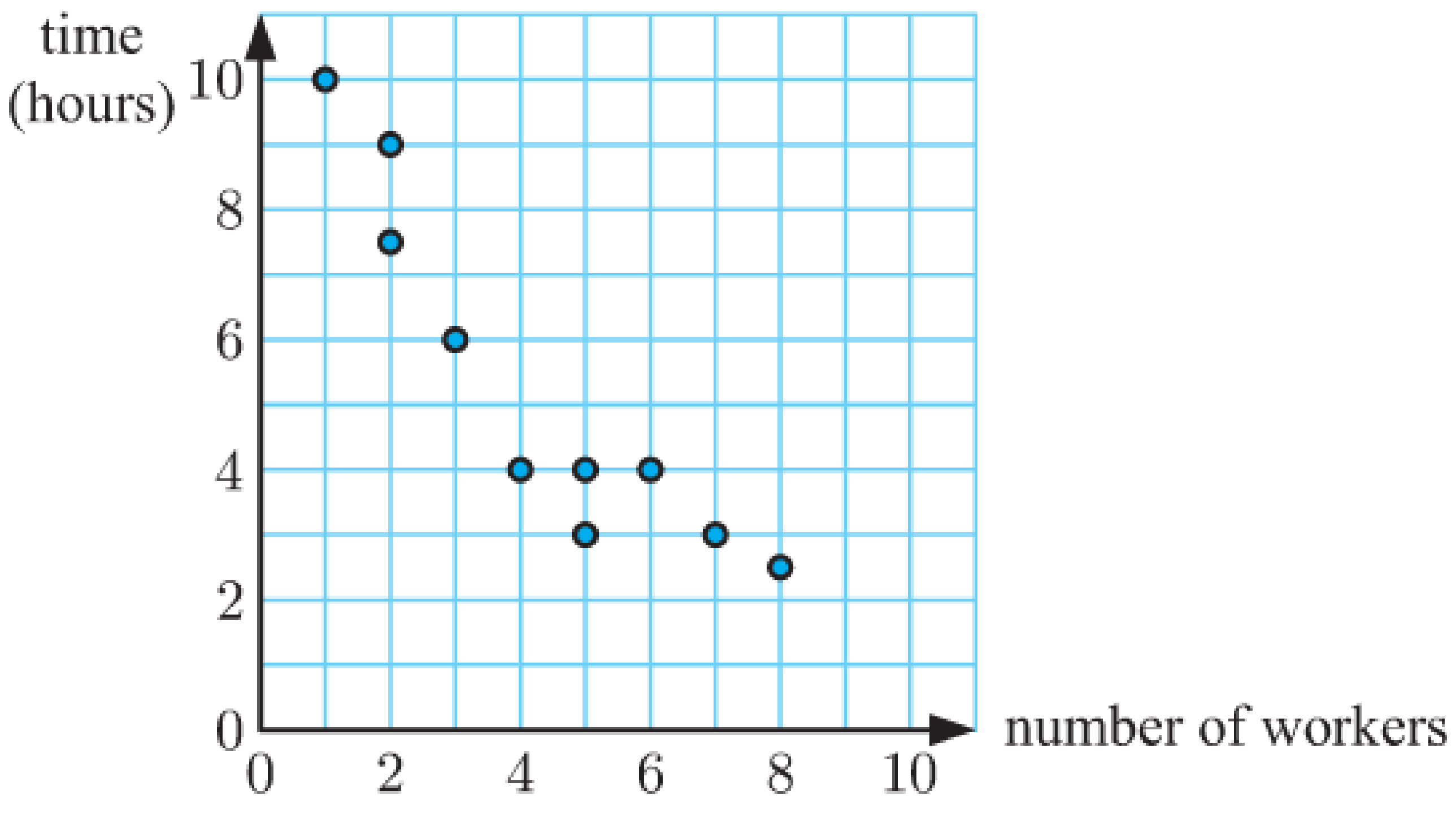


b There appears to be **strong, positive**, **linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.

c No, the scores are related to the quality of the ice skaters' performances.

4 **a** **i** job G **ii** job C

b



c There is a strong, negative, non-linear correlation between *number of workers* and *time*.

5 **a** **D** **b** **A** **c** **B** **d** **C**

6 **a** There is a moderate, positive, linear correlation between *hours of study* and *marks obtained*.

b The test is out of 50 marks, so the outlier (> 50) appears to be an error. It should be discarded.

c Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.

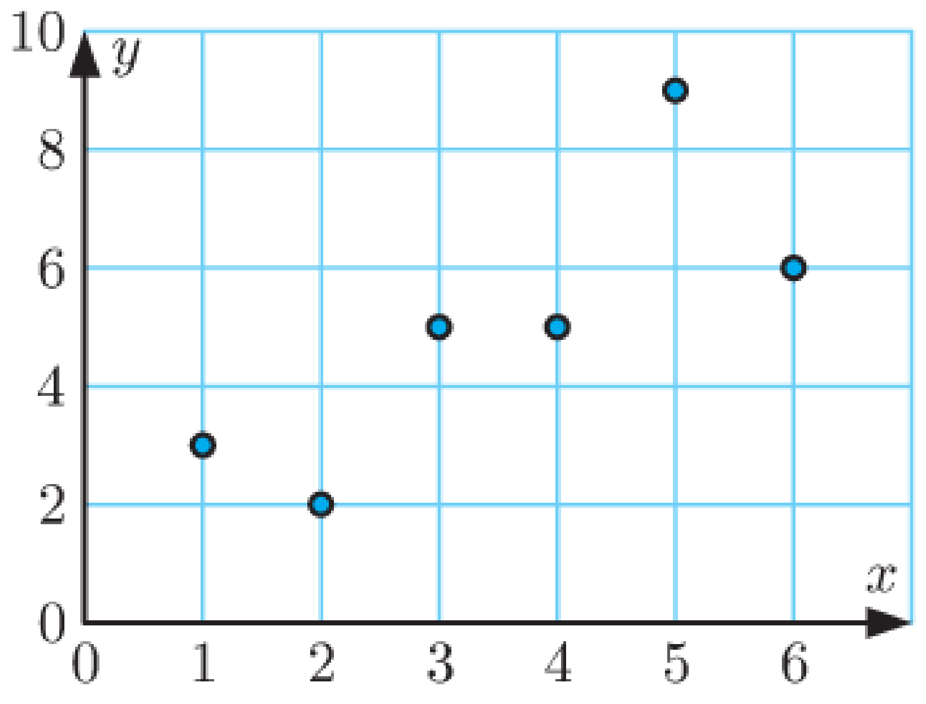
7 **a** Not causal, dependent on genetics and/or age.
b Not causal, dependent on the size/location of the fire.
c Causal, an increase in advertising is likely to cause an increase in sales.
d Causal, the childrens' adult height is determined by the genetics they receive from their parents to a great extent.
e Not causal, dependent on population of town.

EXERCISE 5B

1 weak, positive correlation

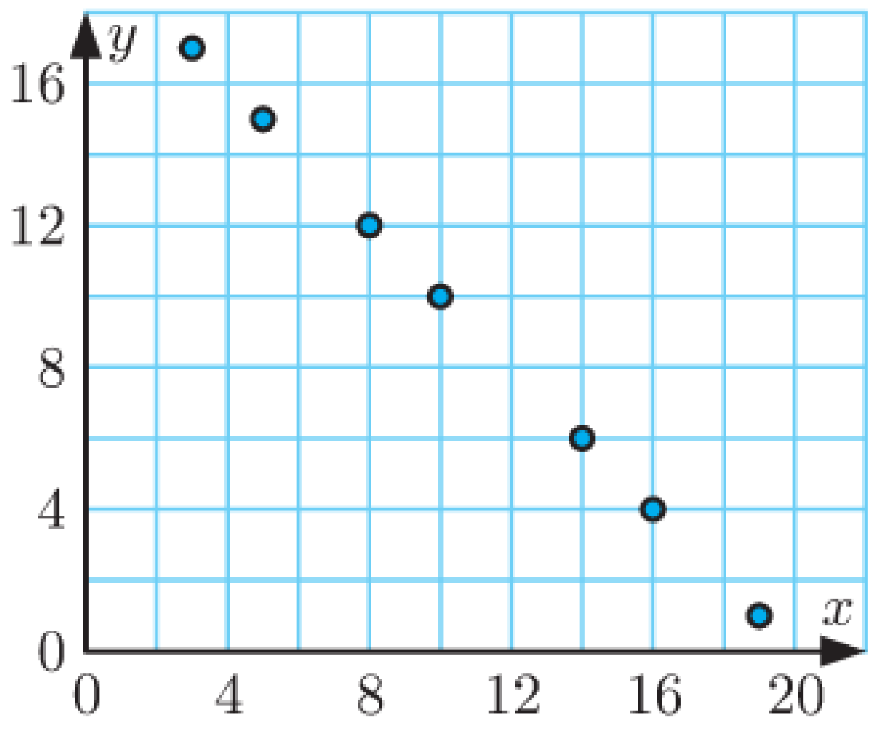
2 **a** B **b** A **c** D **d** C **e** E

3 **a** **i**



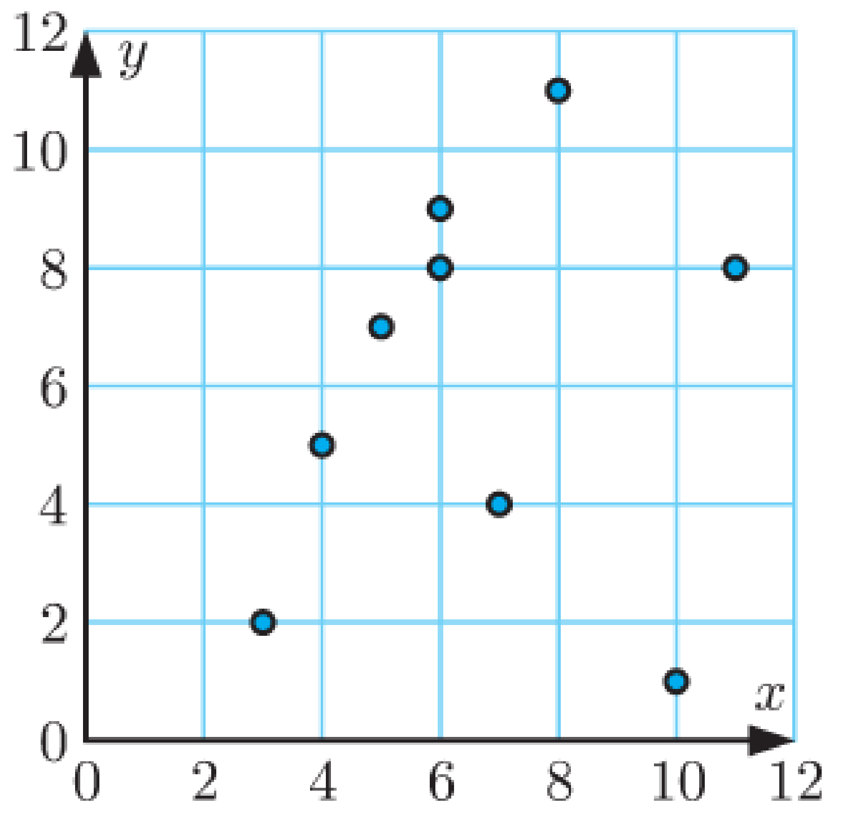
ii $r \approx 0.786$
iii moderate, positive correlation

b **i**



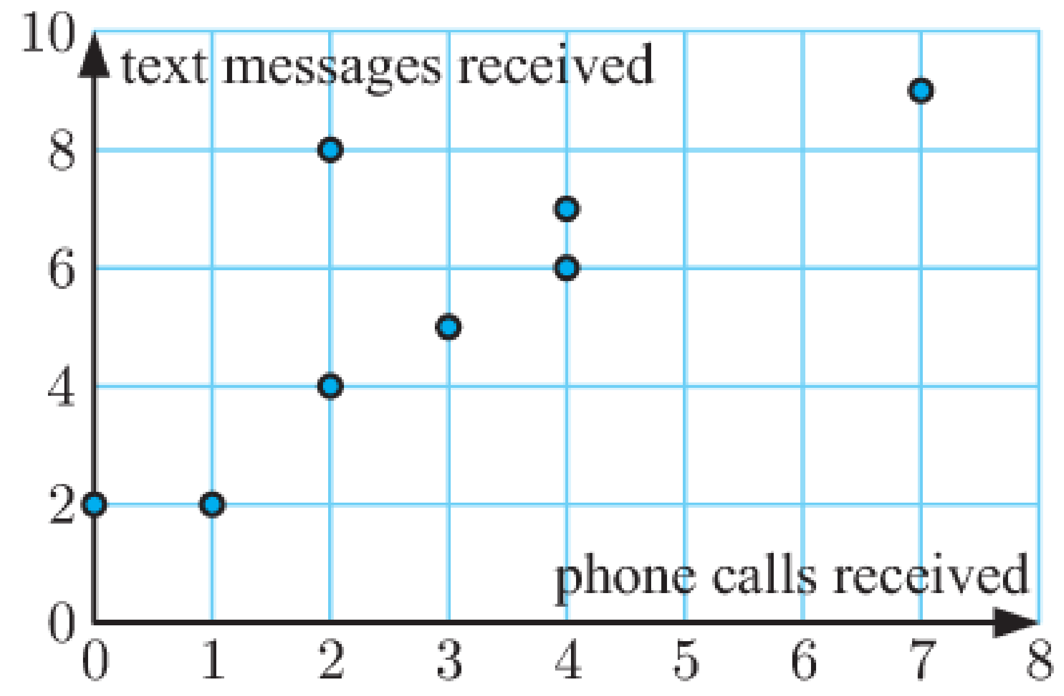
ii $r = -1$
iii perfect, negative correlation

c **i**



ii $r \approx 0.146$
iii very weak, positive correlation

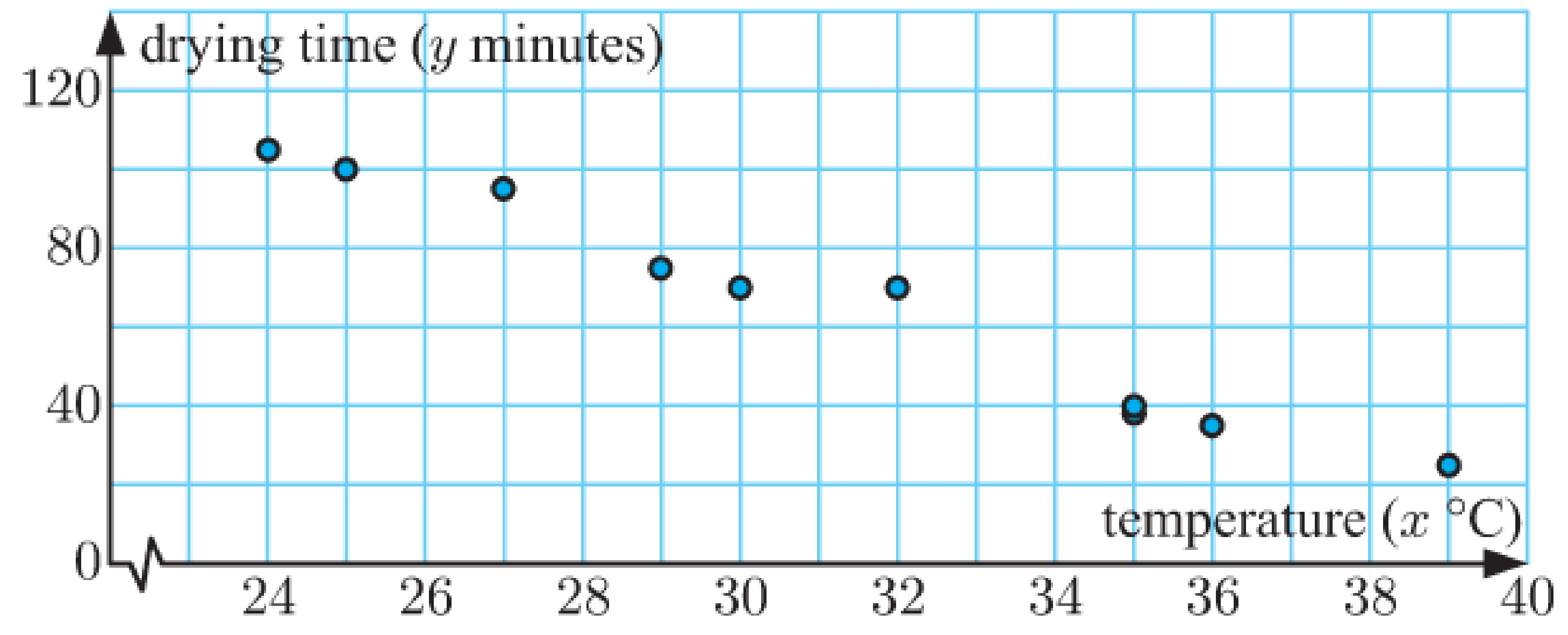
4 **a**



b $r \approx 0.816$
c moderate, positive correlation
d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

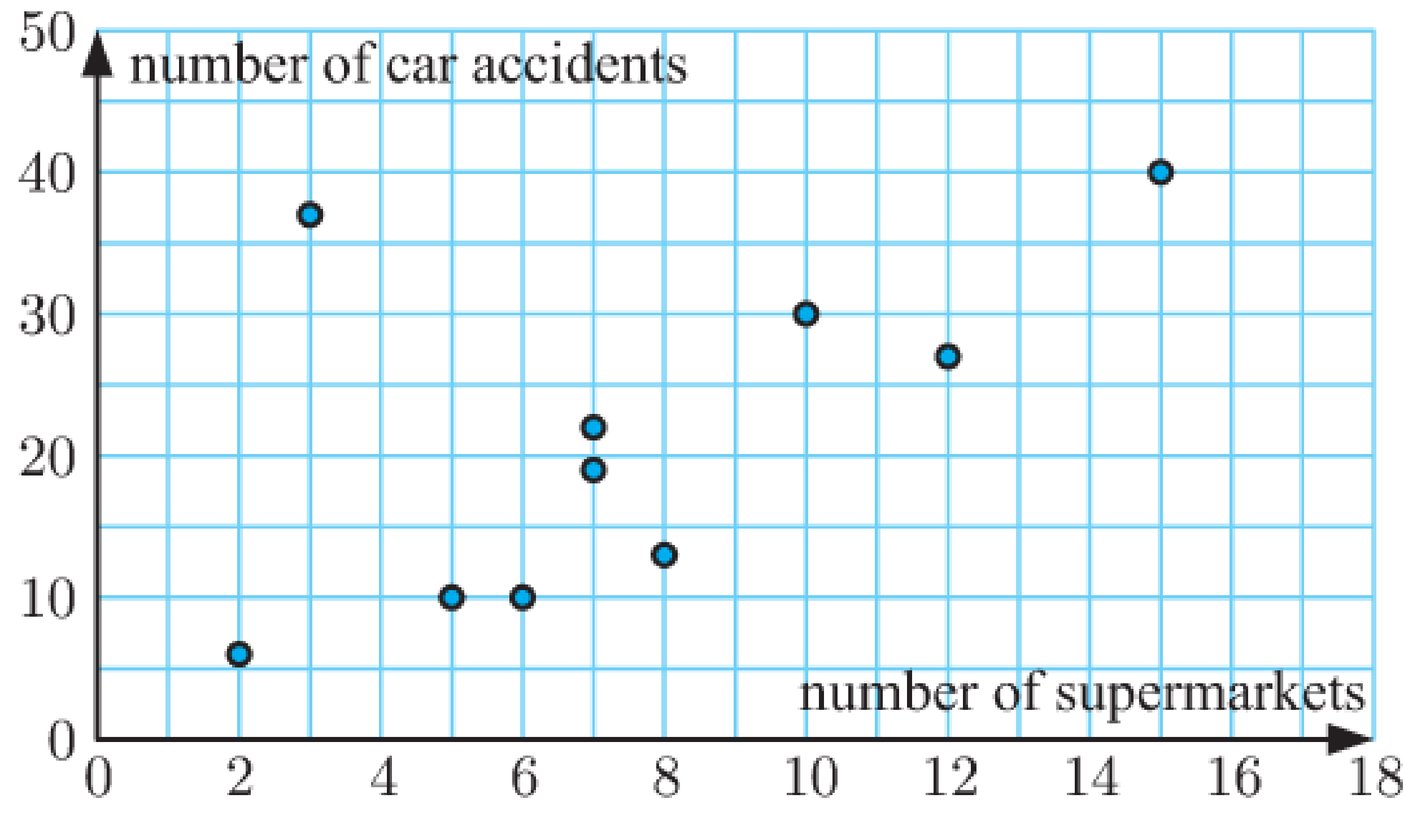
5 **a** $r \approx 0.917$
b strong, positive correlation
In general, the higher the young athlete's age, the further they can throw a discus.

6 **a**



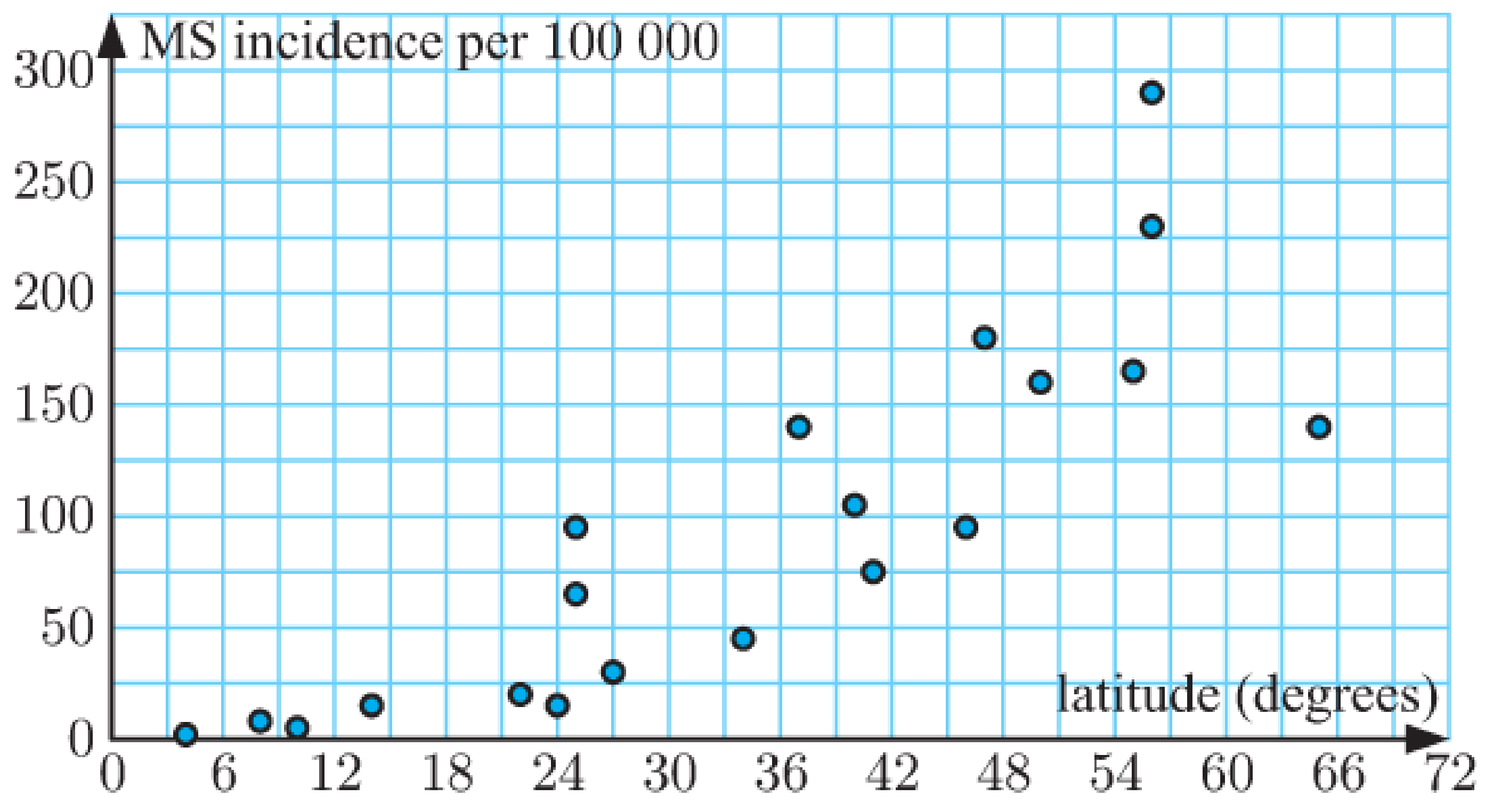
b $r \approx -0.987$ **c** very strong, negative correlation

7 **a**



b $r \approx 0.572$
c The point (3, 37), which represents 37 car accidents in a town with 3 supermarkets, is an outlier.
d **i** $r \approx 0.928$ **ii** strong, positive correlation
iii Removing the outlier had a very significant effect on the value of r.
e No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

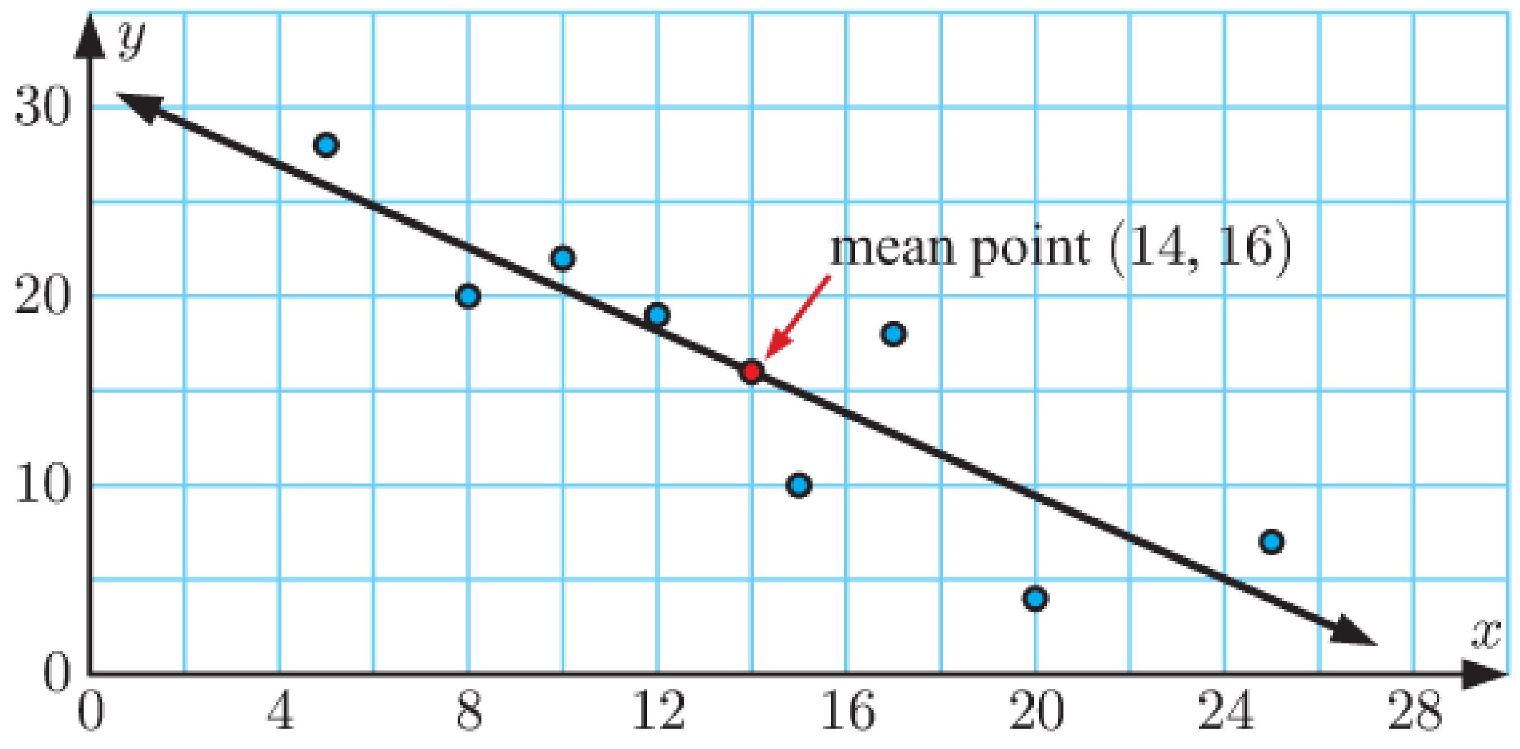
8 **a**



b $r \approx 0.849$ **c** moderate, positive correlation
d The incidence of MS is higher near the poles.

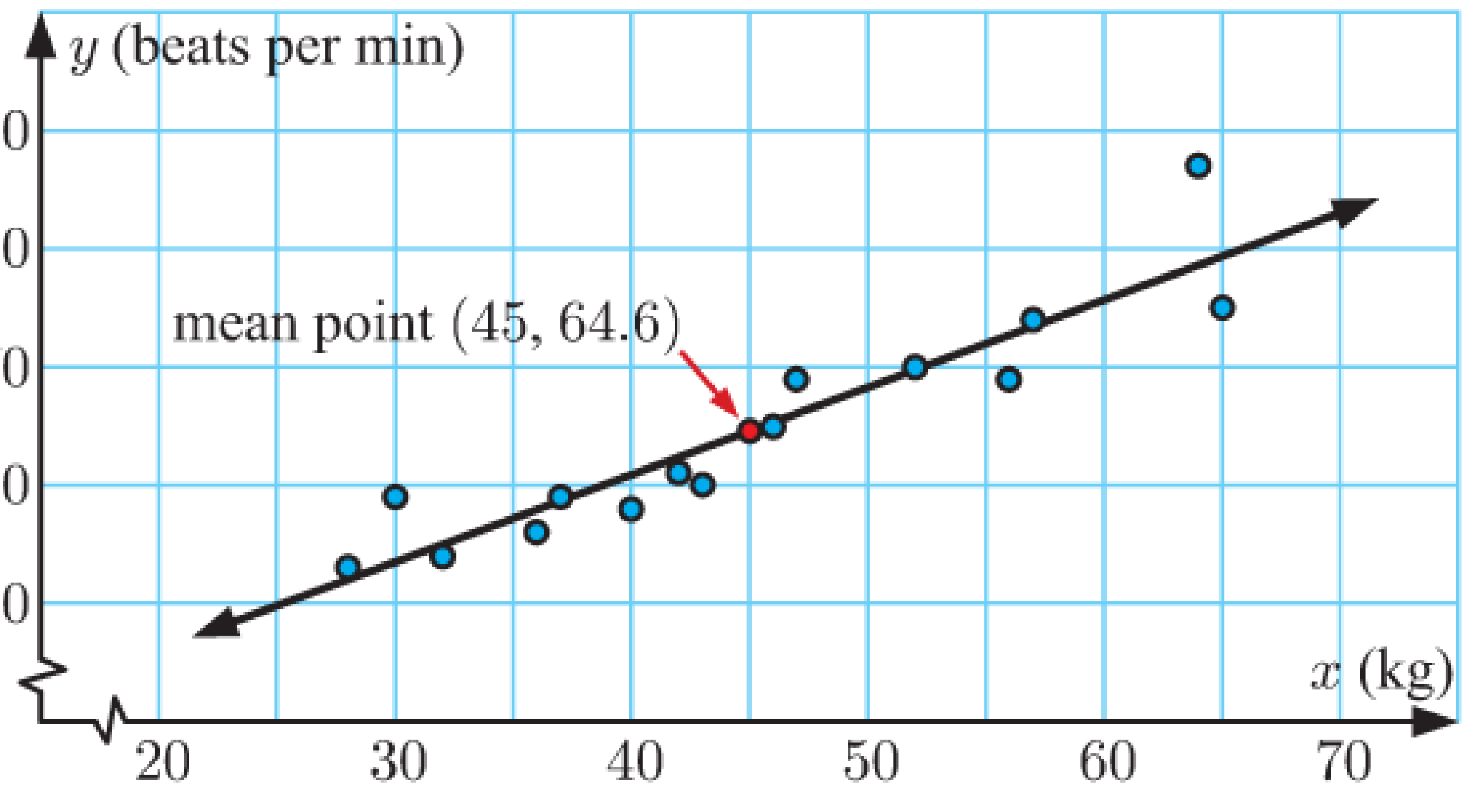
EXERCISE 5C

1 **a, f**



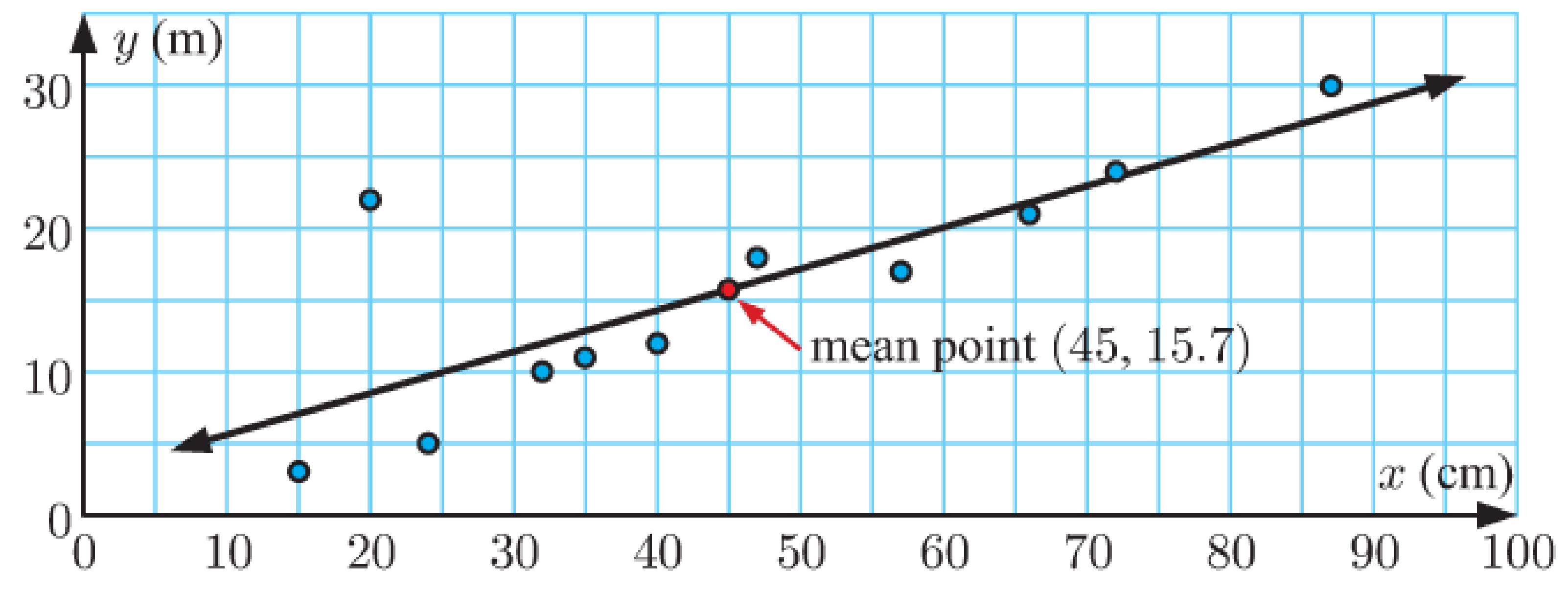
b negatively correlated **c** $r \approx -0.881$
d strong, negative correlation **e** (14, 16) **g** $y \approx 7$

2 **a, e**



b $r \approx 0.929$
c There is a strong, positive correlation between *weight* and *pulse rate*.
d (45, 64.6)
f 68 beats per minute. This is an interpolation, so the estimate is reliable.

3 **a, e**



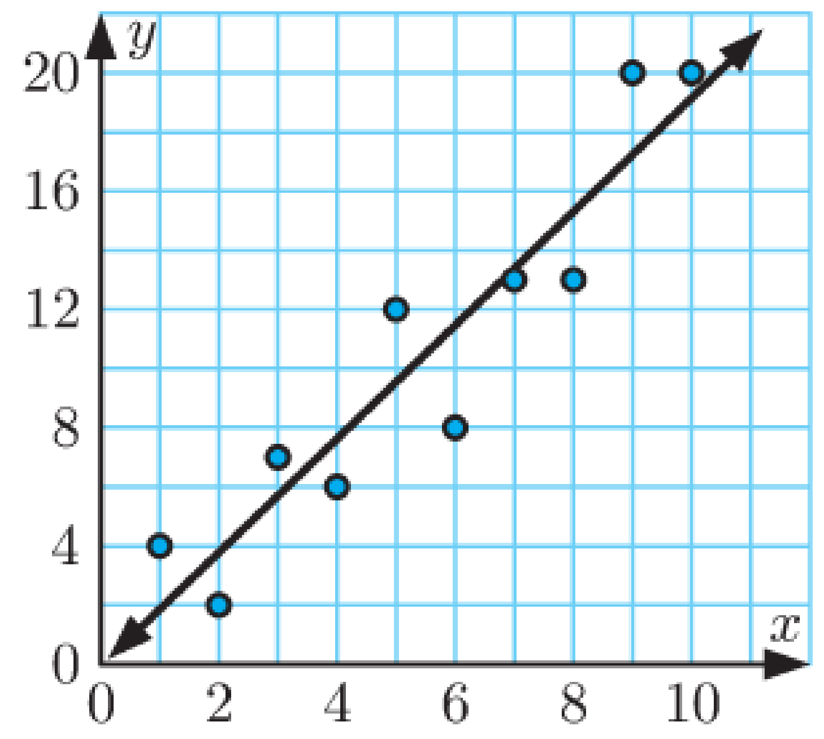
b (20, 22) **c** very tall and thin **d** (45, 15.7)

f ≈ 37 m. This is an extrapolation, so the prediction may not be reliable.

g ≈ 25 cm. This is an interpolation, so the estimate is reliable.

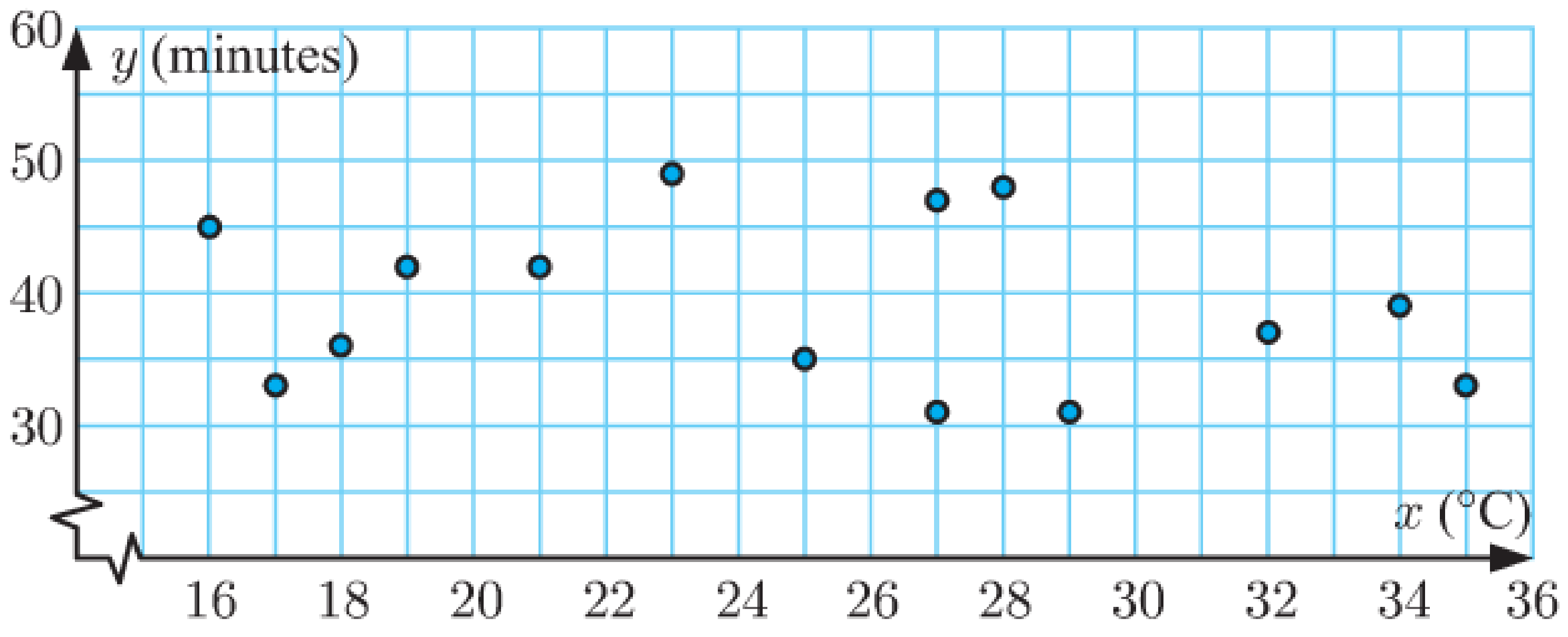
EXERCISE 5D

1 **a, c**



b $y \approx 1.92x - 0.0667$

2 **a**



b $r \approx -0.219$

c There is a very weak, negative correlation between *temperature* and *time*.

d No, as there is almost no correlation.

3 **a** $r \approx -0.924$

b There is a strong, negative, linear correlation between the *petrol price* and the *number of customers*.

c $y \approx -4.27x + 489$

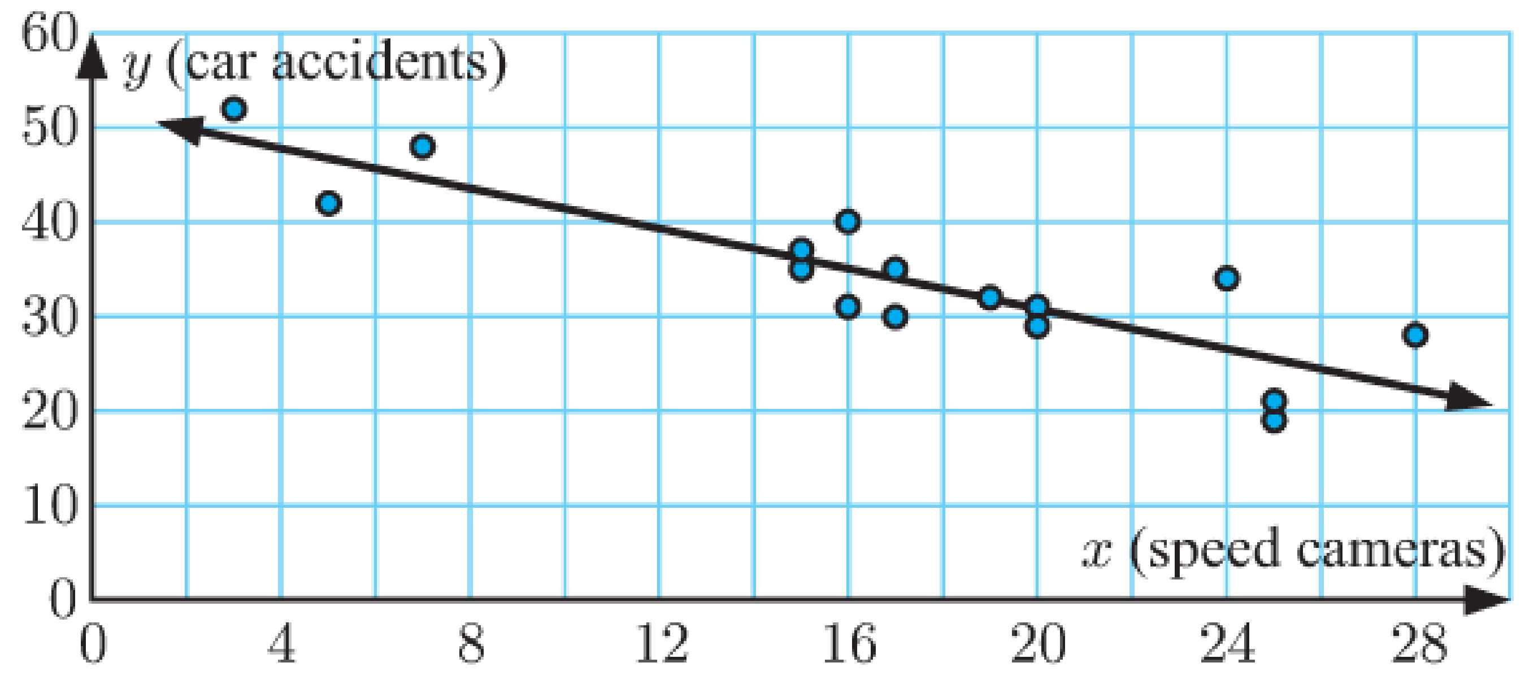
d ≈ -4.27; this indicates that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

e ≈ -5.10 customers **f** ≈ 105.3 cents per litre

g In **e**, it is impossible to have a negative number of customers. This extrapolation is not valid.

In **f**, this is an interpolation, so this estimate is likely to be reliable.

4 **a**



b $r \approx -0.878$

c There is a strong, negative correlation between *number of speed cameras* and *number of car accidents*.

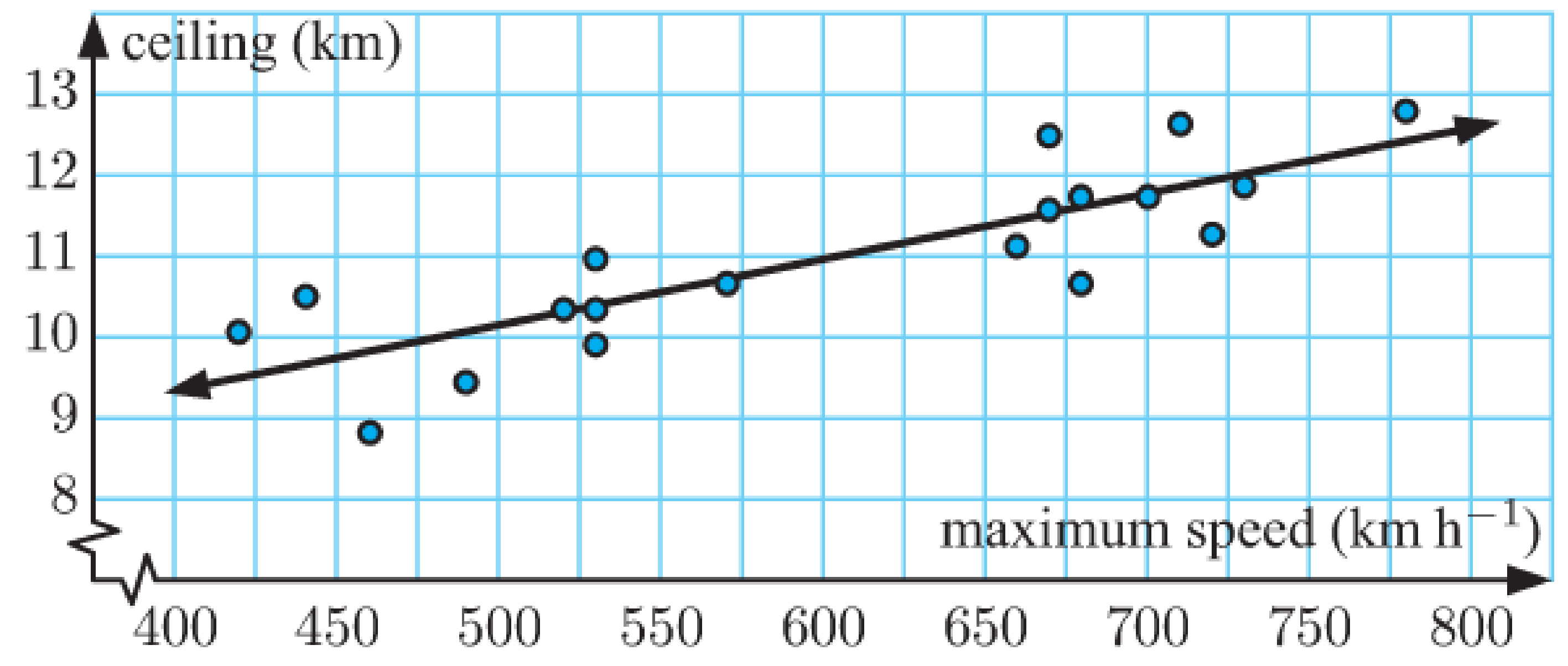
d $y \approx -1.06x + 52.0$

e gradient: ≈ -1.06; this indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.

y-intercept: ≈ 52.0; this indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

f ≈ 41.4 car accidents

5 **a, d**



b $r \approx 0.840$

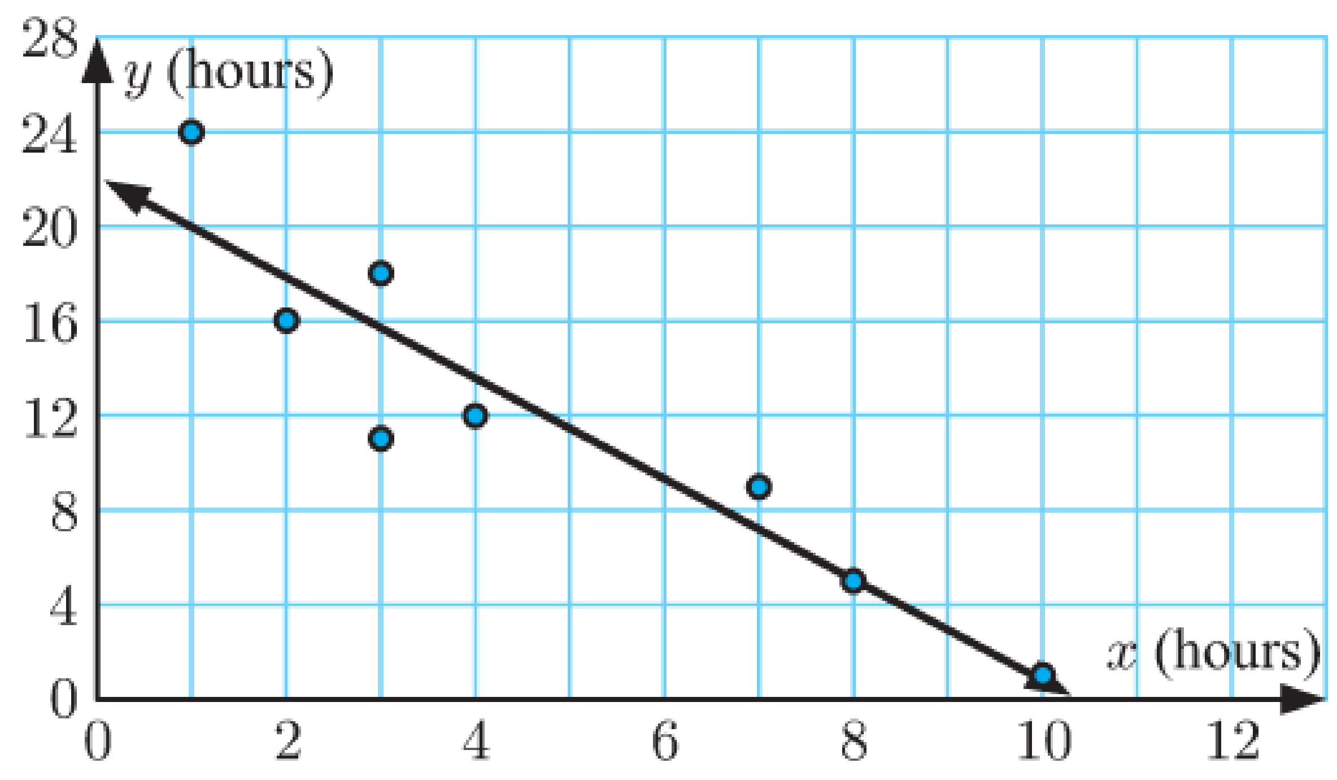
c moderate, positive, linear correlation

d $y \approx 0.008\,12x + 6.09$

e $\approx 0.008\,12$; this indicates that for each additional km h^{-1}, the ceiling increases by an average of 0.008 12 km or 8.12 m.

f ≈ 11.0 km **g** ≈ 605 km h^{-1}

6 **a, d**



b $r \approx -0.927$

c There is a strong, negative, linear correlation between *time exercising* and *time watching television.*

d $y \approx -2.13x + 22.1$

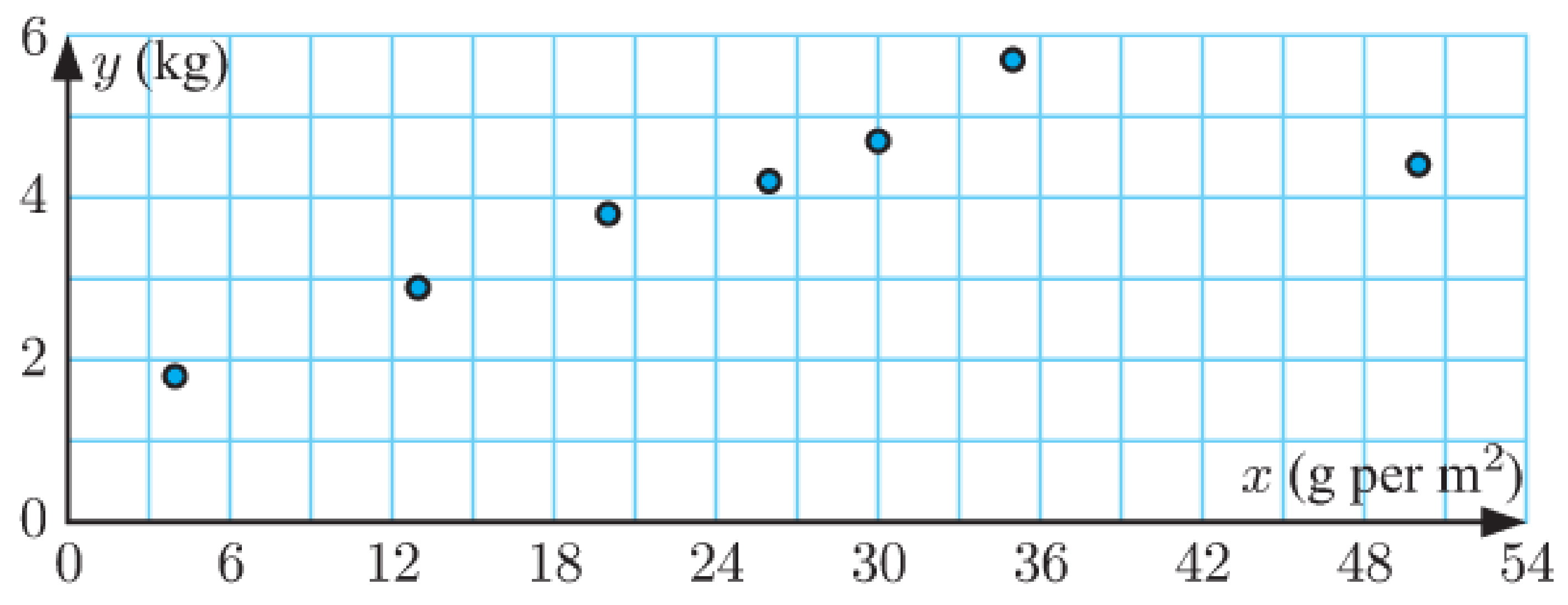
e gradient: ≈ -2.13; this indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.

y-intercept: ≈ 22.1; this indicates that for children who do not spend time exercising, they would watch television for an average of 22.1 hours per week.

f **i** 9 hours per week **ii** ≈ 7.22 hours per week

iii This particular child spent more time watching television than predicted.

7 **a**



(50, 4.4) is the outlier.

b **i** reduces the strength of the correlation

ii decreases the gradient of the regression line

c **i** $r \approx 0.798$ **ii** $r \approx 0.993$

d **i** $y \approx 0.0672x + 2.22$ **ii** $y \approx 0.119x + 1.32$

e The one which excludes the outlier, as this will be more accurate for an interpolation.

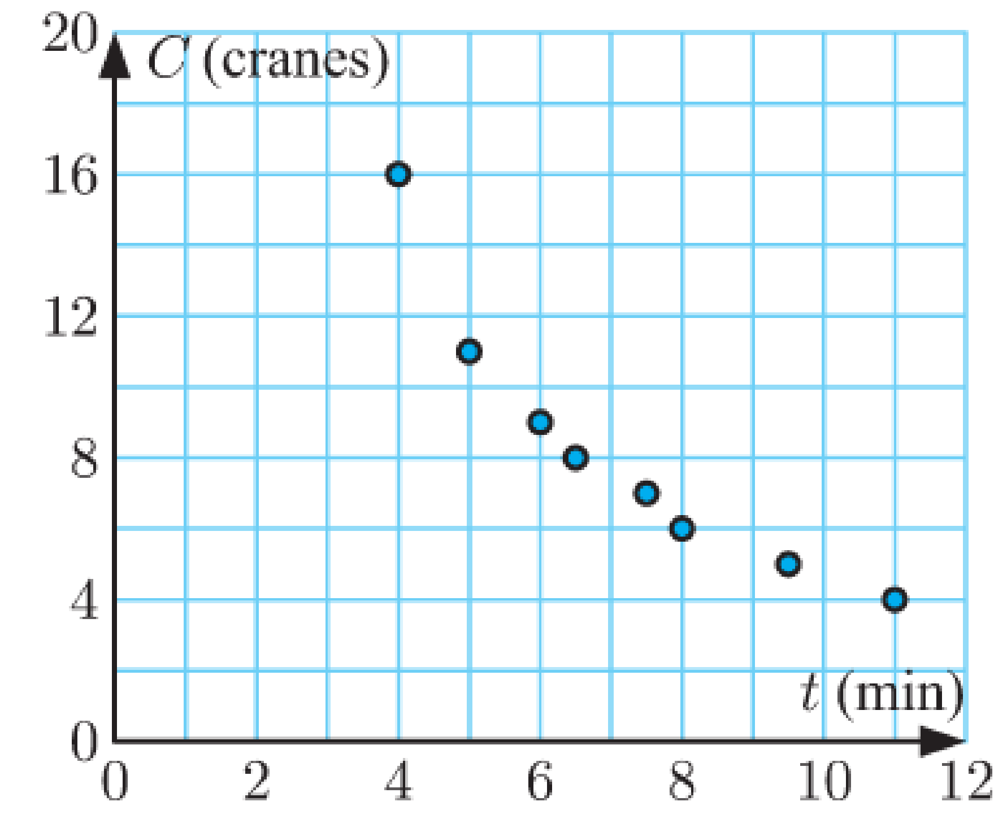
f Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

EXERCISE 5E

1 **a** **B** **b** **C** **c** **A** **2** **a** negative **b** no

3 **a** **A** **b** **A**

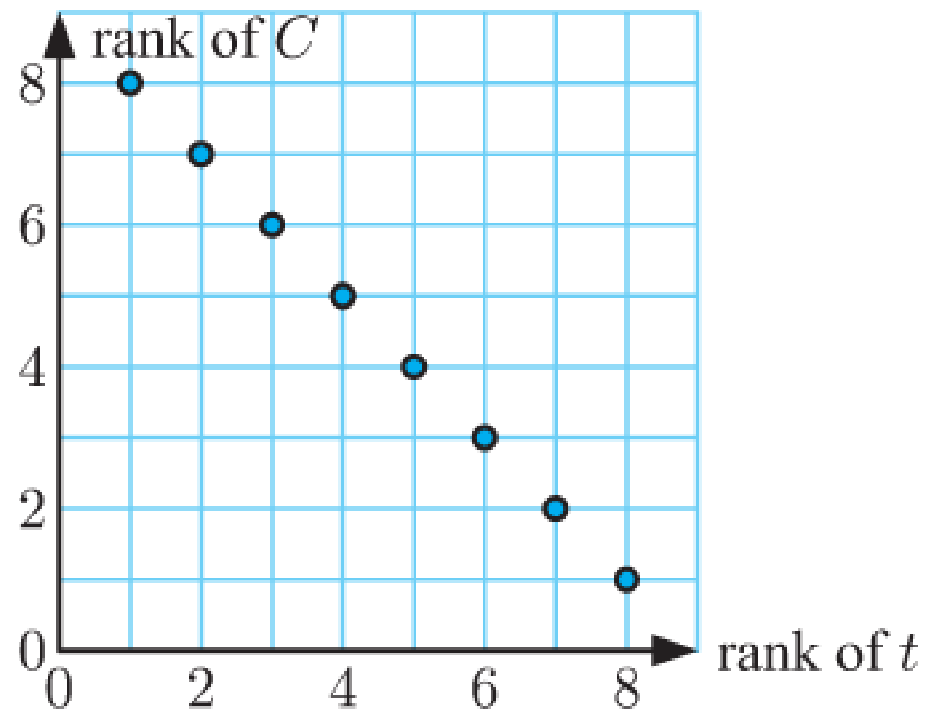
4 **a**



b $r_p \approx -0.921$

c

t	4	5	6	6.5	7.5	8	9.5	11
rank of t	1	2	3	4	5	6	7	8
C	16	11	9	8	7	6	5	4
rank of C	8	7	6	5	4	3	2	1

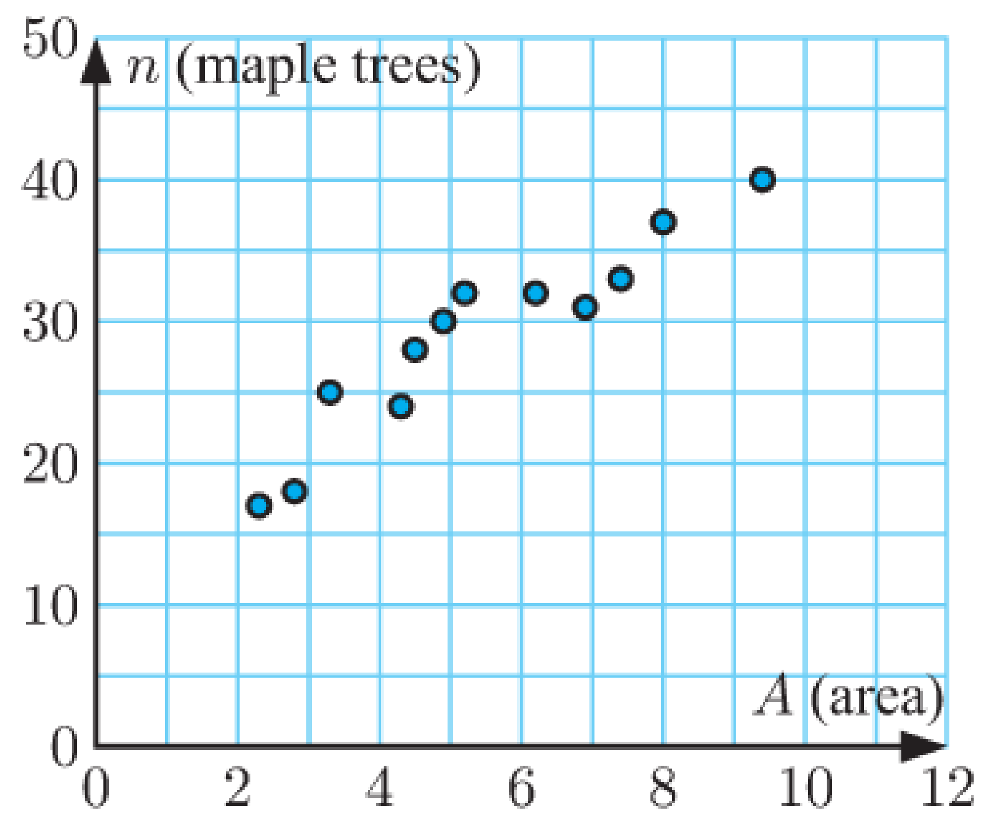


d $r_s = -1$

e The scatter diagram of the raw data shows a non-linear trend. Spearman's rank correlation coefficient is therefore more appropriate.

f strong, negative, linear correlation

5 **a**

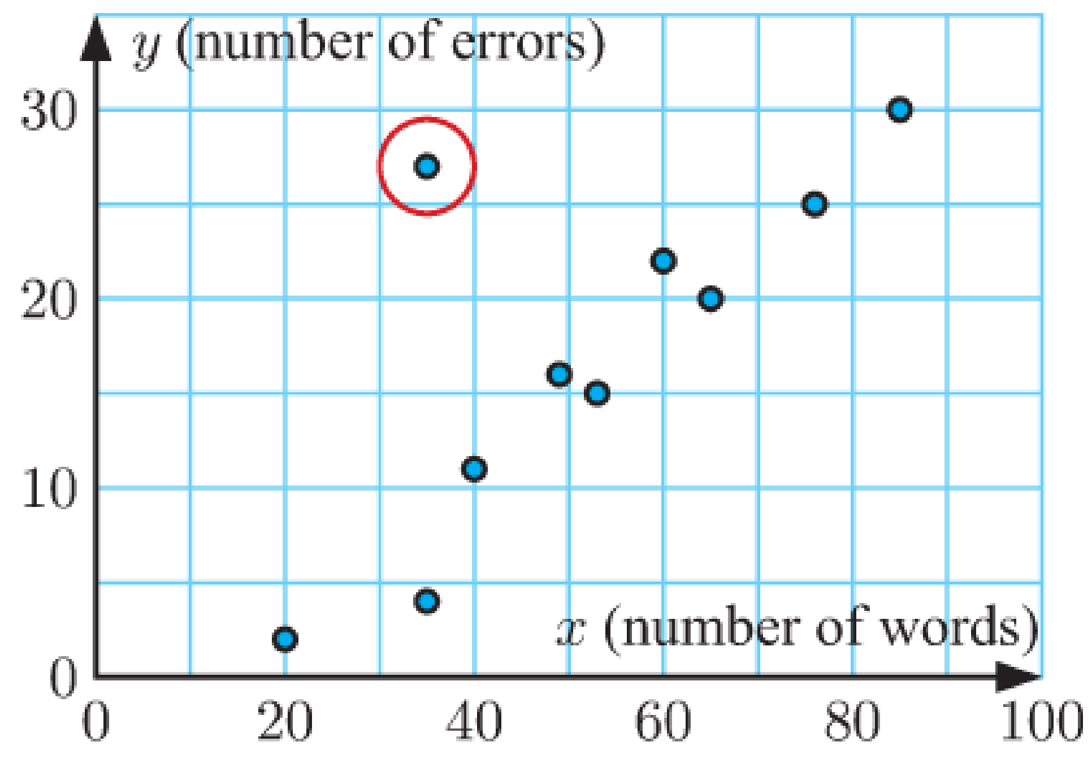


b There is a positive, linear correlation in the scatter diagram of the raw data. We expect to see the same in the rank scatter diagram. r_p and r_s will be very similar.

c $r_p \approx 0.943$, $r_s \approx 0.970$

6 **a** $r_p \approx 0.762$, $r_s \approx 0.681$

b



c $r_p \approx 0.976$, $r_s \approx 0.967$ **d** r_s

7 **a**

Longest jump (y m)	5.29	5.22	4.64	4.62	4.58
Placing	1	2	3	4	5

Longest jump (y m)	4.38	4.31	4.28	3.94	3.89
Placing	6	7	8	9	10

b *longest jump*

c **i** The variable *placing* has the values 1 to 10 which act as a ranking.

ii The longest jump *always* decreases as the placing increases.

REVIEW SET 5A

1 **a** strong, positive, linear correlation, with no outliers

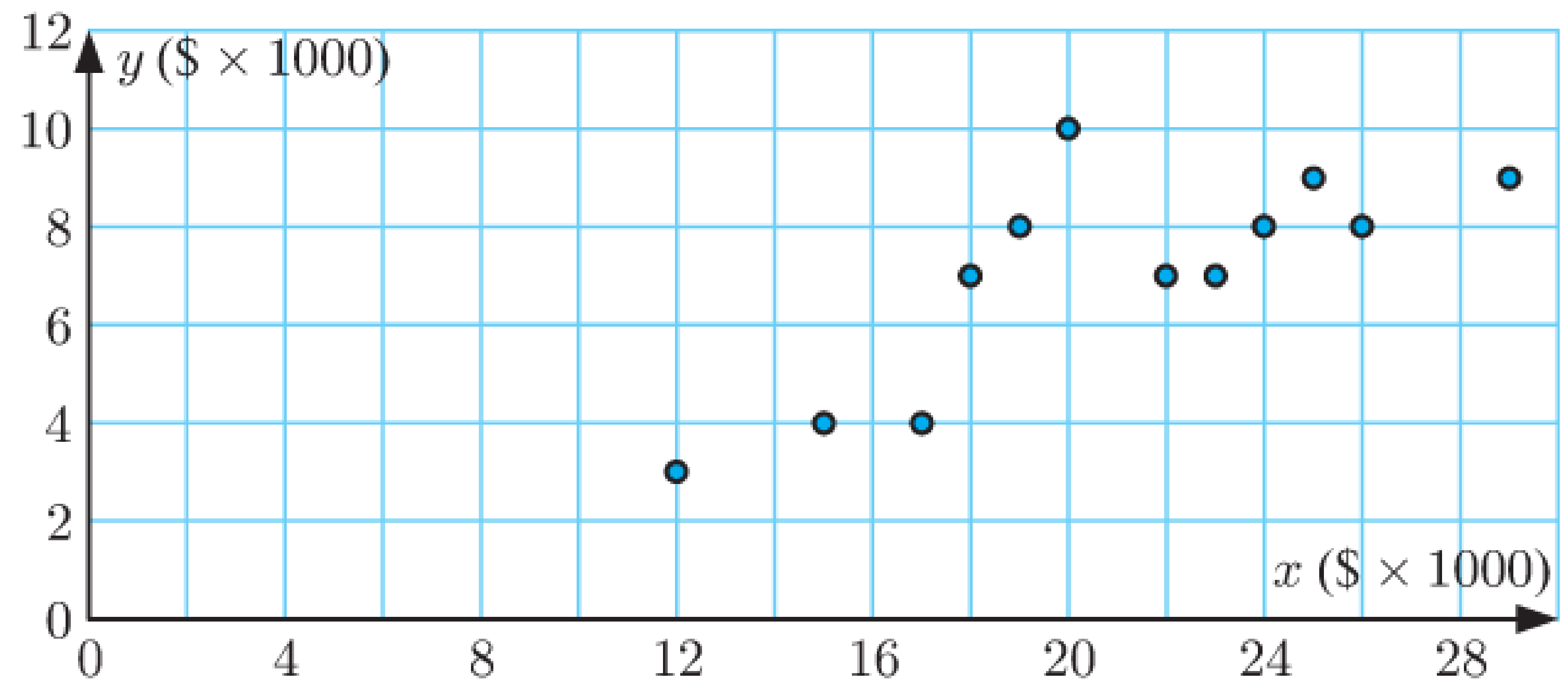
b weak, negative, linear correlation, with one outlier

c strong, negative, non-linear correlation, with no outliers

2 **a** The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.

b No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

3 **a**

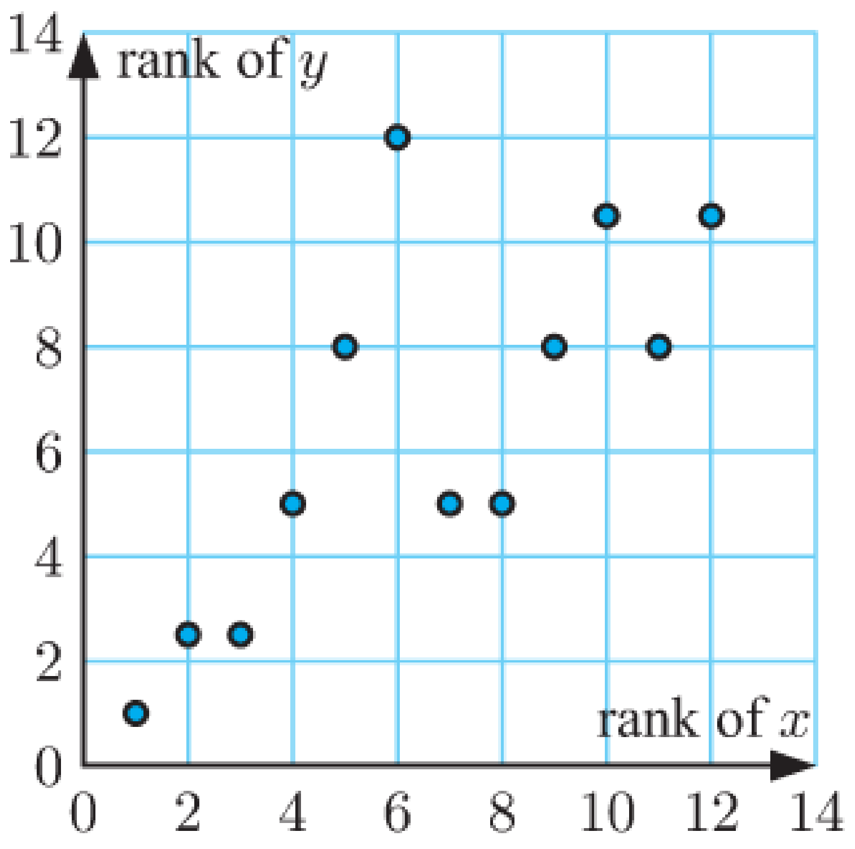


b $r_p \approx 0.776$

c

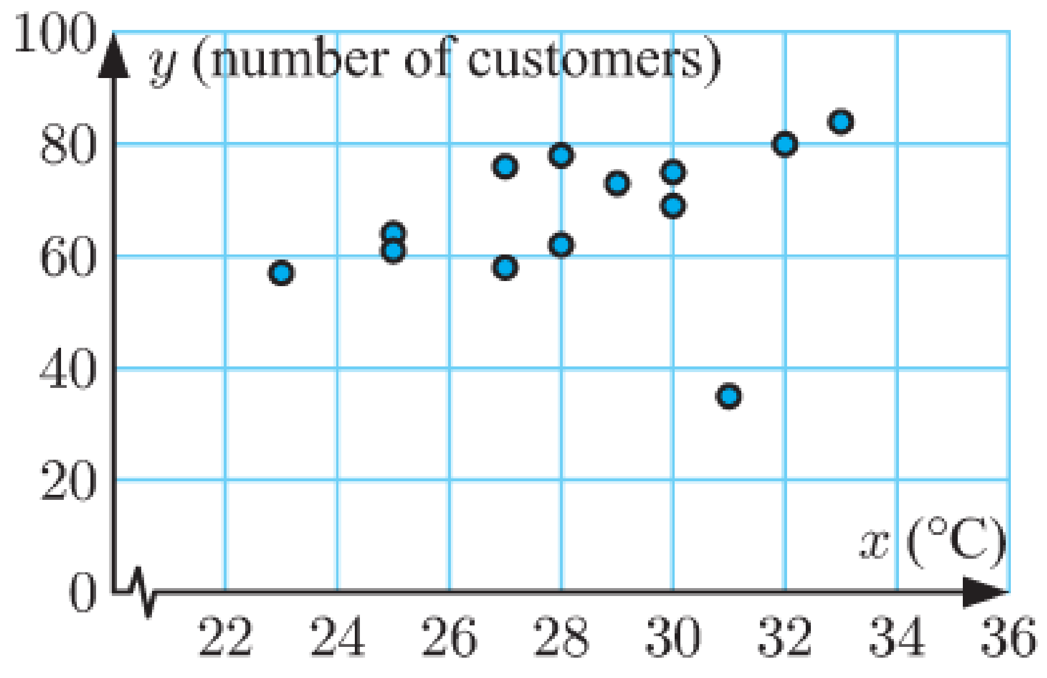
Ticket sales (\$$x$ × 1000)	25	22	15	19	12	17	24	20	18	23	29	26
rank of x	10	7	2	5	1	3	9	6	4	8	12	11

Beverage sales (\$$y$ × 1000)	9	7	4	8	3	4	8	10	7	7	9	8
rank of y	10.5	5	2.5	8	1	2.5	8	12	5	5	10.5	8



d $r_s \approx 0.744$ **e** moderate, positive correlation

4 **a**



b $r_p \approx 0.314$

c $r_s \approx 0.511$

d (31, 35)

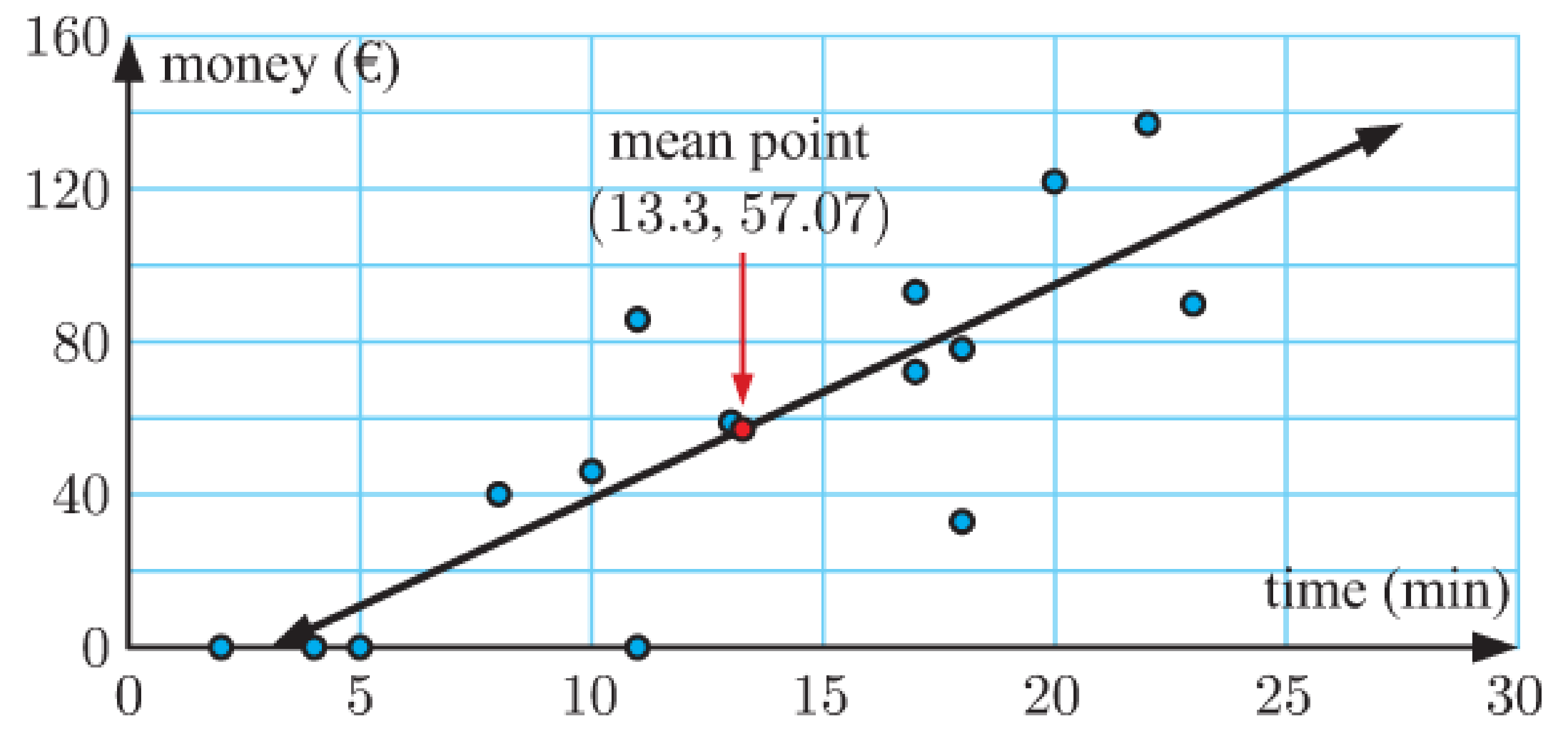
e $r_p \approx 0.801$, $r_s \approx 0.764$

f r_p

g moderate, positive, linear correlation

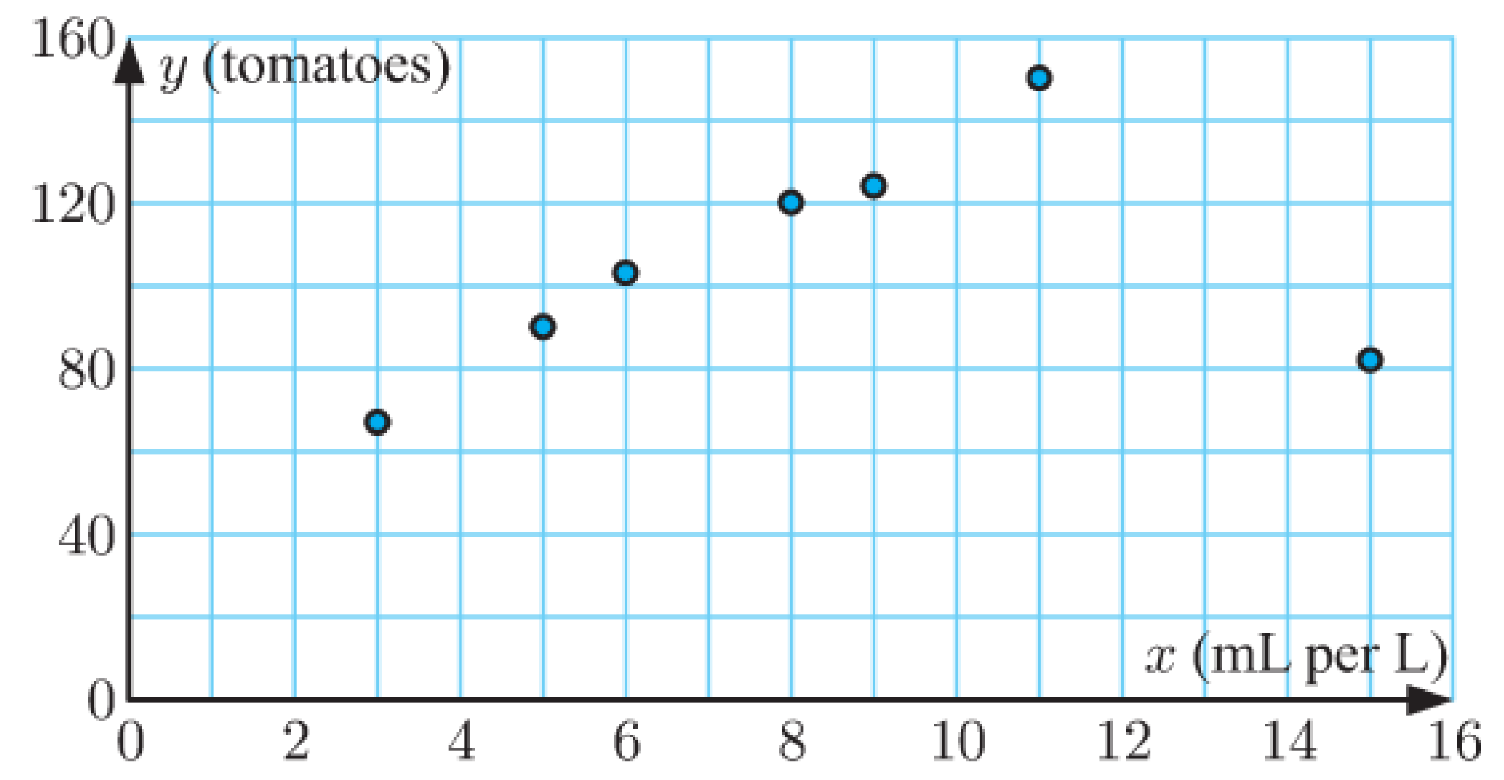
5 **a** mean time ≈ 13.3 min, mean spending ≈ €57.07

b



c There is a moderate, positive, linear correlation between *time in the store* and *money spent*.

6 **a**



b $r \approx 0.340$. There is a very weak, positive, linear correlation between spray concentrations and yield.

c Yes, (15, 82) is an outlier.

d $r \approx 0.994$. Yes it is now reasonable to draw a linear model.

e $y \approx 9.93x + 39.5$

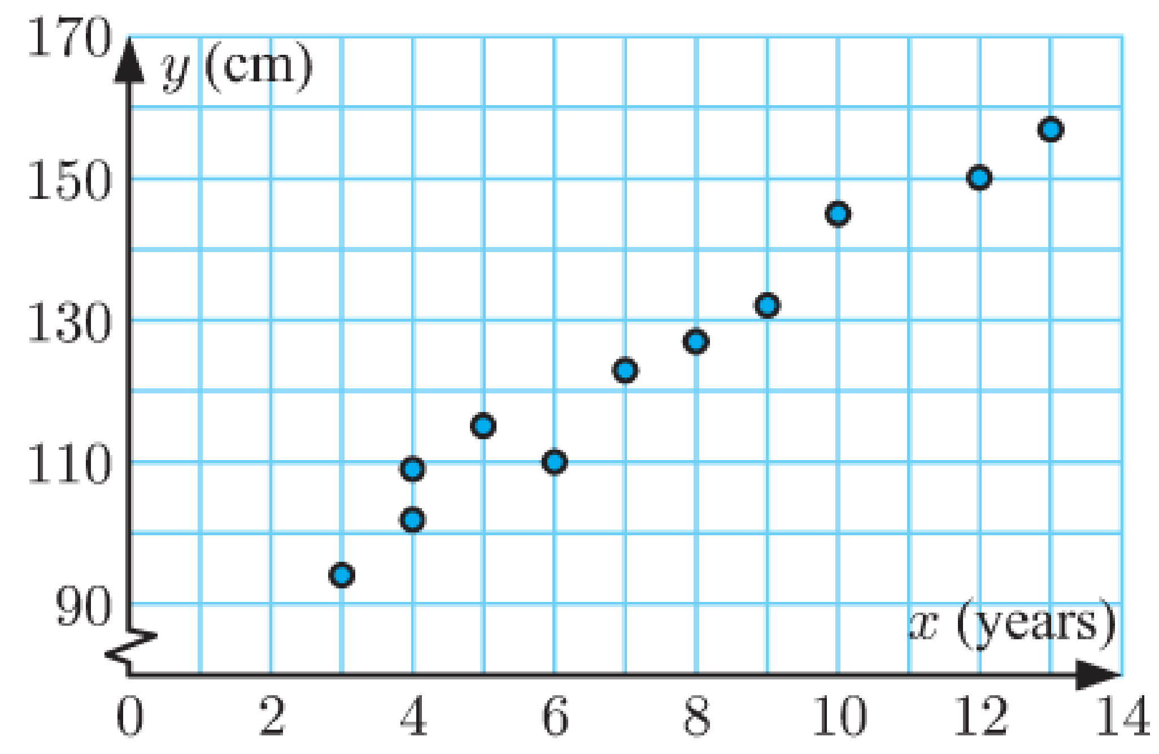
f gradient: ≈ 9.93; this indicates that for every additional mL per L the spray concentration increases by, the yield of tomatoes per bush increases on average by 9.93.

y-intercept: ≈ 39.5; this indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g **i** ≈ 109 tomatoes per bush **ii** ≈ 16.2 mL per L

h In **g i**, this is an interpolation, so this estimate is likely to be reliable. In **g ii**, this is an extrapolation, so this estimate may not be reliable.

7 **a**



b $y \approx 5.98x + 80.0$

c ≈ 5.98; this indicates that each year, a child grows taller by an average of 5.98 cm.

d ≈ 110 cm **e** 10 years old

REVIEW SET 5B

1 **a** Negative correlation. As prices increase, the number of tickets sold is likely to decrease.

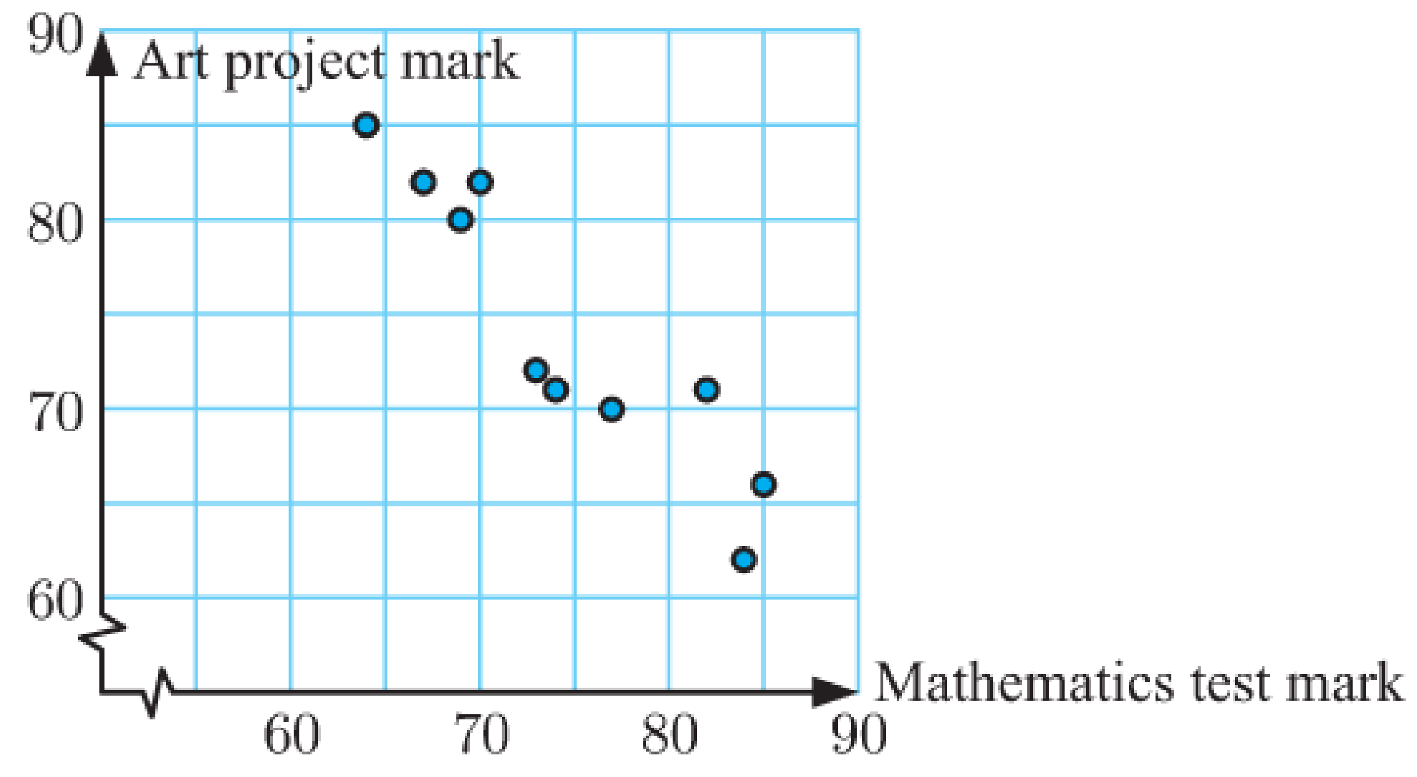
Causal. Less people will be able to afford tickets as the prices increase.

b Positive correlation. As ice cream sales increase, the number of shark attacks is likely to increase.

Not causal. Both of these variables are dependent on the number of people at the beach.

2 **a** **C** **b** **A** **c** **B**

3 **a**

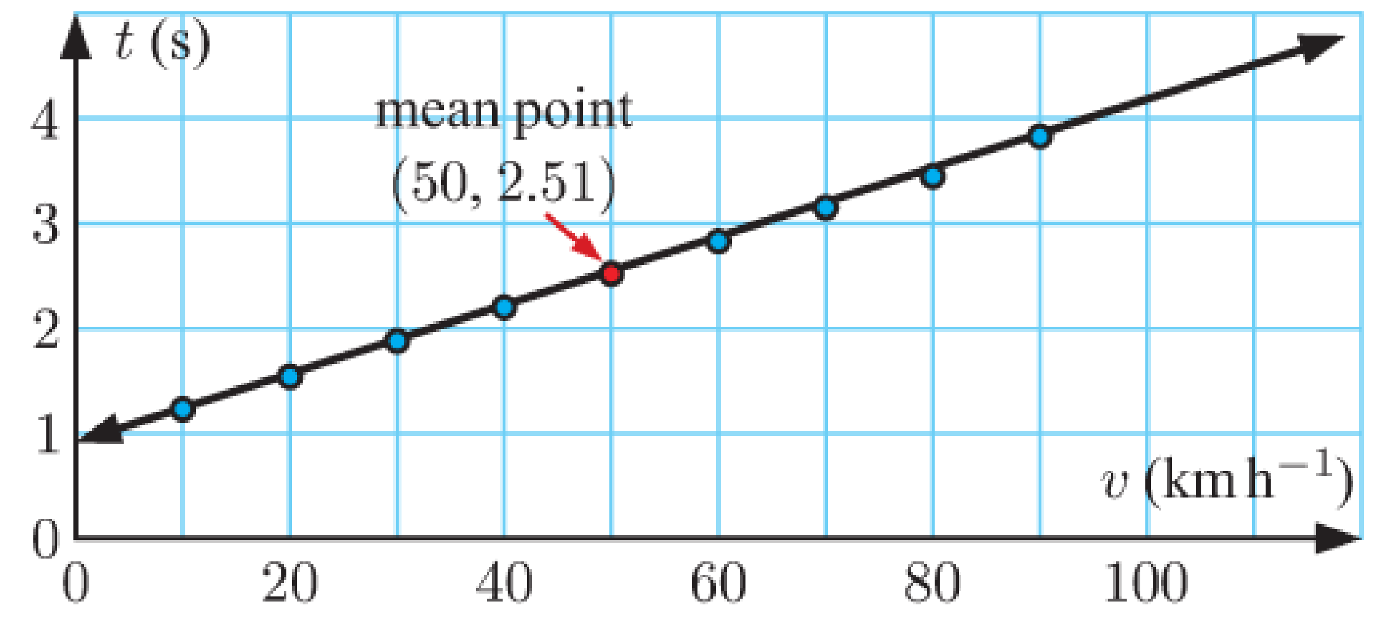


b There is a strong, negative, linear correlation between the Mathematics and Art marks.

c $r \approx -0.930$

4 **a** (50, 2.51)

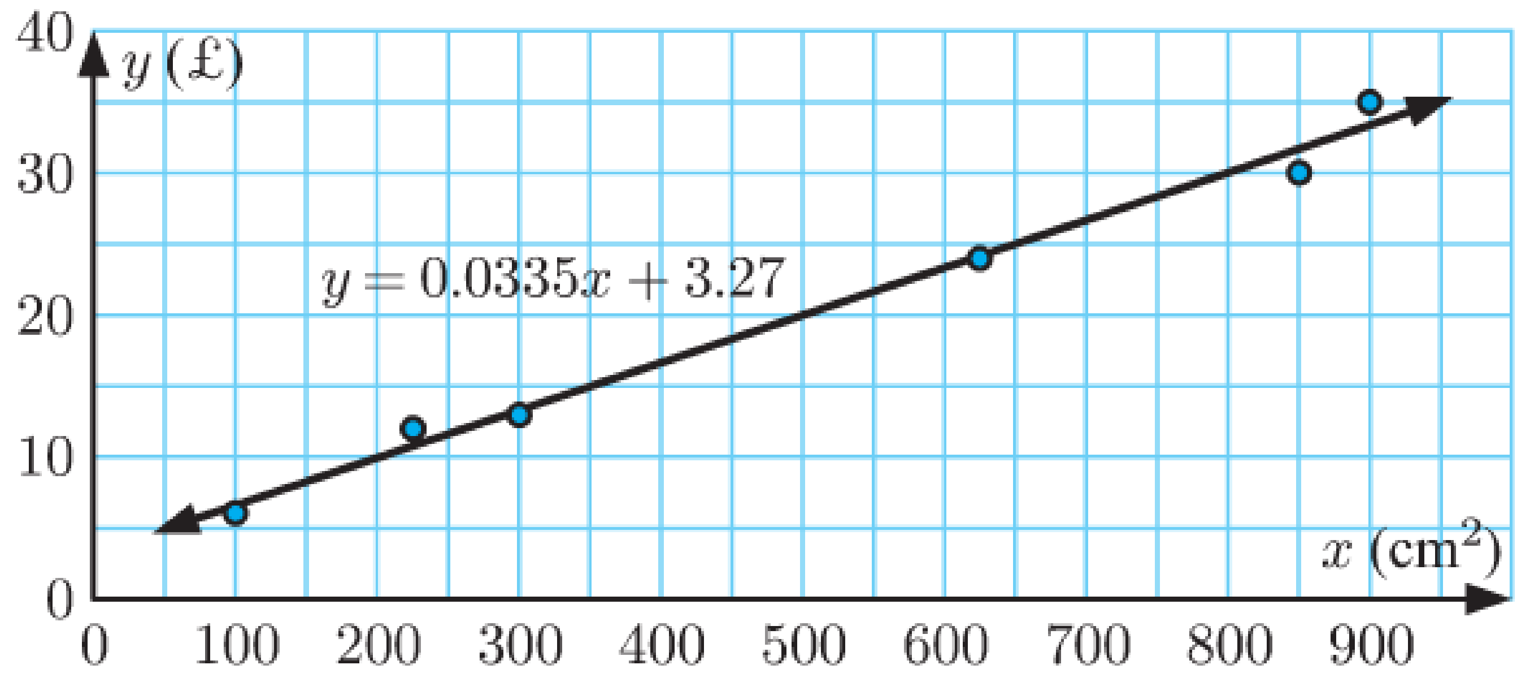
b



c **i** ≈ 2.7 seconds **ii** ≈ 4.4 seconds

d The estimate in **c i**, since it is an interpolation.

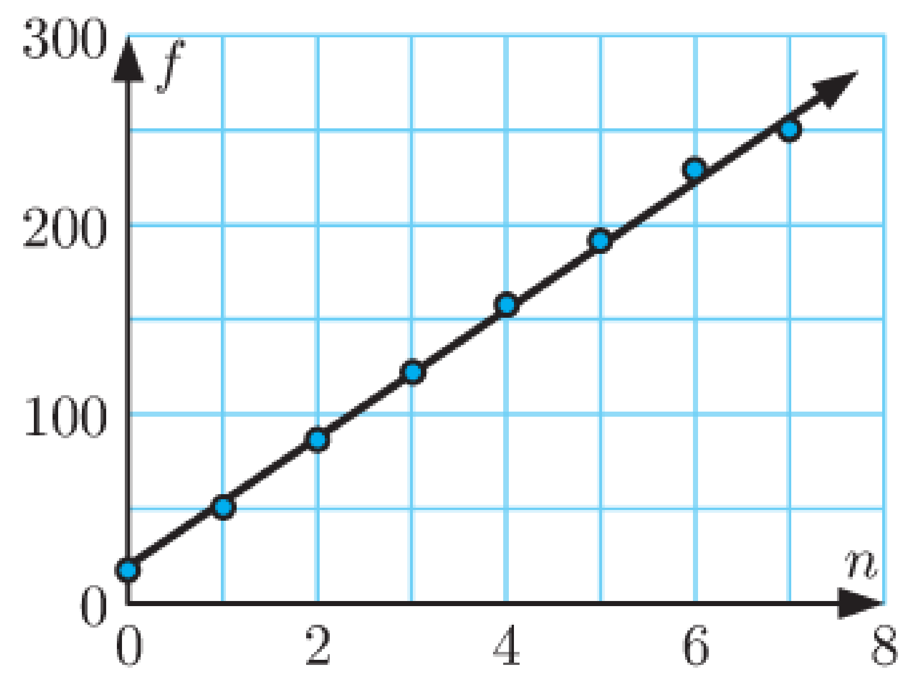
5 **a, d**



b $r \approx 0.994$ **c** There is a very strong, positive, linear correlation between *area* and *price*.

e ≈ £43.42, this is an extrapolation, so it may be unreliable.

6 **a, d**



There is a very strong, positive linear correlation between number of waterings and flowers produced.

b $f \approx 34.0n + 19.3$

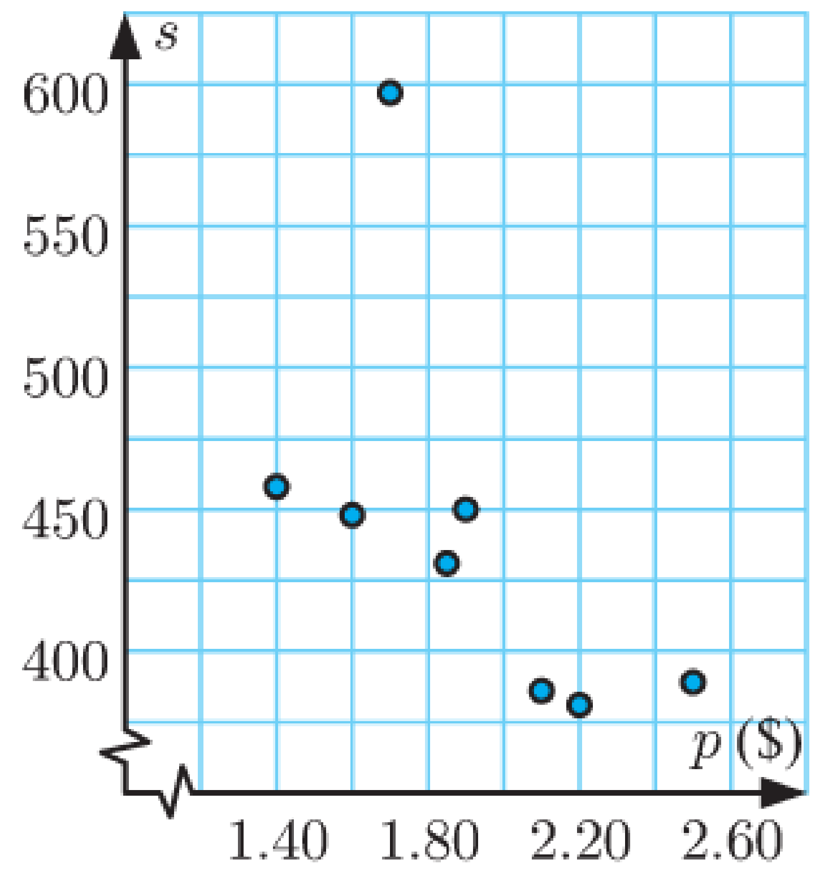
c Yes, plants need water to grow, so it is expected that an increase in watering will result in an increase in flowers.

e **i** 104 flowers ($n = 2.5$), 359 flowers ($n = 10$)

ii $n = 2.5$ is reliable, as it is an interpolation.

$n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

7 **a**



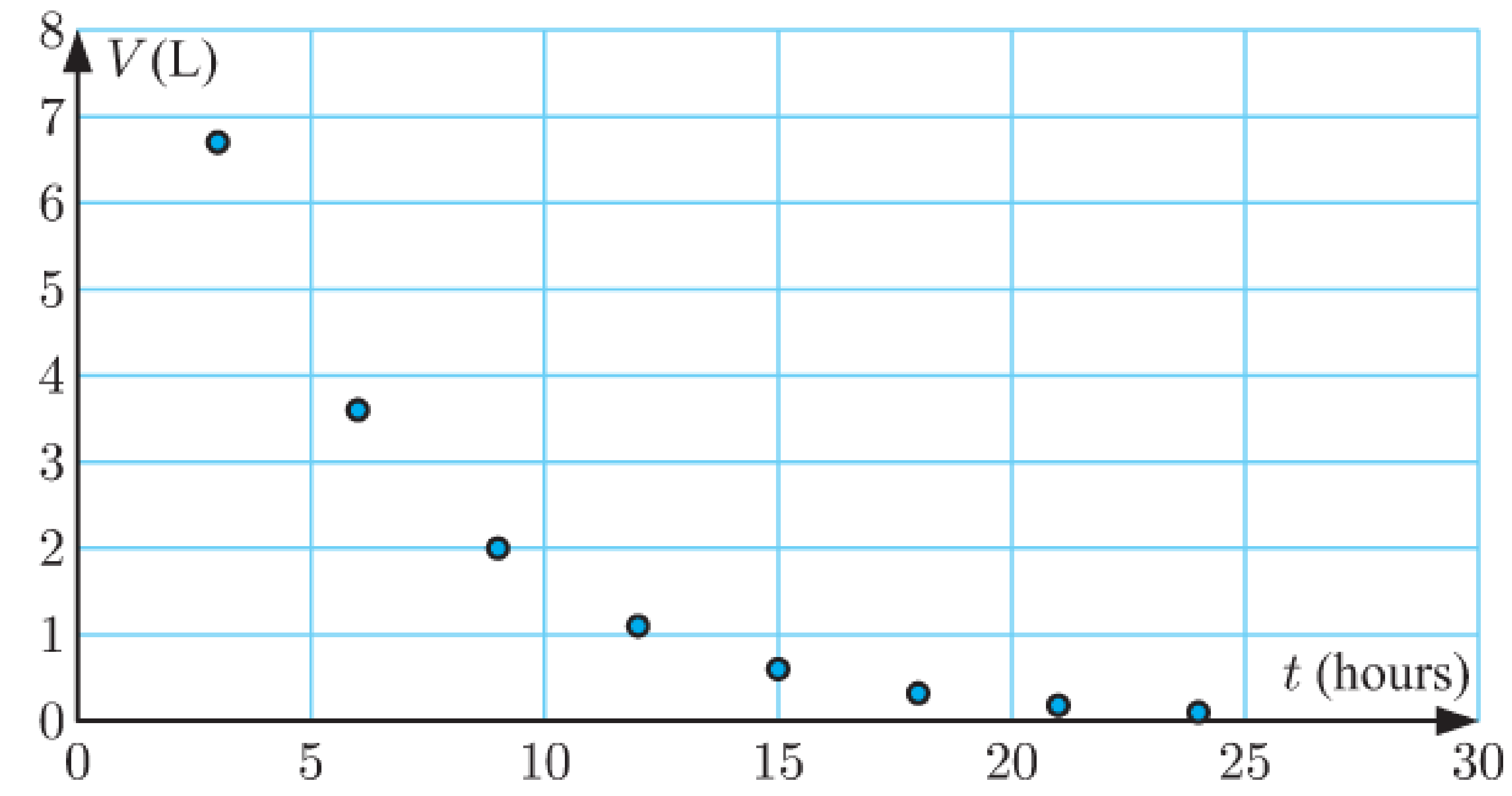
b Yes, the point (1.70, 597) is an outlier. It should not be deleted as there is no evidence that it is a recording error.

c $s \approx -116p + 665$

d ≈ -116; this indicates that with every additional dollar the price increases by, the number of sales decreases by 116.

e No, the prediction would not be accurate, as it is an extrapolation.

8 **a**



b $r_p \approx -0.874$ **c** $r_s = -1$

d From the scatter diagram, the relationship between the variables is clearly non-linear. Spearman's rank correlation coefficient is therefore more appropriate.

EXERCISE 6A

1 **a** quadratic **b** not quadratic **c** quadratic
d quadratic **e** not quadratic **f** quadratic

2 **a** $y = -3$ **b** $y = -16$ **c** $y = 16$ **d** $y = -12$

3 **a**

x	-2	-1	0	1	2
y	11	5	1	-1	-1

b

x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

4 **a** $f(2) = 3$, $f(-1) = -9$ **b** $f(0) = 1$, $f(-3) = 22$
c $g(3) = -29$, $g(-2) = -4$

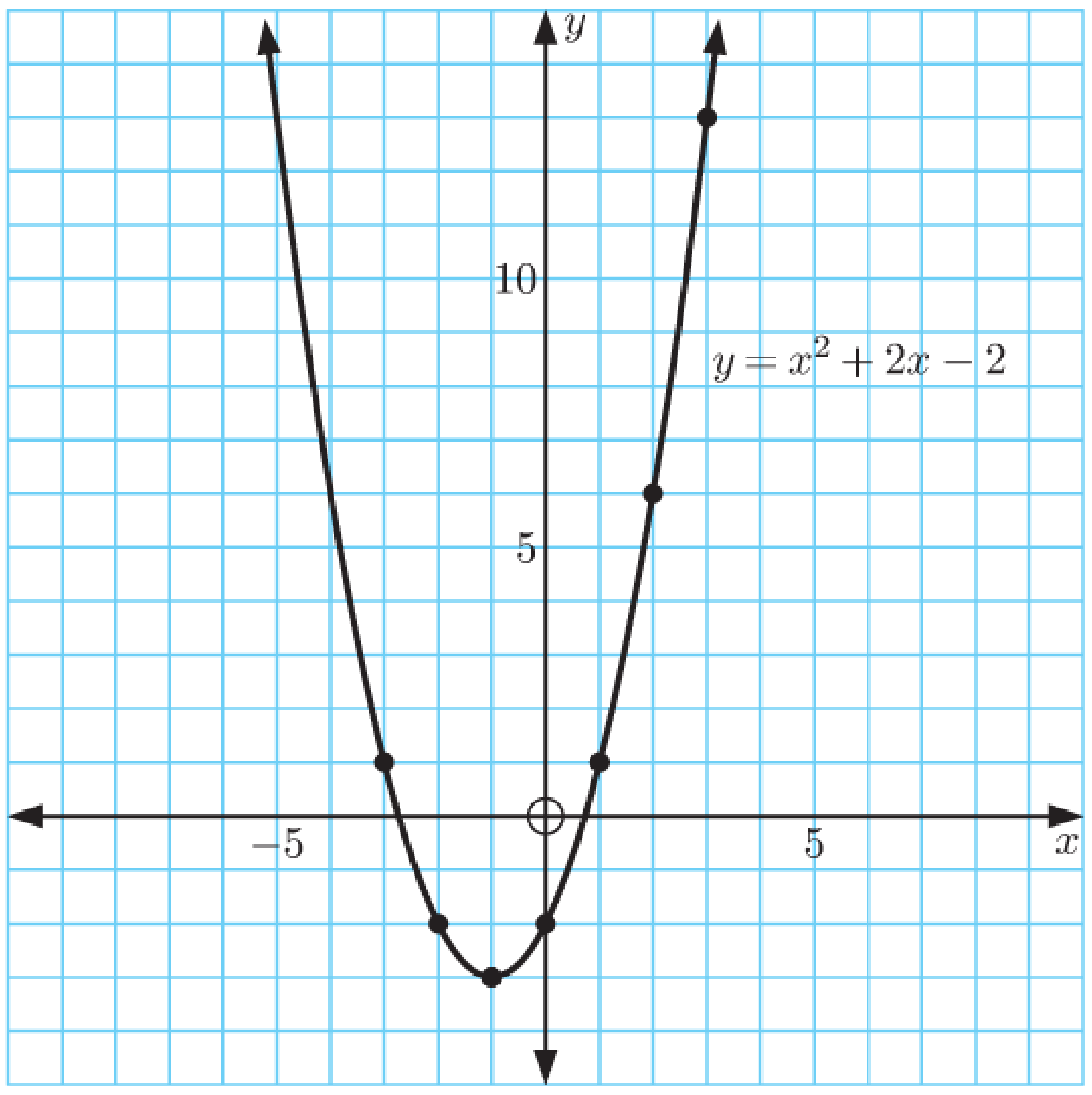
5 **a** no **b** yes **c** yes **d** yes **e** no **f** yes

6 **a** $x = -2$ or -1 **b** $x = 2$ **c** $x = 1$ or 5
d $x = -3$ or $\frac{1}{2}$ **e** $x = -6$ or 1 **f** no real solutions

EXERCISE 6B

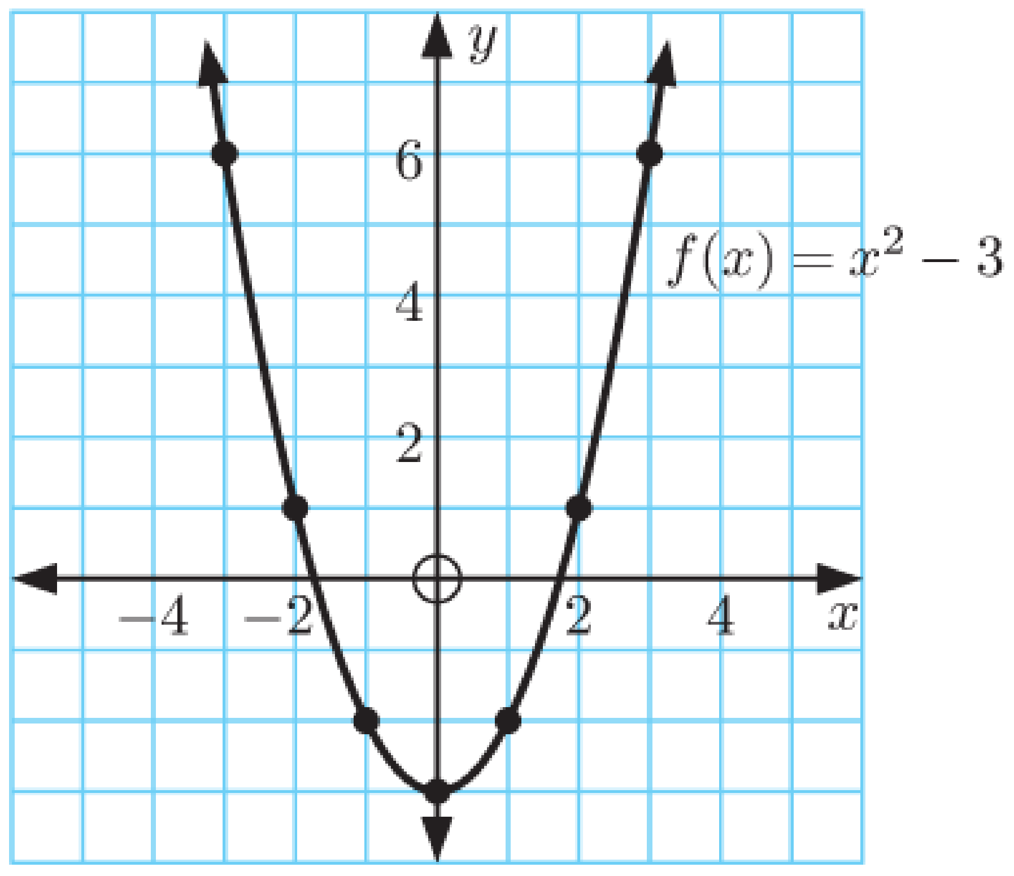
1 **a**

x	-3	-2	-1	0	1	2	3
y	1	-2	-3	-2	1	6	13



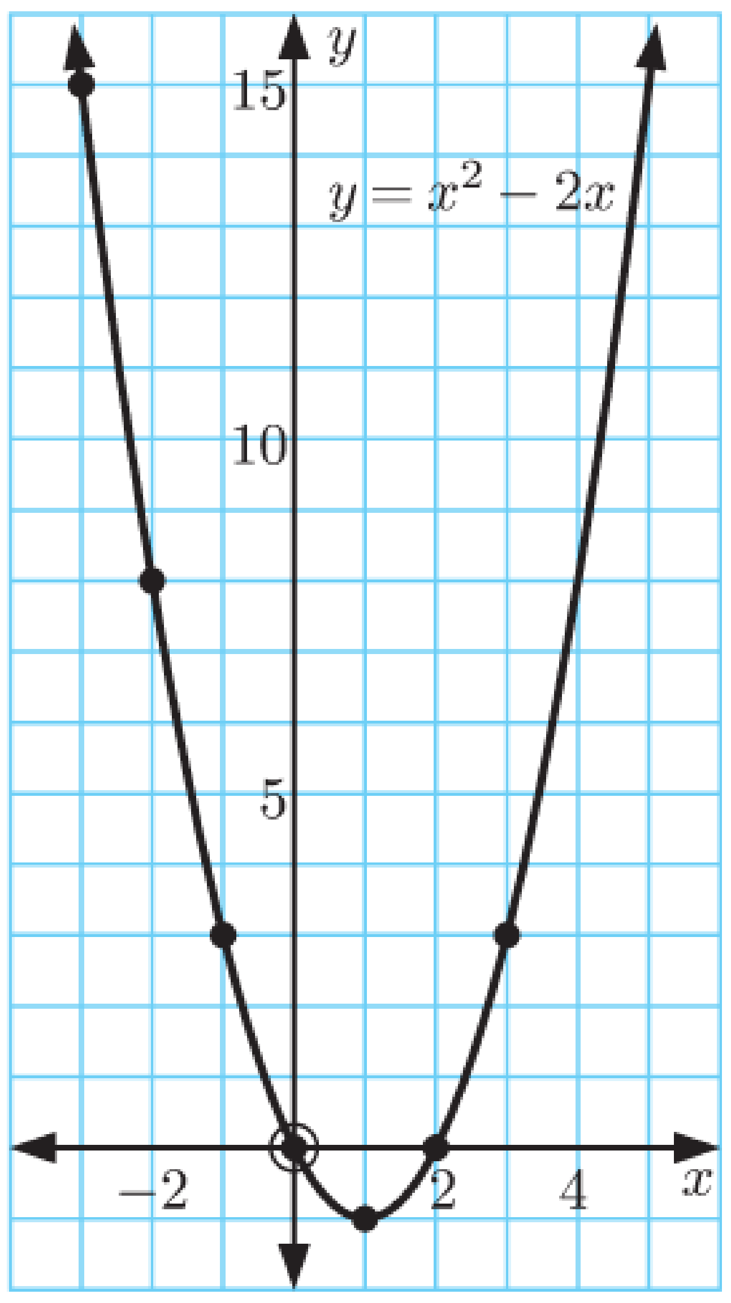
b

x	-3	-2	-1	0	1	2	3
$f(x)$	6	1	-2	-3	-2	1	6



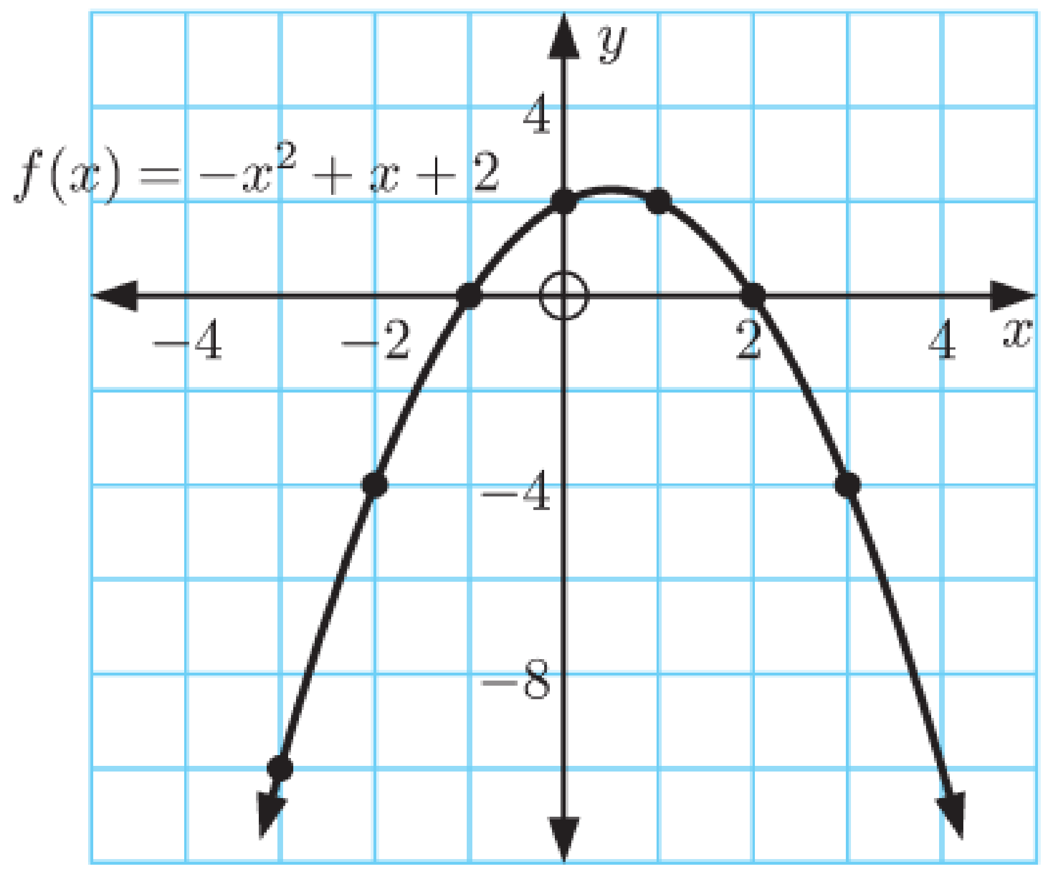
c

x	-3	-2	-1	0	1	2	3
y	15	8	3	0	-1	0	3



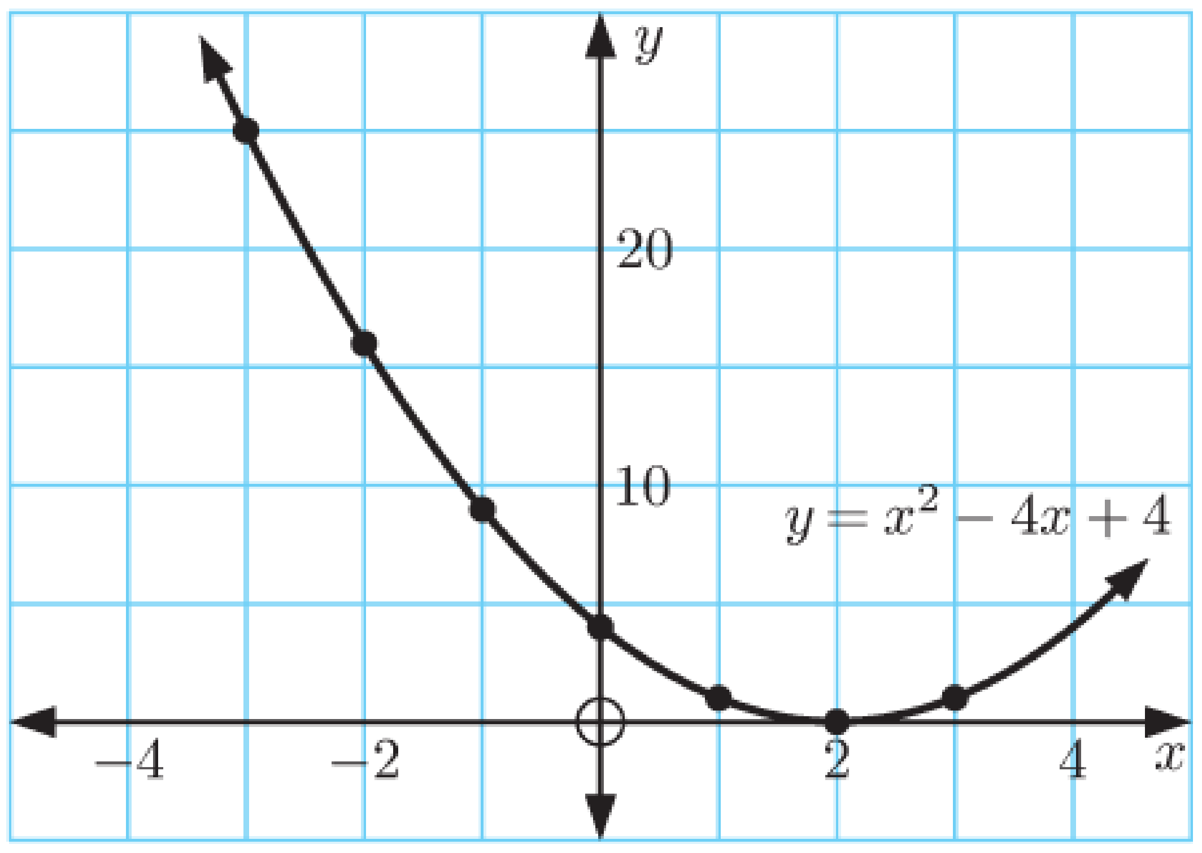
d

x	-3	-2	-1	0	1	2	3
$f(x)$	-10	-4	0	2	2	0	-4



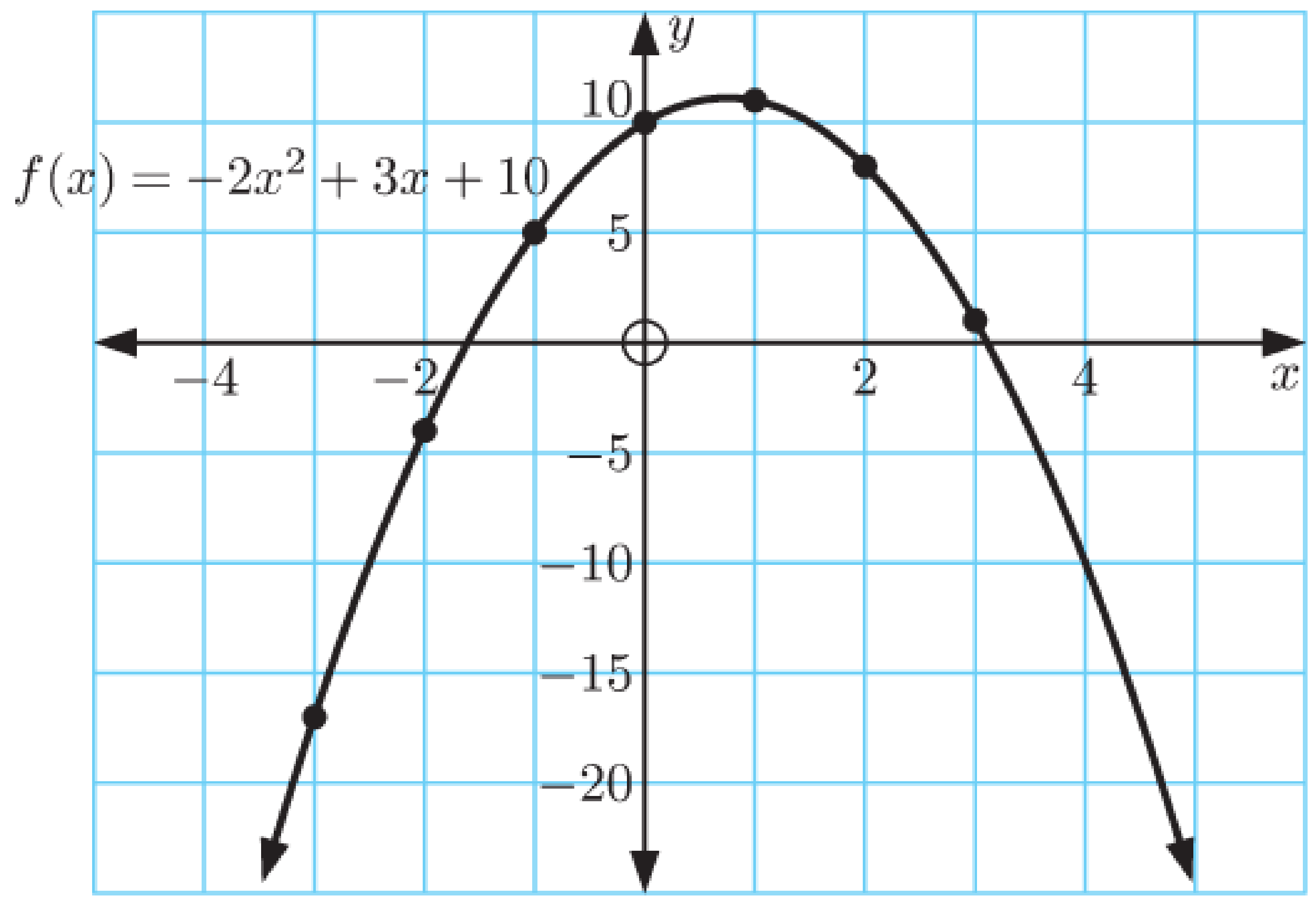
e

x	-3	-2	-1	0	1	2	3
y	25	16	9	4	1	0	1



f

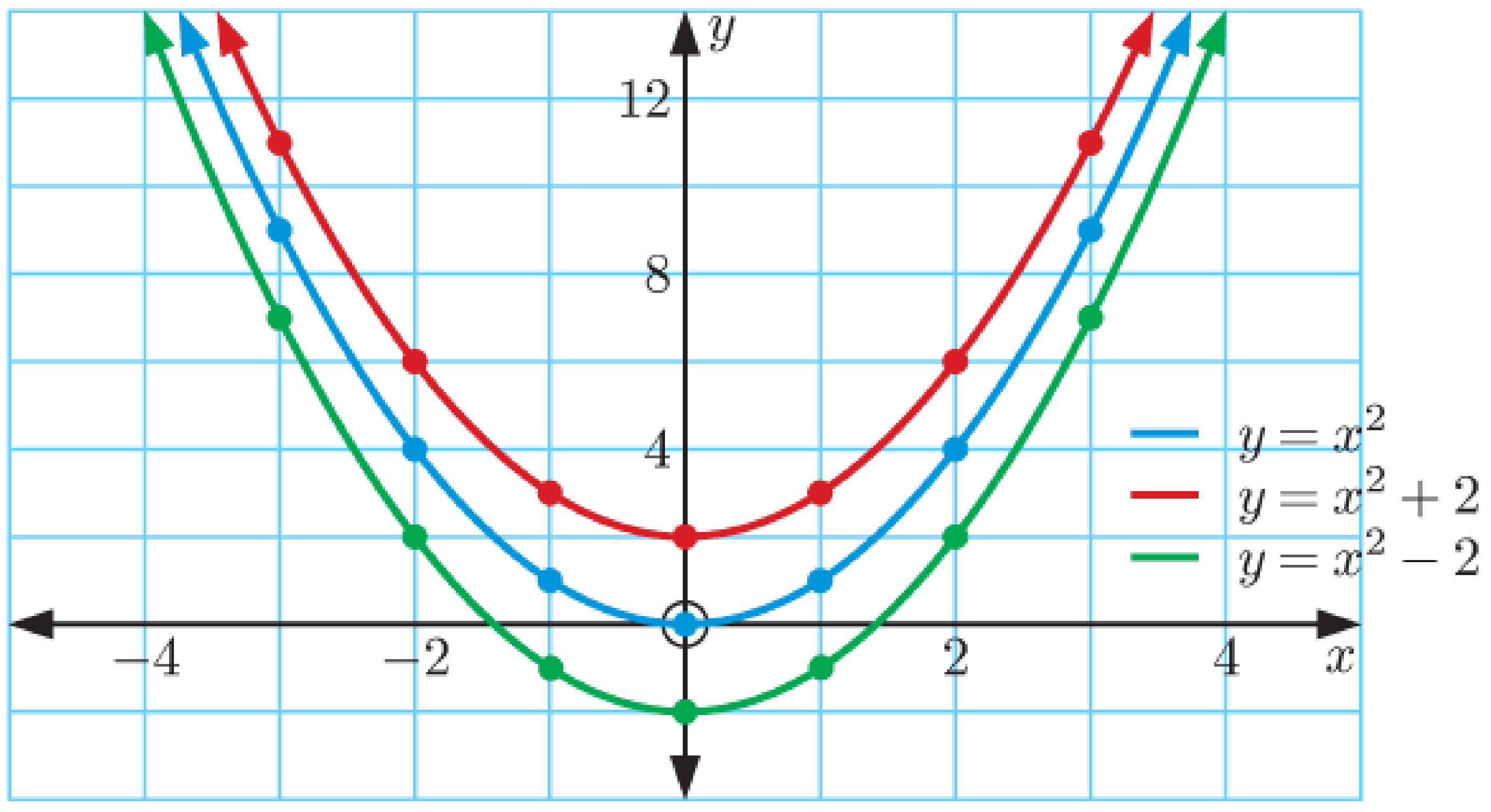
x	-3	-2	-1	0	1	2	3
$f(x)$	-17	-4	5	10	11	8	1



2 **a**

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
x^2+2	11	6	3	2	3	6	11
x^2-2	7	2	-1	-2	-1	2	7

b

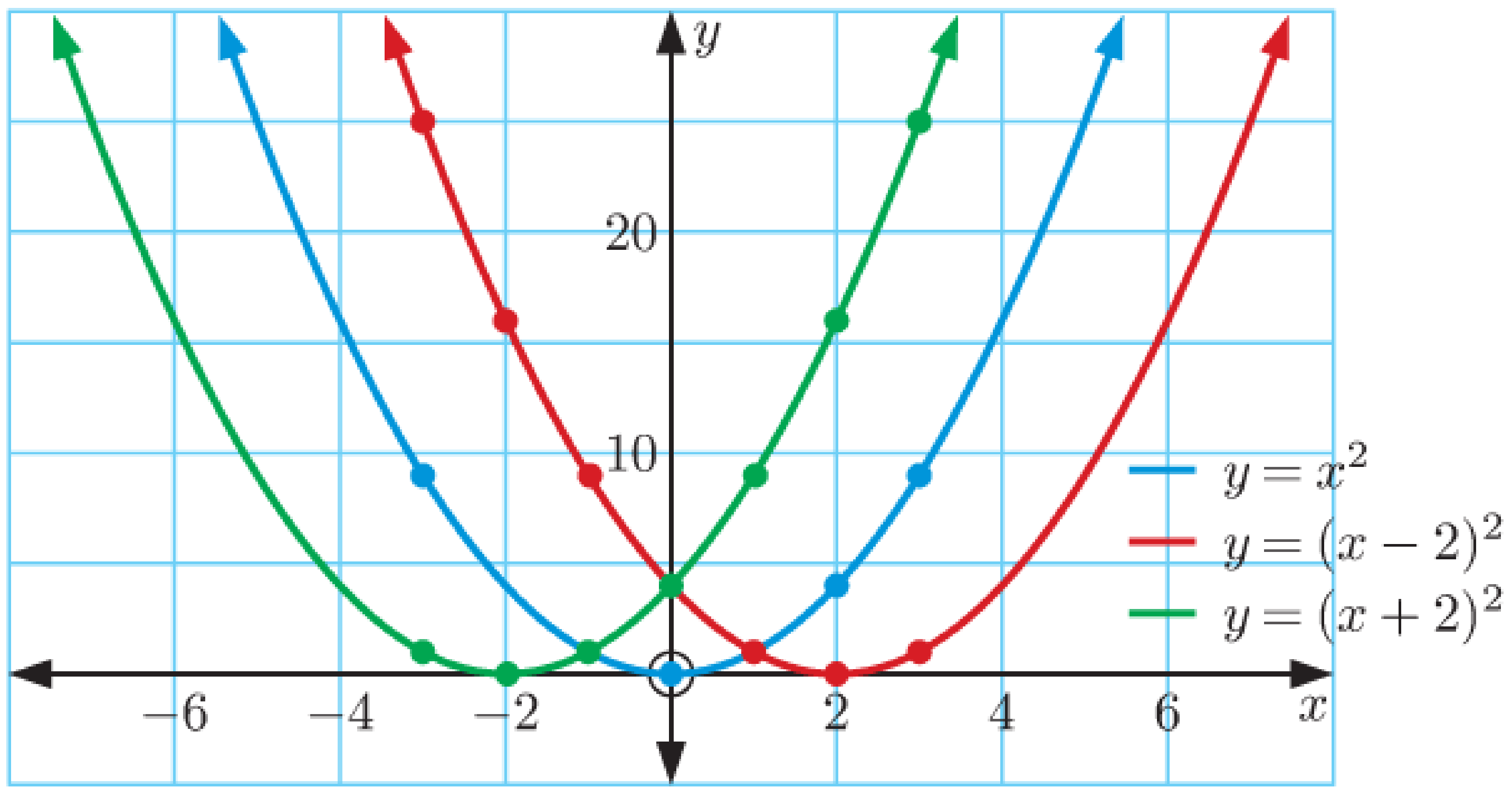


c The graphs have the same shape. $y = x^2 + 2$ is $y = x^2$ translated 2 units up, and $y = x^2 - 2$ is $y = x^2$ translated 2 units down.

3 **a**

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$(x+2)^2$	1	0	1	4	9	16	25
$(x-2)^2$	25	16	9	4	1	0	1

b

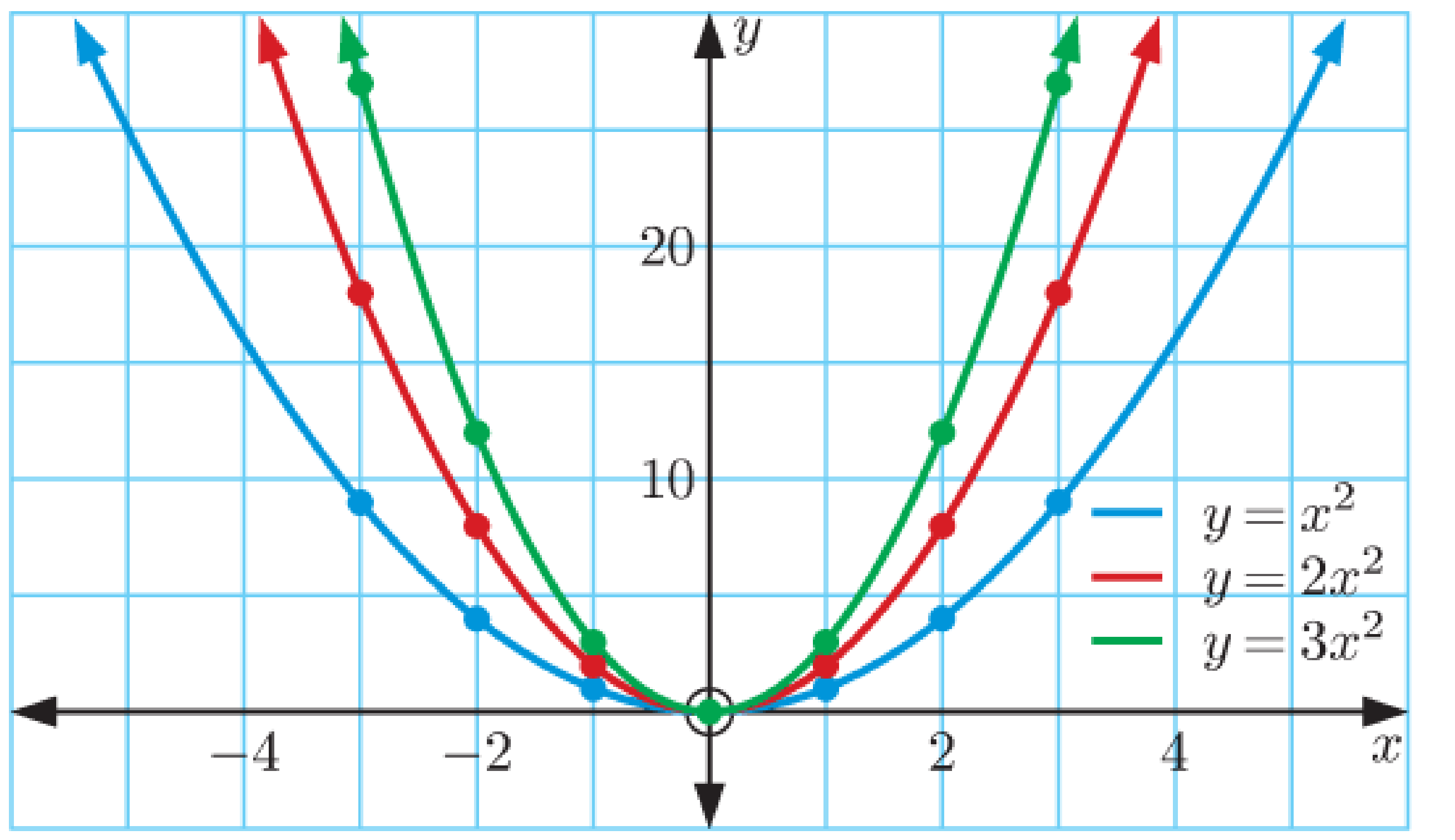


c The graphs have the same shape. $y = (x+2)^2$ is $y = x^2$ translated 2 units left, and $y = (x-2)^2$ is $y = x^2$ translated 2 units right.

4 **a**

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
$3x^2$	27	12	3	0	3	12	27

b

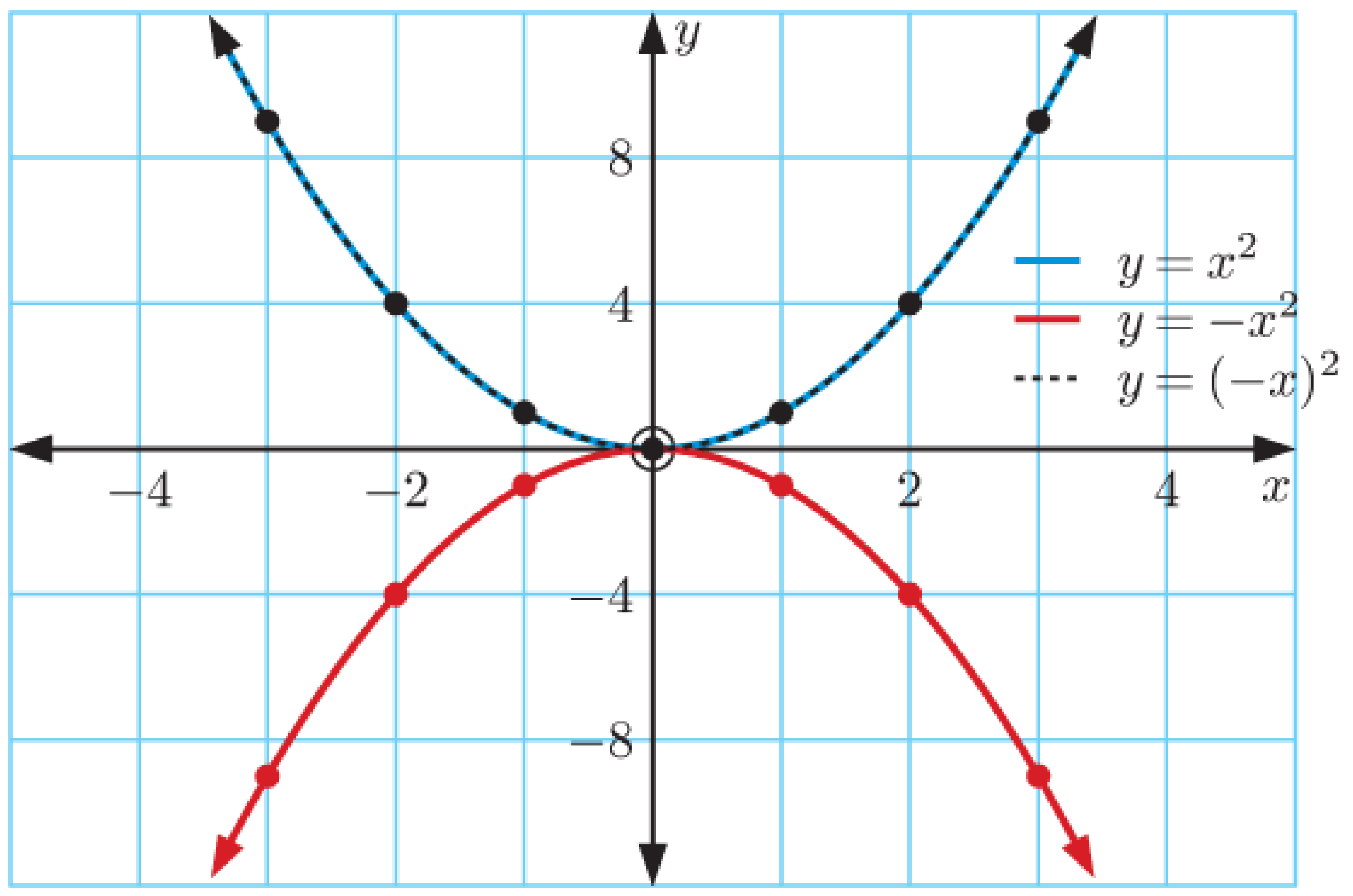


c As the coefficient of x^2 increases, the graphs become narrower.

5 **a**

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-x^2$	-9	-4	-1	0	-1	-4	-9
$(-x)^2$	9	4	1	0	1	4	9

b



c $y = -x^2$ is a reflection of $y = x^2$ (or $y = (-x)^2$) in the x-axis.

EXERCISE 6C

1 **a** 3 **b** -1 **c** -4 **d** 1 **e** 5 **f** 0
g 8 **h** -5 **i** 2

2 **a** 3 **b** -6 **c** 49 **d** 15 **e** 0 **f** 20

3 **a** 2 and 5 **b** 3 and -4 **c** -6 and -3 **d** 7 and -1
e 0 and 8 **f** -5 and 5 **g** $\frac{3}{2}$ and -1 **h** $-\frac{1}{3}$ and $\frac{5}{2}$
i -4 **j** 2 **k** -1 **l** $-\frac{3}{4}$

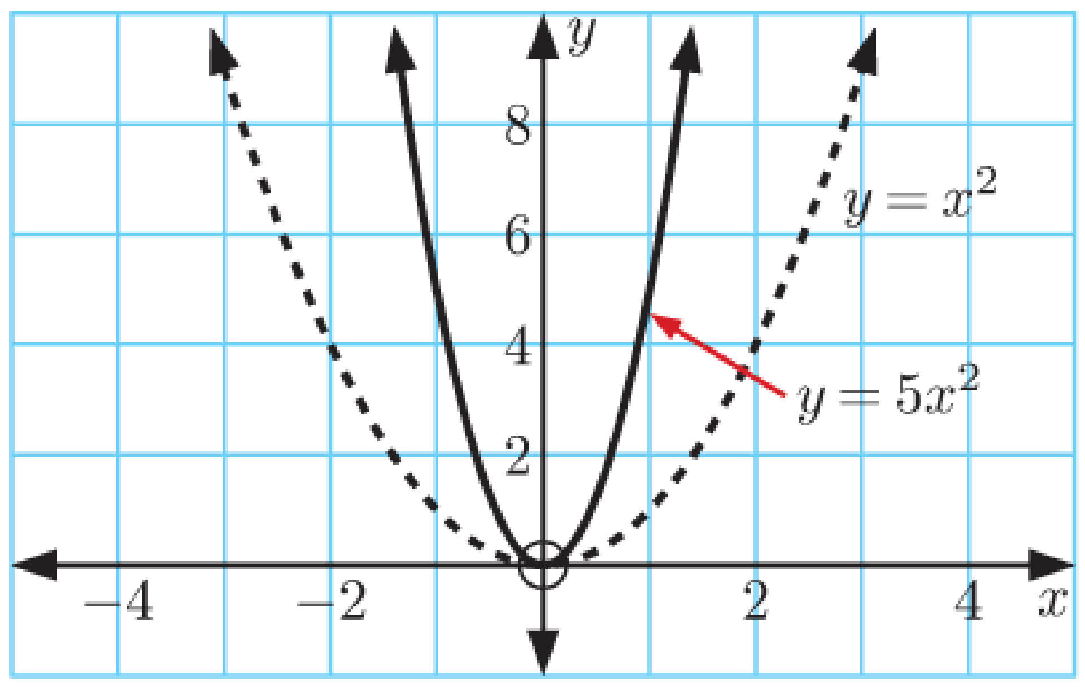
4 **a** 2 **b** 1 **c** 0

5 **a** -2 and 3 **b** 4 and -4 **c** no zeros
d 0 and 3 **e** 6 **f** ≈ 2.19 and ≈ -3.19
g no zeros **h** 1 and $-\frac{5}{6}$ **i** ≈ 0.434 and ≈ -0.768

6 **a** x-intercepts -2 and 1, y-intercept -2
b x-intercept -3, y-intercept 9
c x-intercepts -5 and 2, y-intercept -10
d x-intecepts $\frac{2}{3}$ and 5, y-intercept 10
e x-intercepts ≈ 1.44 and ≈ 5.56, y-intercept -8
f x-intercept -4, y-intercept -16
g x-intercepts 0 and 7, y-intercept 0
h x-intercepts ≈ -1.27 and ≈ 2.77, y-intercept 7
i x-intercepts -3 and 3, y-intercept -18
j no x-intercepts, y-intercept -9
k x-intercepts $-\frac{1}{2}$ and $\frac{3}{2}$, y-intercept -3
l x-intercepts ≈ -1.30 and ≈ 1.70, y-intercept 11

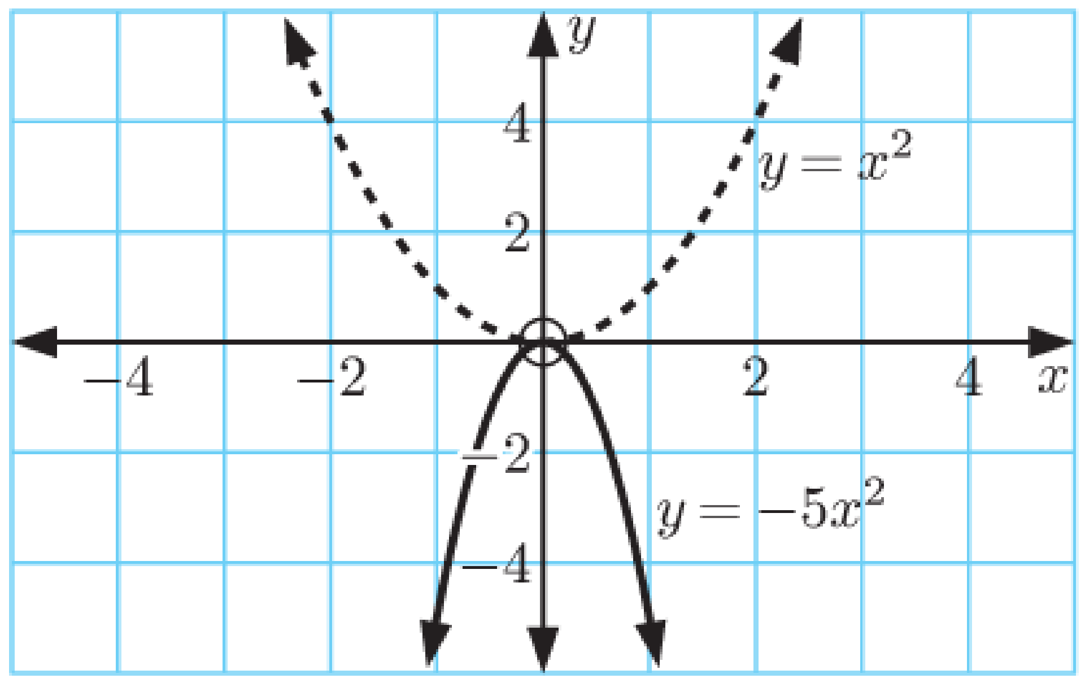
EXERCISE 6D

1 **a**



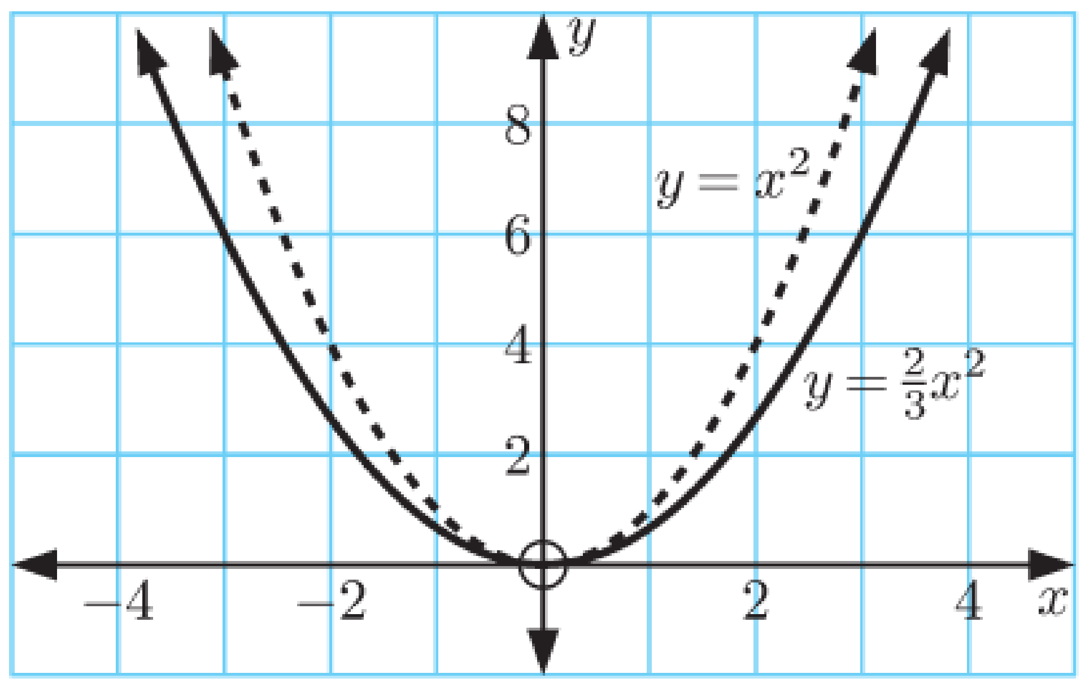
$y = 5x^2$ opens upwards and is "thinner".

b



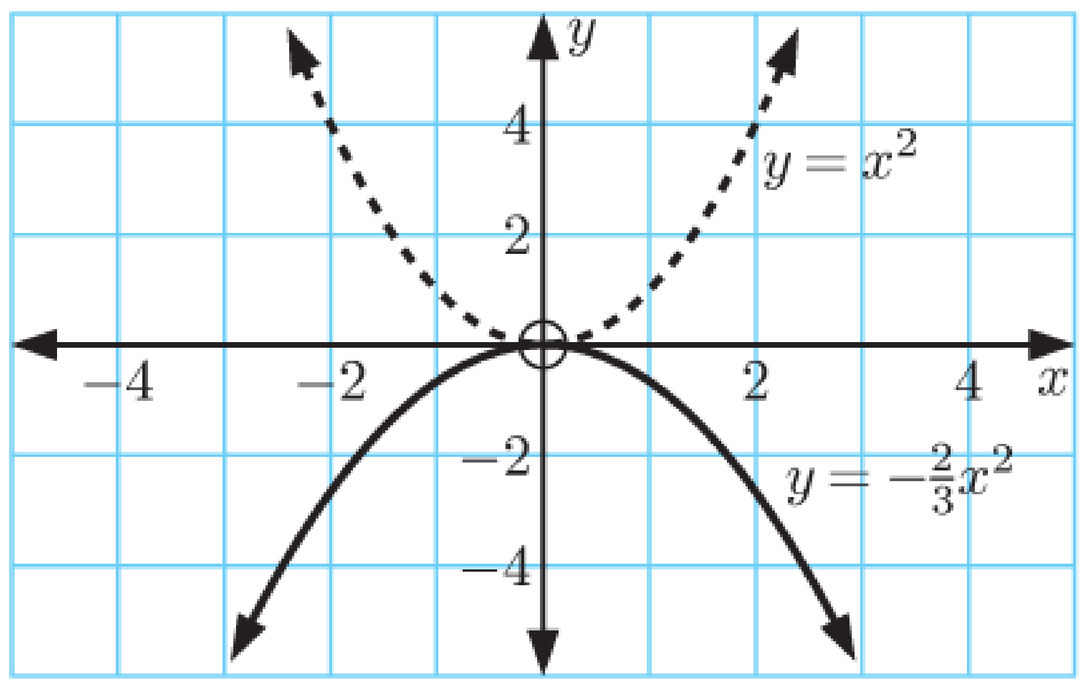
$y = -5x^2$ opens downwards and is "thinner".

c



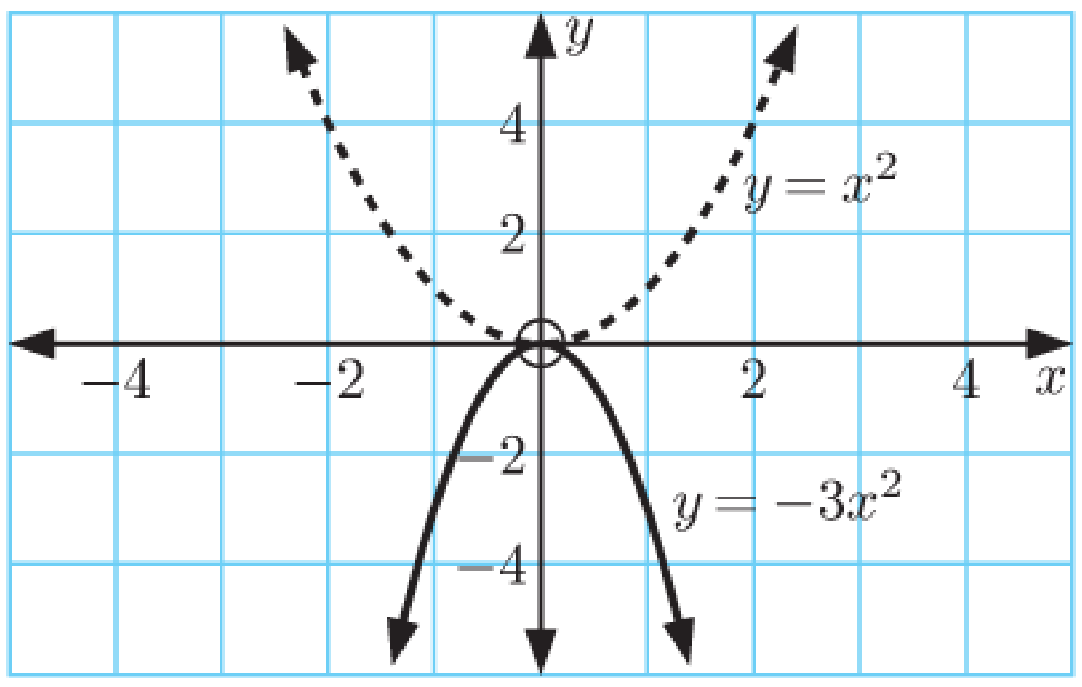
$y = \frac{2}{3}x^2$ opens upwards and is "wider".

d



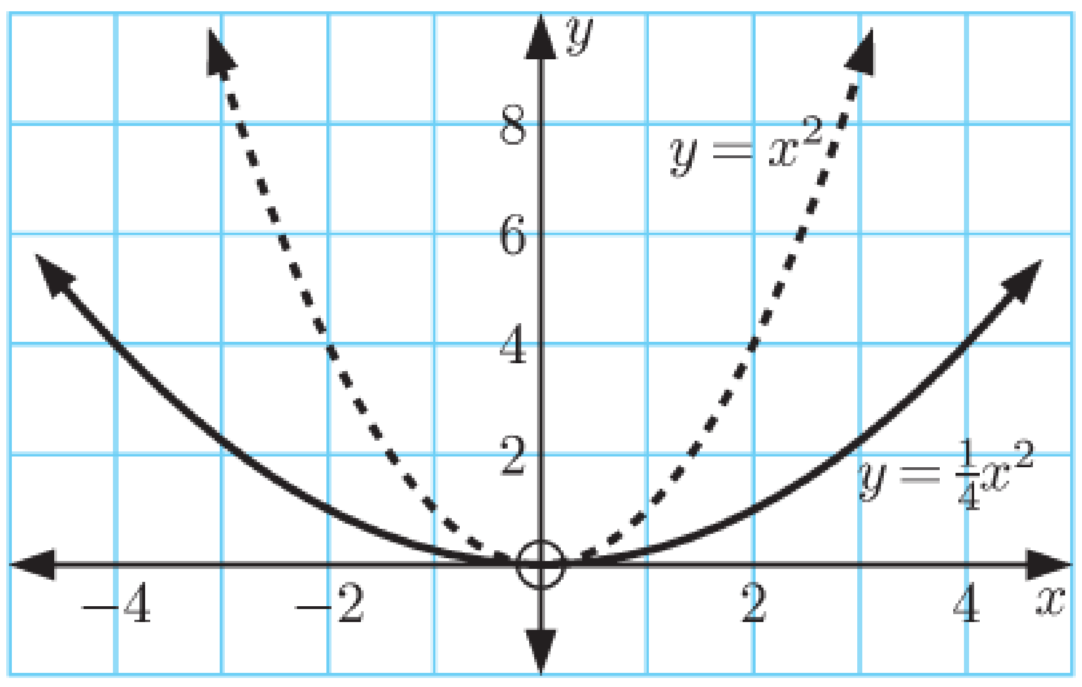
$y = -\frac{2}{3}x^2$ opens downwards and is "wider".

e



$y = -3x^2$ opens downwards and is "thinner".

f

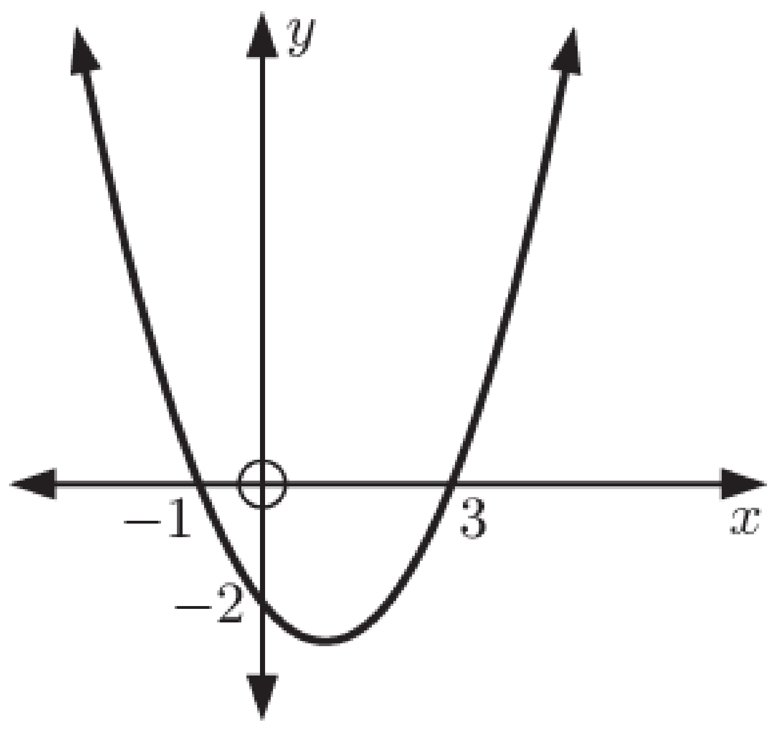


$y = \frac{1}{4}x^2$ opens upwards and is "wider".

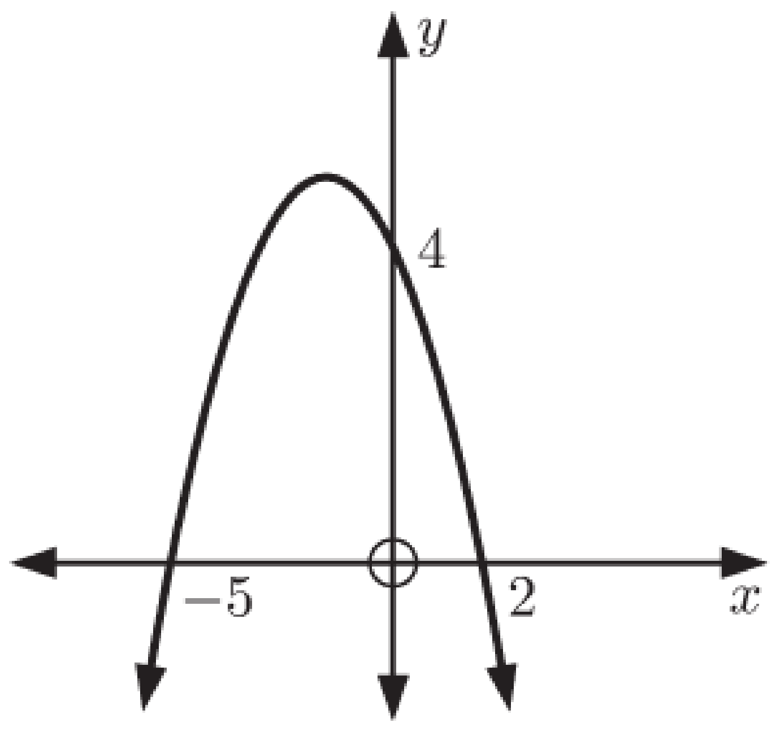
2 **a** (0, 0), minimum turning point
b (0, 0), maximum turning point
c (0, 0), maximum turning point

EXERCISE 6E

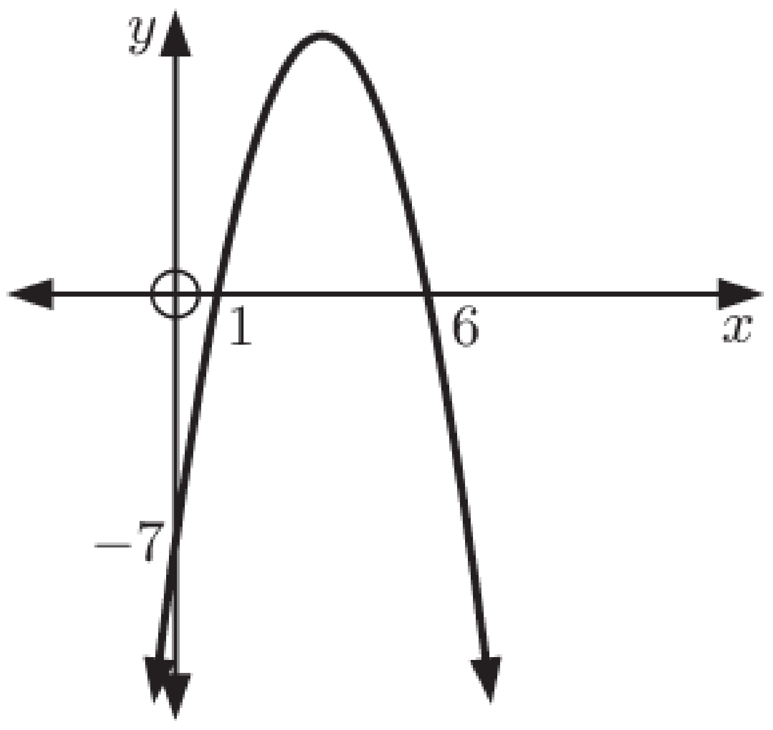
1 **a**



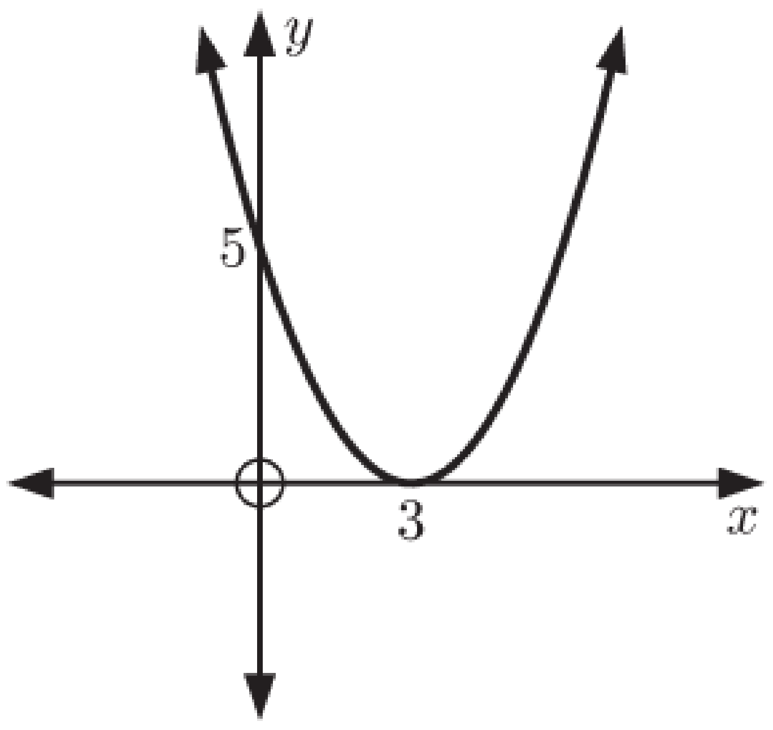
b



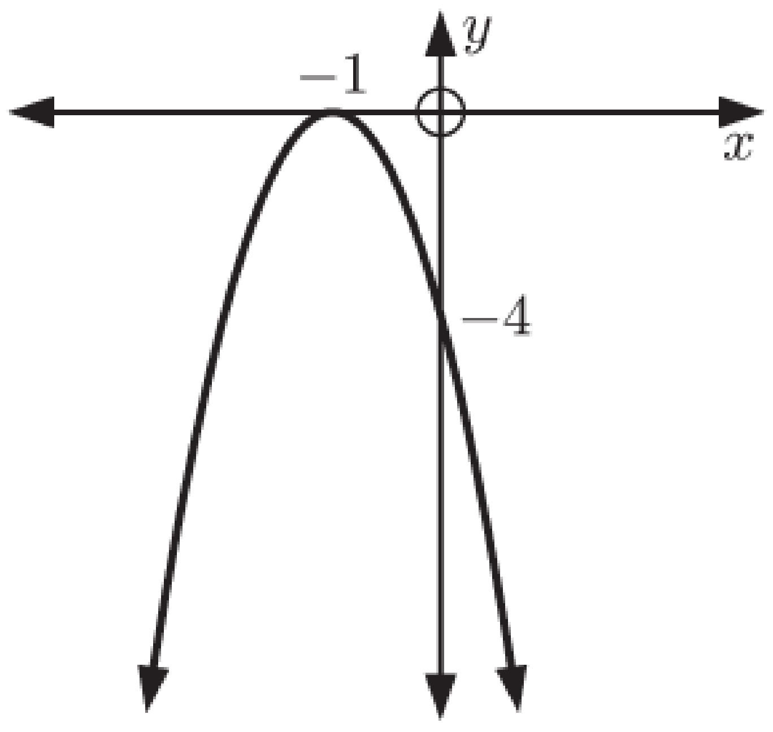
c



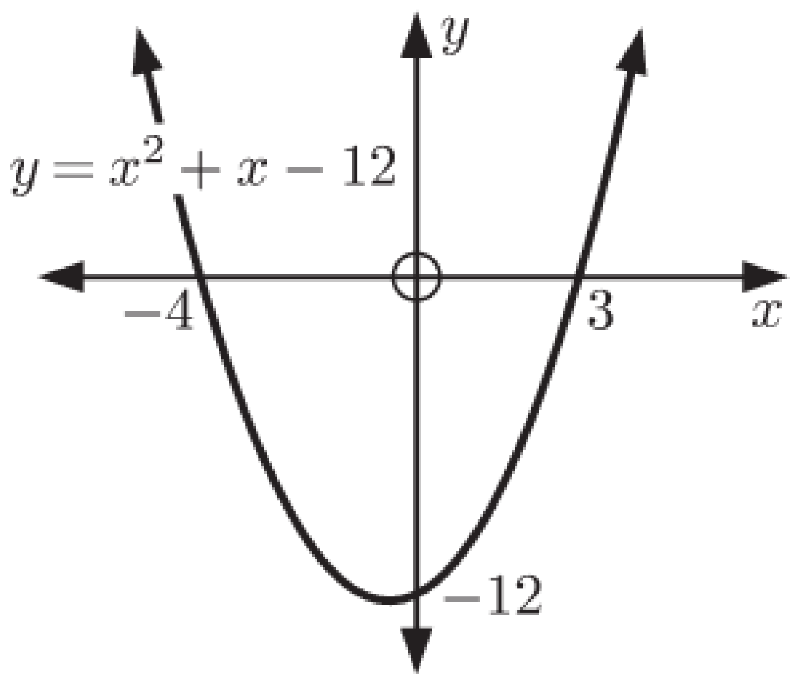
d



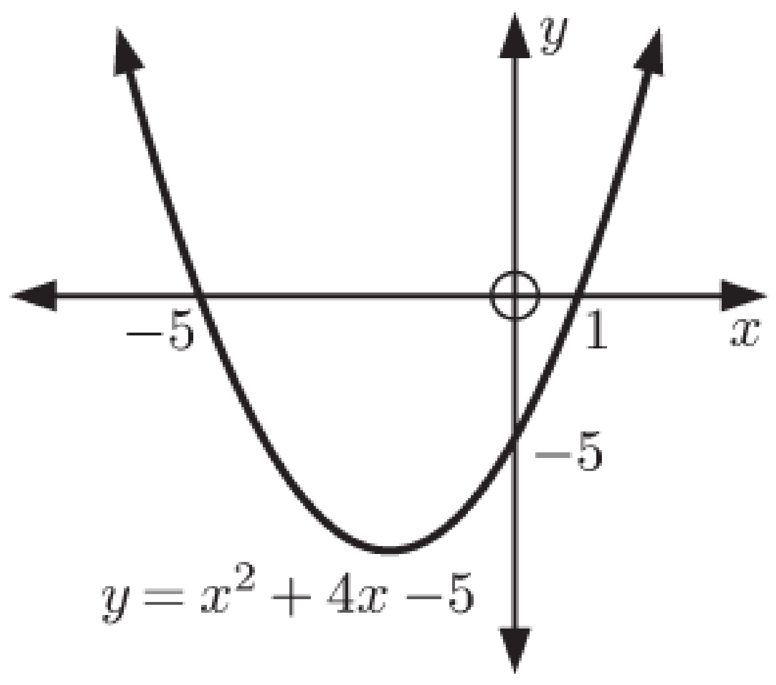
e



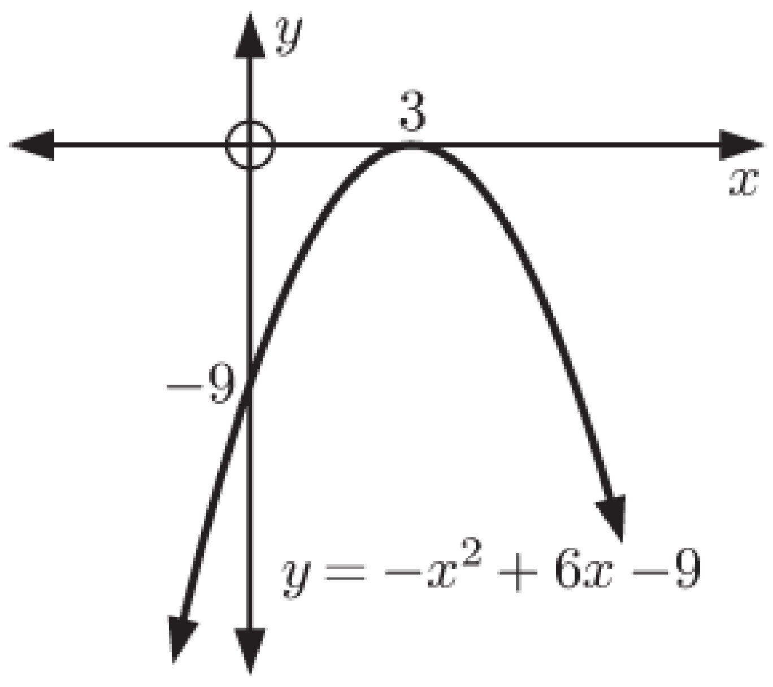
2 **a**



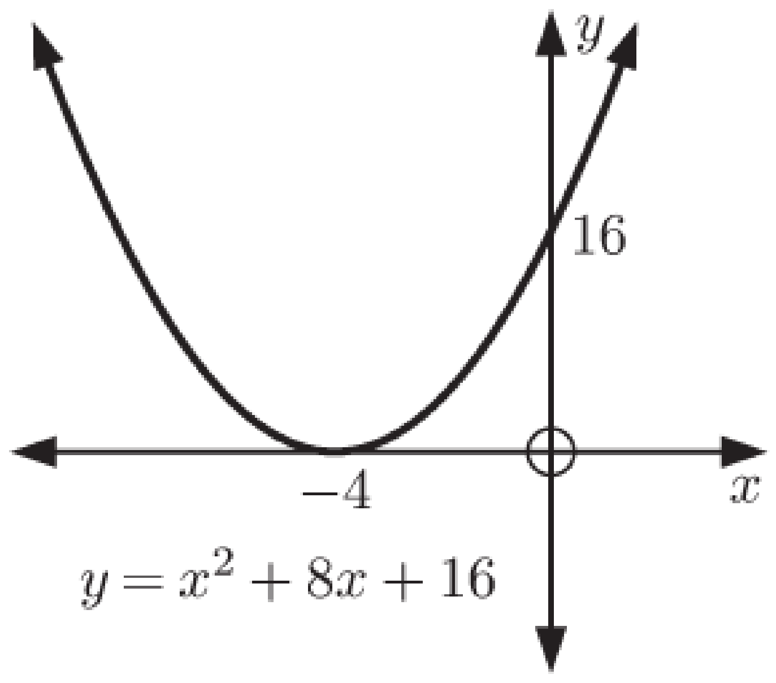
b



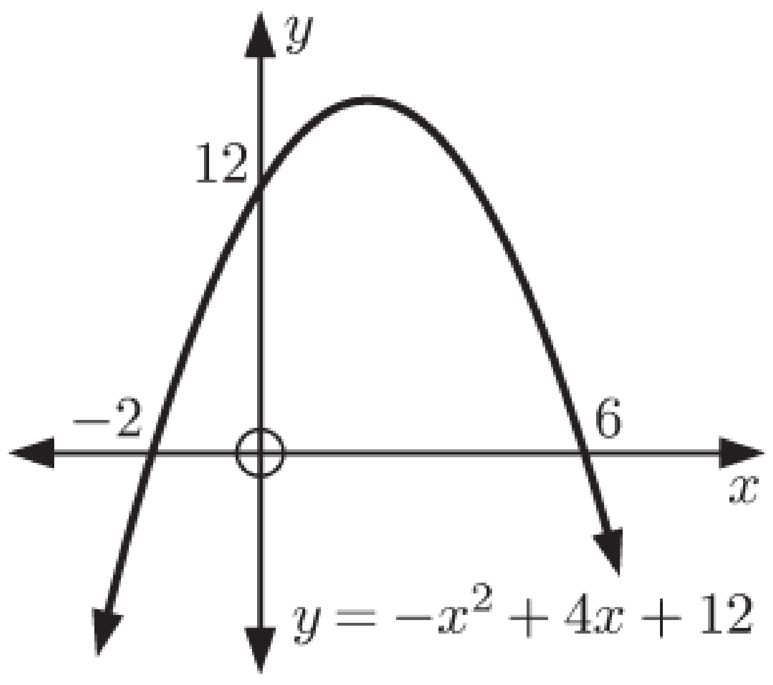
c



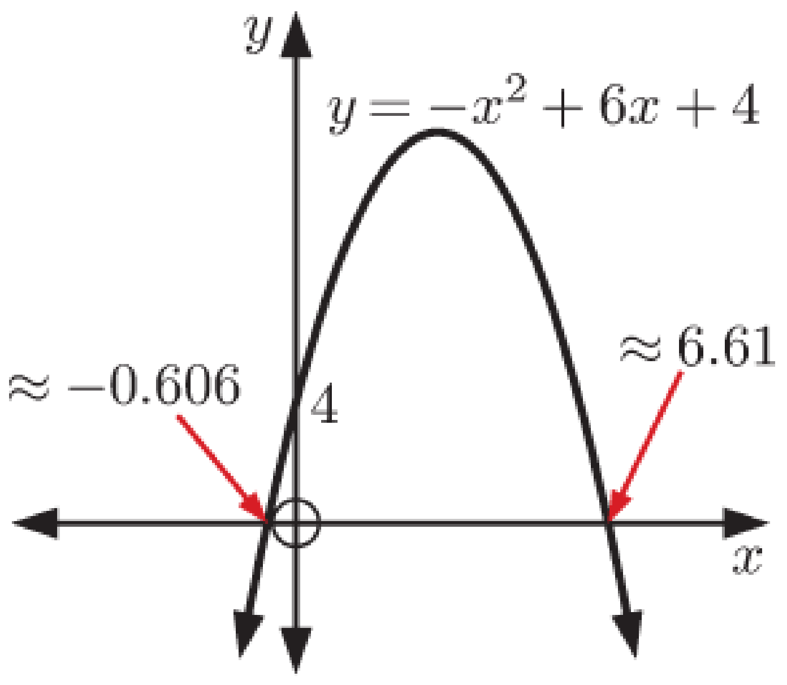
d



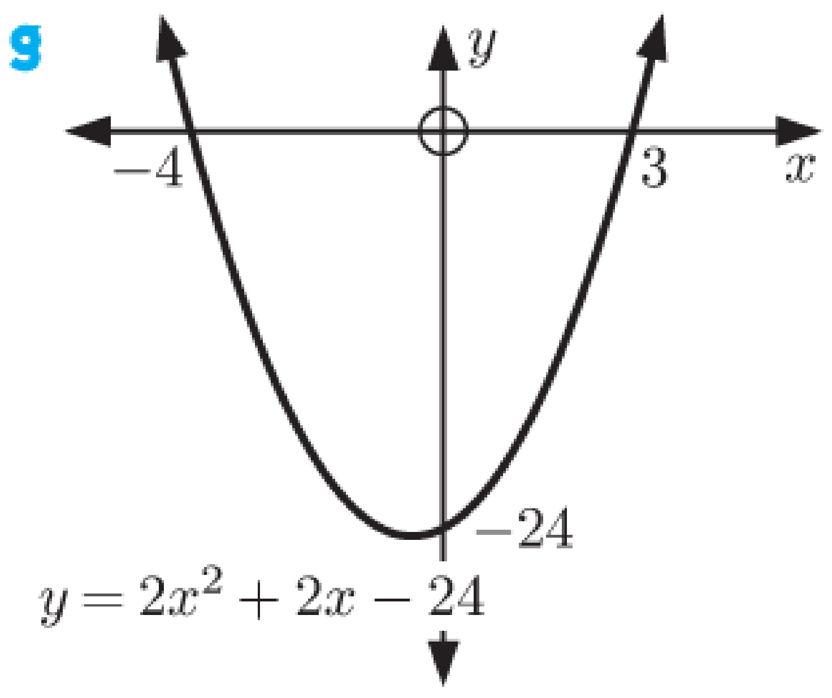
e



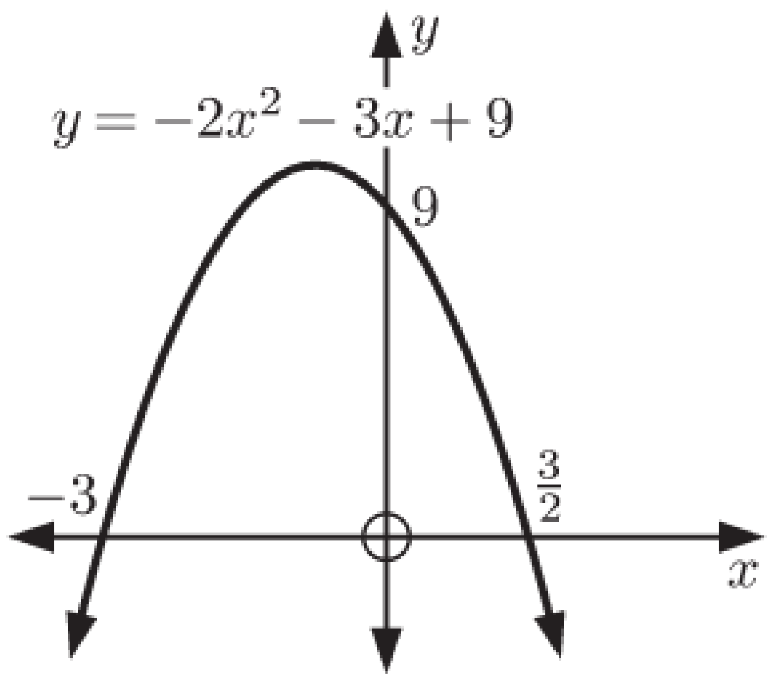
f



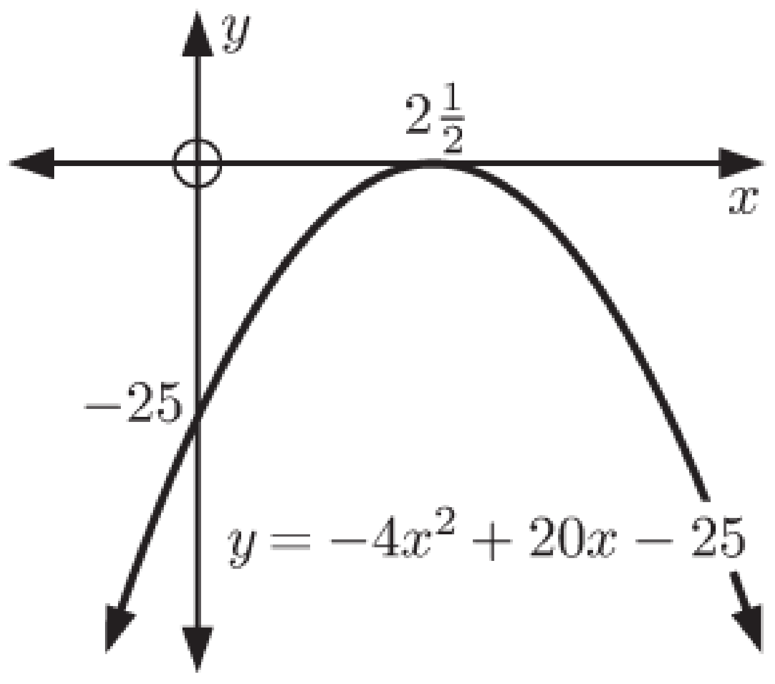
g



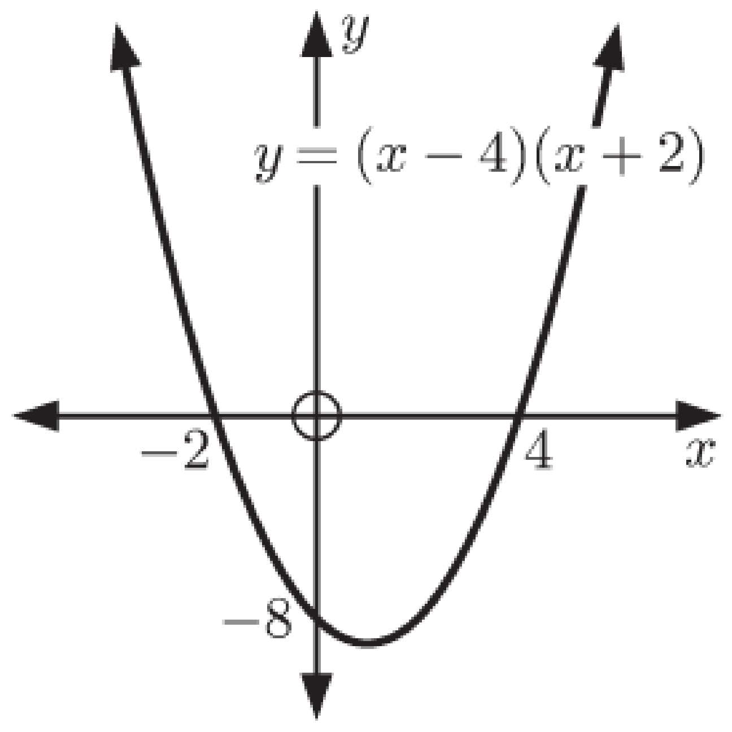
h



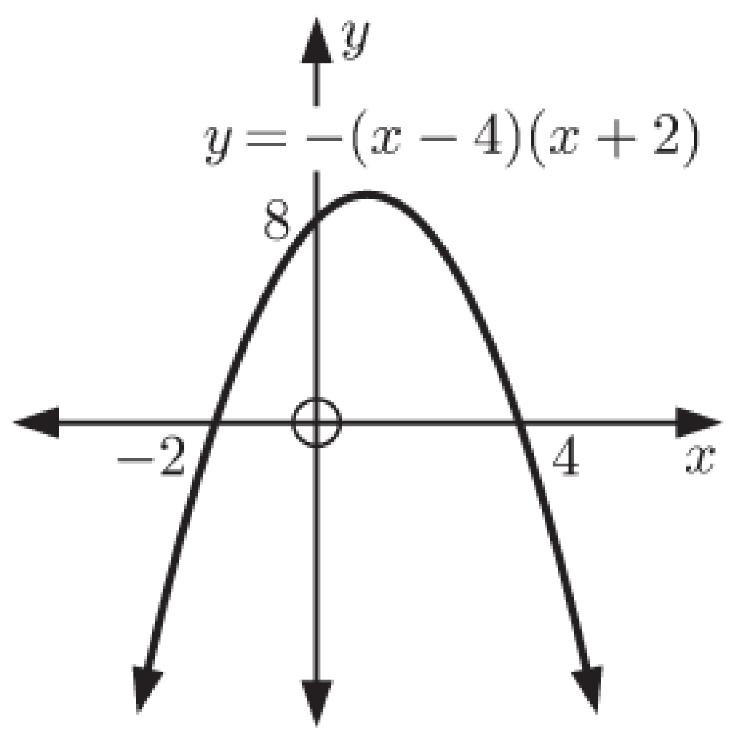
i



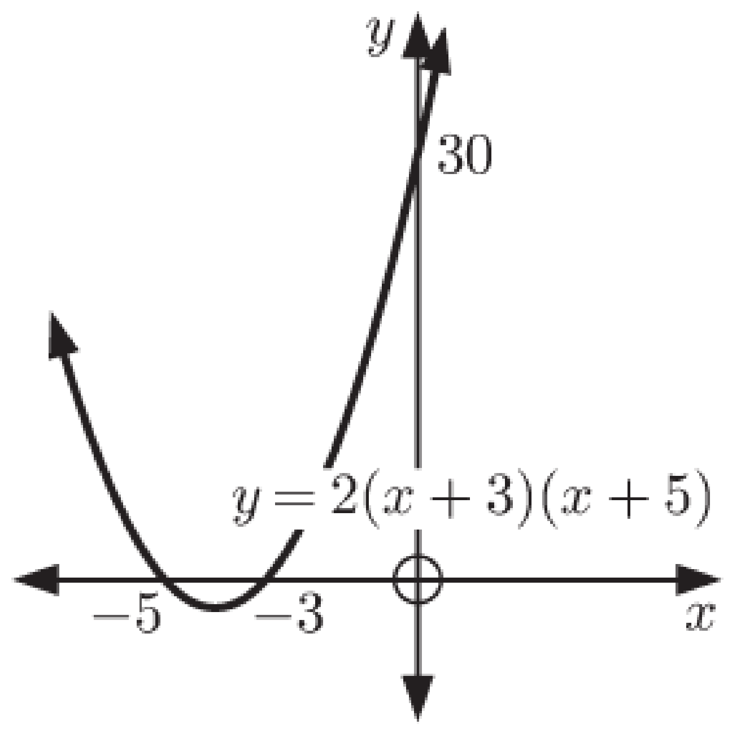
3 **a**



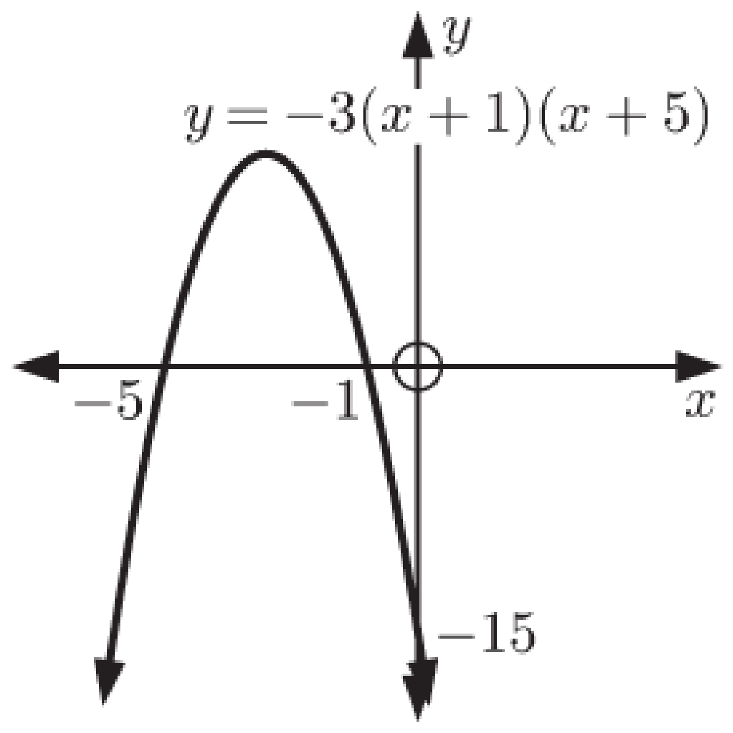
b



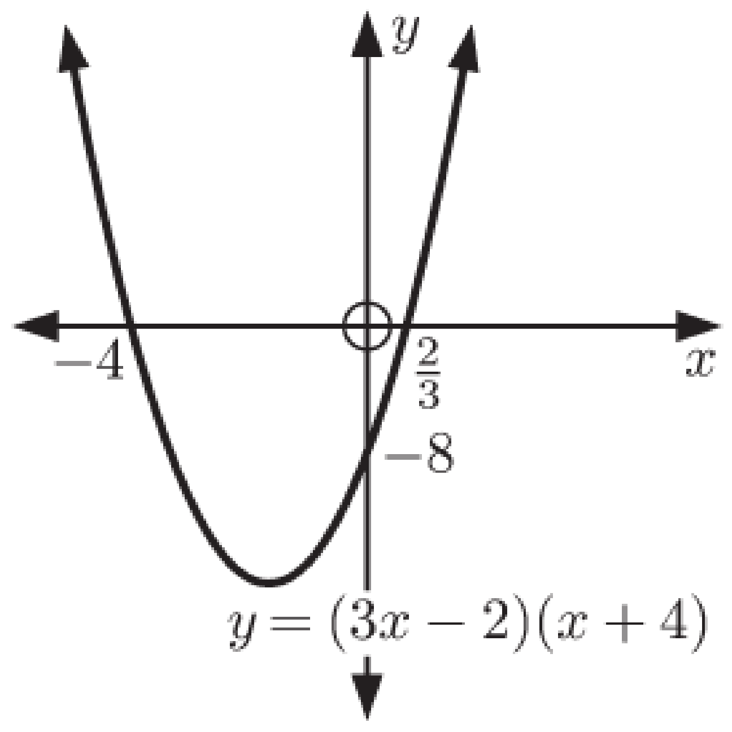
c



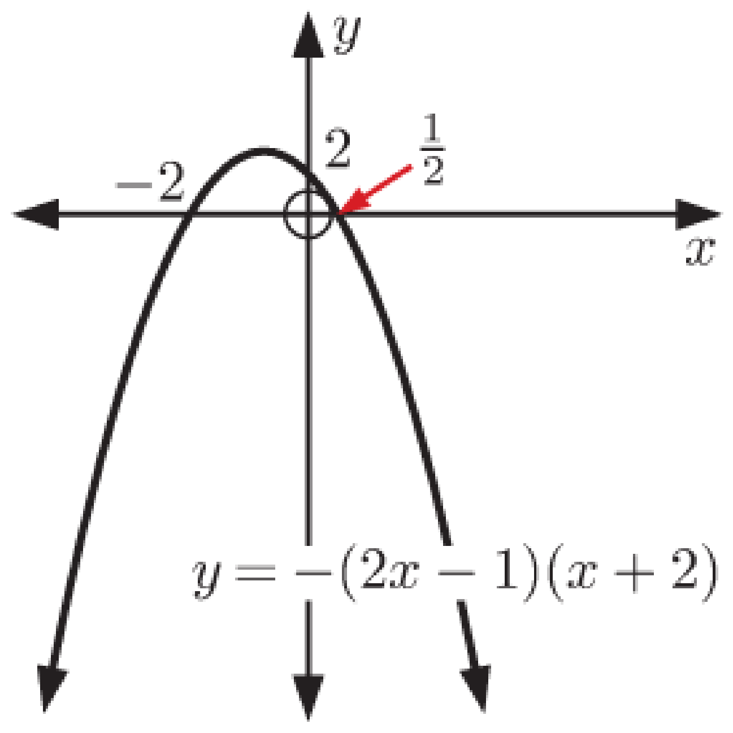
d



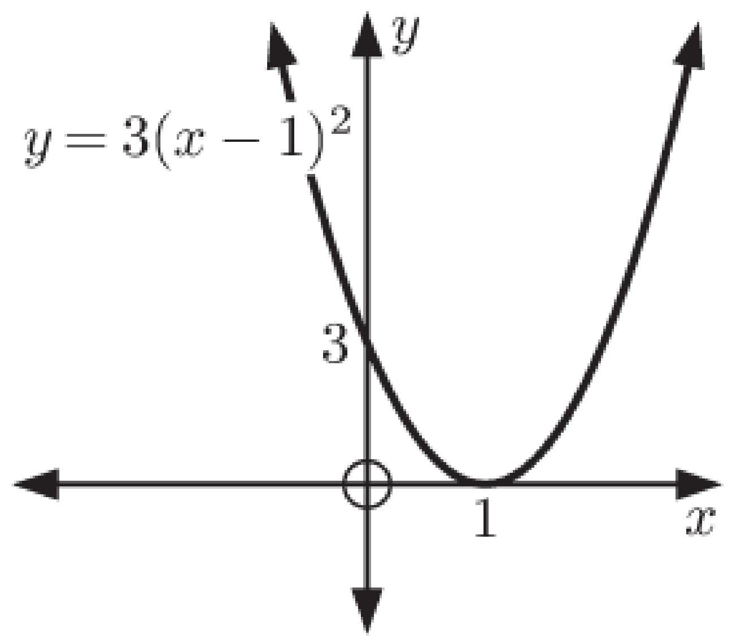
e



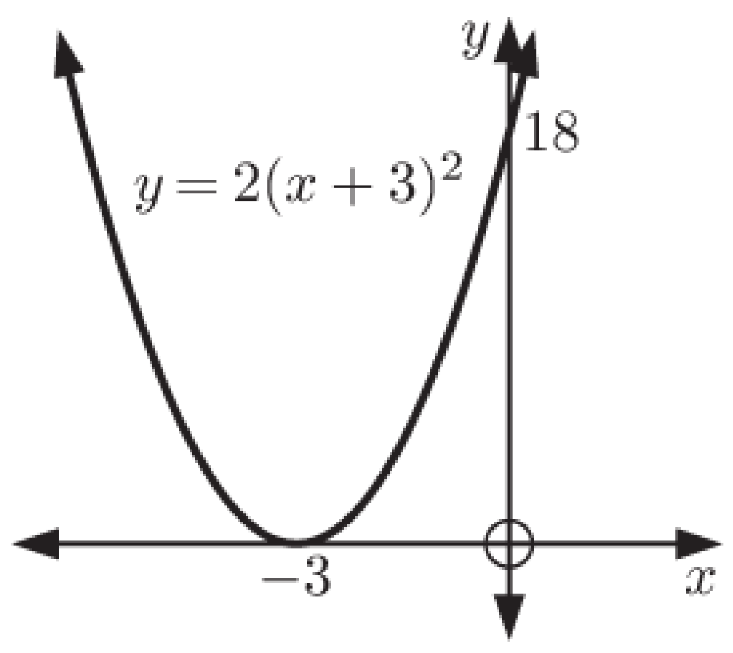
f



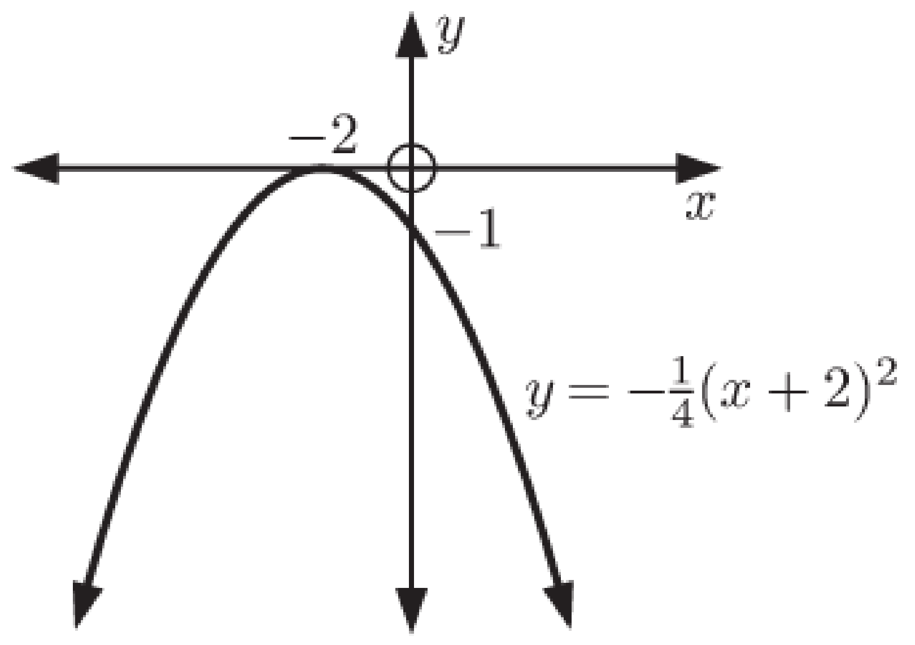
4 **a**



b



c



EXERCISE 6F

1 **a** $x = 3$ **b** $x = -\frac{5}{2}$ **c** $x = 1$
d $x = -4$ **e** $x = 3$ **f** $x = -4$

2 **a** $x = 4$ **b** $x = -2$ **c** $x = 1$
d $x = \frac{11}{2}$ **e** $x = 5$ **f** $x = -2$

3 -10

4 **a** $x = -3$ **b** $x = 4$ **c** $x = -\frac{5}{4}$ **d** $x = \frac{3}{2}$
e $x = 0$ **f** $x = \frac{7}{10}$ **g** $x = -\frac{1}{6}$ **h** $x = \frac{5}{3}$
i $x = -4$

5 **a** $x = \frac{3}{2}$ **b** $y = x^2 - 3x - 10$,
axis of symmetry is $x = -\frac{(-3)}{2(1)} = \frac{3}{2}$ ✓

6 $a = \frac{3}{2}$

7 **a** **i** $x = -\frac{5}{4}$ **ii** $x = -\frac{5}{4}$ **iii** $x = -\frac{5}{4}$
The axes of symmetry are the same.

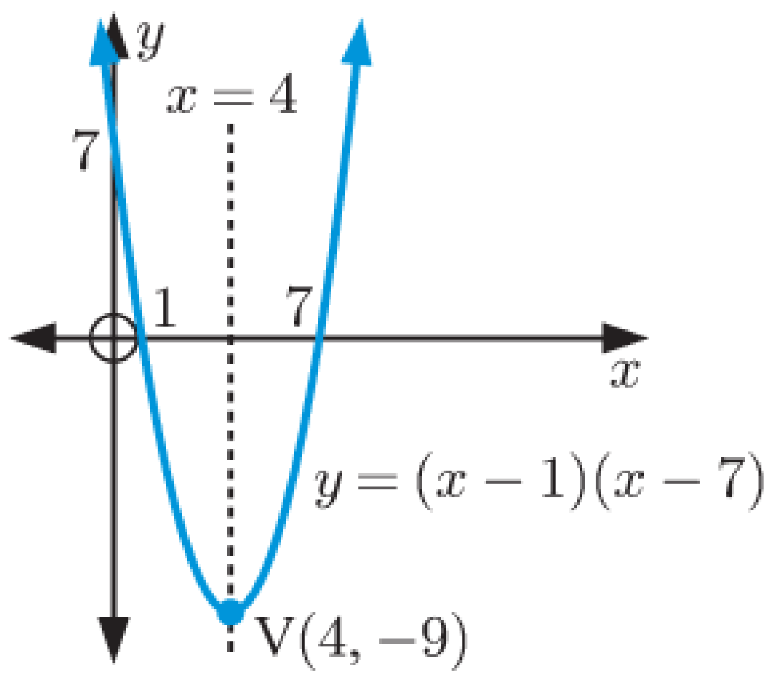
b The value of c only affects the vertical translation of the graph. It does not affect the axis of symmetry (since it is a vertical line). Hence $x = -\dfrac{b}{2a}$ depends only on a and b.

EXERCISE 6G

1 **a** $(2, -2)$, minimum turning point
b $(-1, -4)$, minimum turning point
c $(0, 4)$, minimum turning point
d $(0, 1)$, maximum turning point
e $(1, -9)$, minimum turning point
f $(-2, -5)$, maximum turning point
g $(-\frac{3}{2}, -\frac{11}{2})$, minimum turning point
h $(\frac{1}{2}, \frac{49}{2})$, maximum turning point
i $(1, -\frac{9}{2})$, maximum turning point
j $(14, -43)$, minimum turning point

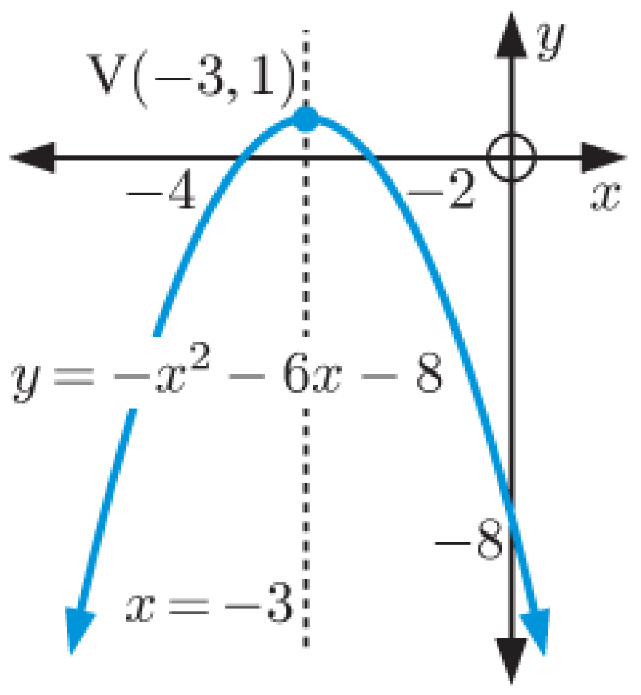
2 **a** **i** x-intercepts 1 and 7
y-intercept 7
ii $x = 4$
iii $(4, -9)$
v Domain is
$\{x \mid x \in \mathbb{R}\}$
Range is
$\{y \mid y \geqslant -9\}$

iv



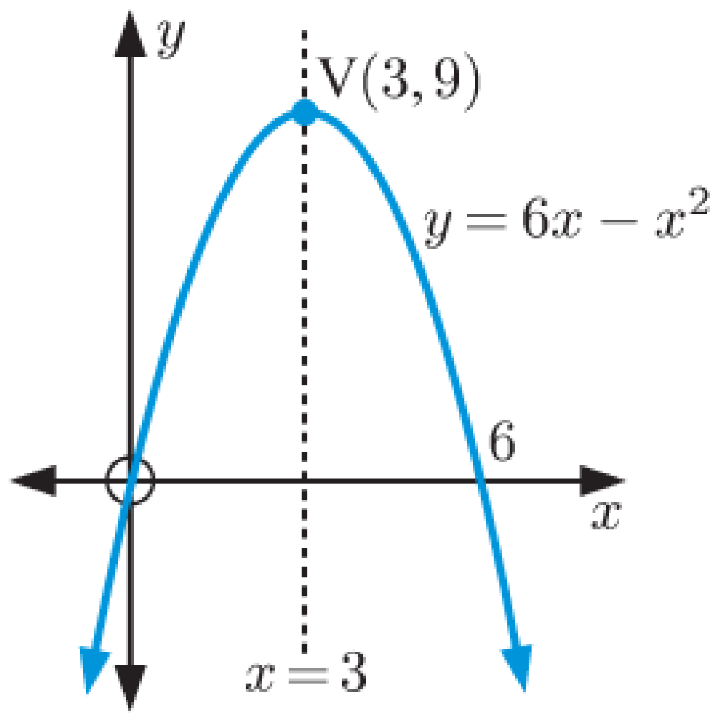
b **i** x-intercepts -2
and -4
y-intercept -8
ii $x = -3$
iii $(-3, 1)$
v Domain is
$\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 1\}$

iv



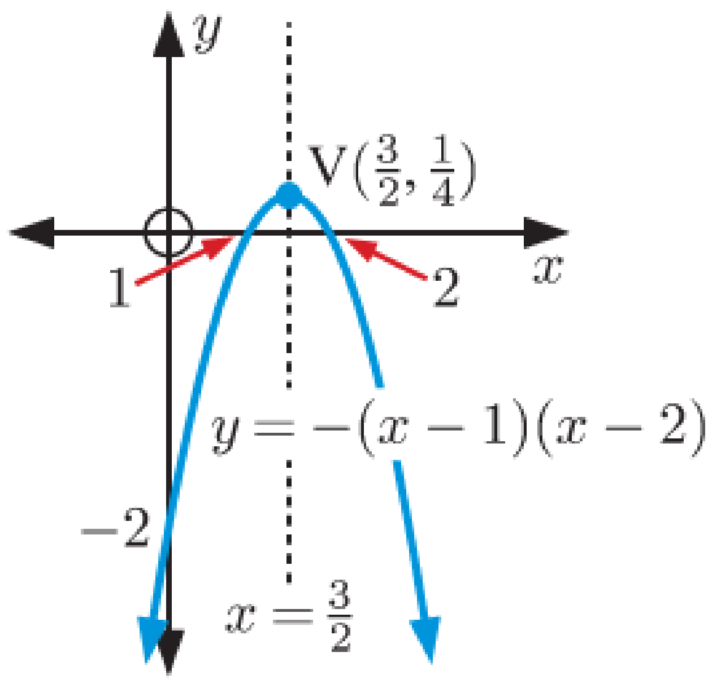
c **i** x-intercepts 0 and 6
y-intercept 0
ii $x = 3$
iii $(3, 9)$
v Domain is
$\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 9\}$

iv



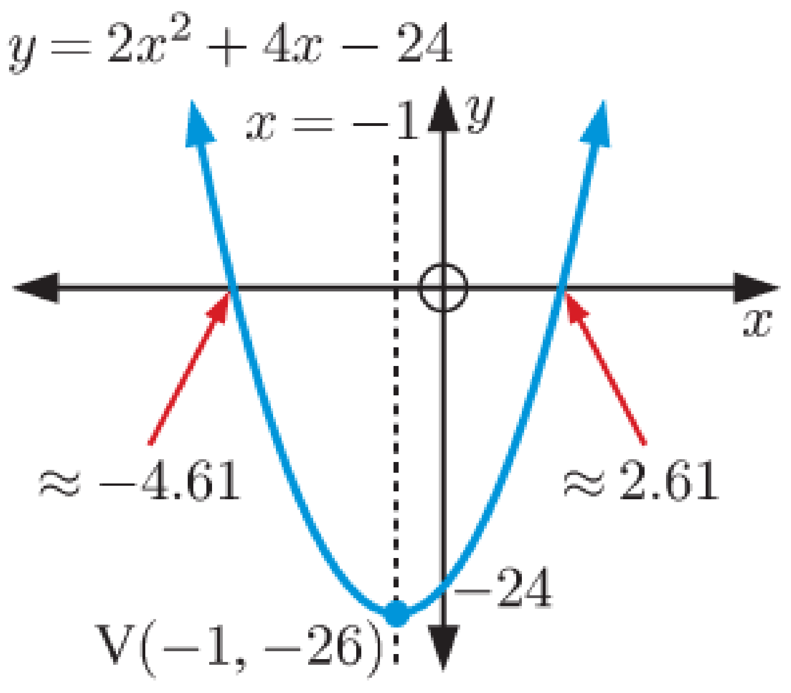
d **i** x-intercepts 1 and 2
y-intercept -2
ii $x = \frac{3}{2}$
iii $(\frac{3}{2}, \frac{1}{4})$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant \frac{1}{4}\}$

iv



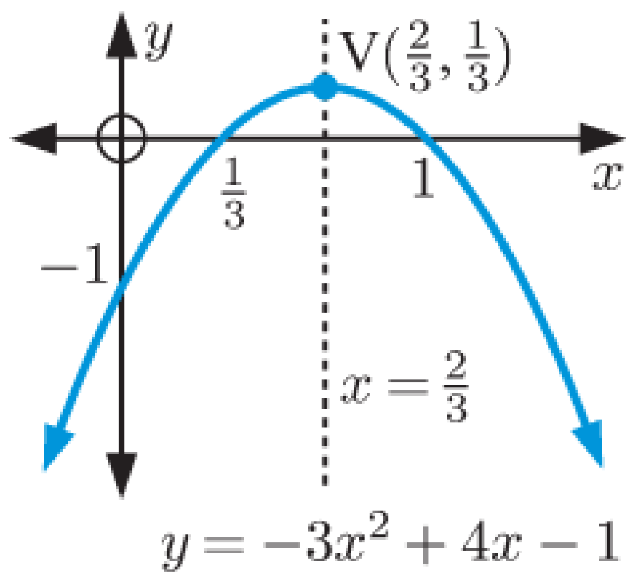
e **i** x-intercepts ≈ -4.61 and ≈ 2.61
y-intercept -24
ii $x = -1$
iii $(-1, -26)$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -26\}$

iv



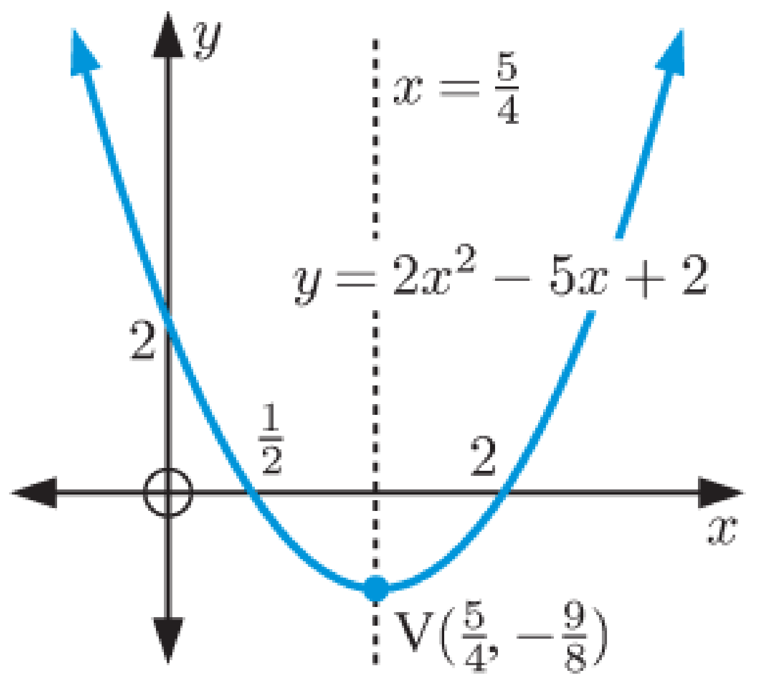
f **i** x-intercepts $\frac{1}{3}$ and 1
y-intercept -1
ii $x = \frac{2}{3}$
iii $(\frac{2}{3}, \frac{1}{3})$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant \frac{1}{3}\}$

iv



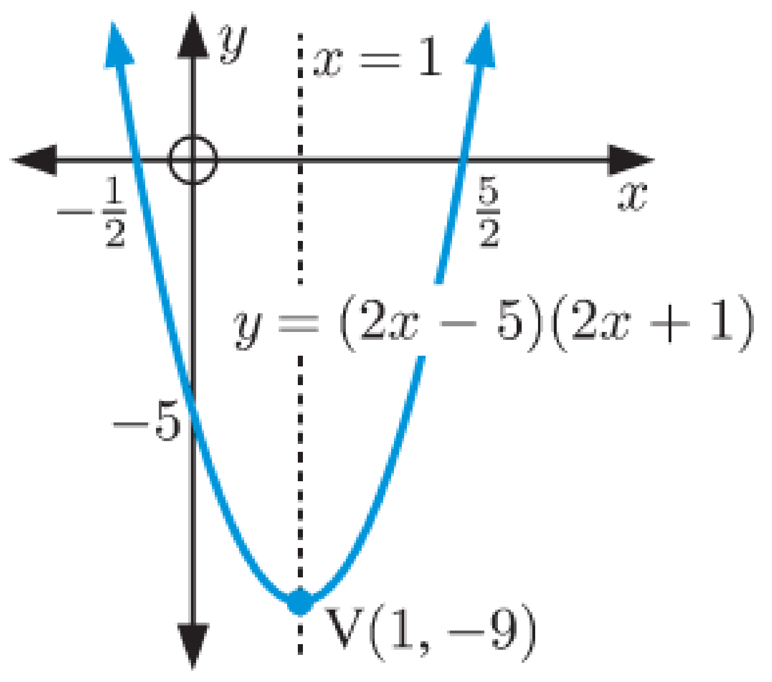
g **i** x-intercepts $\frac{1}{2}$ and 2
y-intercept 2
ii $x = \frac{5}{4}$
iii $(\frac{5}{4}, -\frac{9}{8})$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -\frac{9}{8}\}$

iv



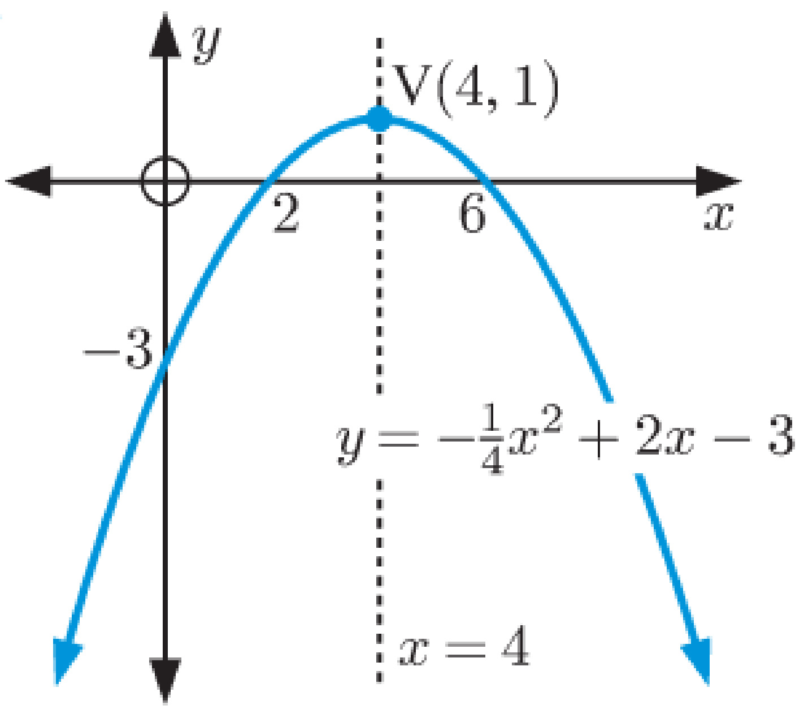
h **i** x-intercepts $-\frac{1}{2}$ and $\frac{5}{2}$
y-intercept -5
ii $x = 1$
iii $(1, -9)$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -9\}$

iv



i **i** x-intercepts 2 and 6
y-intercept -3
ii $x = 4$
iii $(4, 1)$
v Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant 1\}$

iv



3 **a** $y = -11$ when $x = 4$ **b** $y = 29$ when $x = -5$
c $y = -\frac{11}{4}$ when $x = -\frac{1}{2}$ **d** $y = \frac{41}{2}$ when $x = 7$

EXERCISE 6H

1 **a** $y = 2(x-1)(x-2)$ **b** $y = 3(x-2)^2$
c $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

2 **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

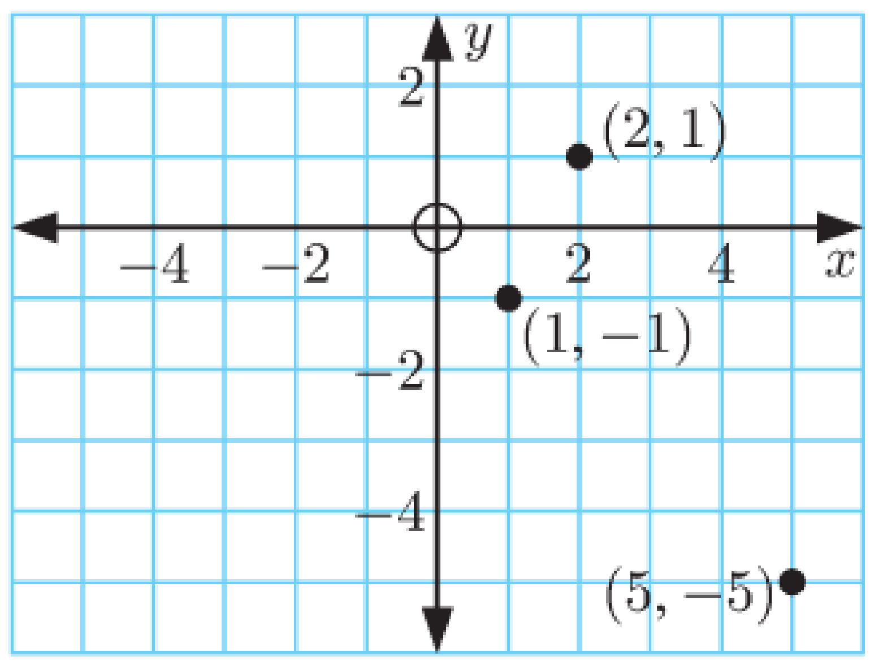
3 **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$
e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **f** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

4 **a** $c = 2$ **b** $a + b = -1,\ 2a + b = 2$
c $a = 3,\ b = -4,\ f(x) = 3x^2 - 4x + 2$

5 **a** $y = x^2 + 3x - 6$ **b** $y = -2x^2 + 4x + 9$
c $y = \frac{1}{2}x^2 + 3x - 4$

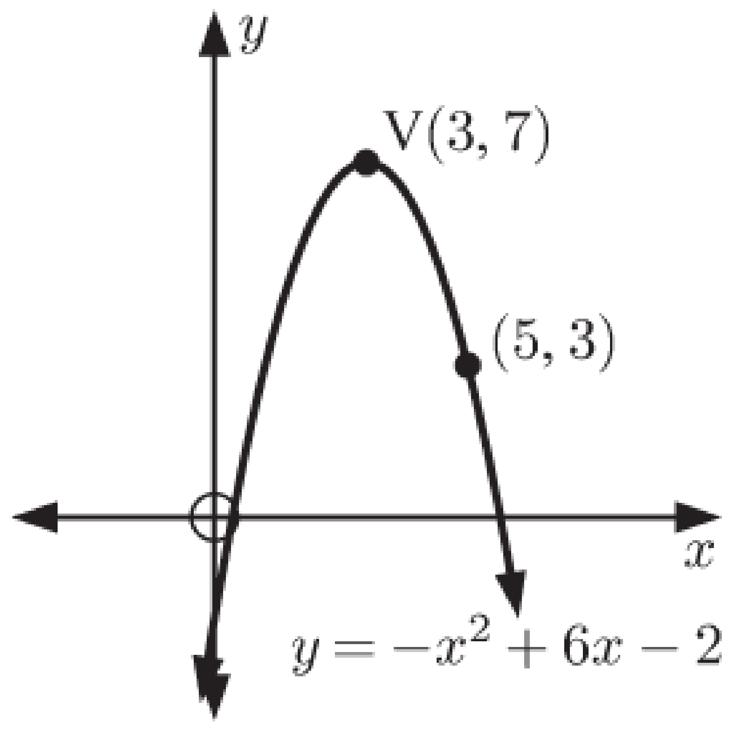
6 We find that $a = 0$, $b = -2$, and $c = 5$. So the three points lie on the line $y = -2x + 5$, but not on a quadratic.

7 **a**



b **i** The graph of the quadratic through these points must open downwards, so $a < 0$.
ii The y-intercept is below the x-axis, so $c < 0$.
iii The vertex has x-coordinate $-\dfrac{b}{2a}$ and is > 0. Since $a < 0$, then $b > 0$.
c $a = -1,\ b = 5,\ c = -5$ **d** $(\frac{5}{2}, \frac{5}{4})$

8 **a** $6a + b = 0,\ 9a + 3b + c = 7,\ 25a + 5b + c = 3$
b $f(x) = -x^2 + 6x - 2$ **c**



EXERCISE 6I

1 **a** $(1, 1)$ and $(0, -1)$ **b** $(-2, -5)$ and $(2, 3)$
c $(1, 7)$ and $(2, 8)$ **d** $(-3, -9)$ and $(4, 5)$
e $(3, 0)$ **f** graphs do not intersect

2 **a** $(0.586, 5.59)$ and $(3.41, 8.41)$ **b** $(3, -4)$
c graphs do not intersect
d $(-2.56, -18.8)$ and $(1.56, 1.81)$
e $(0.176, -3.09)$ and $(-5.68, -0.162)$
f $(9.90, -27.7)$ and $(0.101, 1.70)$ **g** $(0.5, -1.75)$

3 **a** do not intersect **b** $(1, 2)$
c do not intersect **d** $(2, 1)$ and $(-5, -20)$
e $(-0.0981, 6.52)$ and $(5.10, 58.5)$
f $(-2.29, 68.7)$ and $(2.62, 11.4)$

EXERCISE 6J

1 **a** 3 m **b** 0.5 seconds **c** 4 m **d** 1.5 seconds
2 **a** \$91 **b** 10 necklaces **c** \$100

4 500 m by 250 m **5** **c** 100 m by 112.5 m

6 **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m

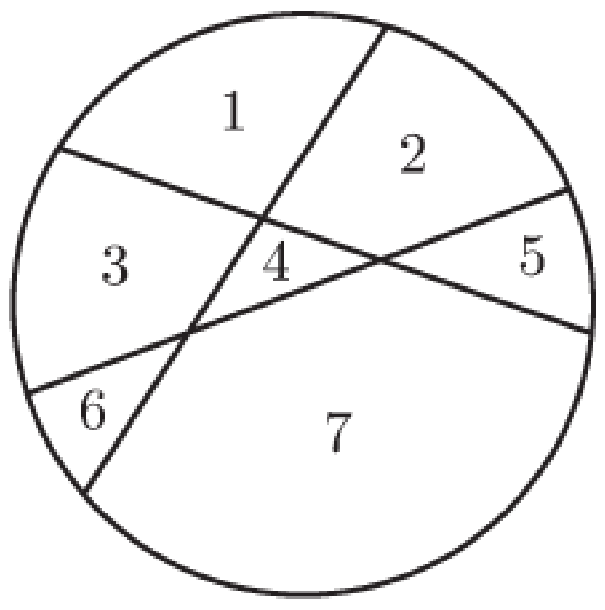
7 **a** The graph of $H(x)$ is a parabola that opens downwards.
b 1.7 m **c** 15.7 m **d** ≈ 18.4 m
e $0 \leqslant x \leqslant 68.3$ **f** ≈ 64.9 m

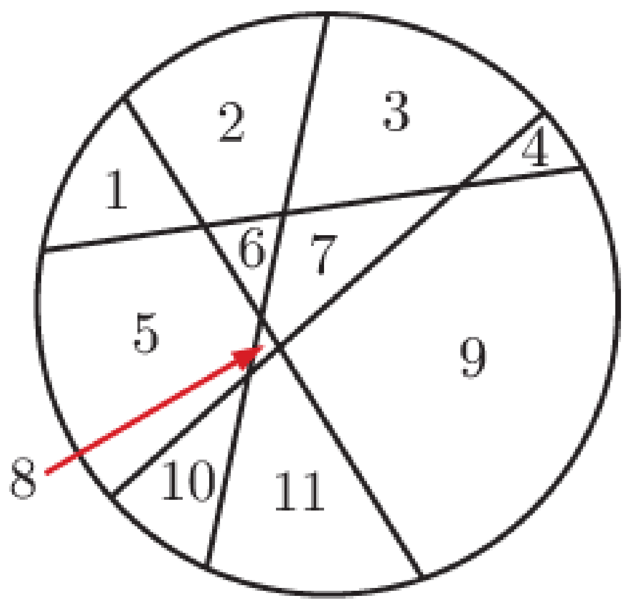
8 **a** **i** $c = 8$ **ii** $b = 0$ **iii** $a = -\frac{8}{9}$
$y = -\frac{8}{9}x^2 + 8$
b no

9 **a** $y = -\frac{1}{80}x^2 + x + \frac{1}{4}$ **b** 20.25 m **c** yes

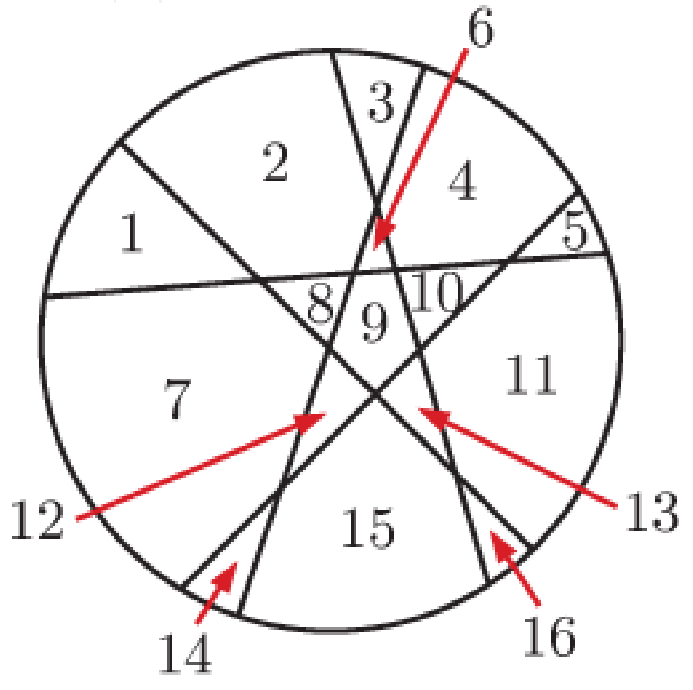
10 **a** $c = 60$ **b** $4a + b = 0, \quad 2a + b = 10$
c $a = -5, \quad b = 20$ **d** 75 m **e** 6 seconds

11 **a** $P(3) = 7$ $P(4) = 11$





b $a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = 1$ **c** $P(5) = 16$
d 79 pieces
e $n \geqslant 0, \quad n \in \mathbb{Z}$



12 **a** $y = \frac{1}{125}x^2 - \frac{4}{5}x + 50$ **b** 37.2 m **c** $0 \leqslant x \leqslant 100$

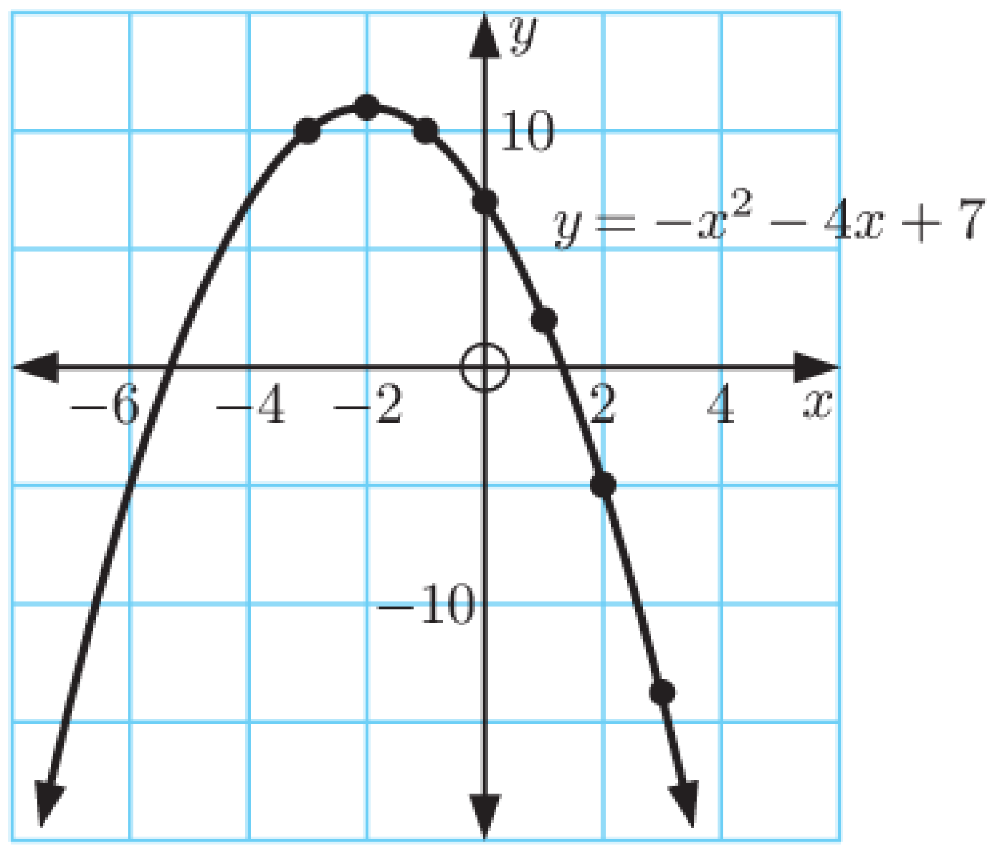
REVIEW SET 6A

1 **a** $f(0) = -15$ **b** $f(1) = -17$ **c** $x = -3$ or 6

2 **a**

x	-3	-2	-1	0	1	2	3
y	10	11	10	7	2	-5	-14

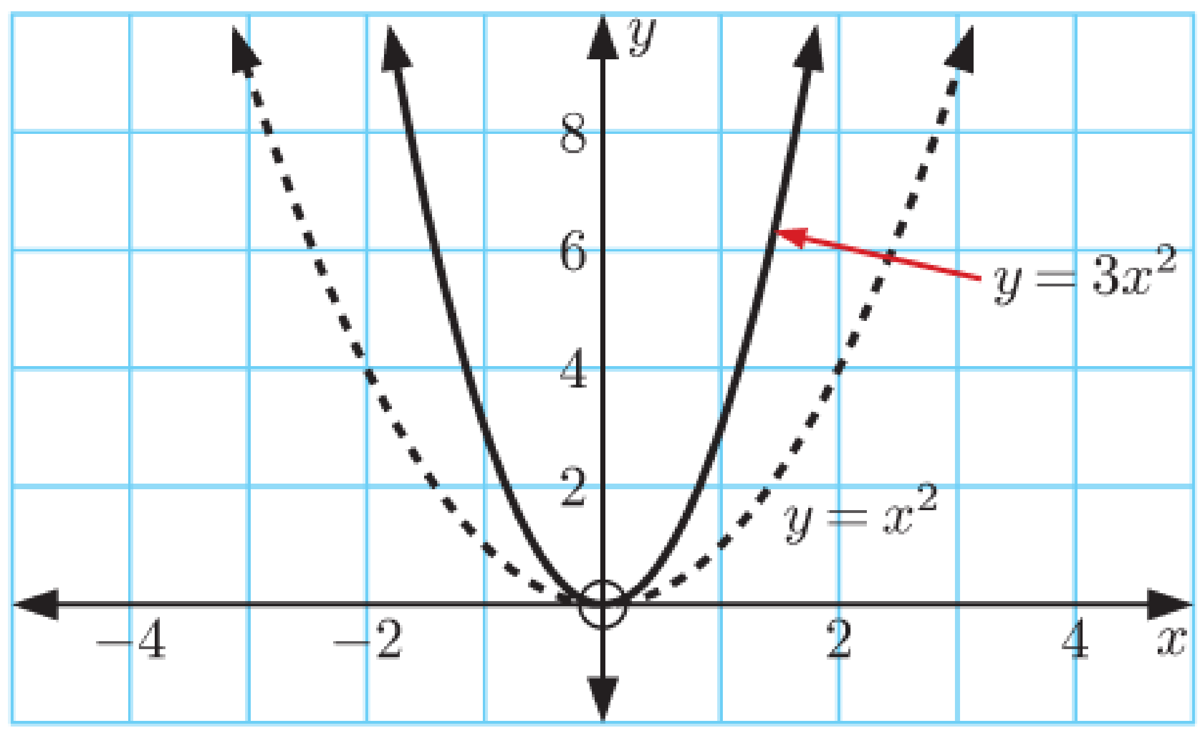
b



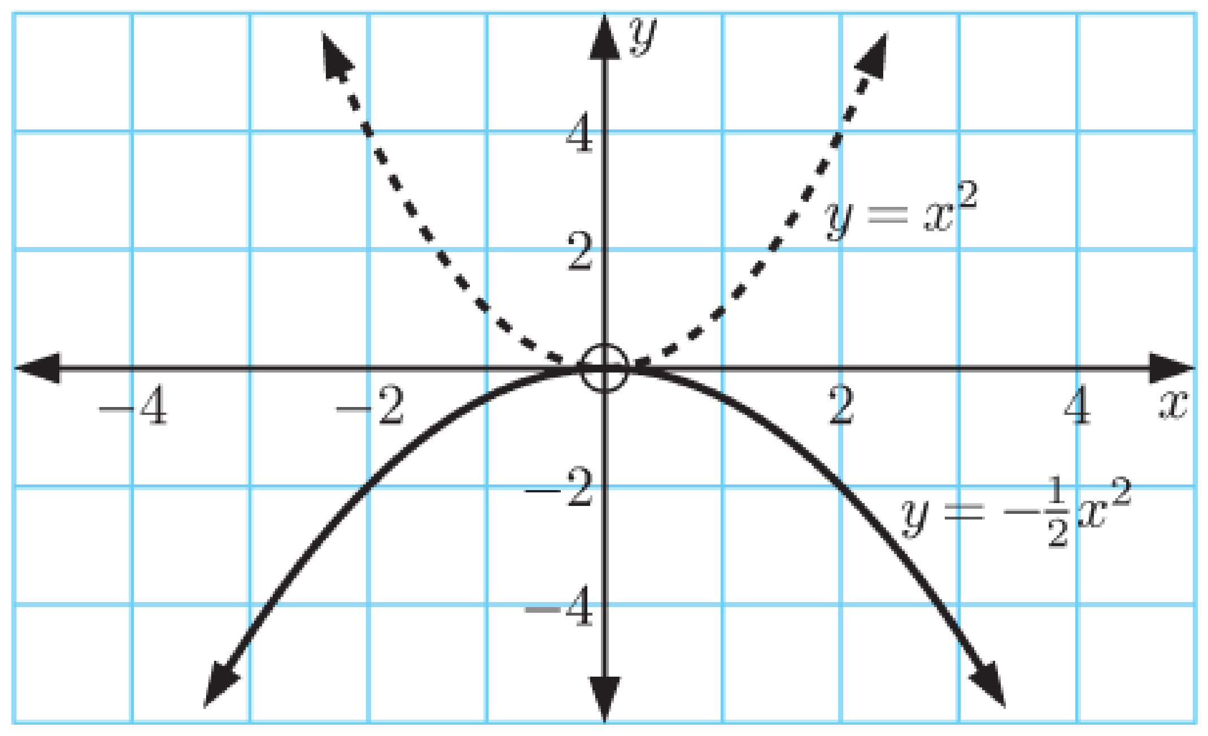
3 **a** 9 **b** -30 **c** -16

4 **a** $\frac{5}{3}$ and -2 **b** 7 **c** -6 and 2

5 **a**

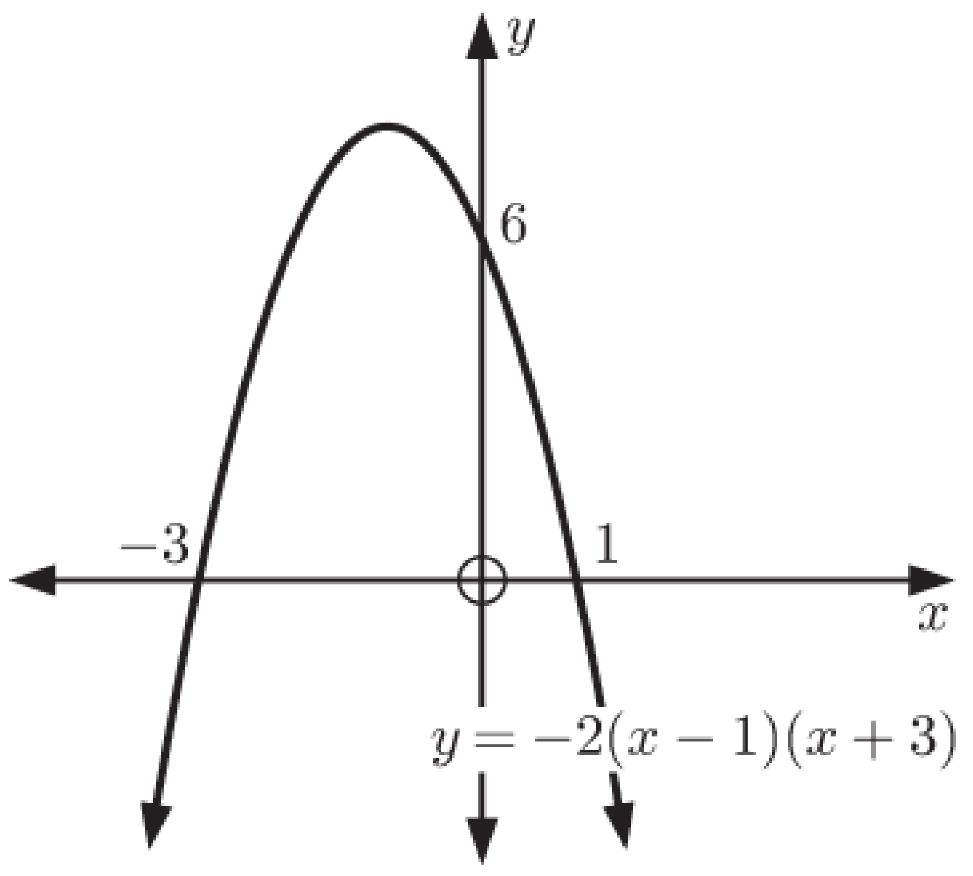


b



6 **a** $x = 4$ **b** $x = \frac{1}{2}$ **c** $x = -2$

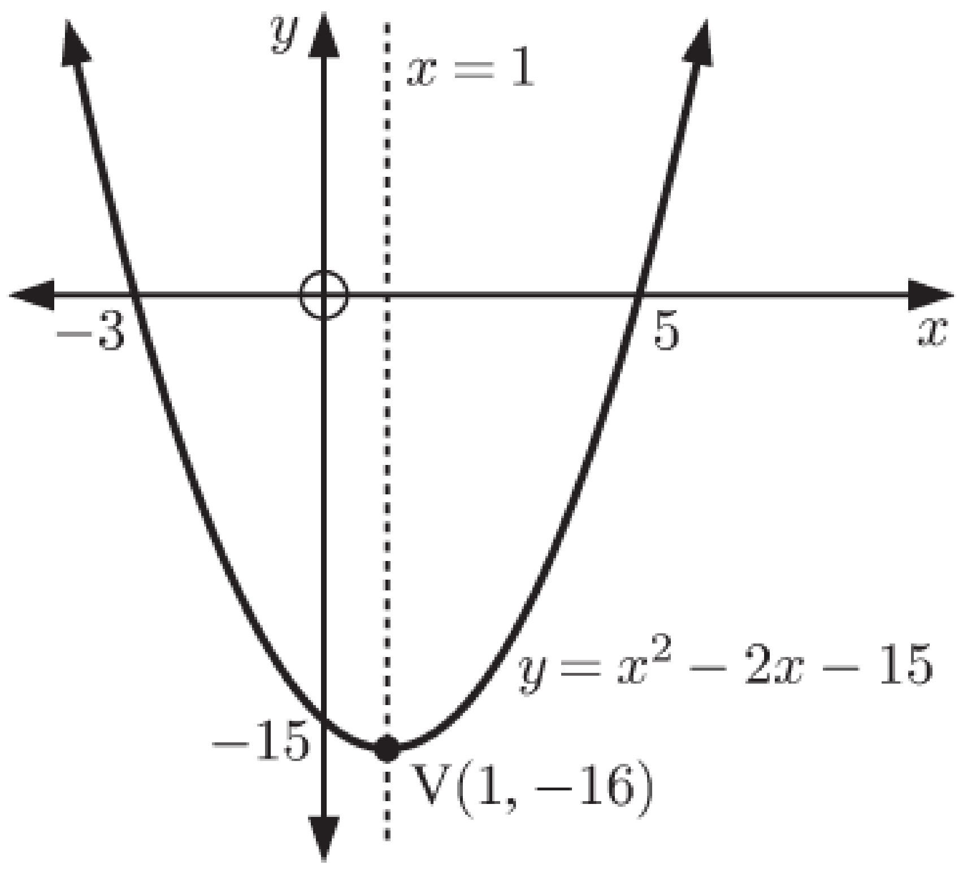
7 **a** $a = -2$ which is < 0, so the parabola opens downwards.
b 6
c 1 and -3
d



8 **a** $x = \frac{3}{2}$ **b** $x = \frac{7}{2}$ **c** $x = \frac{3}{4}$

9 **a** $(4, 21)$ **b** $(\frac{1}{2}, -\frac{81}{4})$ **c** $(-2, -16)$

10 **a** **i** -15
ii -3 and 5
iii $x = 1$
iv $(1, -16)$
b



11 **a** $y = \frac{20}{9}(x + 1)(x - 5)$ **b** $y = -\frac{2}{7}(x - 1)(x - 7)$
c $y = \frac{2}{9}(x + 3)^2$

12 **a** $y = 3x^2 - 3x - 6$ **b** $y = 3x^2 - 24x + 48$

13 **a** $c = 8$ **b** $a - b = -5, \quad 3a + b = -3$
c $a = -2, \quad b = 3, \quad f(x) = -2x^2 + 3x + 8$

14 $y = 3x^2 - x - 4$ **15** $(-3, 18)$ and $(4, 4)$

16 **a** 16.1 m **b** 21 m

17 **a** $A(x) = (1000x - \frac{3}{2}x^2)$ m^2
b $166\,666\frac{2}{3}$ m^2 when the fields are 250 m by $333\frac{1}{3}$ m.

18 **a** $S(3) = 6$ **b** $a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = 0$

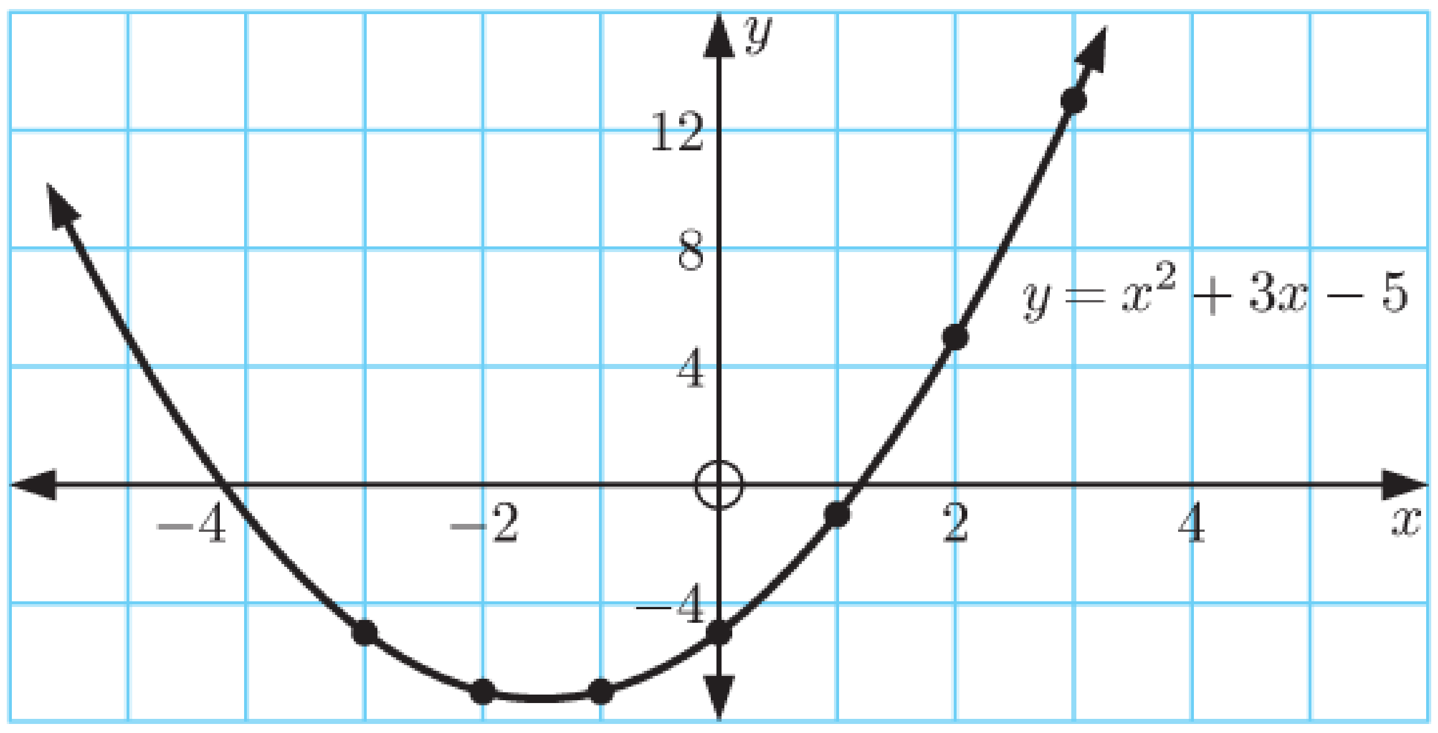
c $S(6) = \frac{1}{2}(6)^2 + \frac{1}{2}(6) = 18 + 3 = 21$
and $1 + 2 + 3 + 4 + 5 + 6 = 21$ ✓
d $S(60) = 1830$
e $S(2\frac{1}{2}) = 4.375$
This result is not meaningful as we cannot find the sum of the first "two and a half" positive integers.
f $n \in \mathbb{Z}^+$

REVIEW SET 6B

1 no **2** $x = -2$ and $x = 7$

3

x	-3	-2	-1	0	1	2	3
y	-5	-7	-7	-5	-1	5	13



4 **a** x-intercepts -5 and 1, y-intercept -5

b x-intercepts $\frac{3}{2}$ and -4, y-intercept 24

c x-intercepts $-\frac{1}{2}$ and $\frac{3}{4}$, y-intercept -3

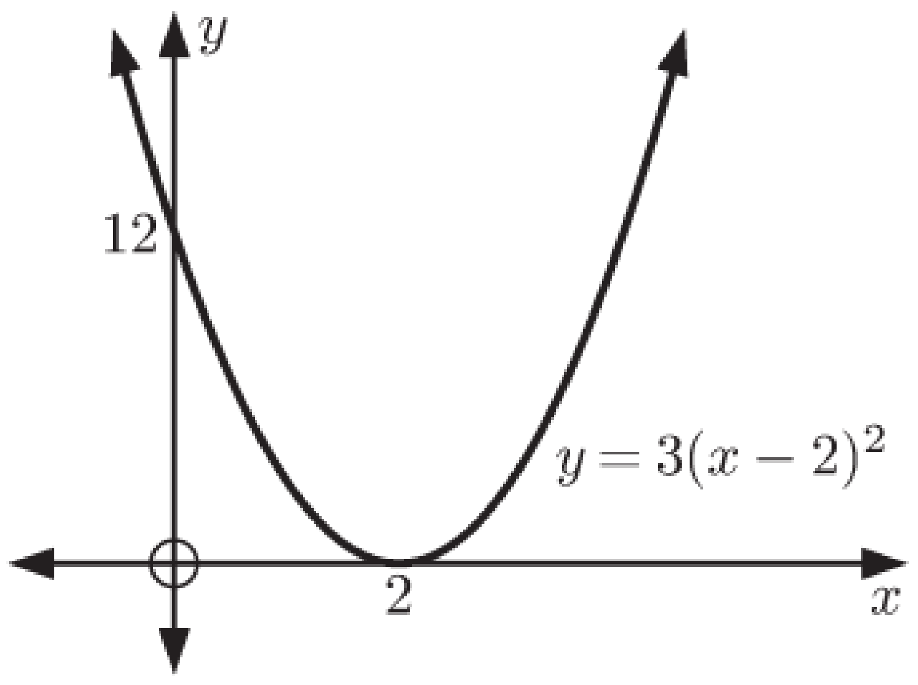
5 **a** **B** **b** **C** **c** **A** **d** **D**

6 **a** $a = 3$ which is > 0, so the parabola opens upwards.

b 12

c 2

d



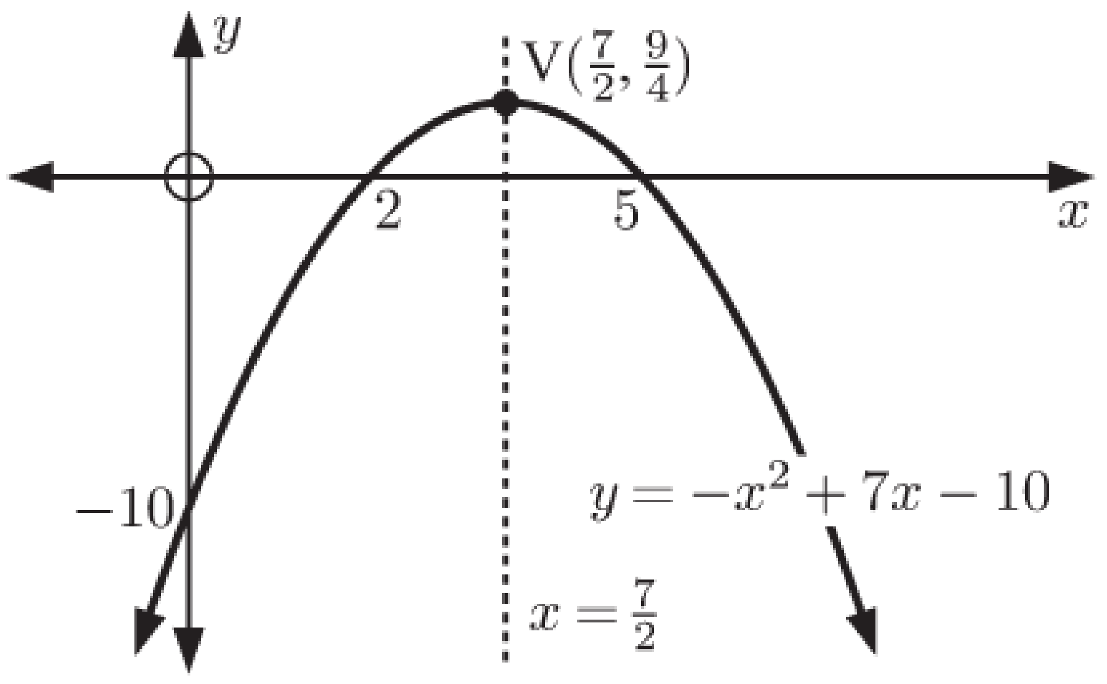
7 **a** $x = \frac{11}{2}$ **b** $x = 4$ **c** $x = \frac{3}{4}$

8 15 **9** $(\frac{4}{3}, \frac{37}{3})$, maximum turning point

10 **a** $y = 16$ when $x = 5$ **b** $y = \frac{9}{2}$ when $x = \frac{1}{2}$

11 **a** **i** -10 **ii** 2 and 5 **iii** $x = \frac{7}{2}$ **iv** $(\frac{7}{2}, \frac{9}{4})$

b



c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant \frac{9}{4}\}$

12 **a** $y = -\frac{2}{5}(x+5)(x-1)$

b vertex $(-2, \frac{18}{5})$, axis of symmetry $x = -2$

13 **a** $y = 2x^2 - 12x + 18$ **b** $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

c $y = x^2 + 7x - 3$ **d** $y = -2x^2 + 12x - 3$

14 There is no such quadratic. The x-coordinate 2 corresponds to two different points, which is not possible for a quadratic.

15 $(-9, -21)$ and $(-2, -7)$

16 **a** There are 8 sections of fence with length x m, and 9 sections of fence with length y m, with total length 600 m.
$\therefore$ $8x + 9y = 600$

c $37\frac{1}{2}$ m by $33\frac{1}{3}$ m, maximum area 1250 m^2

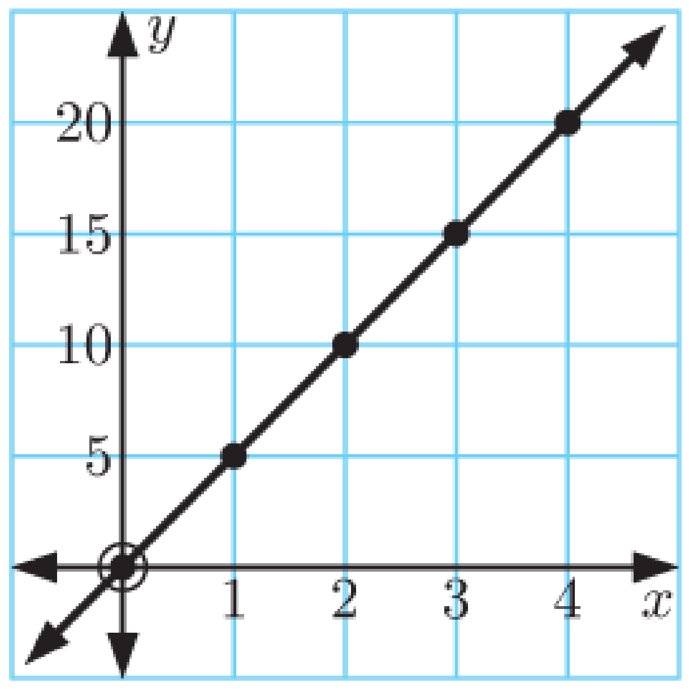
17 **b** $R = -\frac{2}{3}x^2 + 20x + 2250$

c selling price \$60, maximum daily revenue \$2400

EXERCISE 7A

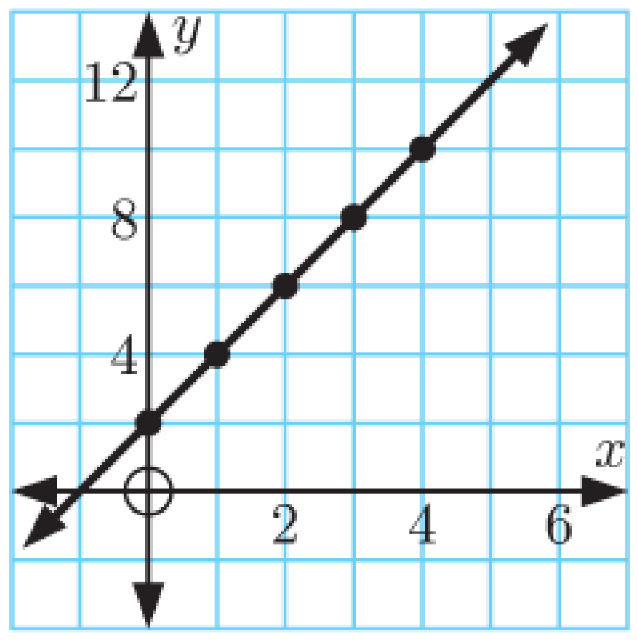
1 **D**

2 **a** **i**



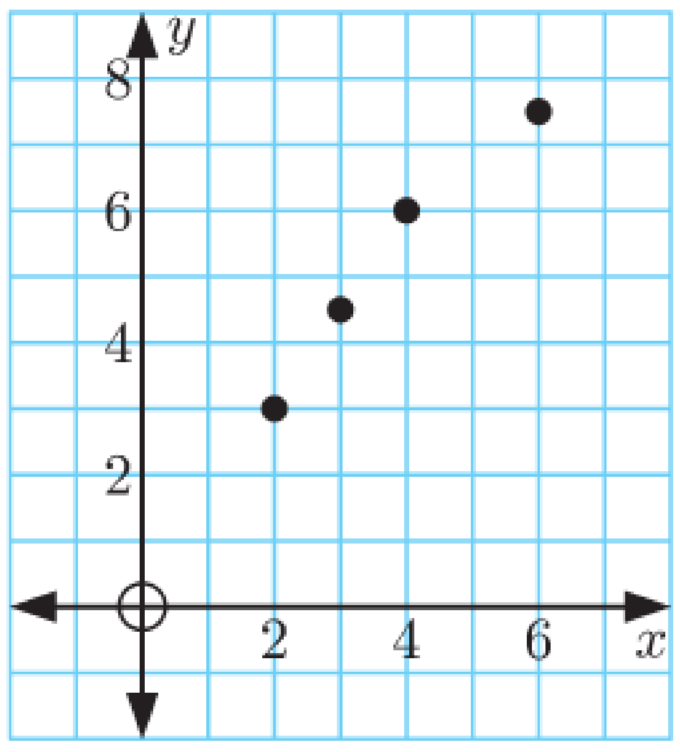
ii $y \propto x$, $k = 5$

b **i**



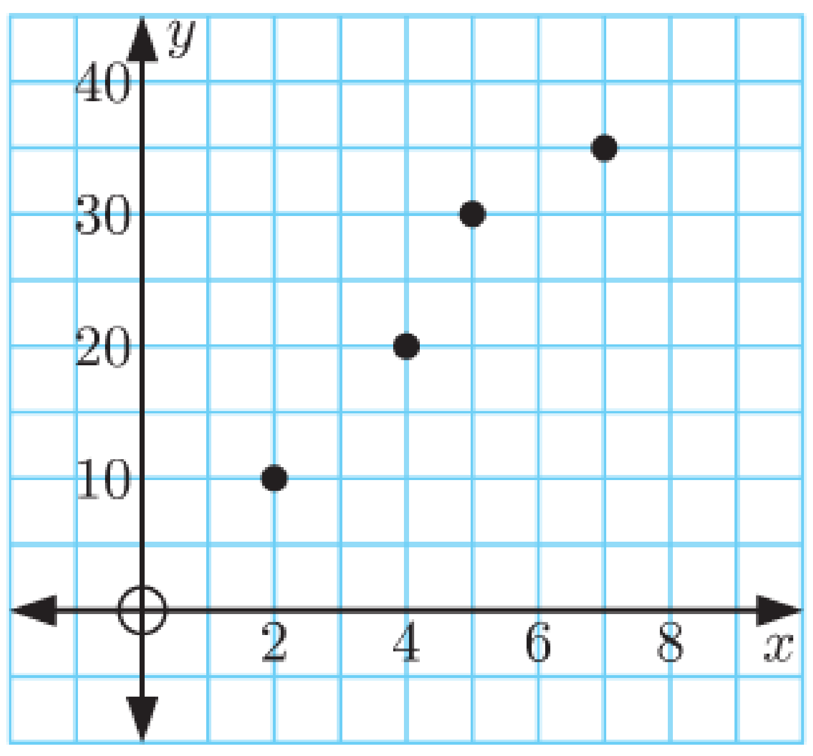
ii y is not directly proportional to x.

c **i**



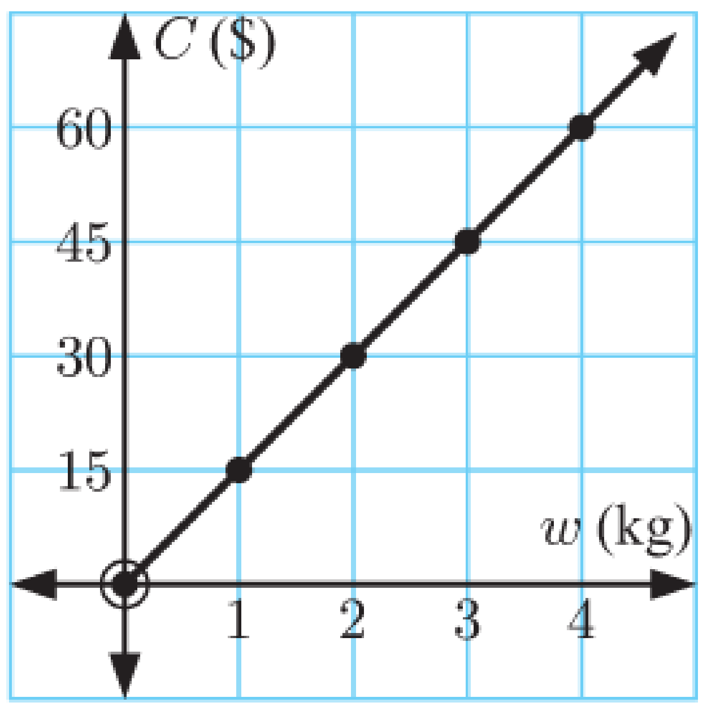
ii y is not directly proportional to x.

d **i**



ii y is not directly proportional to x.

3 **a**



b The graph is a straight line which passes through the origin.

c $C = 15w$

4 **a** **i** €6 **ii** €12 **iii** €18 **iv** €24

b The graph of C against t is not a straight line through the origin.
$\therefore$ C and t are not directly proportional.

5 **a** y is doubled **b** x is multiplied by 6

c y is divided by 3 **d** x is increased by 20%

e y is decreased by 70% **f** x is increased by $\frac{10}{k}$

6 When the temperature in °C is 0, the temperature in °F is 32, so the graph of these variables does not pass through the origin.
$\therefore$ these variables are not directly proportional.

7 **a** directly proportional **b** not directly proportional

c directly proportional **d** not directly proportional

e not directly proportional

8 **a** $y = 72$ **b** $x = \frac{35}{4}$

9 **a** $M = 200$ **b** $l = \frac{9}{5}$

10 **a** $k = 1.8$, the heater uses 1.8 kWh of energy each hour.
b 2.7 kWh **c** 3 h 20 min

11 **a** $k = 20$, 1 cm on the graph represents an actual length of 20 m.
b 5.5 cm **c** 300 m

12 **a** 366 kg m s^{-1} **b** ≈ 3.44 m s^{-1}

13 **a** **i** The graph of C against t has C-intercept 30, so does not pass through the origin.
$\therefore$ C and t are not directly proportional.
ii The graph of $(C - 30)$ against t passes through the origin and is a straight line.
$\therefore$ $(C - 30)$ and t are directly proportional.
b £330

EXERCISE 7B

1 **a** $A \propto r^2$, $k = \pi$ **b** $V \propto r^3$, $k = \frac{4}{3}\pi$
c $T \propto n^4$, $k = \frac{3}{4}$

2 **a** y is multiplied by 8 **b** y is divided by 1000
c y is increased by 72.8% **d** x is multiplied by $\sqrt[3]{2.5}$

3 **a** $M = 90$ **b** $t = 6\sqrt{3} \approx 10.4$

4 **a** $V = 1920$ **b** $y = 3\sqrt[3]{6} \approx 5.45$

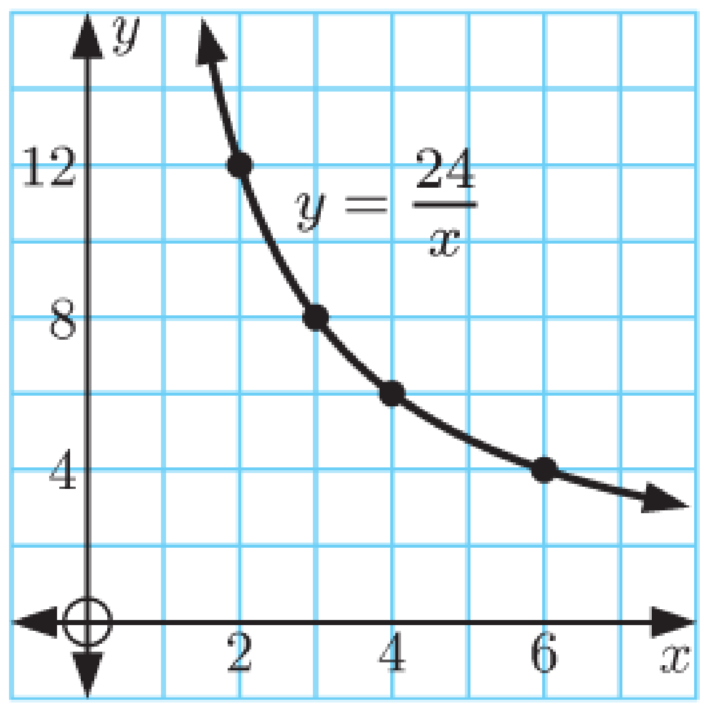
5 2500 g **6** **a** ≈ 11.9 mL **b** ≈ 5.45 cm

7 **a** $\approx 15.8\%$ increase **b** $\approx 26.0\%$ increase

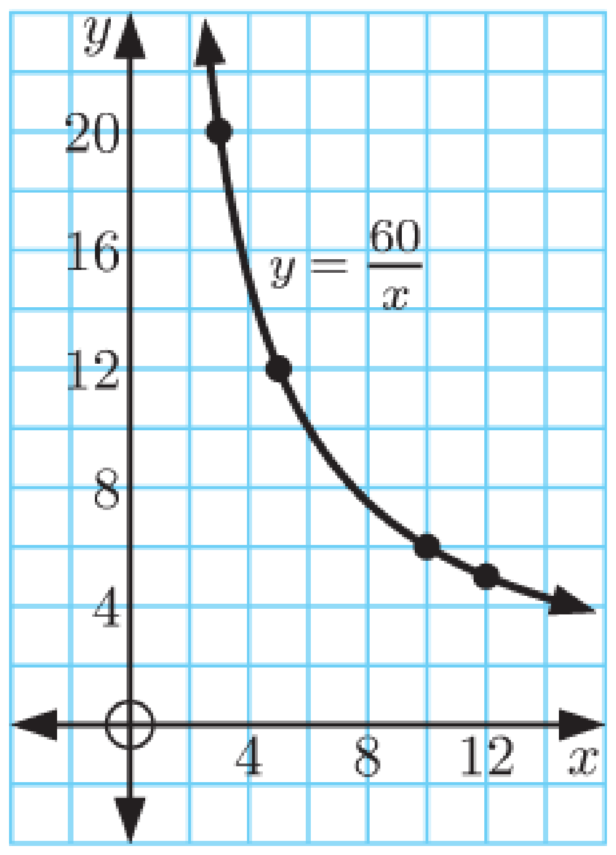
8 **a** $E \propto m$ **b** $E \propto v^2$ **c** E decreases by 19%
d Stopping distance $d \propto E$
and $E \propto v^2$
$\therefore$ $d \propto v^2$

EXERCISE 7C

1 **a** $xy = 24$ for each point.
$\therefore$ x and y are inversely proportional, $y = \dfrac{24}{x}$.

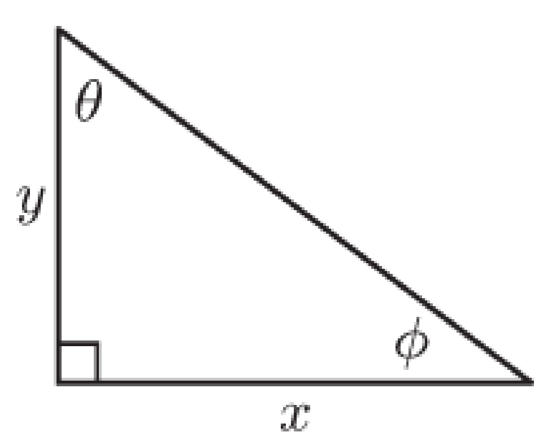


b $xy = 20$ when $x = 1$, 2, or 5, and $xy = 24$ when $x = 4$.
$\therefore$ x and y are not inversely proportional.
c $xy = 60$ for each point.
$\therefore$ x and y are inversely proportional, $y = \dfrac{60}{x}$.



2 **B**

3 **a** No, $\theta = 90 - \phi$. There is no constant k such that $\theta \times \phi = k$.
b $\tan\theta = \dfrac{x}{y}$ and $\tan\phi = \dfrac{y}{x}$
$\therefore$ $\tan\theta = \dfrac{1}{\tan\phi}$
$\therefore$ $\tan\theta$ and $\tan\phi$ are inversely proportional.



4 **a** y is halved **b** y is multiplied by 7
c y is multiplied by $\frac{5}{9}$ **d** y is decreased by $\approx 23.1\%$

5 **a** $C = 5$ **b** $t = 4.5$ **6** 10 hours

7 **a** 3.75 m s^{-2} **b** 0.75 kg

8 **a** If the *share price* is multiplied by a constant, then the *number of shares* Wendy can buy will be divided by the same constant.
$\therefore$ *number of shares* is inversely proportional to *share price*.
b **i** $\frac{125}{122}n$ shares (rounded down) **ii** $33\frac{1}{3}\%$

EXERCISE 7D

1 **a** y is divided by 8 **b** y is multiplied by $\frac{125}{27}$
c x is divided by 4

2 **a** $y = 3$ **b** $x = \frac{24}{5}$ **3** **a** $M = 216$ **b** $c = 48$

4 **a**
$$V = \pi r^2 h$$
$$\therefore \quad h = \frac{V}{\pi} \times \frac{1}{r^2}$$
$$\therefore \quad h \propto \frac{1}{r^2} \quad \{\text{since } V \text{ and } \pi \text{ are constants}\}$$
b ≈ 10.4 cm **c** ≈ 2.79 cm
d The cans would otherwise be too small or too large to be practical for use.

5 **a** $\approx 24.9\%$ decrease **b** $\approx 30.7\%$ decrease

EXERCISE 7E

1 **a** The graph of y against x is a straight line and passes through the origin.
b $y = \frac{7}{6}x$

2 **a** $H = \dfrac{d^2}{16}$ **b** 16 m

3 **a** $m = \dfrac{l^3}{20\,000}$ **b** 6.25 kg **c** ≈ 27.1 cm

4 **a** $xy = 10$ for all 3 points, so it is reasonable to assume that y varies inversely with x.
b $y = \dfrac{10}{x}$ **c** $y = \frac{5}{4}$

5 **a** $x^2 y = 5$ for all values in the table **b** $x = \sqrt{10}$
$\therefore$ $y = \dfrac{5}{x^2}$
so $k = 5$

6 **a** **i** $R = \dfrac{v^2}{200}$
ii When $v = 20$, $R = \dfrac{20^2}{200} = 2 \neq 4$
When $v = 30$, $R = \dfrac{30^2}{200} = 4.5 \neq 13.5$
When $v = 40$, $R = \dfrac{40^2}{200} = 8 \neq 32$
$\therefore$ this model is incorrect.

b When $v = 10$ and $R = 0.5$, $\frac{R}{v^3} = \frac{0.5}{10^3} = \frac{1}{2000}$

When $v = 20$ and $R = 4$, $\frac{R}{v^3} = \frac{4}{20^3} = \frac{1}{2000}$

When $v = 30$ and $R = 13.5$, $\frac{R}{v^3} = \frac{13.5}{30^3} = \frac{1}{2000}$

When $v = 40$ and $R = 32$, $\frac{R}{v^3} = \frac{32}{40^3} = \frac{1}{2000}$

$\frac{R}{v^3} = \frac{1}{2000}$ for all data points, so $R = \frac{1}{2000}v^3$.

c 62.5 units

EXERCISE 7F

1 **a** $y \approx 0.602x^4$ **b** $y \approx \frac{799}{x^3}$

2 **a** Inverse variation, as the points appear to lie on a curve which is asymptotic to both axes.

b $P \approx \frac{451}{x^2}$ **c** $P \approx 28.2$

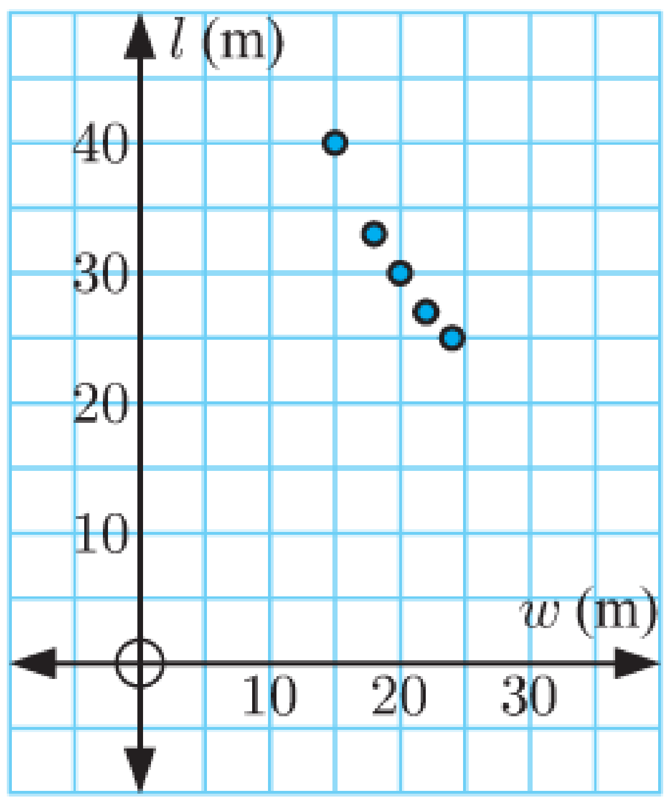
3 **a** Direct variation, as the graph of C against t should pass through the origin, and we expect the charge to increase as time increases.

b $C \approx 1.25t$ **c** $\approx 70\%$

d $0 \leqslant t \leqslant 80$, as the battery charge must lie between 0 and 100%.

4 **a** Inverse variation, as $lw \approx 600$ for all l and w.

b



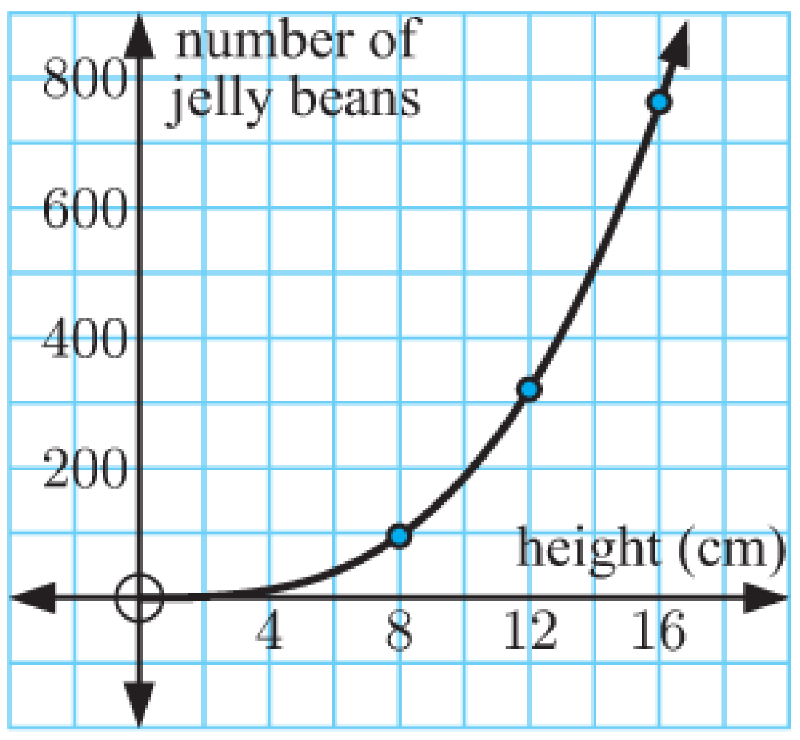
Yes, the points appear to lie on a curve which is asymptotic to both axes, as expected for inverse variation.

c $l \approx \frac{602}{w}$ **d** ≈ 26.2 m

e $15 \leqslant w \leqslant 24.5$, so the width of the block is practical for building a house, and the block is no wider than it is long.

5 **a** $R \approx 0.0197s^{2.29}$ **b** ≈ 3.84 m **c** $\approx 10.2\ \text{m s}^{-1}$

6 **a**



b no

c $V = \pi r^2 h = \frac{\pi h^3}{4}$ (since $r = \frac{h}{2}$)

$\therefore\ V \propto h^3$ $\{\frac{\pi}{4}$ is constant$\}$

The volume, and hence number of jelly beans, is proportional to the cube of the jar height.

d number of jelly beans, $N \approx 0.183h^3$ **e** 1464 jelly beans

7 **a** $F \approx \frac{5.63}{d^2}$ **b** ≈ 35.2 N **c** ≈ 0.0931 m

8 **a** $P \approx 2.93 \times T^{0.0521}$

Since multiplying the values of P does *not* result in the values of t being multiplied by the same number, then P is *not* directly proportional to T.

b **i**

Temperature (T K)	278.15	283.15	288.15	293.15
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39

Temperature (T K)	298.15	303.15	308.15
Pressure ($P \times 10^5$ Pa)	3.45	3.51	3.57

ii $P \approx 0.0112T$, $r \approx 0.9997$ which suggests this is an almost perfect fit for the data as it is almost equal to 1. So this power model is a much better fit for the data.

iii Yes, our power model suggests that $P \propto T$, where T is measured in kelvin.

REVIEW SET 7A

1 **C** **2** **a** A is multiplied by 4 **b** t is increased by 5%

3 ≈ 106 N

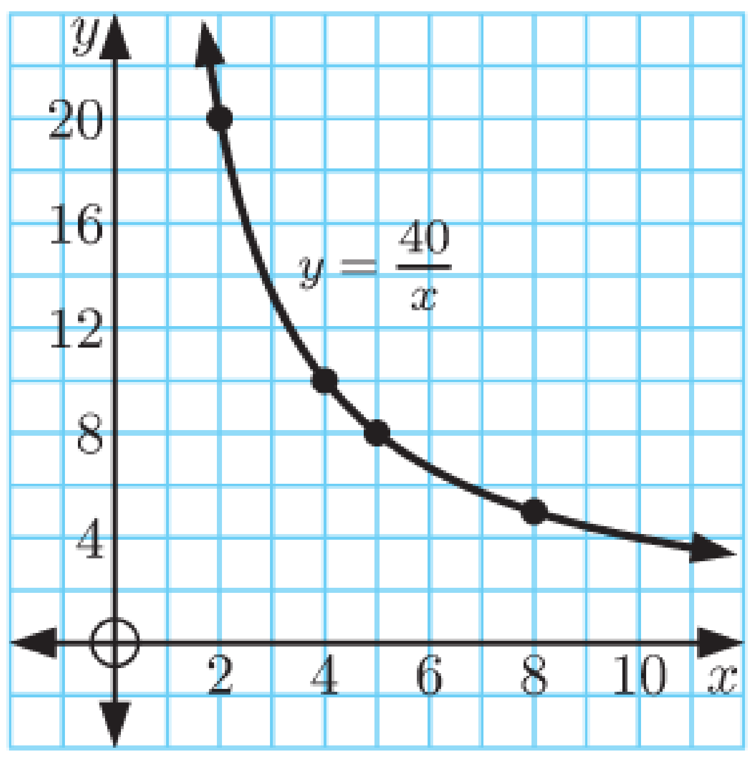
4 **a** $y \propto x^2$, $k = 5$ **b** $P \propto n^4$, $k = \frac{2}{3}$

c $V \propto a^3$, $k = \frac{\sqrt{5}}{4}$

5 **a** $y = \frac{15}{32}x^2$ **b** **i** $y = 7.5$ **ii** $x = 8\sqrt{5} \approx 17.9$

6 **a** $xy = 40$ for each point.

$\therefore$ x and y are inversely proportional, $y = \frac{40}{x}$.



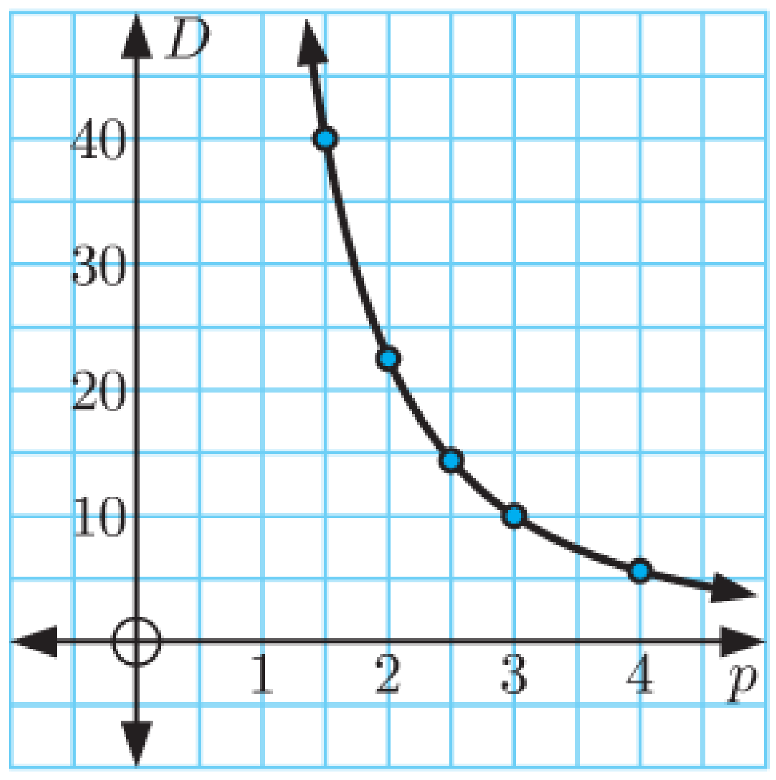
b $xy = 60$ when $x = 3$, 5, or 10, and $xy = 64$ when $x = 8$.

$\therefore$ x and y are not inversely proportional.

7 625 THz **8** **a** ≈ 0.516 ohms **b** ≈ 0.321 cm

9 **a** $y = \frac{2}{3}x^2$ **b** $y = 80\frac{2}{3}$

10 **a**



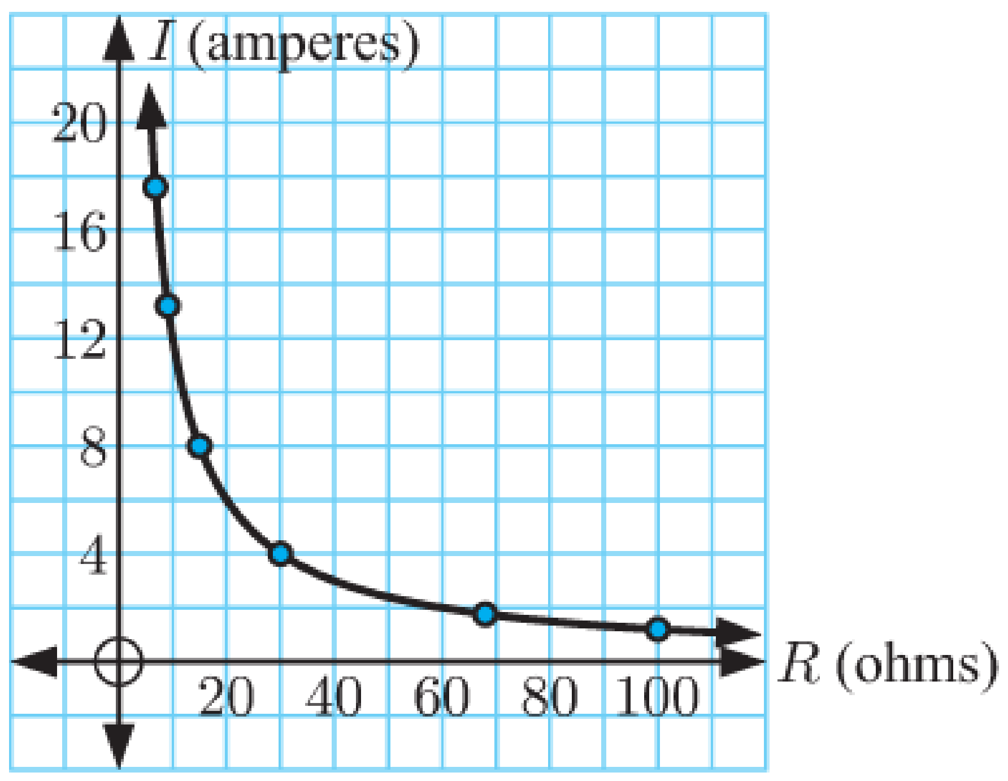
b We see that $Dp^2 = 90$ for each point.

The points appear to lie on a curve which is asymptotic to both axes, so the model $D = \frac{k}{p^2}$ seems appropriate.

c $k = 90$ **d** $D = 3.6$

11 **a** $y \approx 1.50x^3$ **b** $y \approx \frac{500}{x^4}$

12 **a**

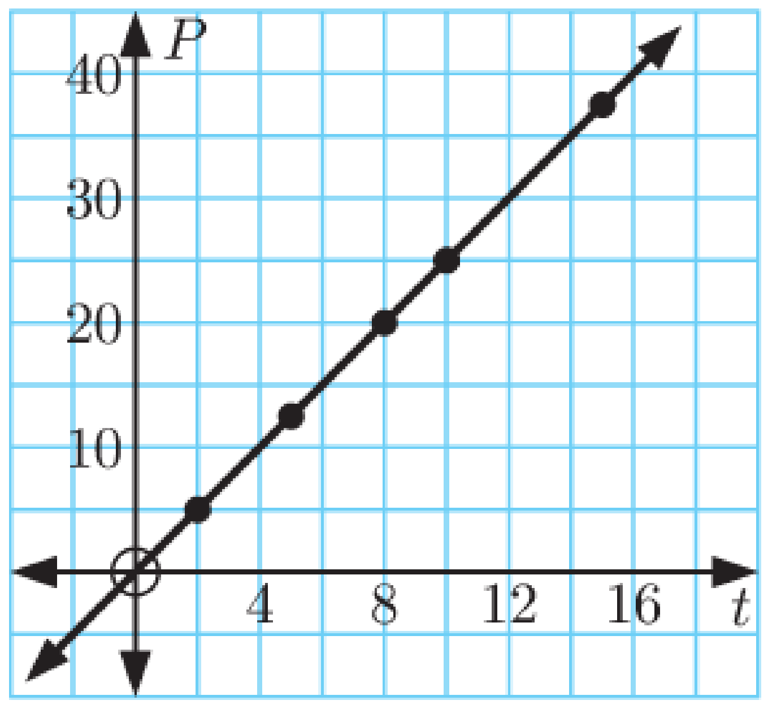


b Inverse variation, as $RI \approx 120$ for each point. The points appear to lie on a curve which is asymptotic to both axes.

c $I \approx \dfrac{120}{R}$ **d** ≈ 0.48 amperes

REVIEW SET 7B

1 **a**



b The graph of P against t is a straight line which passes through the origin.

c $P = 2.5t$

2 **a** 2.9 N **b** 1.8 A **3** **B**

4 **a** $A = \pi r^2$, π is a constant

$\therefore\ A \propto r^2$

b 80.625 kg **c** ≈ 23.1 cm

5 **a** $y = \dfrac{3456}{x^3}$ **b** **i** $y = 54$ **ii** $x = -12$

6 6.75 days **7** The orbital speed decreases by $\approx 8.71\%$.

8 **a** $y = \dfrac{28}{x}$ **b** $y = 280$

9 **a** **Hint:** Write the base lengths l in terms of h.

b $V = \frac{2}{5}h^3$ **c** $\frac{2}{5} \times 4^3 = 25.6$ ✓ $\frac{2}{5} \times 6^3 = 86.4$ ✓

d **i** $V = 204.8$ **ii** $h = 5$

10 **a** $y \approx \dfrac{180}{x^2}$ **b** $y \approx 7.2$

11 **a** Direct variation, as the graph of y against x appears to pass through the origin and is increasing.

b $y \approx 0.753x^2$

12 **a**

d (m)	1	5	10	15	20
I (W m^{-2})	63.7	2.55	0.637	0.283	0.159
$I \times d$	63.7	12.75	6.37	4.245	3.18

b $I \approx \dfrac{63.7}{d^2}$ **c** $\approx 49.0\%$ decrease

EXERCISE 8A

1 **a** exponential **b** exponential **c** not exponential
d exponential **e** not exponential **f** exponential

2 **a** 1 **b** -1 **c** -2 **d** $-2\frac{1}{2}$ **e** $-2\frac{3}{4}$

3 **a** 15 **b** 135 **c** 5 **d** $\frac{5}{81}$ **e** $\frac{5}{3}$

4 **a** 32 **b** 2 **c** 4 **d** 1 **e** $\frac{1}{16}$

5 **a** $\frac{1}{5}$ **b** $\frac{1}{125}$ **c** 1 **d** 25 **e** 125

6 **a** 3 **b** 3.3 **c** ≈ 4.83 **d** ≈ 2.48 **e** ≈ 4.31

7 **a** yes **b** no **c** yes **d** yes **e** no **f** yes

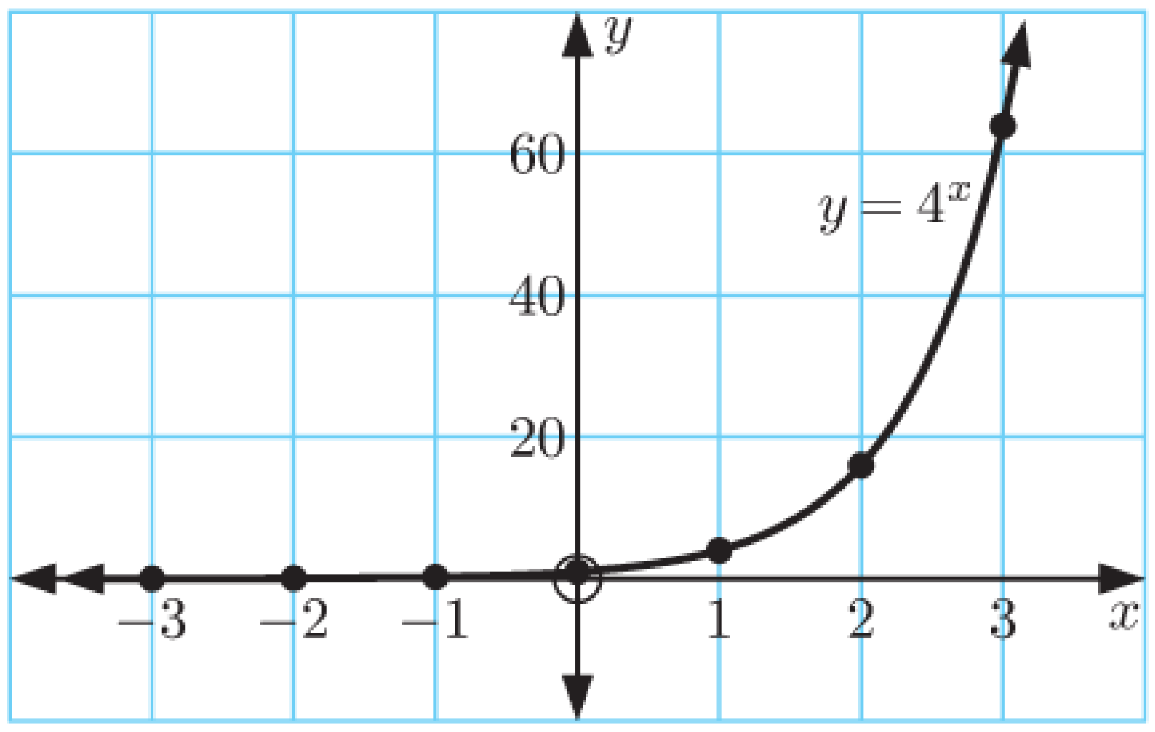
EXERCISE 8B

1 **a**

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

b **i** If x is increased by 1, the value of y is quadrupled.
ii If x is decreased by 1, the value of y is divided by 4.

c



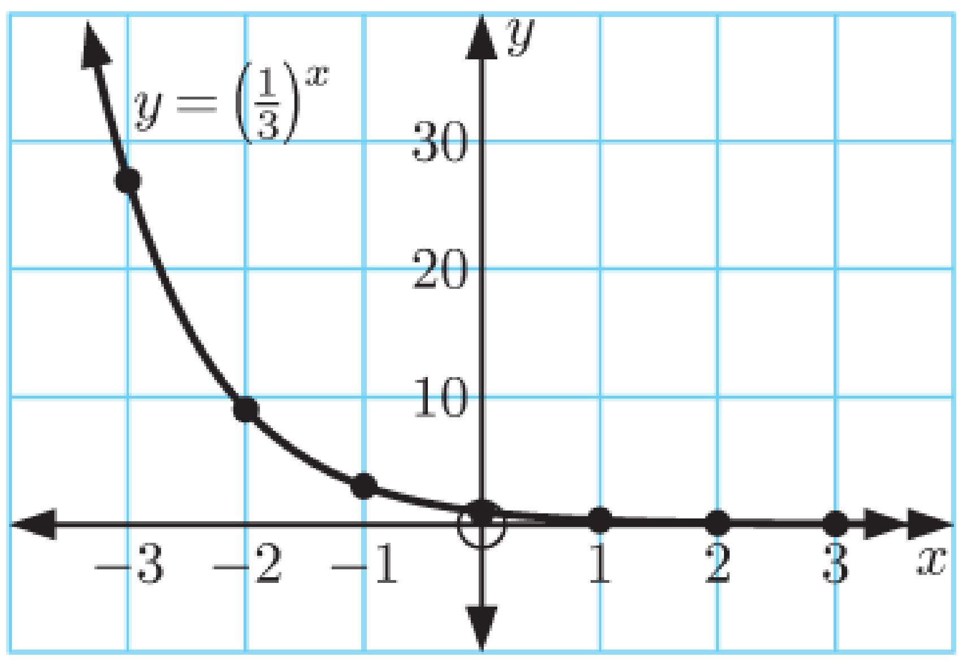
d **i** as $x \to \infty$, $y \to \infty$ **ii** as $x \to -\infty$, $y \to 0^+$

e $y = 0$

2 **a**

x	-3	-2	-1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

b



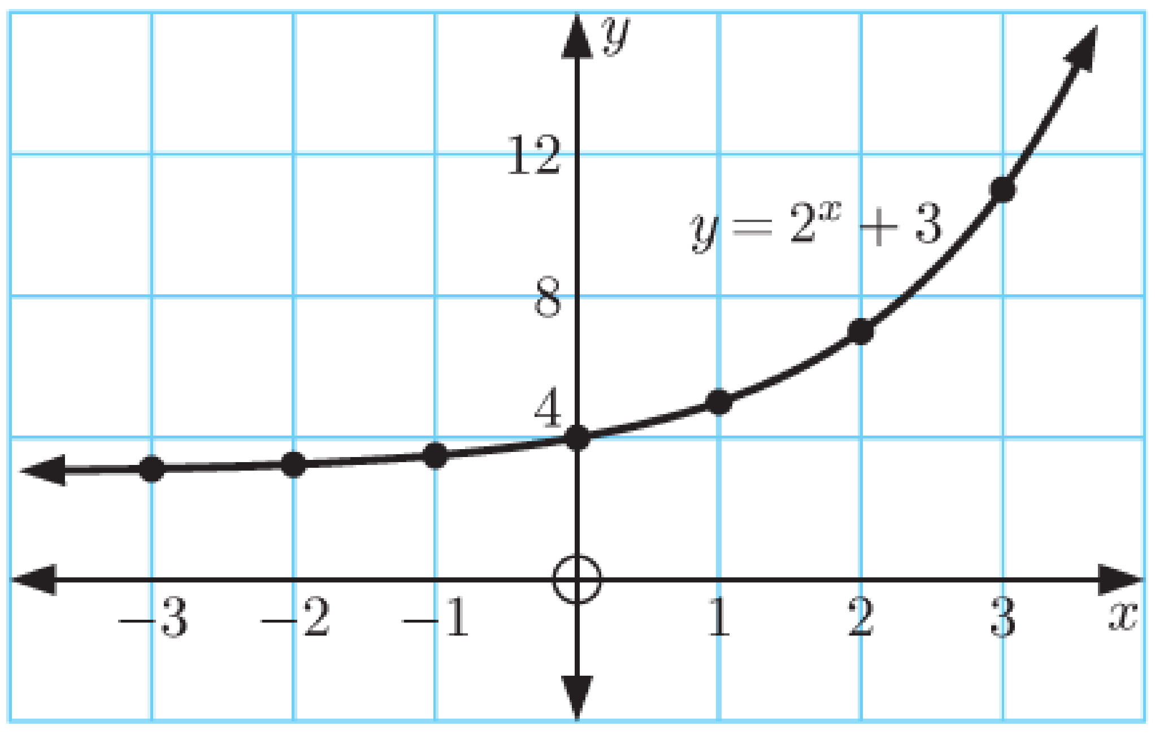
c decreasing

d **i** as $x \to \infty$, $y \to 0^+$ **ii** as $x \to -\infty$, $y \to \infty$

e $y = 0$

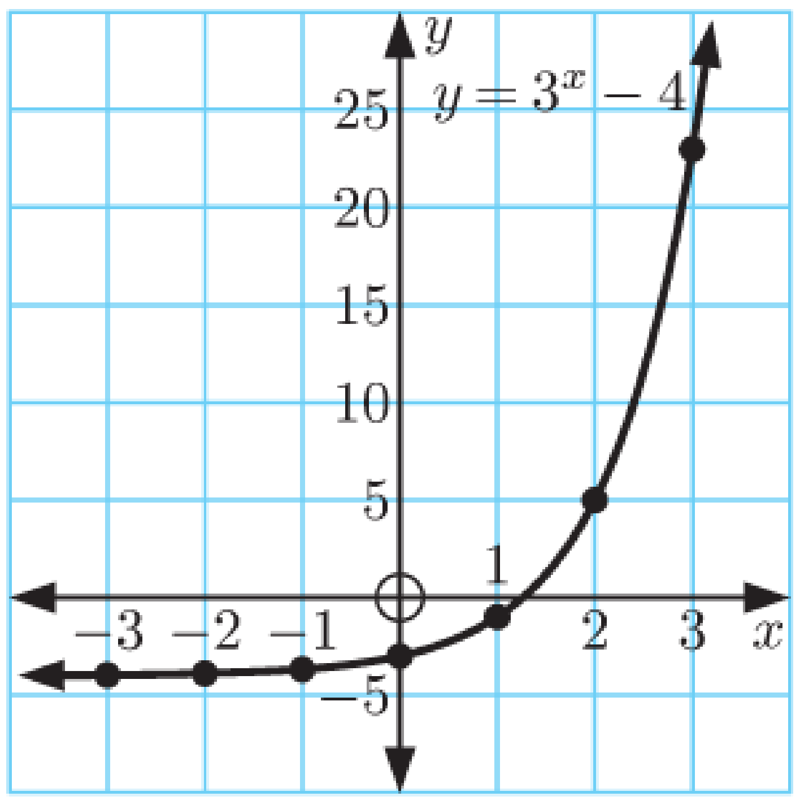
3 **a**

x	-3	-2	-1	0	1	2	3
y	$3\frac{1}{8}$	$3\frac{1}{4}$	$3\frac{1}{2}$	4	5	7	11



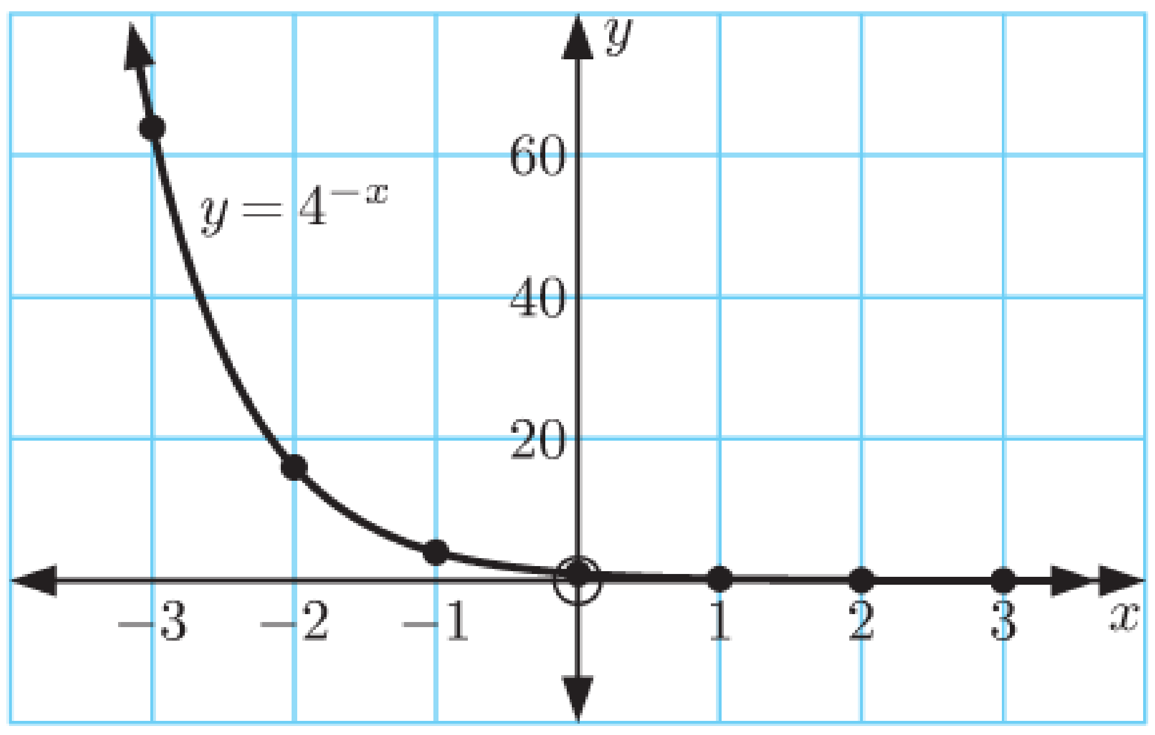
b

x	-3	-2	-1	0	1	2	3
y	$-3\frac{26}{27}$	$-3\frac{8}{9}$	$-3\frac{2}{3}$	-3	-1	5	23



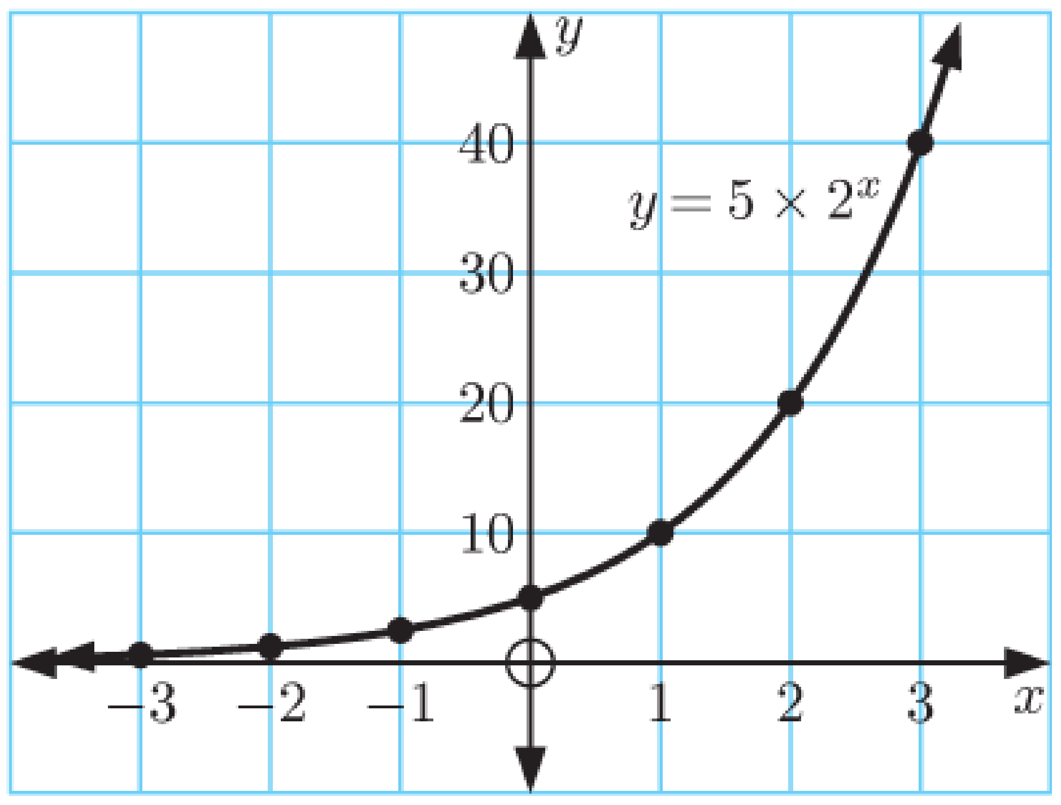
c

x	-3	-2	-1	0	1	2	3
y	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$



d

x	-3	-2	-1	0	1	2	3
y	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{5}{2}$	5	10	20	40



4 **a** **i**

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

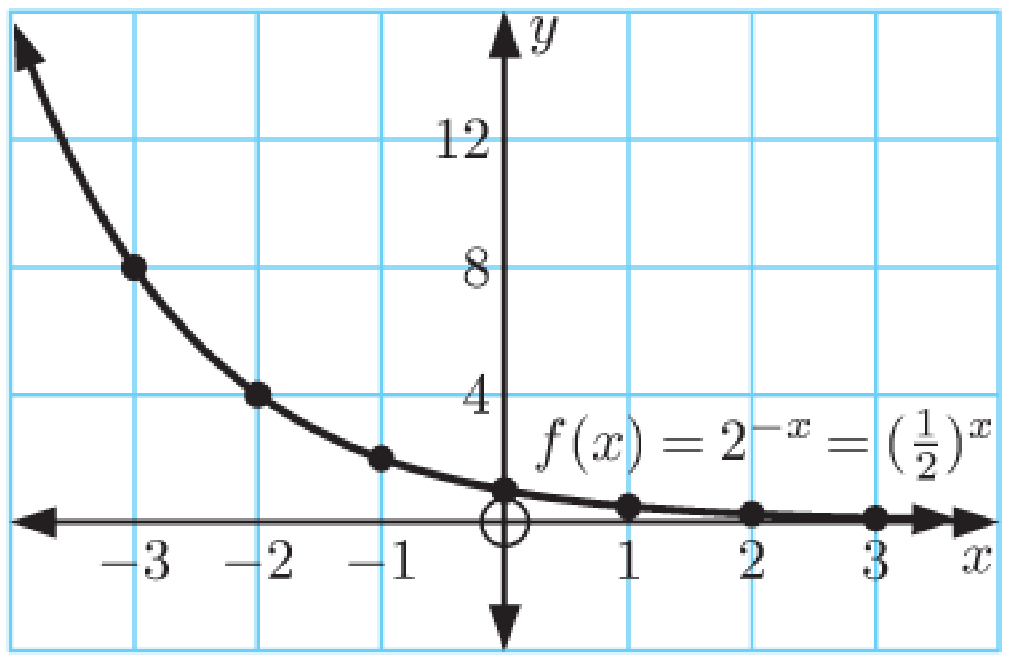
ii

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

b $2^{-x} = (2^{-1})^x$

$= (\frac{1}{2})^x$

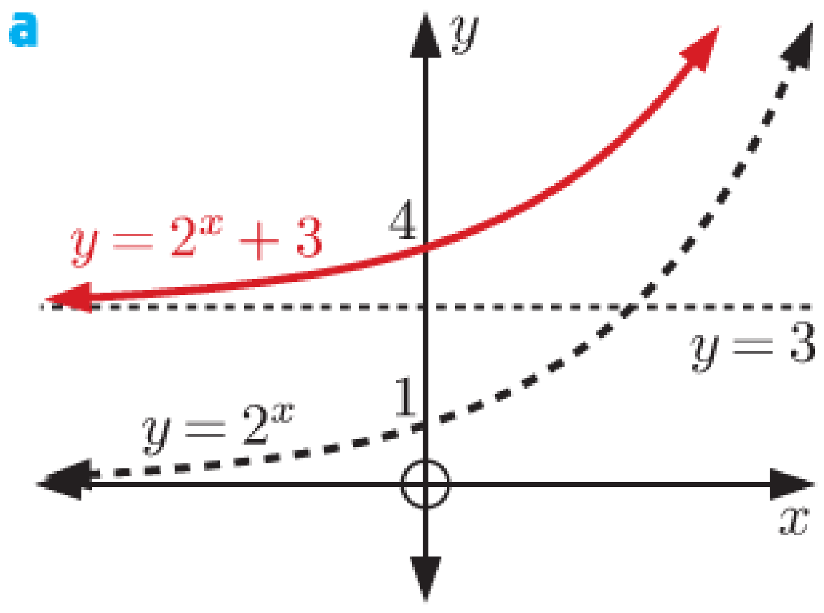
c



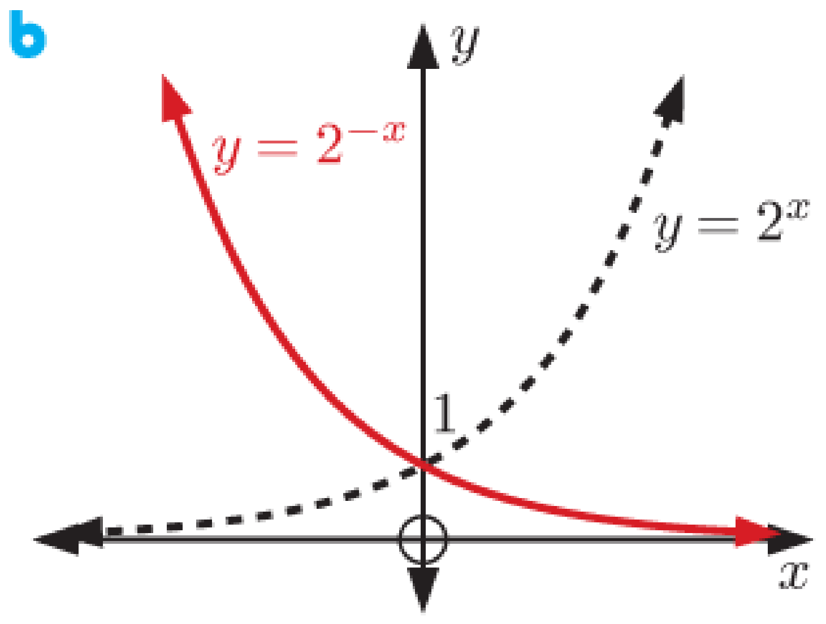
EXERCISE 8C

1 **a** C **b** B **c** E **d** A **e** D

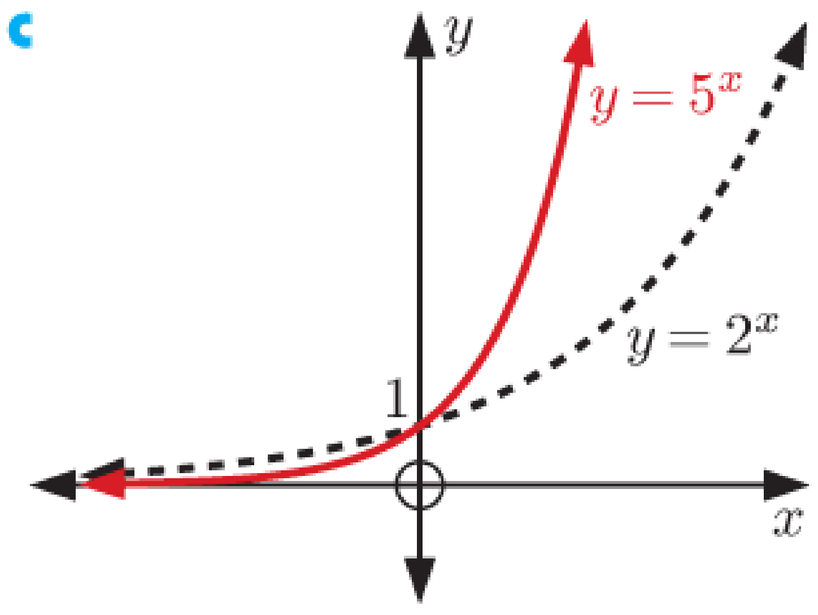
2 **a**



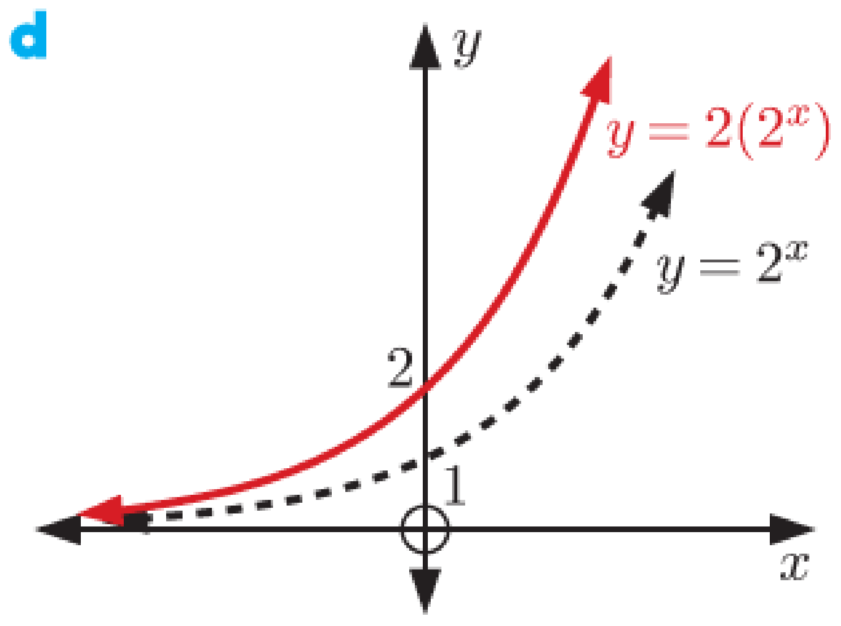
b



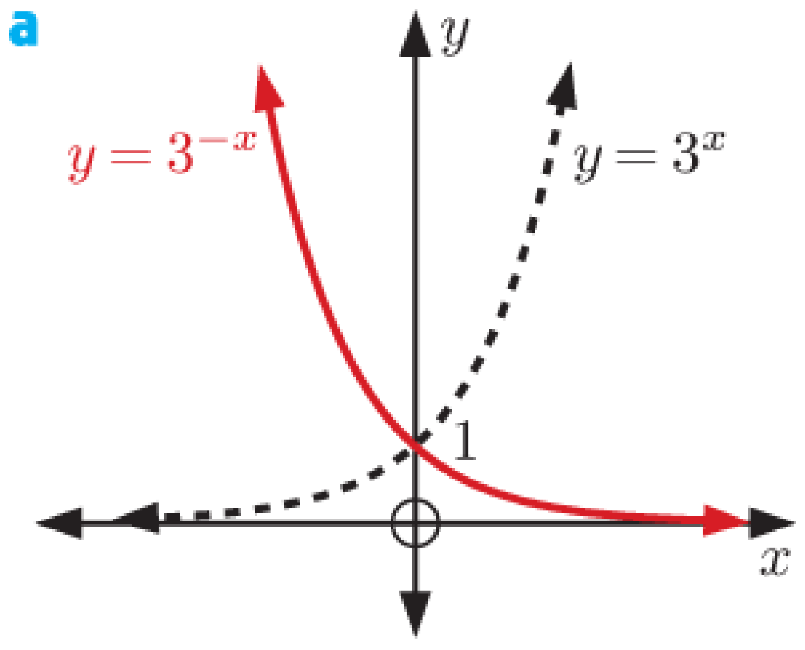
c



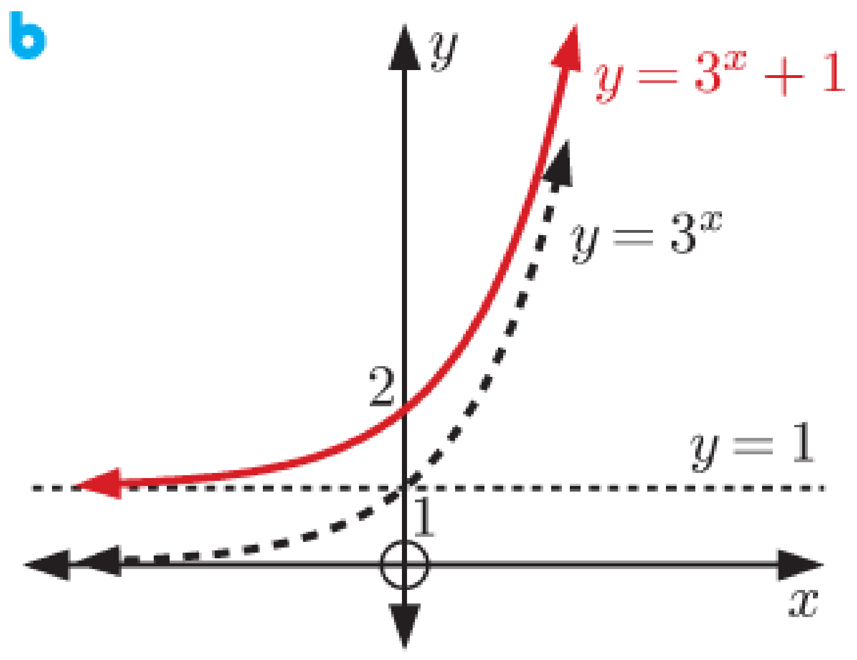
d



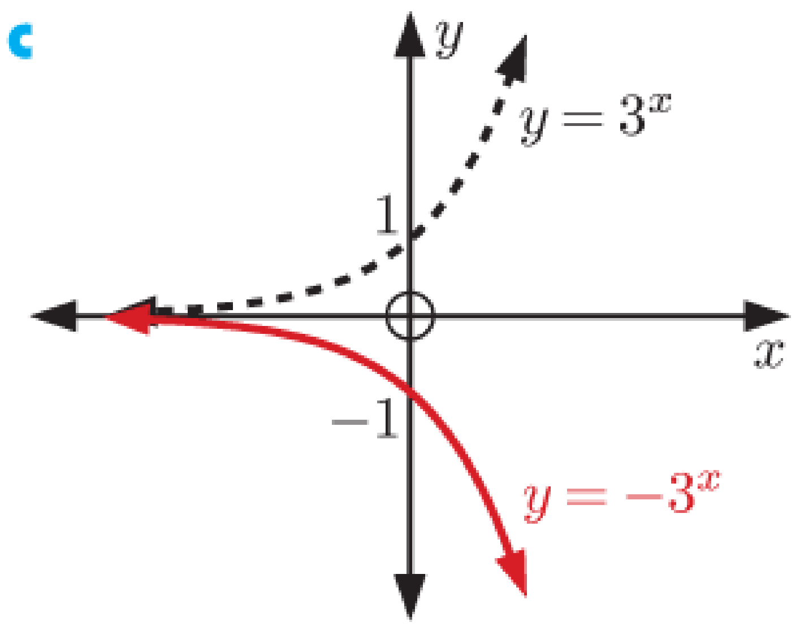
3 **a**



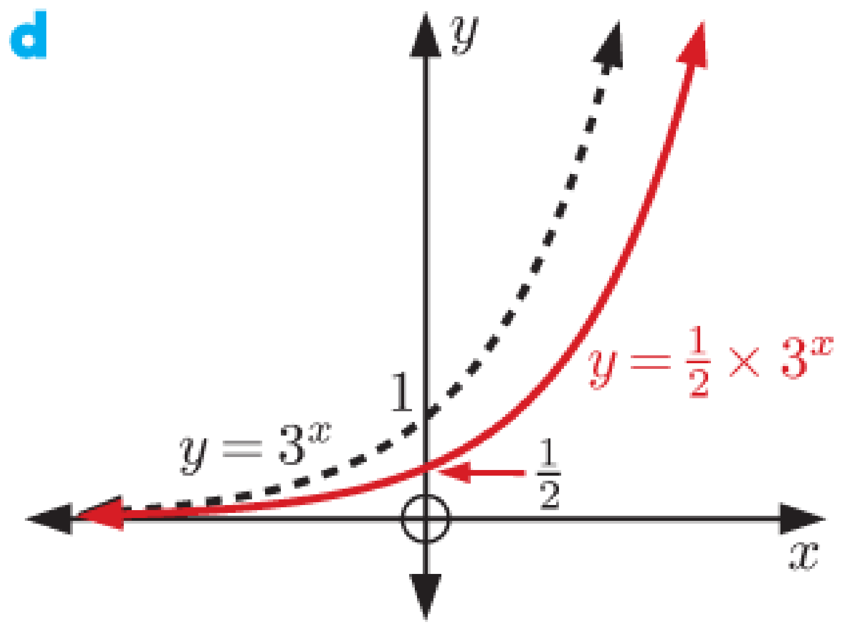
b



c



d

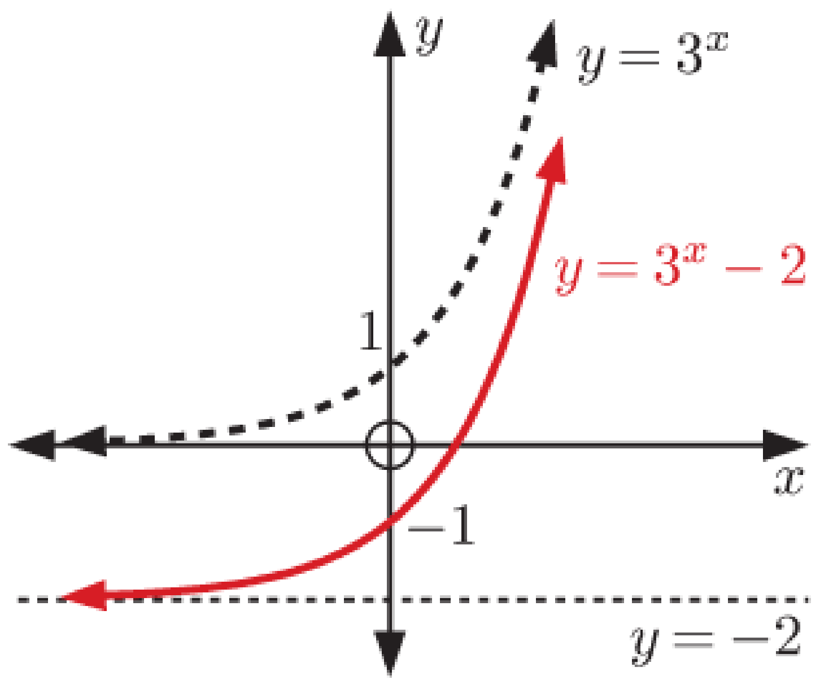


4 **a** $y = -1$ **b** $y = 4$ **c** $y = 1$ **d** $y = -5$

5 **a** 5 **b** -1 **c** 10 **d** $5\frac{1}{2}$

6 **a** **i** -1 **ii** 7 **iii** $-1\frac{8}{9}$ **b** $y = -2$

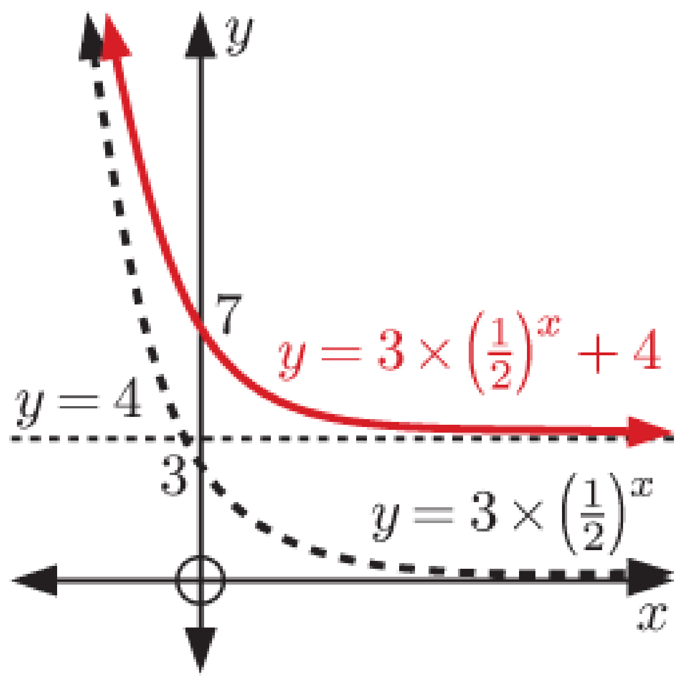
c



d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -2\}$

7 **a** **i** 7 **ii** $4\frac{3}{4}$ **iii** 16 **b** $y = 4$

c



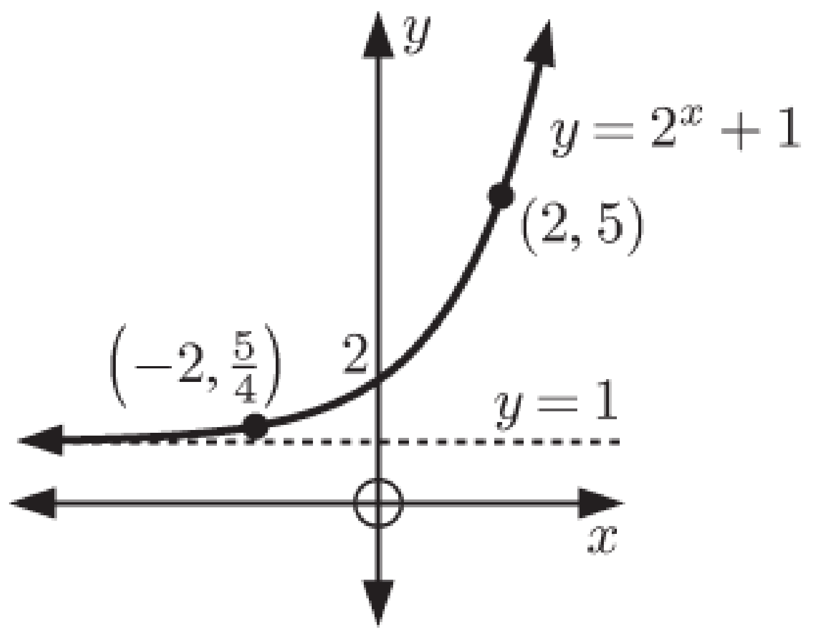
d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 4\}$

8 **a** **i** 2

ii $y = 1$

iii When $x = 2$,
$y = 5$
When $x = -2$,
$y = \frac{5}{4}$

iv



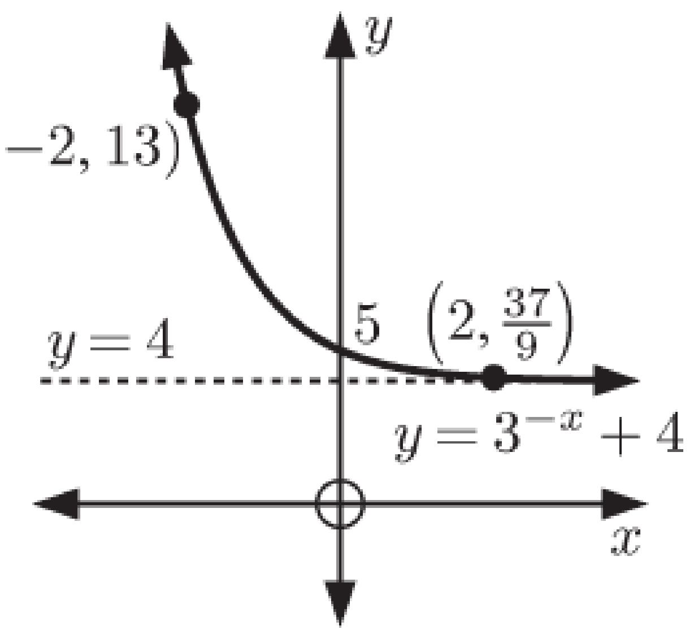
v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 1\}$

b **i** 5

ii $y = 4$

iii When $x = 2$,
$y = \frac{37}{9}$
When $x = -2$,
$y = 13$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 4\}$

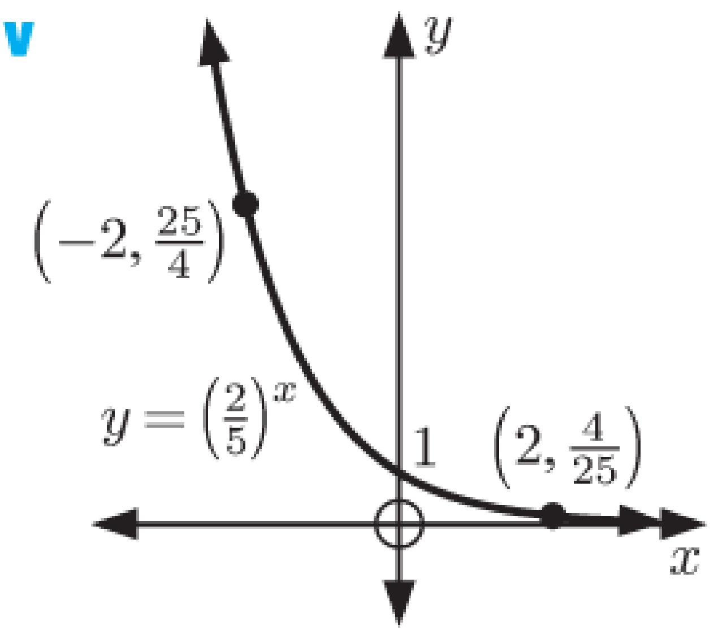
c **i** 1

ii $y = 0$

iii When $x = 2$, $y = \frac{4}{25}$

When $x = -2$, $y = \frac{25}{4}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 0\}$

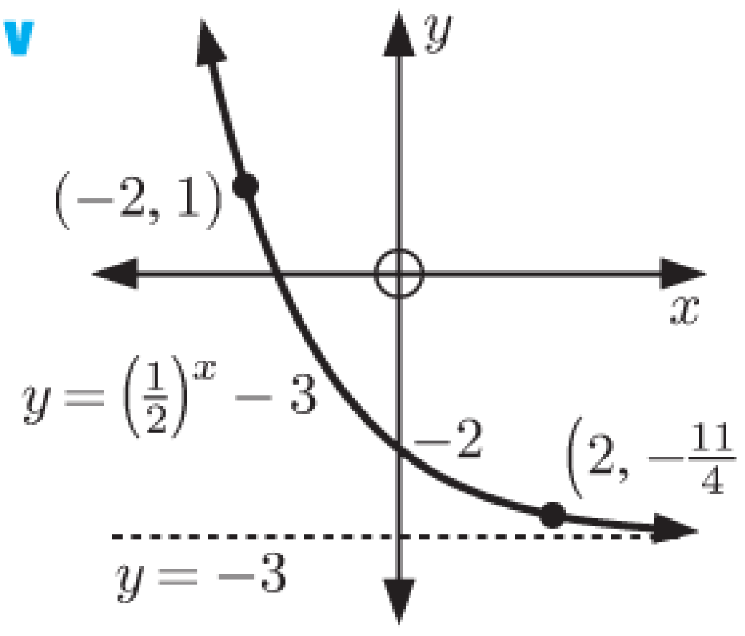
d **i** -2

ii $y = -3$

iii When $x = 2$, $y = -\frac{11}{4}$

When $x = -2$, $y = 1$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -3\}$

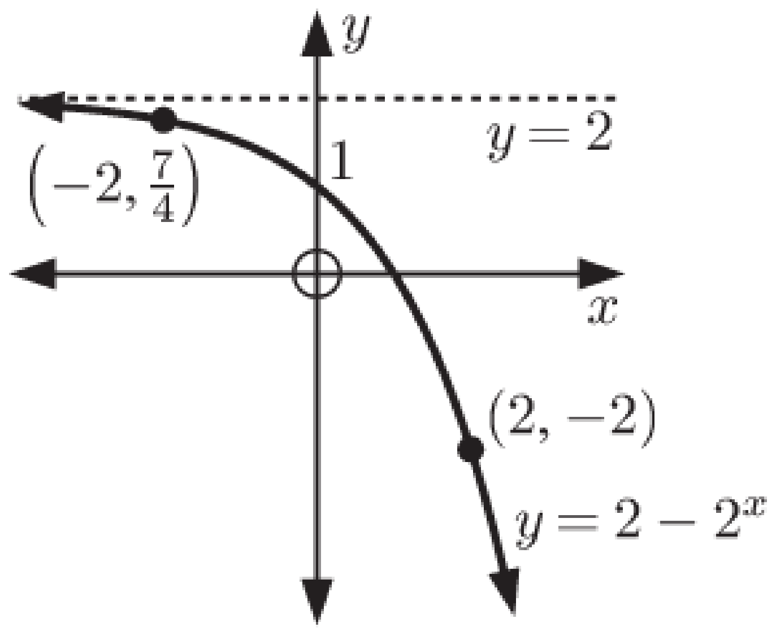
e **i** 1

ii $y = 2$

iii When $x = 2$, $y = -2$

When $x = -2$, $y = \frac{7}{4}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 2\}$

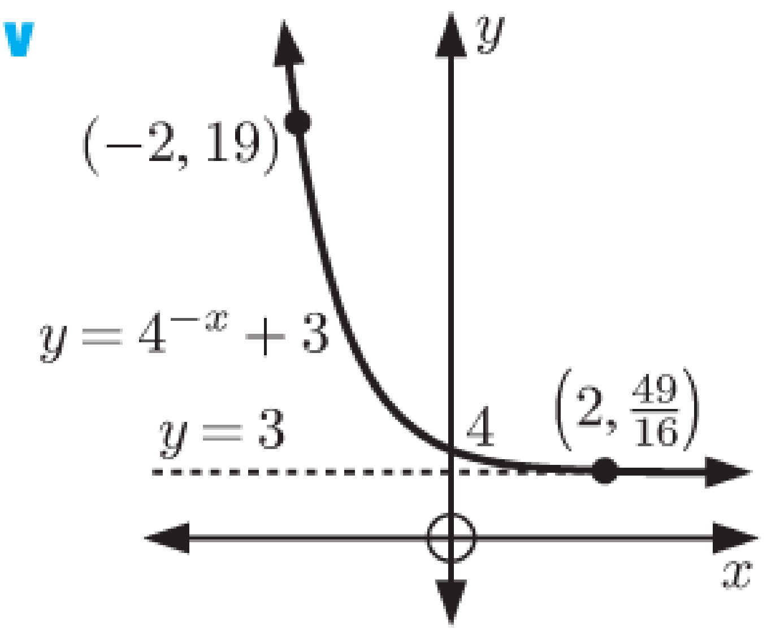
f **i** 4

ii $y = 3$

iii When $x = 2$, $y = \frac{49}{16}$

When $x = -2$, $y = 19$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 3\}$

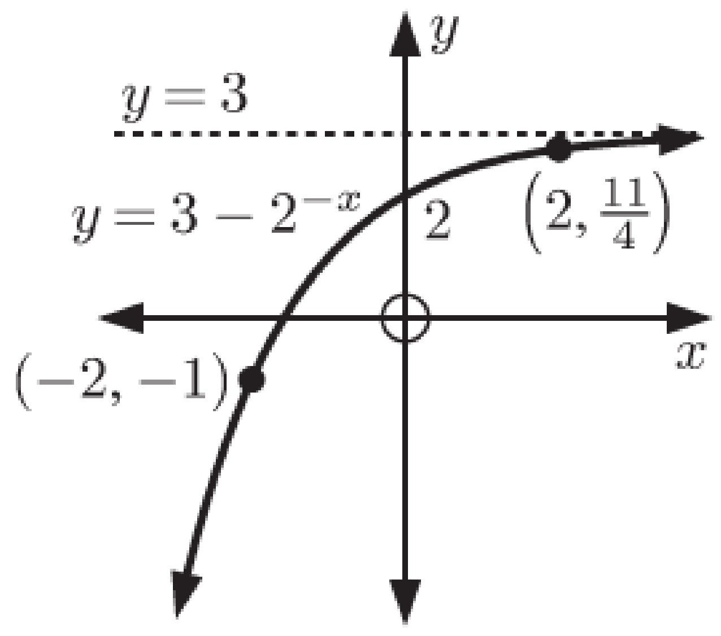
g **i** 2

ii $y = 3$

iii When $x = 2$, $y = \frac{11}{4}$

When $x = -2$, $y = -1$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

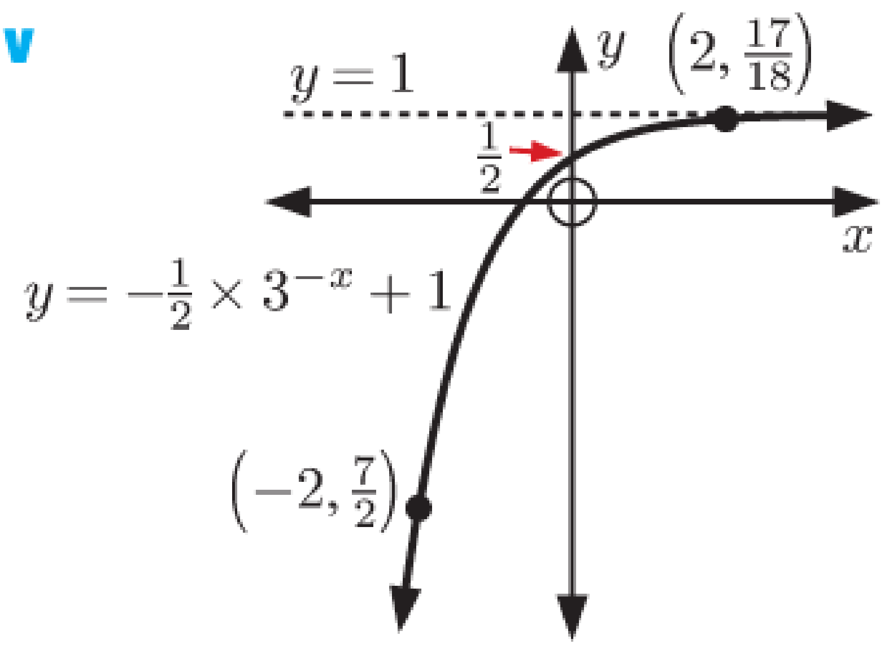
h **i** $\frac{1}{2}$

ii $y = 1$

iii When $x = 2$, $y = \frac{17}{18}$

When $x = -2$, $y = -\frac{7}{2}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 1\}$

9 **a** $k = 5$, $c = -10$ **b** $y = 310$

10 **a** $P(0, 2.5)$ **b** $a = 1.5$ **c** $y = 3.5$

11 **a** $y = 2^x + 3$ **b** $y = -6 \times 3^x + 5$ **c** $y = 4 \times 2^{-x} - 7$

12 **a** **i** k must be positive as the function is increasing and lies above the asymptote.

ii c must be negative as the horizontal asymptote is below the x-axis.

b $y = \frac{1}{2} \times 4^x - 6$ **c** $-\frac{11}{2}$ **d** $y = -6$

13 $f(x) = 8 \times 2^x + 2$ **14** $f(x) = -3 \times \left(\frac{3}{2}\right)^{-x} + 4$

EXERCISE 8D

1 **a** **i** $x \approx 1.6$ **ii** $x \approx -0.7$

2 **a** $x \approx 4.32$ **b** $x \approx 3.32$ **c** $x \approx 3.10$

d $x \approx 6.03$ **e** $x \approx 36.8$ **f** $x \approx 6.58$

3 **a** $x \approx 4.95$ **b** $x \approx 6.21$ **c** $x \approx 2.46$

d $x \approx 9.88$ **e** $x \approx 6.95$ **f** $x \approx 5.36$

g $x \approx 3.46$ **h** $x \approx 1.71$ **i** $x \approx 7.23$

4 **a** $k < 10$ **b** $k \geqslant 10$

EXERCISE 8E.1

1 **a** 3 m^2

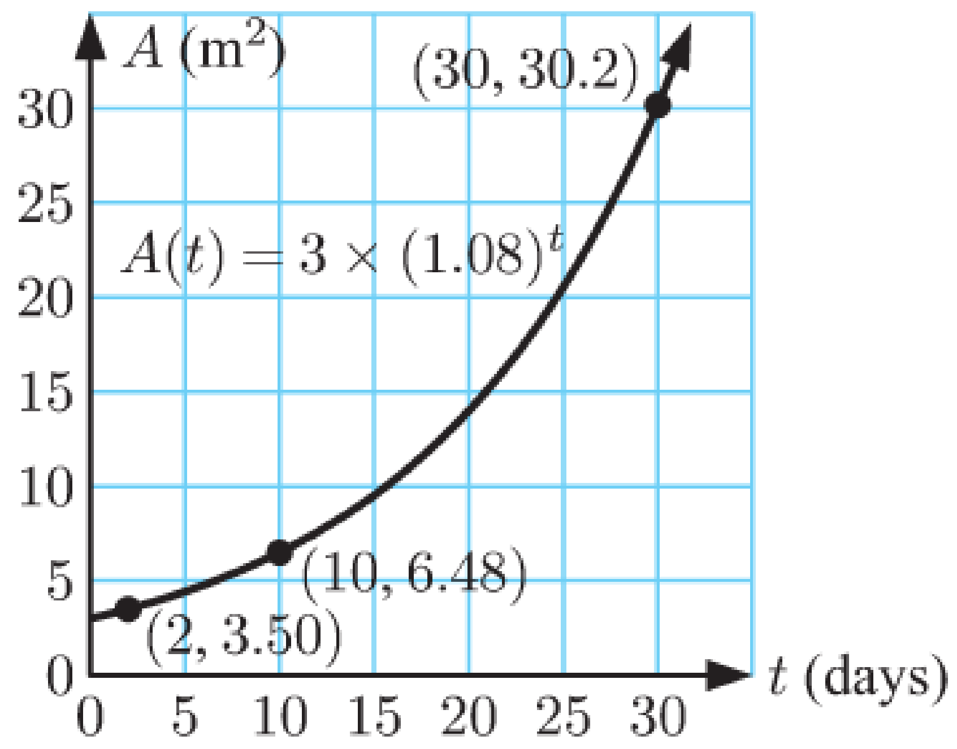
b 8%

c **i** ≈ 3.50 m^2

ii ≈ 6.48 m^2

iii ≈ 30.2 m^2

d



2 **a** 100 grams

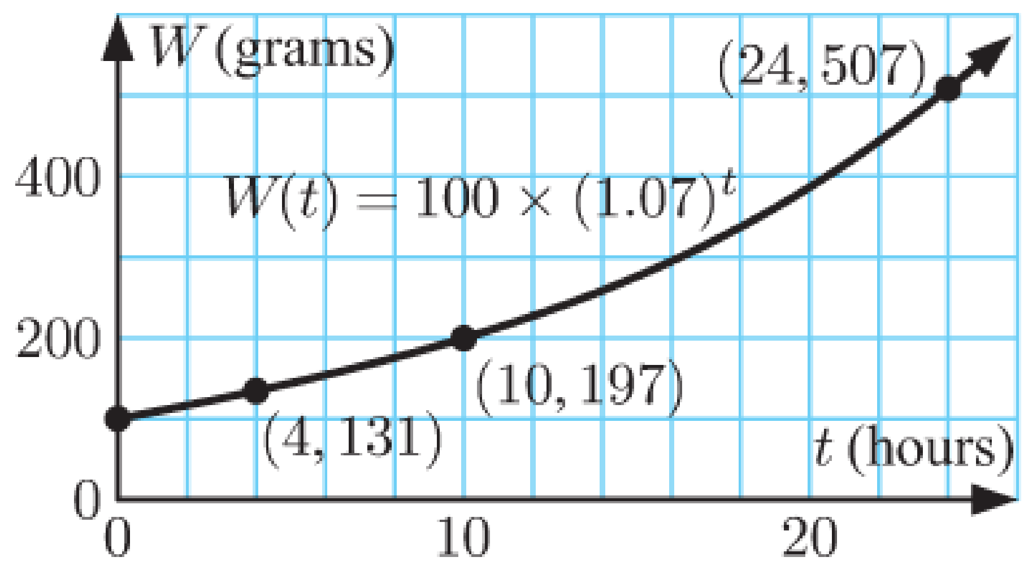
b The weight is increasing by 7% every hour.

c **i** ≈ 131 g

ii ≈ 197 g

iii ≈ 507 g

d

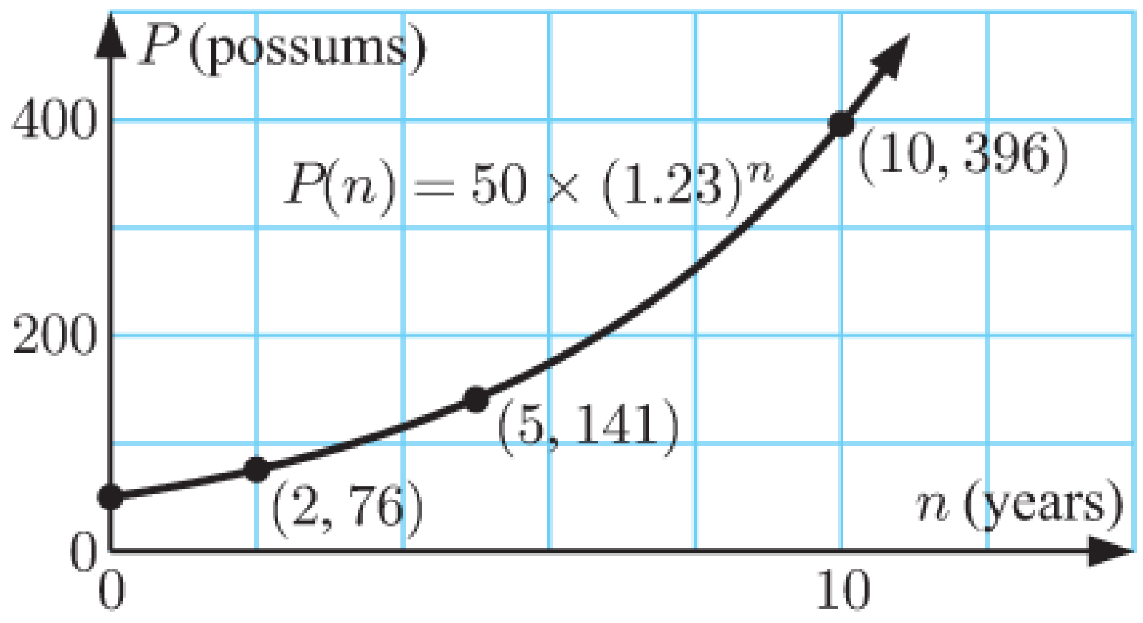


3 **a** $P_0 = 50$

b **i** ≈ 76 possums **ii** ≈ 141 possums

iii ≈ 396 possums

c



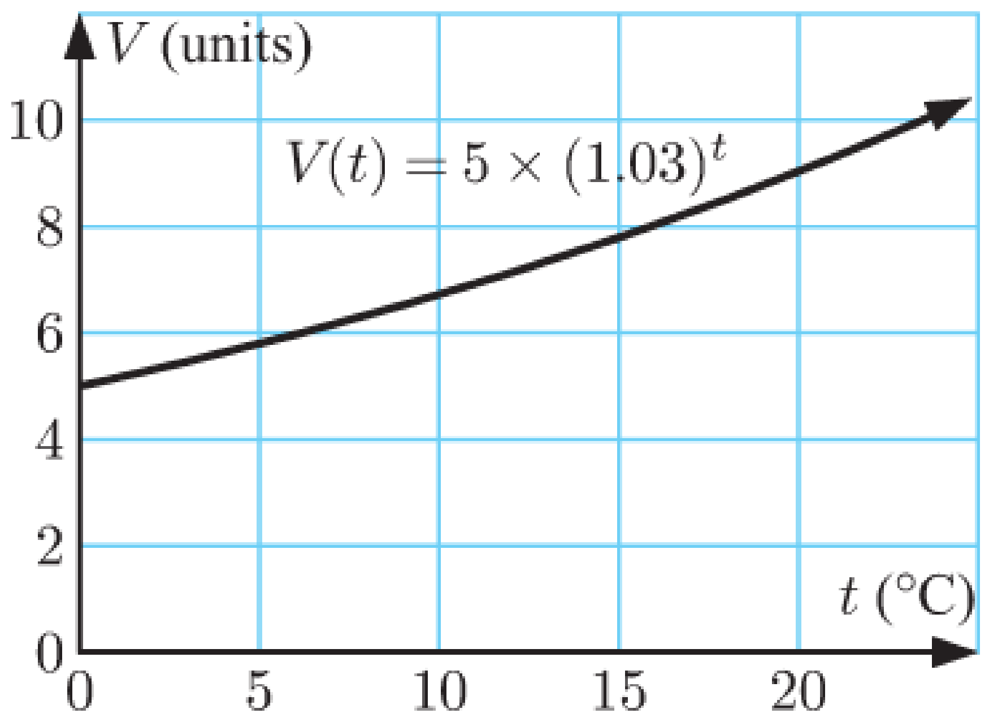
d ≈ 11.1 years

4 **a** **i** 5 units

ii ≈ 9.03 units

b $\approx 80.6\%$ increase

c



d ≈ 37.2°C

5 **a** 4 people **b** ≈ 393 people **c** ≈ 19.9 days

d $0 \leqslant t \leqslant 19.9$

6 **a** $B_0 = 200$

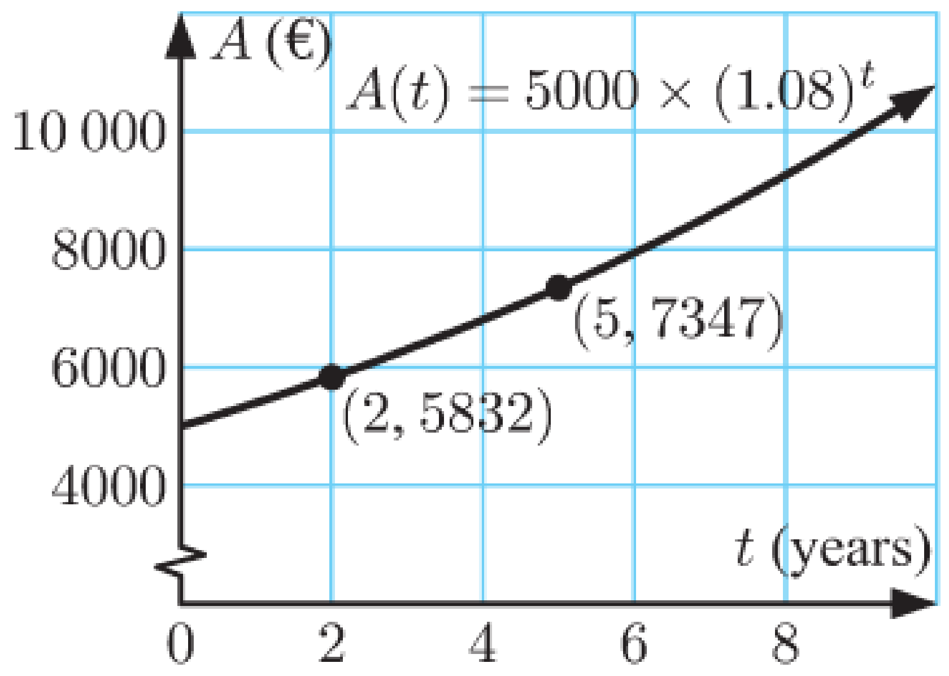
b $a = 1.1$, the bear population is increasing by 10% every year.

c ≈ 1350 bears **d** $\approx 159\%$ increase **e** ≈ 24.2 years

7 **a** $A(t) = 5000 \times (1.08)^t$ euros

b **i** €5832 **ii** €7346.64

c



8 **a** $a = 1.075$

b $k = 360\,000$, the original value of the house was $360 000.

c ≈ 5.86 years

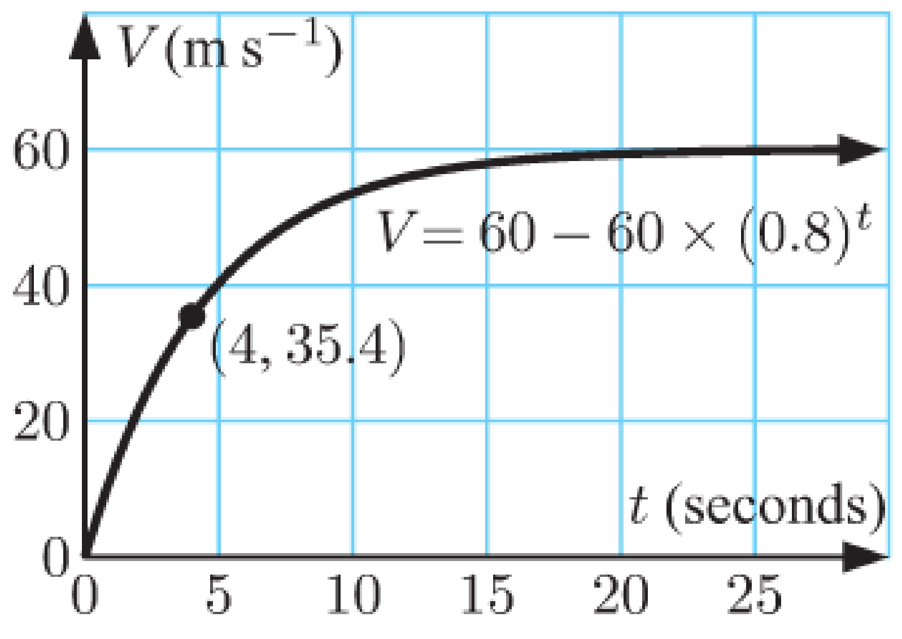
9 **a** $k = 160$, $c = 40$ **b** ≈ 4.42 weeks

10 **a** When $t = 0$, $V = c - k = 0$

$\therefore\ c = k$

b $V = 60 - 60 \times (0.8)^t$ **c** $\approx 35.4\ \text{m}\,\text{s}^{-1}$

d



e The parachutist accelerates rapidly until he approaches his terminal velocity of 60 $\text{m}\,\text{s}^{-1}$.

EXERCISE 8E.2

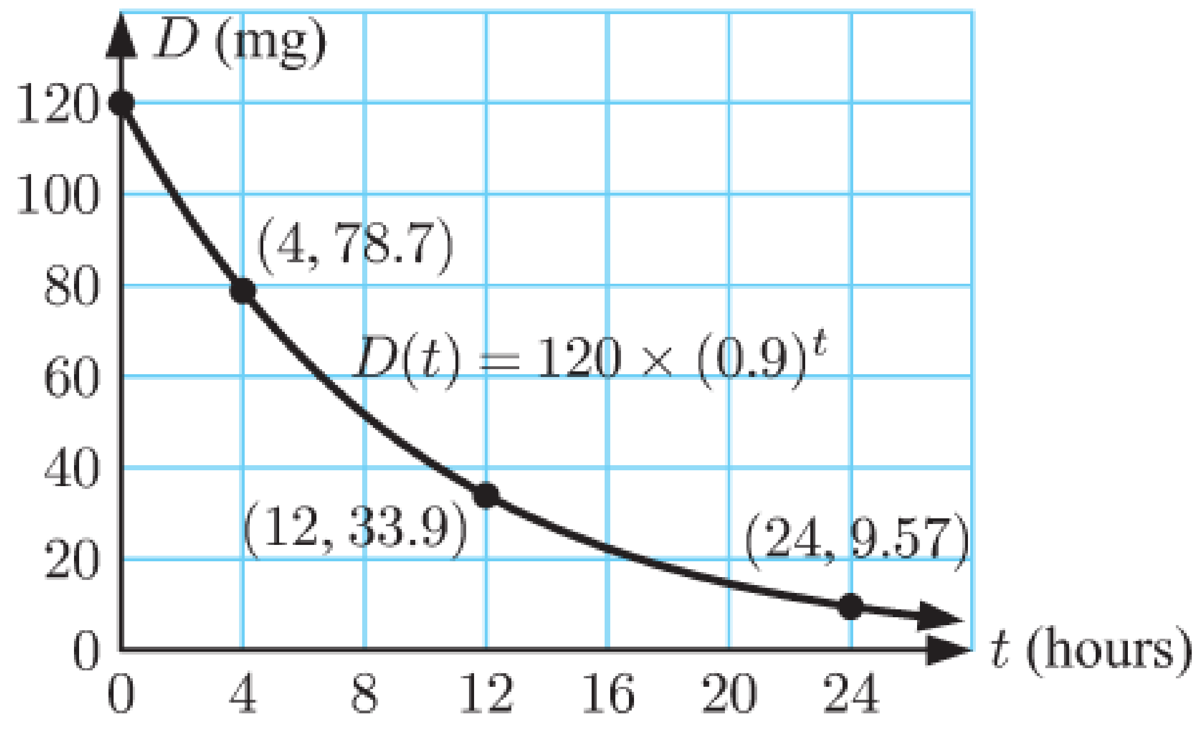
1 **a** The amount of the drug in the body decreases by 10% each hour.

b

t (hours)	0	4	12	24
$D(t)$ (mg)	120	≈ 78.7	≈ 33.9	≈ 9.57

c 120 mg

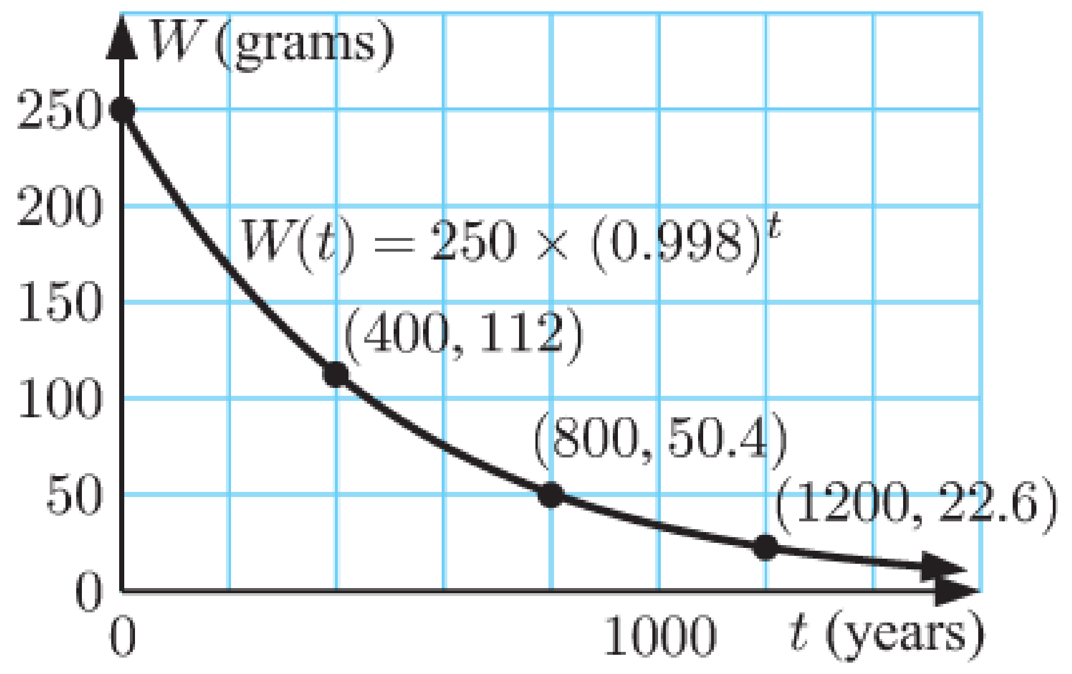
d



e ≈ 14.9 hours

2 **a** 250 g **b** **i** ≈ 112 g **ii** ≈ 50.4 g **iii** ≈ 22.6 g

c



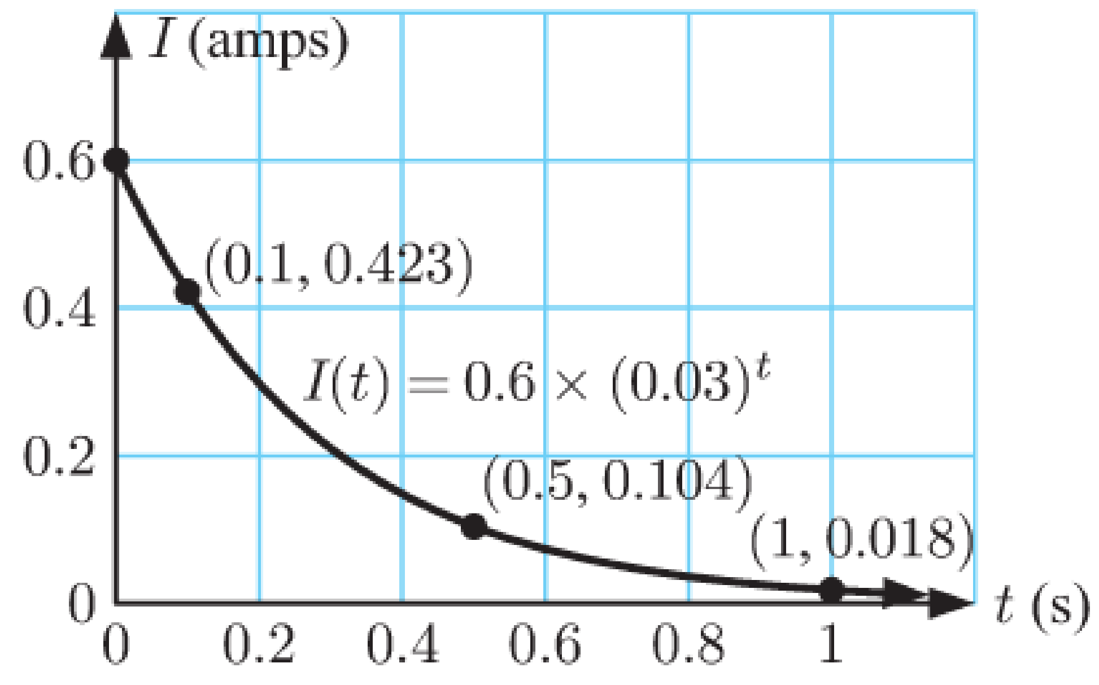
d ≈ 346 years

3 **a** 0.6 amps

b **i** ≈ 0.423 amps **ii** ≈ 0.104 amps

iii 0.018 amps

c



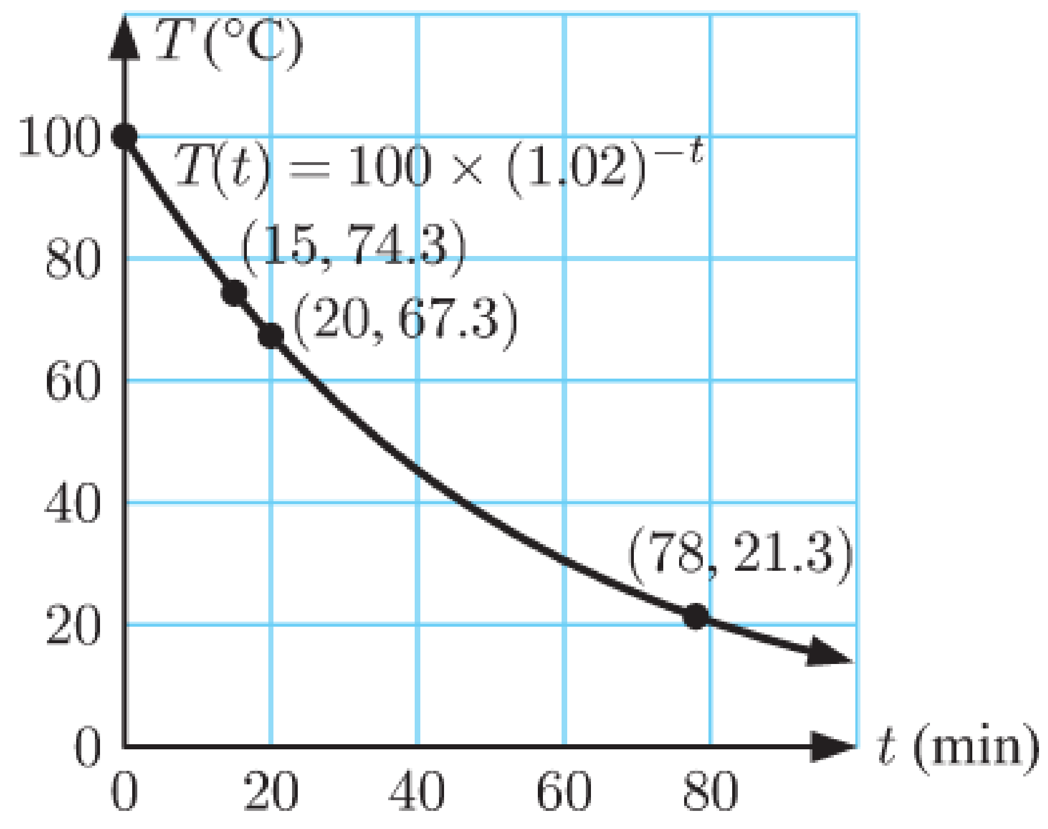
4 **a** 100°C

b **i** ≈ 74.3°C

ii ≈ 67.3°C

iii ≈ 21.3°C

c

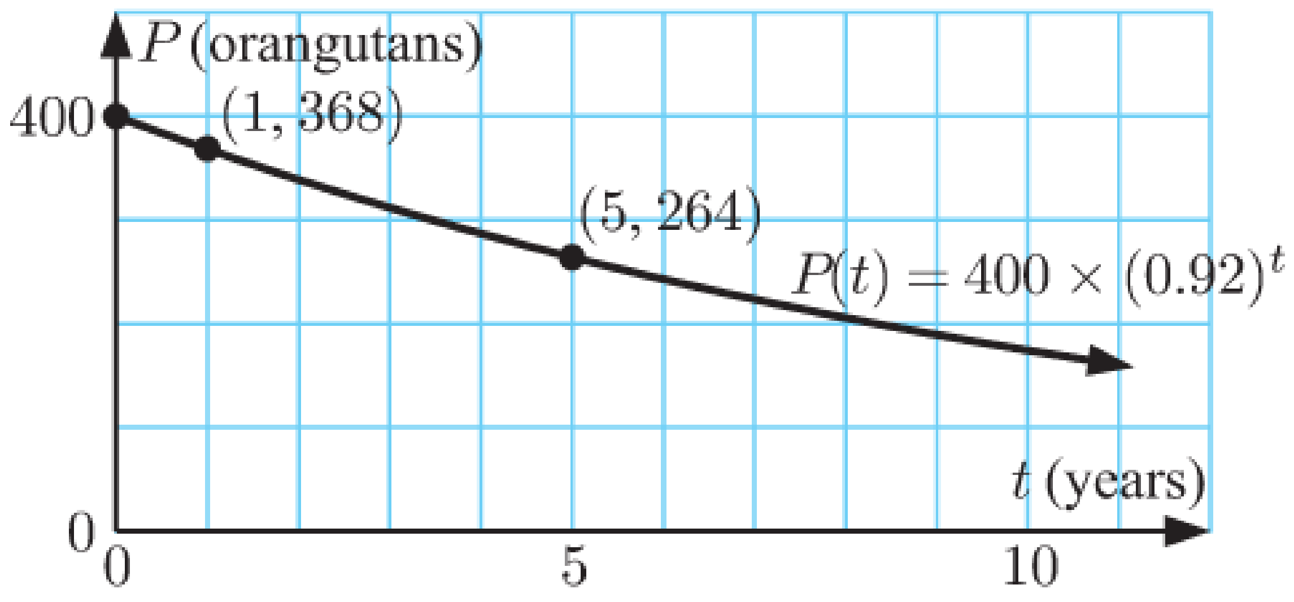


d ≈ 95.8 minutes

5 **a** $P(t) = 400 \times (0.92)^t$

b **i** 368 orangutans **ii** ≈ 264 orangutans

c



d ≈ 8.31 years, or ≈ 8 years 114 days

6 **a** As the intensity of light diminishes as the depth increases, we would expect that $0 < a < 1$.

b $a = 0.95$ **c** ≈ 2.77 units **d** ≈ 17.9 m

e $23.5 \leqslant d \leqslant 44.9$

7 **a** $a = 0.95$

b $k = 340$, the initial population of turtles was 340.

c ≈ 204 turtles

d No, the model is based on data collected since 2005. Before 2005 would be an extrapolation and therefore unreliable.

8 **a** $c = 1000$, the value of the car approaches a minimum "salvage value" of $1000.

b $k = 23\,000$ **c** $24 000 **d** ≈ 5 years

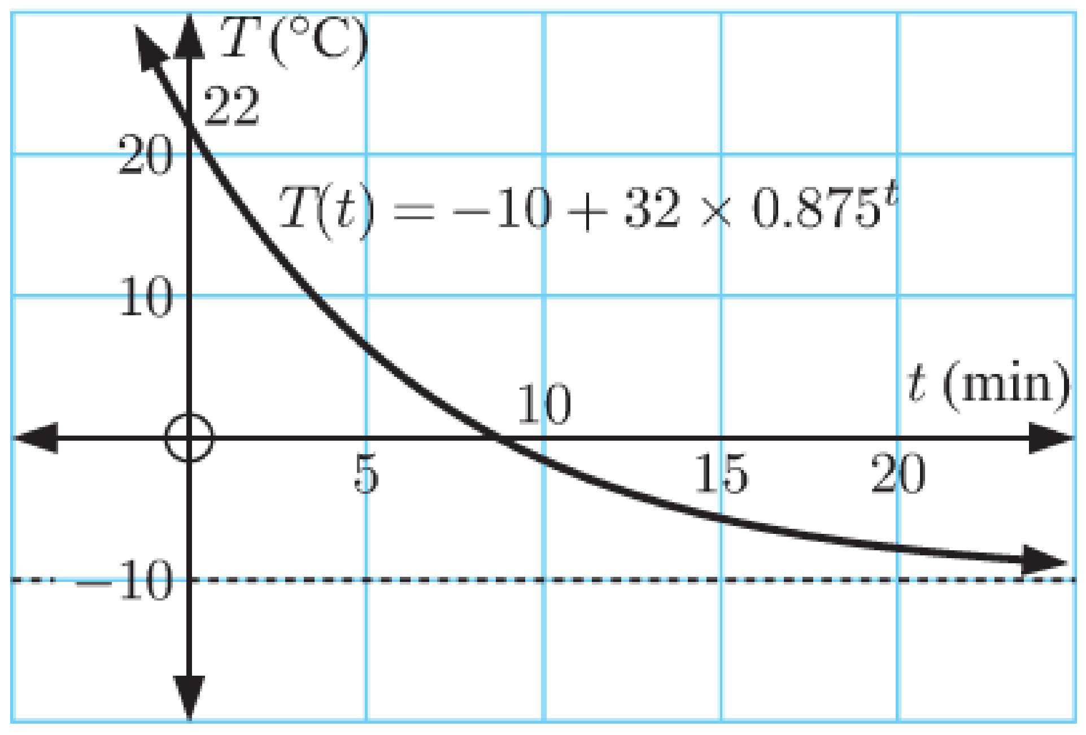
e No, the car depreciates by the same percentage each year *above* the salvage value of $1000.

9 **a** $c = -10$, $k = 32$

b -10°C, as the horizontal asymptote is $T = -10$ which suggests that the temperature in the freezer is -10°C.

c **i** 22°C **ii** ≈ 6.41°C **iii** ≈ -1.58°C

d



e ≈ 8.71 minutes

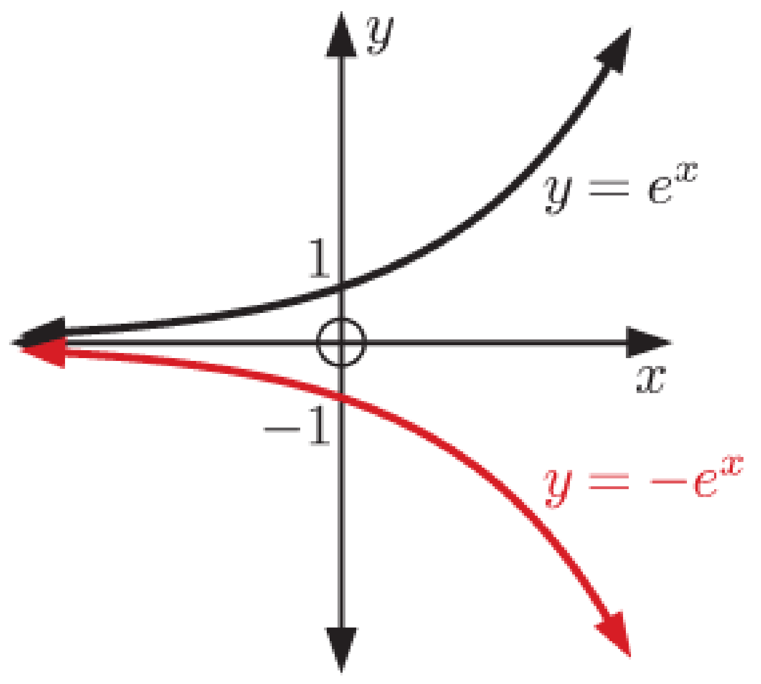
10 **a** The initial weight of the isotope is 10 mg.

b $a \approx 0.7937$; each day the isotope's weight is decreasing by $\approx 20.63\%$.

c ≈ 6.30 mg **d** **i** ≈ 5.21 days **ii** ≈ 9.00 days

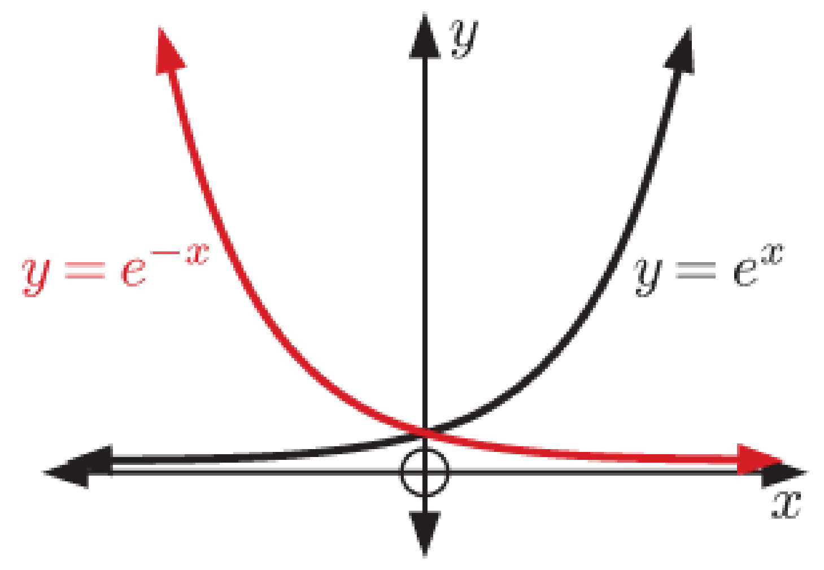
EXERCISE 8F

1



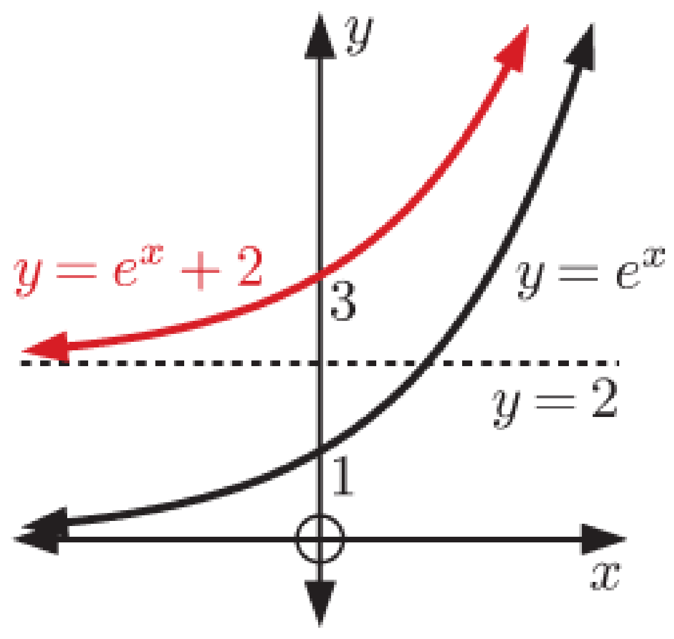
One is the other reflected in the x-axis.

2

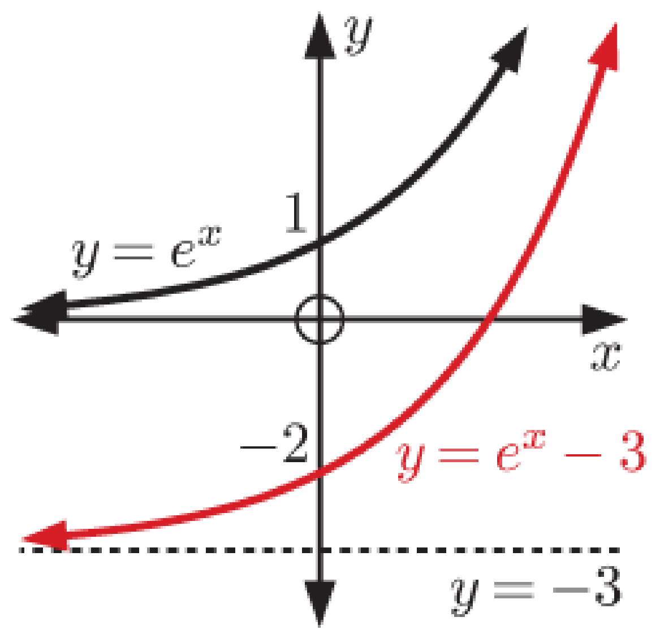


One is the other reflected in the y-axis.

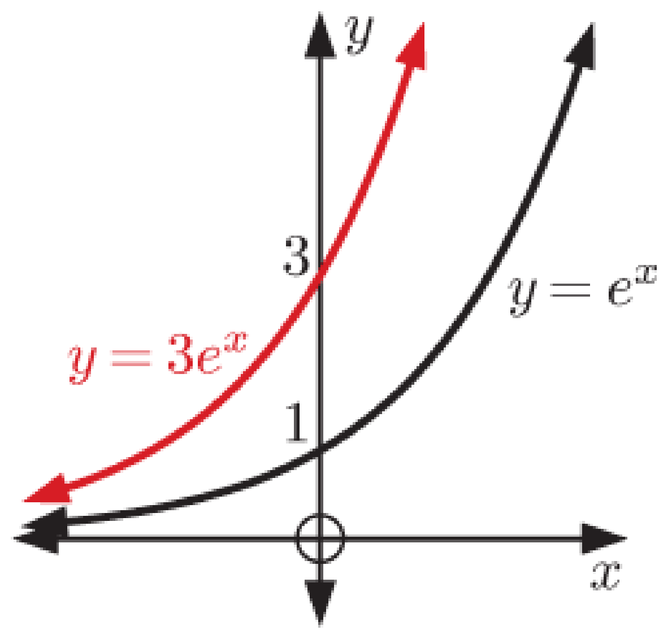
3 **a**



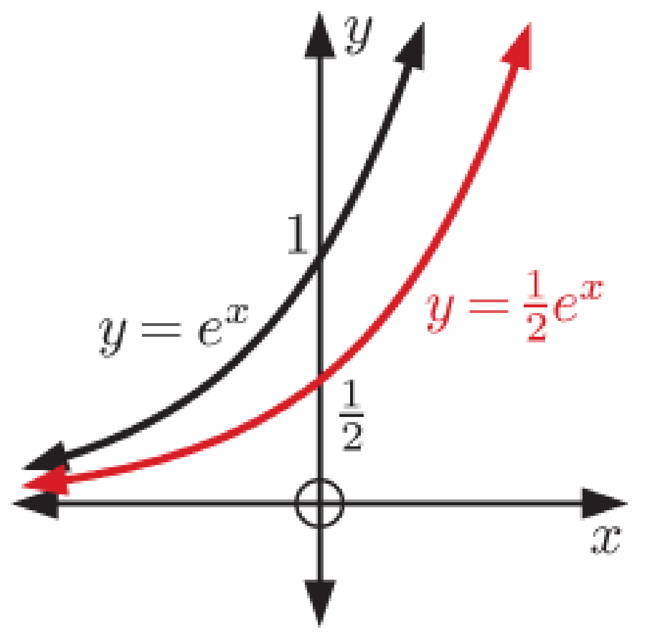
b



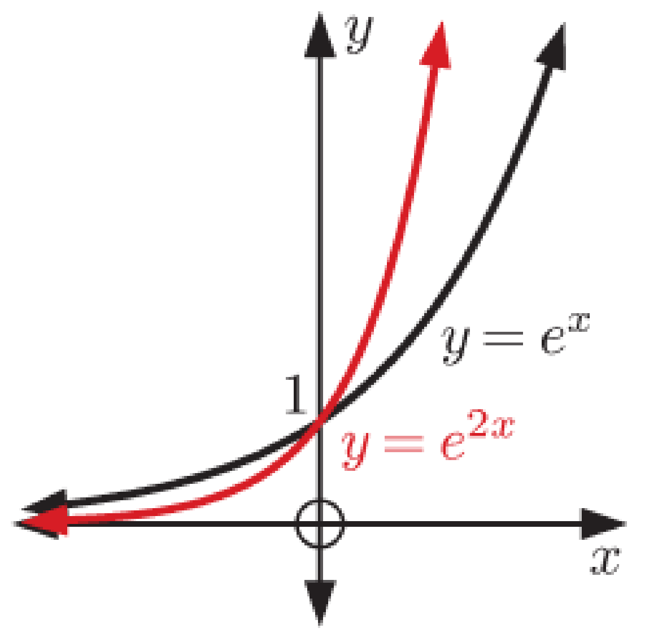
c



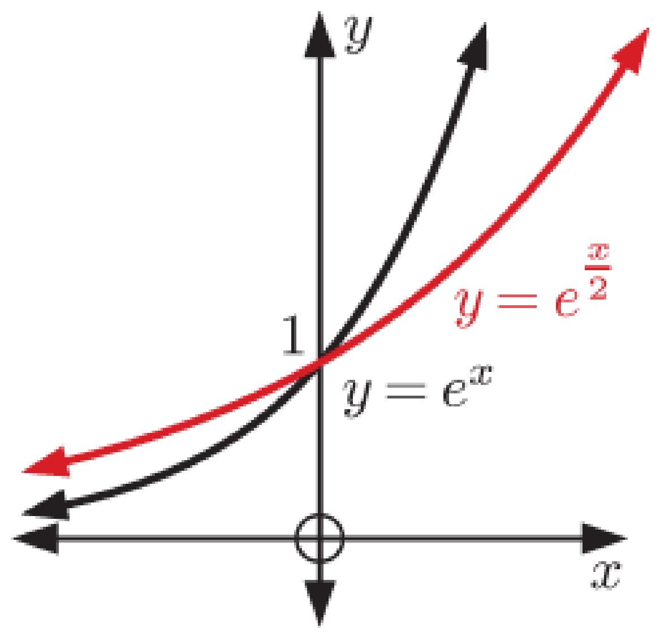
d



e



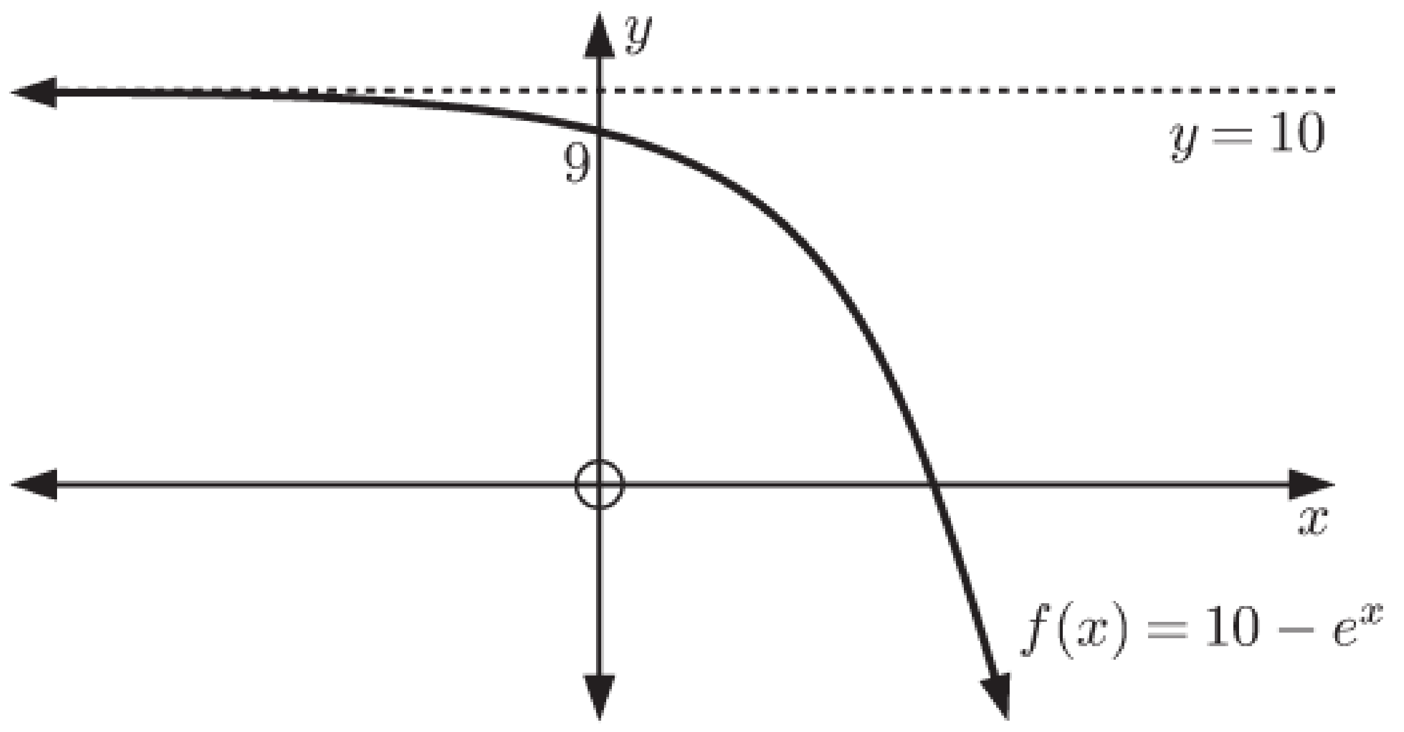
f



4 **a** ≈ 7.3891 **b** ≈ 20.086 **c** ≈ 1.6487
d $\approx 0.367\,88$ **e** ≈ 10.074 **f** $\approx 0.099\,261$
g ≈ 125.09 **h** $\approx 0.007\,994\,5$ **i** ≈ 41.914
j ≈ 42.429 **k** ≈ 3540.3 **l** $\approx 0.006\,342\,4$

5 **a** $x \approx 2.30$ **b** $x \approx 1.35$ **c** $x \approx 2.81$ **d** $x \approx 0.366$

6 **a**

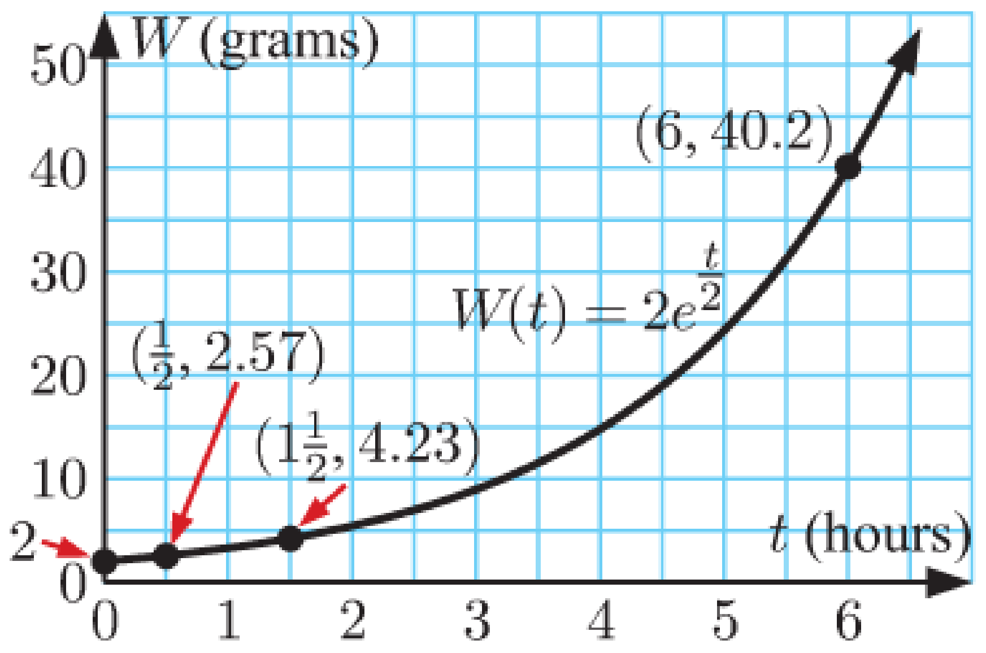


b Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 10\}$

c as $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to 10^-$

7 **a** **i** 2 g
ii ≈ 2.57 g
iii ≈ 4.23 g
iv ≈ 40.2 g

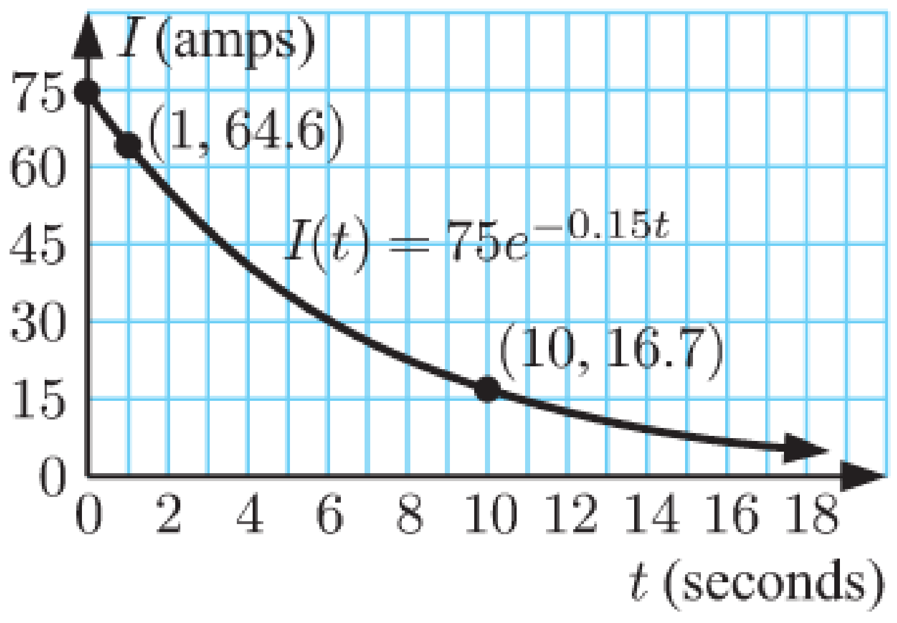
b



8 **a** **i** ≈ 64.6 amps
ii ≈ 16.7 amps

c ≈ 28.8 seconds

b



9 **a** $A = 7e^{-0.5t} + 3$
b The side as the tank never completely empties.
c ≈ 5.58 kL **d** ≈ 3.89 min

10 **a** decreasing **b** **i** 3900 m s^{-1} **ii** ≈ 2600 m s^{-1}
c ≈ 11.8 s

EXERCISE 8G

1 **a** 4 **b** -3 **c** 1 **d** 0

2 **a** n **b** $a+2$ **c** $1-m$ **d** $a-b$

3 **a** $100 < 237 < 1000$ **b** ≈ 2.37
$\therefore \log 100 < \log 237 < \log 1000$
$\therefore 2 < \log 237 < 3$

4 **a** $-1 < \log 0.6 < 0$ **b** ≈ -0.22

5 **a** ≈ 1.88 **b** ≈ 2.06 **c** ≈ 0.48 **d** ≈ 2.92
e ≈ -0.40 **f** ≈ 3.51 **g** ≈ -2.10 **h** does not exist

6 **a** $x > 1$ **b** $x = 1$ **c** $0 < x < 1$ **d** $x \leqslant 0$

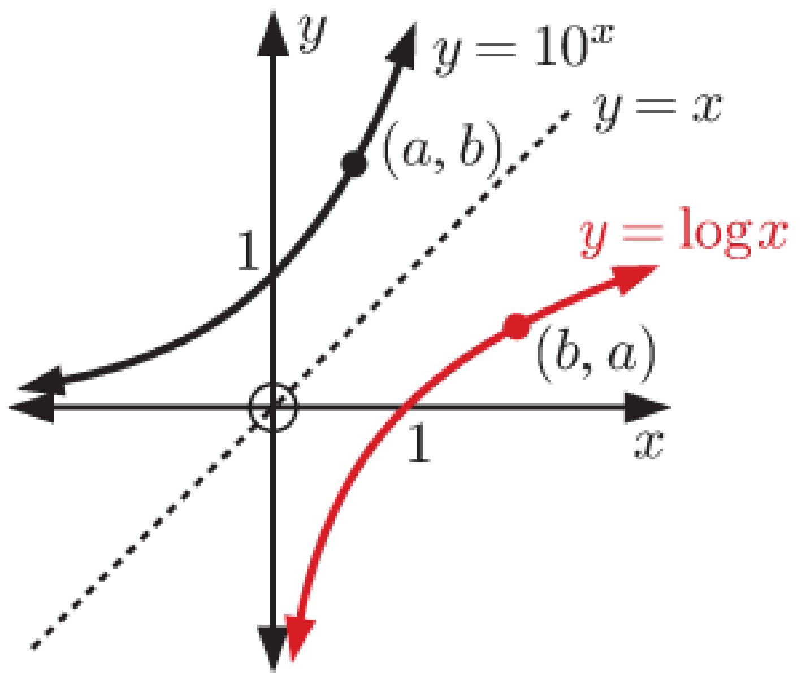
7 **a** ≈ 1.9191 **b** $\approx 10^{1.9191}$

8 **a** $\approx 10^{0.7782}$ **b** $\approx 10^{1.7782}$ **c** $\approx 10^{3.7782}$
d $\approx 10^{-0.2218}$ **e** $\approx 10^{-2.2218}$ **f** $\approx 10^{1.1761}$
g $\approx 10^{3.1761}$ **h** $\approx 10^{0.1761}$ **i** $\approx 10^{-0.8239}$
j $\approx 10^{-3.8239}$

9 **a** **i** ≈ 0.477 **ii** ≈ 2.477
b $\log 300 = \log(3 \times 10^2) = \log(10^{\log 3} \times 10^2) = \ldots.$

10 **a** **i** ≈ 0.699 **ii** ≈ -1.301
b $\log 0.05 = \log(5 \times 10^{-2}) = \log(10^{\log 5} \times 10^{-2}) = \ldots.$

11 **a** $b = 10^a$
c When $x = b$, $y = \log b = a$ {from **b**}
$\therefore$ (b, a) lies on the graph of $y = \log x$.
d They are inverse functions.
e



f 1 **g** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

12 **a** ≈ 5.99 **b** $\approx 2.82 \times 10^{12}$ joules

13 **a** ≈ 3.63 **b** $\approx 6.31 \times 10^{-7}$ mol L^{-1}

14 **a** 261.6 Hz **b** ≈ 1.58 octaves
c **i** ≈ 2090 Hz **ii** ≈ 131 Hz **d** $\approx 1.06 : 1$

EXERCISE 8H

1 **a** 2 **b** 4 **c** 0 **d** −2

2 **a** a **b** $1+a$ **c** $a+b$ **d** ab **e** 3 **f** 9 **g** $\frac{1}{5}$ **h** $\frac{1}{4}$

3 **a** $e \approx 2.718$, $e^2 \approx 7.389$

$\therefore\ e < 5 < e^2$

$\therefore\ \ln e < \ln 5 < \ln(e^2)$

$\therefore\ 1 < \ln 5 < 2$

b $\ln 5 \approx 1.609$

4 **a** $e^3 \approx 20.0855$, $e^4 \approx 54.5982$ **b** 3 and 4 **c** $\ln 40 \approx 3.689$

5 **a** ≈ 2.485 **b** ≈ 4.220 **c** ≈ 0.336 **d** ≈ -0.357 **e** ≈ 6.215 **f** ≈ 6.745 **g** ≈ -3.912 **h** ≈ 11.513 **i** ≈ -5.116 **j** ≈ 8.923

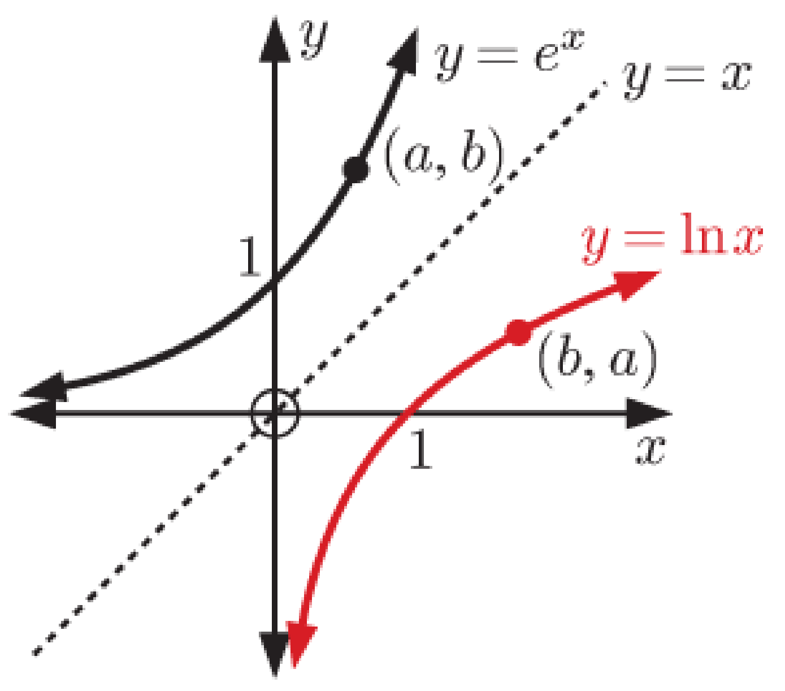
6 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.

7 **a** ≈ 3.1781 **b** $24 \approx e^{3.1781}$

8 **a** $\approx e^{1.7918}$ **b** $\approx e^{4.0943}$ **c** $\approx e^{8.6995}$ **d** $\approx e^{-0.5108}$ **e** $\approx e^{-5.1160}$ **f** $\approx e^{2.7081}$ **g** $\approx e^{7.3132}$ **h** $\approx e^{0.4055}$ **i** $\approx e^{-1.8971}$ **j** $\approx e^{-8.8049}$

9 **b** They are inverse functions.

c



d Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

10 **a** **i** ≈ 3.58 seconds **ii** ≈ 5.55 seconds **b** ≈ 1.34 seconds longer

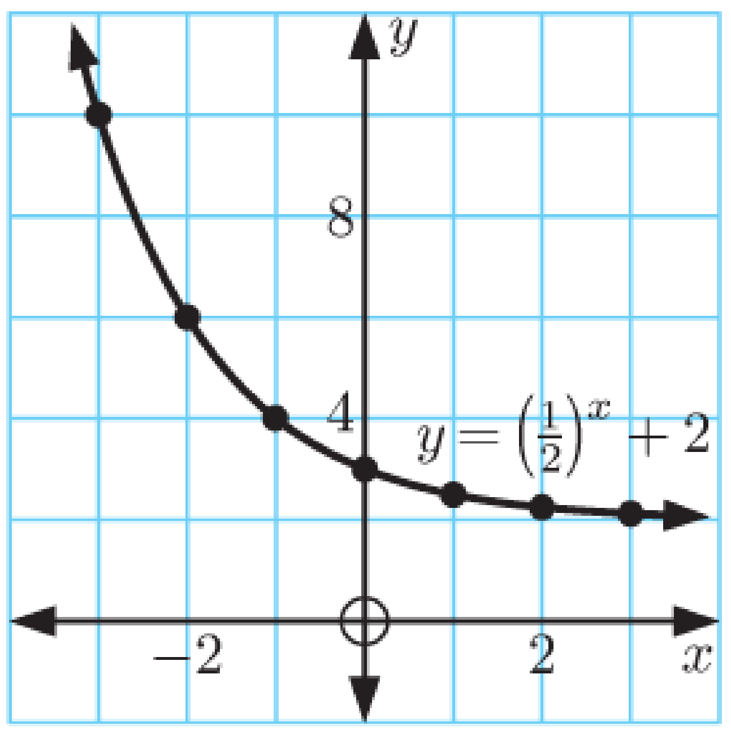
REVIEW SET 8A

1 **a** 3 **b** 24 **c** $\frac{3}{4}$ **2** **a** yes **b** no

3 **a**

x	−3	−2	−1	0	1	2	3
y	10	6	4	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{1}{8}$

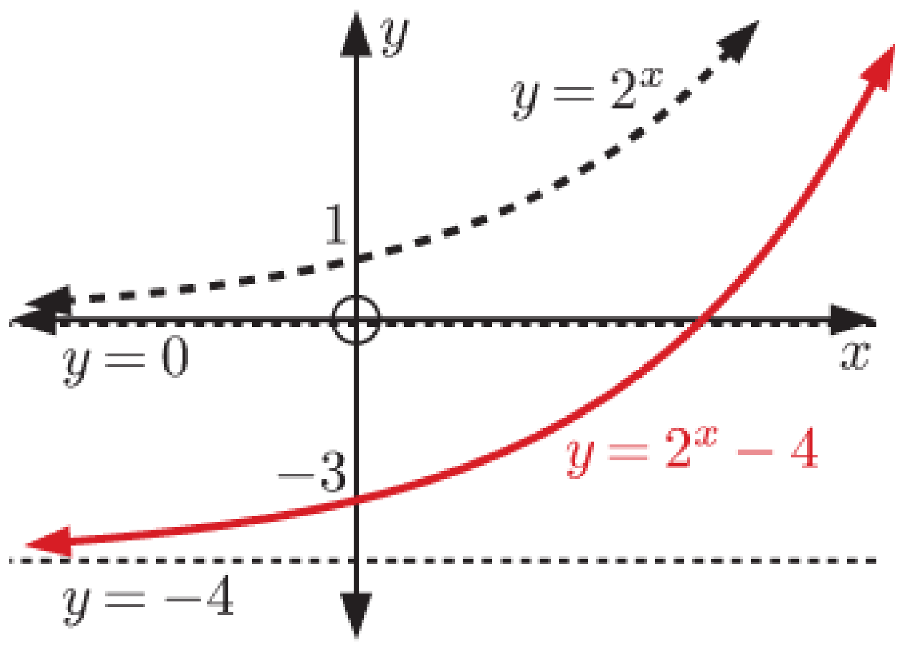
b



c **i** as $x \to \infty$, $y \to 2^+$ **ii** as $x \to -\infty$, $y \to \infty$

d $y = 2$

4



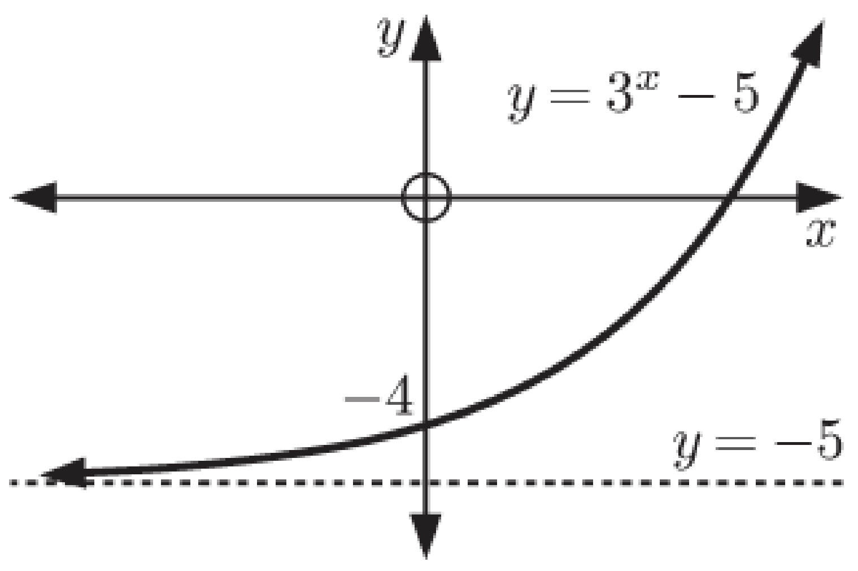
5 **a** 3 **b** −4 **c** $4\frac{2}{3}$

6 **a**

x	−2	−1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	−4	−2	4

b as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to -5^+$

c, d



d $y = -5$

7 **a** $k = 8$, $a = \frac{4}{3}$ **b** $y = 0$ **c** $y = \frac{128}{9}$

8 **a** **i** ≈ 2.2 **ii** ≈ 0.6 **b** **i** $x \approx 1.5$ **ii** $x \approx -0.6$

9 **a** $x \approx 4.29$ **b** $x \approx 3.56$ **c** $x \approx 86.3$

10 **a** 80 seals **b** ≈ 4.96 years **c** $\approx 74.9\%$

11 **a** $k = 700$, $c = 100$

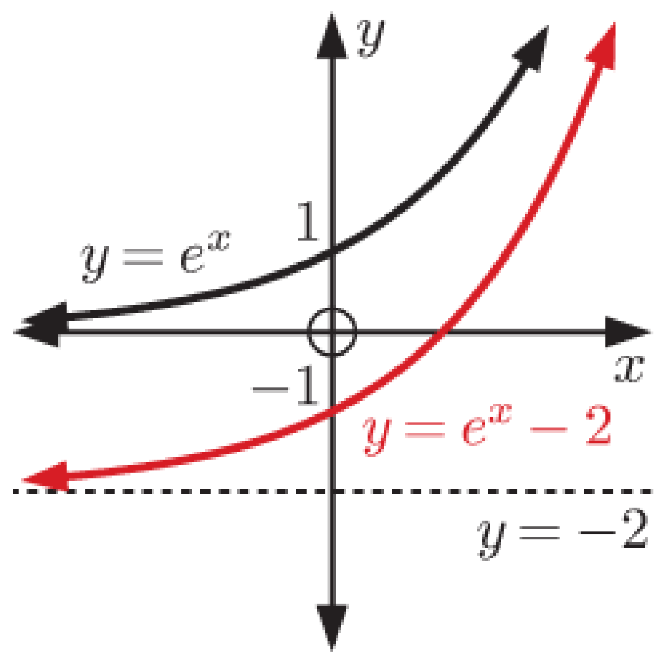
b No, the computer depreciates by 40% each year above the computer's *minimum* price, but not overall.

c £251.20

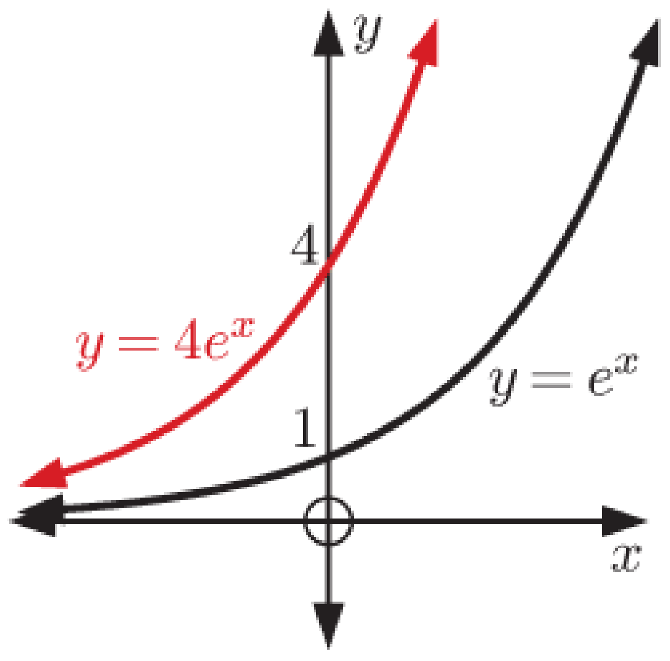
d $V = 100$, the computer will never be worth less than £100.

12 **a** ≈ 54.6 **b** ≈ 0.135 **c** ≈ 331 **d** ≈ 3.19

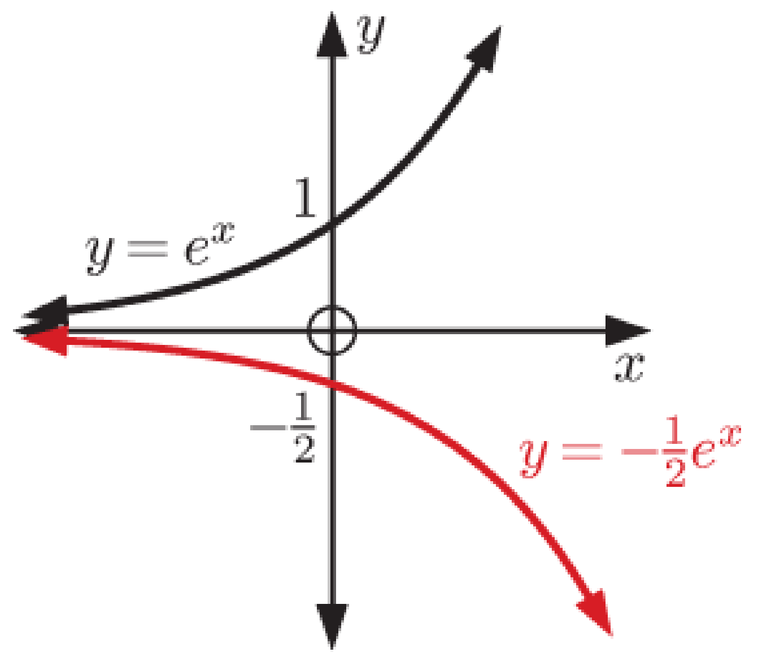
13 **a**



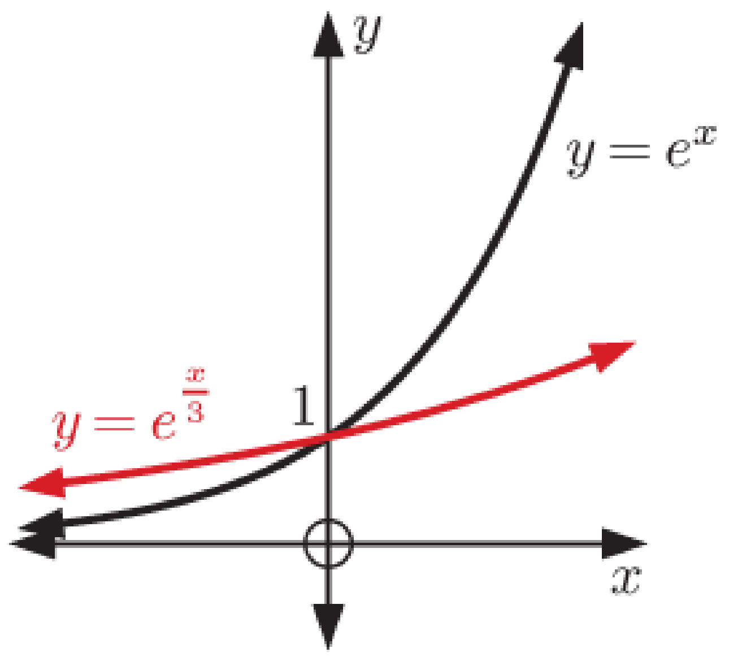
b



c



d



14 **a** $T = 60e^{-0.1t} + 20$ **b** $\approx 42.1°\text{C}$ **c** ≈ 17.9 min

15 **a** ≈ 1.431 **b** ≈ -0.237 **c** ≈ 2.602 **d** ≈ 3.689

16 **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

17 e and 10 **18** **a** 100 **b** (100, 2)

19 **a** **i** ≈ 13.5 years **ii** ≈ 31.2 years **b** ≈ 52.8 years longer **c** $\approx 2.28\%$

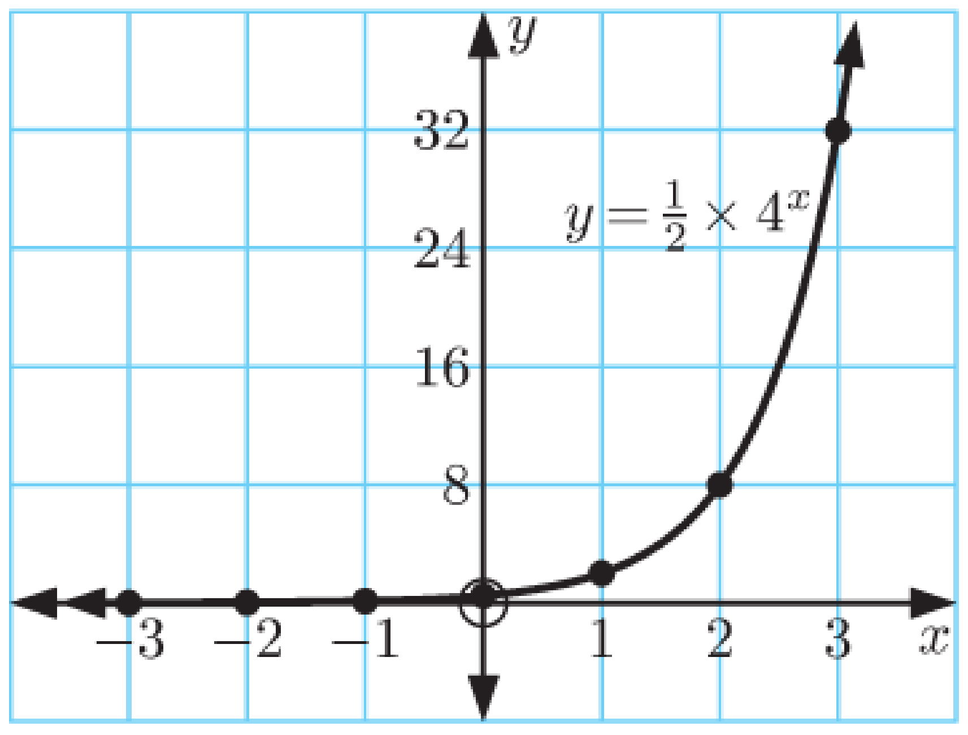
REVIEW SET 8B

1 **a** −1 **b** $-1\frac{2}{3}$ **c** 1 **d** $-1\frac{26}{27}$

3 **a**

x	−3	−2	−1	0	1	2	3
y	$\frac{1}{128}$	$\frac{1}{32}$	$\frac{1}{8}$	$\frac{1}{2}$	2	8	32

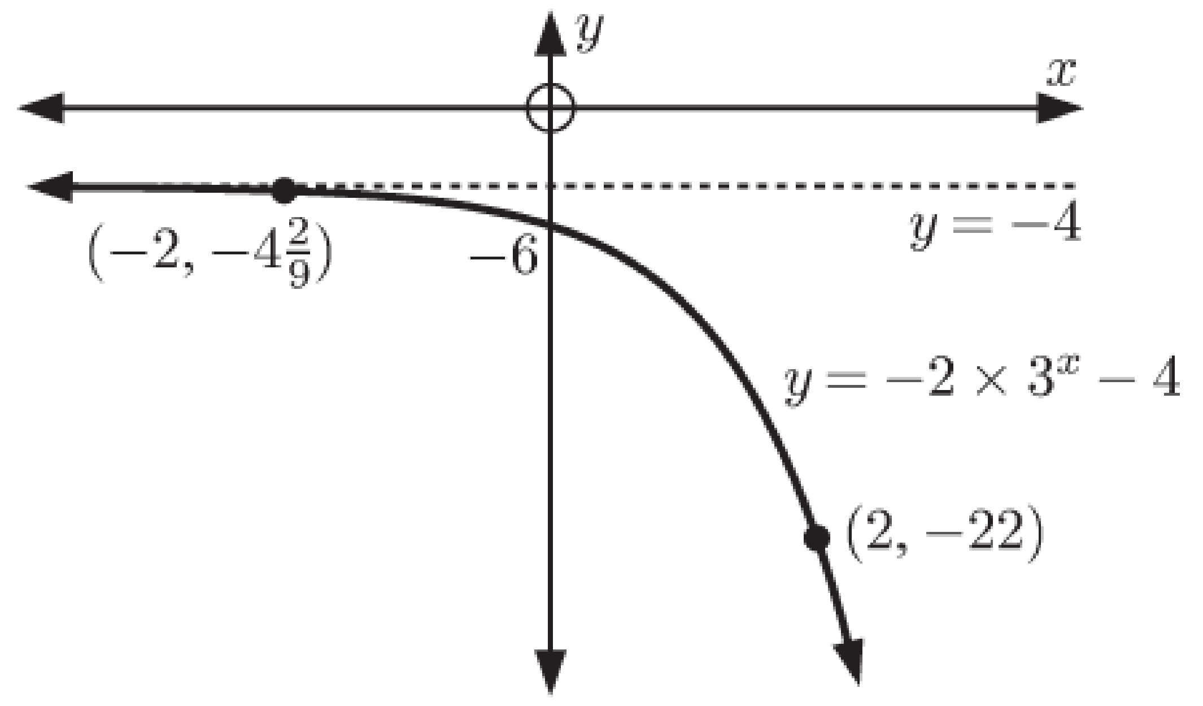
b



4 **a** $y = 3$ **b** $y = -4$ **c** $y = 6$

5 **a** $f(-2) = -4\frac{2}{9}$, $f(2) = -22$ **b** **i** -6 **ii** $y = -4$

c



d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < -4\}$

6 **a** $y = 6 \times 2^{-x} + 4$ **b** $y = -3 \times (\frac{1}{2})^x + 5$

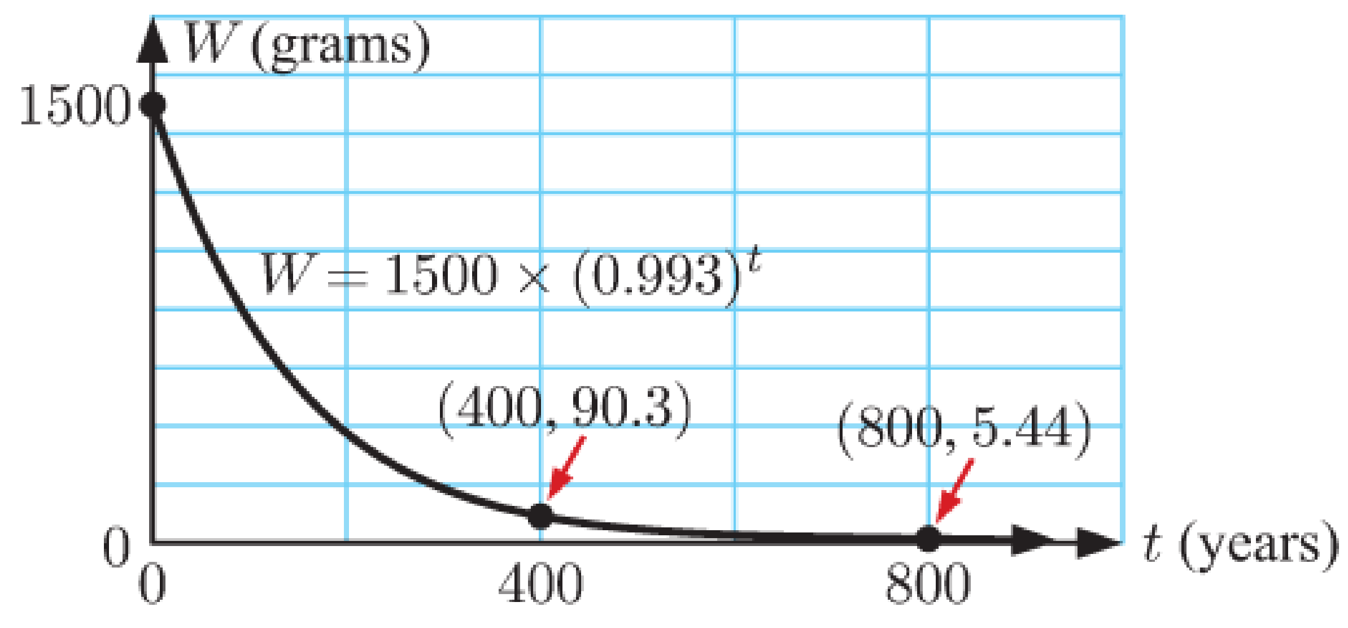
7 **a** $y = 10 \times 5^{-x} - 2$ **b** $y = 48$

8 **a** $x \approx 2.45$ **b** $x \approx 2.67$ **c** $x \approx 2.79$

9 **a** $k > -2$ **b** $k \leqslant -2$

10 **a** 1500 grams **b** **i** ≈ 90.3 grams **ii** ≈ 5.44 grams

c



d ≈ 386 years

11 **a** $a = 1.15$

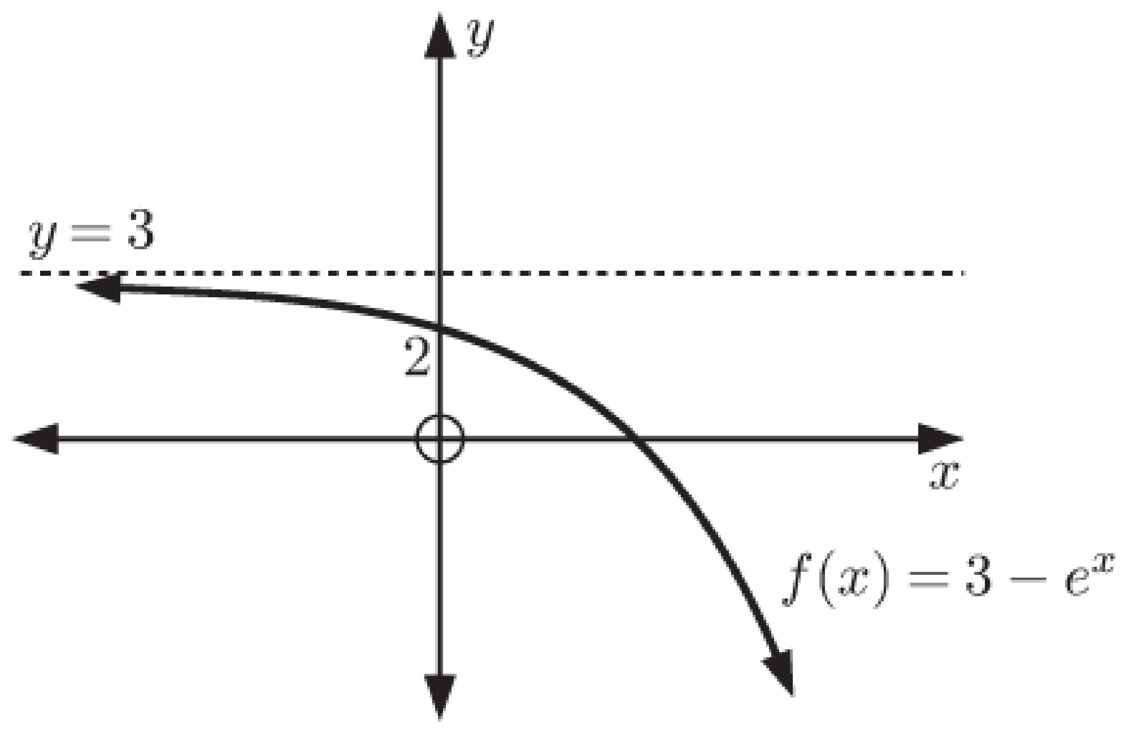
b $k = 800$, the initial population was 800 birds.

c ≈ 1610 birds

d No, the population would be unrealistically large. The birds would eventually reach a limiting population.

12 **a** C **b** E **c** A **d** B **e** D

13 **a**



b Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

c as $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to 3^-$

14 **a** $k = -50$, $c = 50$ **b** ≈ 27.5 m s^{-1} **c** ≈ 8.05 s

15 **a** 4 **b** -5 **c** $c - 3$ **d** $\frac{1}{4}$

16 **a** ≈ 2.097 **b** ≈ -1.523 **c** ≈ 2.944 **d** ≈ -0.4055

17 **a** ≈ -26.7 **b** $\approx 3.64 \times 10^{-8}$ Watts per m^2

18 **a** $\approx e^{2.9957}$ **b** $\approx e^{8.0064}$ **c** $\approx e^{-2.5903}$

19 **a** 2 **b** Q(2, ln 2)

EXERCISE 9A

1 **a** **i** A(cos 26°, sin 26°), B(cos 146°, sin 146°), C(cos 199°, sin 199°)

ii A(0.899, 0.438), B(−0.829, 0.559), C(−0.946, −0.326)

b **i** A(cos 123°, sin 123°), B(cos 251°, sin 251°), C(cos(−35°), sin(−35°))

ii A(−0.545, 0.839), B(−0.326, −0.946), C(0.819, −0.574)

c **i** A(cos(−162°), sin(−162°)), B(cos 298°, sin 298°), C(cos 486°, sin 486°)

ii A(−0.951, −0.309), B(0.469, −0.883), C(−0.588, 0.809)

2

θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1

3 **a** **i** ≈ 0.707 **ii** ≈ 0.866

b

θ (degrees)	30°	45°	60°	135°	150°	240°	315°
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$

4 **a**

Quadrant	*Degree measure*	$\cos\theta$	$\sin\theta$
1	$0° < \theta < 90°$	positive	positive
2	$90° < \theta < 180°$	negative	positive
3	$180° < \theta < 270°$	negative	negative
4	$270° < \theta < 360°$	positive	negative

b **i** quadrants 1 and 4 **ii** quadrants 2 and 3

iii quadrant 3 **iv** quadrant 2

5 **a** $0 \leqslant \sin\theta \leqslant 1$

b **i** $\theta = 90°$ **ii** $\theta = 270°$ **iii** $\theta = 0°, 180°$, or $360°$

iv $0° < \theta < 180°$ **v** $180° < \theta < 360°$

6 **a** $\cos 400° = \cos(360° + 40°) = \cos 40°$

b $\sin 130° = \sin(360° - 230°) = \sin(-230°)$

c $\cos 790° = \cos(720° + 70°) = \cos 70°$

7 **a** 0.891 **b** 0.454 **c** 0.755 **d** -0.208

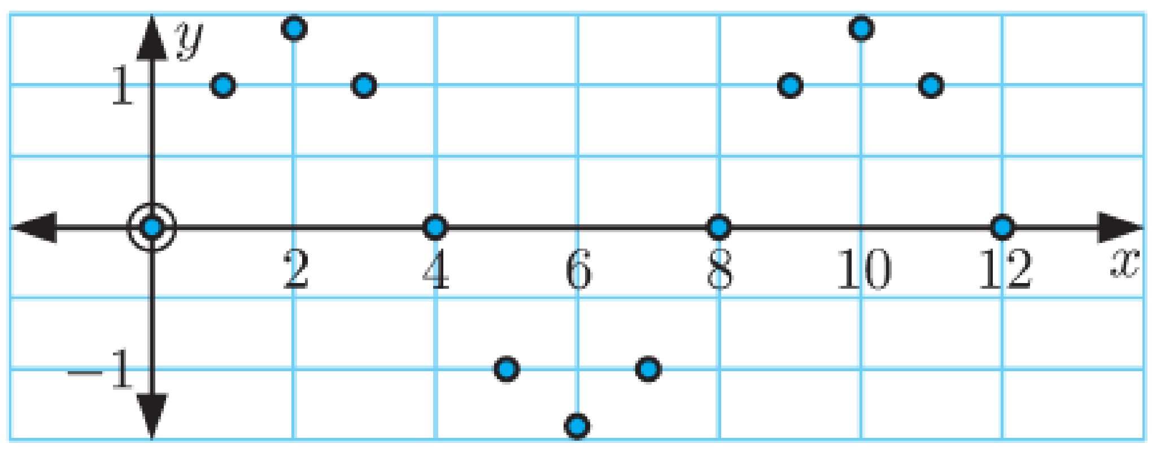
e 0.891 **f** -0.656 **g** -0.978 **h** 0.454

EXERCISE 9B

1 **a** periodic **b** periodic **c** periodic **d** not periodic

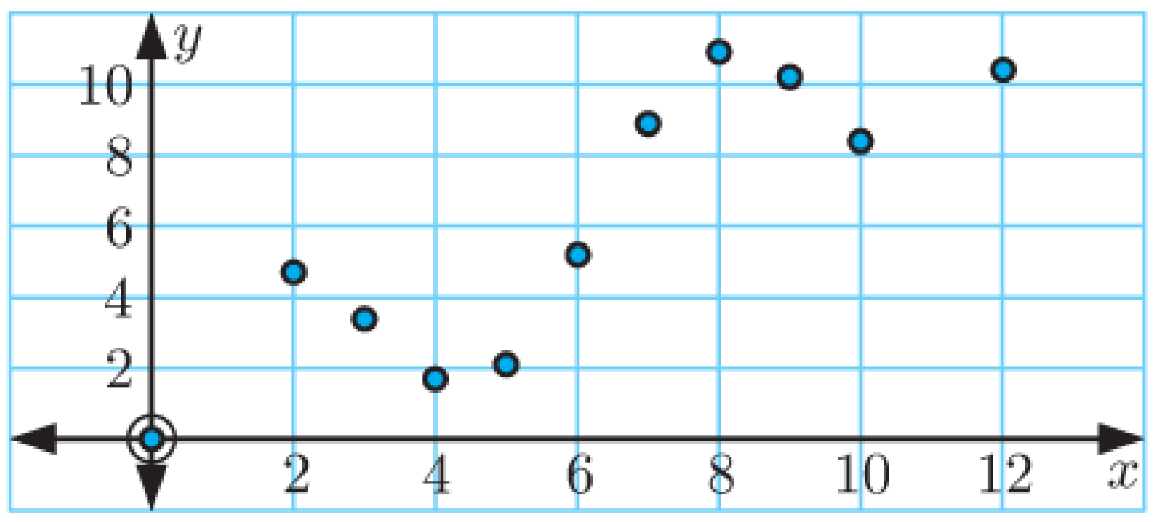
e periodic **f** periodic **g** not periodic **h** not periodic

2 **a**



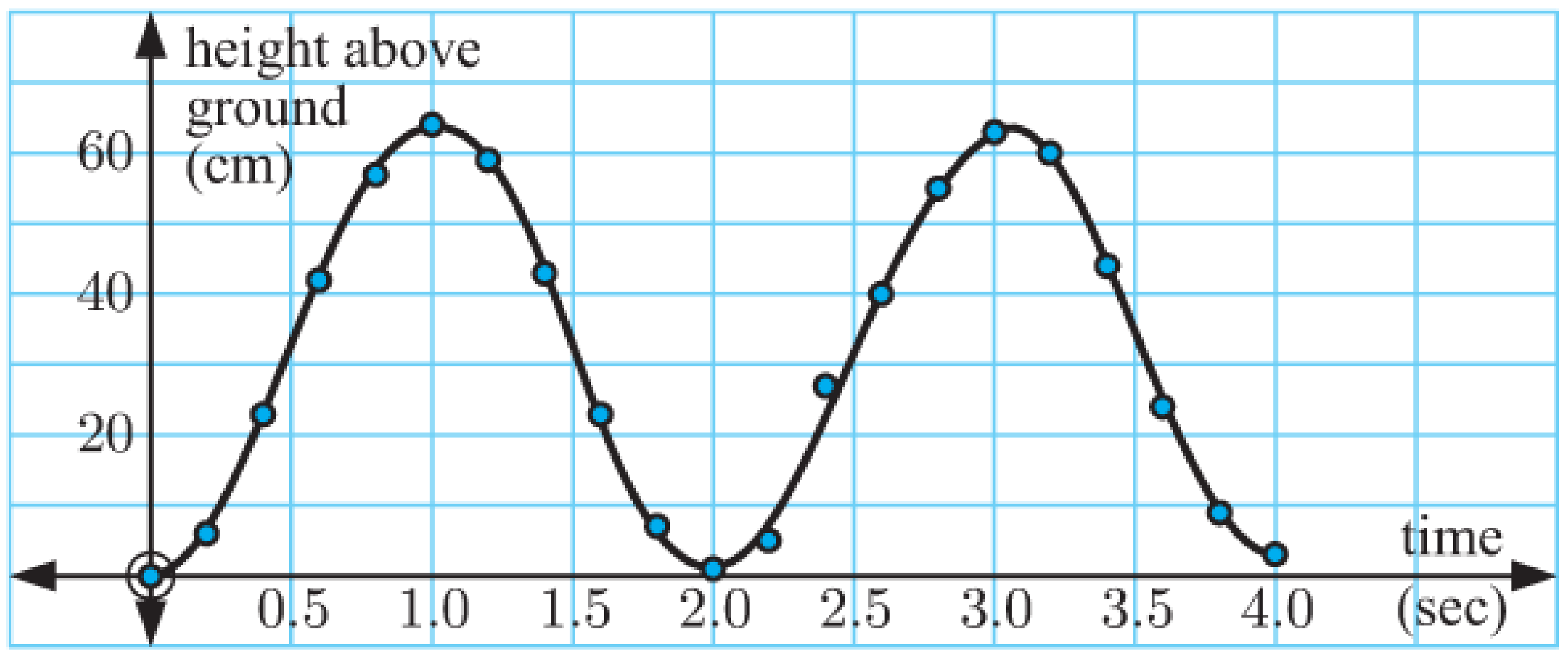
Data exhibits periodic behaviour.

b



Not enough information to say data is periodic.

3 **a**



b A curve can be fitted to the data.

c Yes, the data is periodic.

i $y = 32$ (approximately) **ii** ≈ 64 cm

iii ≈ 2 seconds **iv** ≈ 32 cm

EXERCISE 9C

1 **a** 0

b **i** $\theta = 0°, 180°, 360°, 540°, 720°$

ii $\theta = 270°, 630°$ **iii** $\theta = 30°, 150°, 390°, 510°$

iv $\theta = 60°, 120°, 420°, 480°$

c $\theta \approx 17°, 163°, 377°, 523°$

d **i** $0° < \theta < 180°$, $360° < \theta < 540°$

ii $180° < \theta < 360°$, $540° < \theta < 720°$

e $\{y \mid -1 \leqslant y \leqslant 1\}$

2 **a** 1

b **i** $\theta = 90°, 270°, 450°, 630°$ **ii** $\theta = 0°, 360°, 720°$

iii $\theta = 120°, 240°, 480°, 600°$

iv $\theta = 135°, 225°, 495°, 585°$

c $\theta \approx 73°, 287°, 433°, 647°$

d **i** $0° \leqslant \theta < 90°$, $270° < \theta < 450°$, $630° < \theta \leqslant 720°$

ii $90° < \theta < 270°$, $450° < \theta < 630°$

e $\{y \mid -1 \leqslant y \leqslant 1\}$

EXERCISE 9D

1 **a** 4 **b** 2 **c** $\frac{1}{3}$ **2** **a** 120° **b** 90° **c** 720°

3 **a** $y = -3$ **b** $y = 5$ **c** $y = 0$

4 **a** $b = 4$ **b** $b = 15$ **c** $b = \frac{1}{4}$

5 **a** maximum 4, minimum -4

b maximum 8, minimum 2

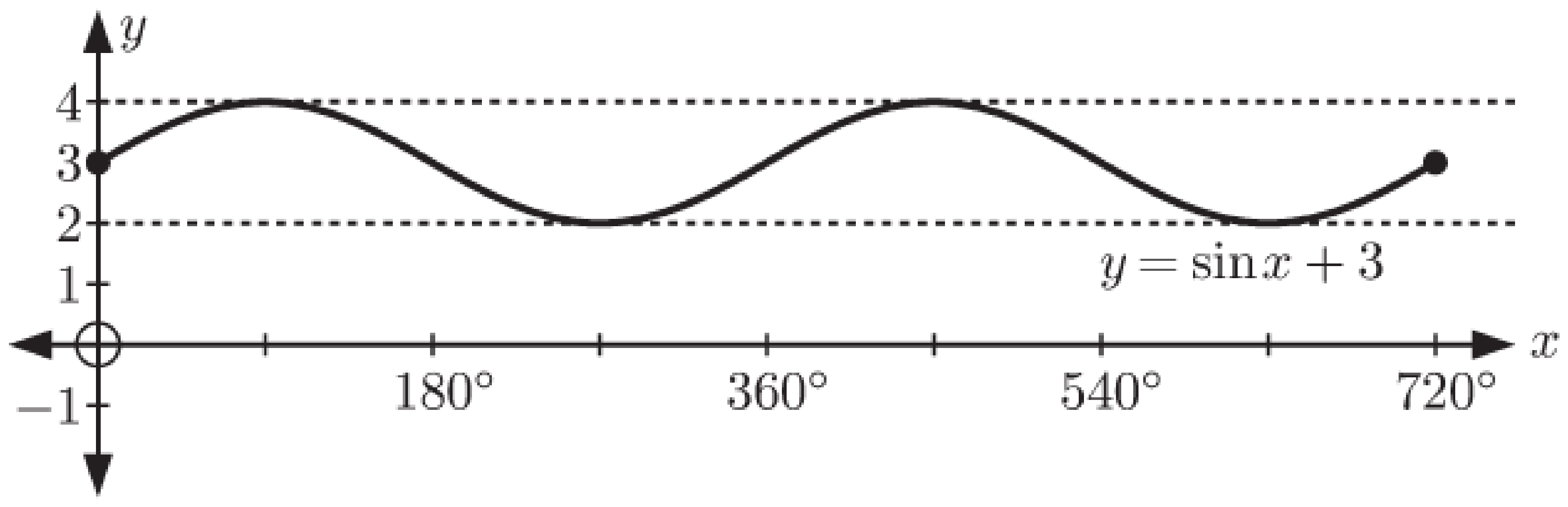
c maximum -2, minimum -6

6 **a** A translation 1 unit downwards.

b A vertical stretch with scale factor 2.

c A horizontal stretch with scale factor $\frac{1}{4}$.

7 **a**



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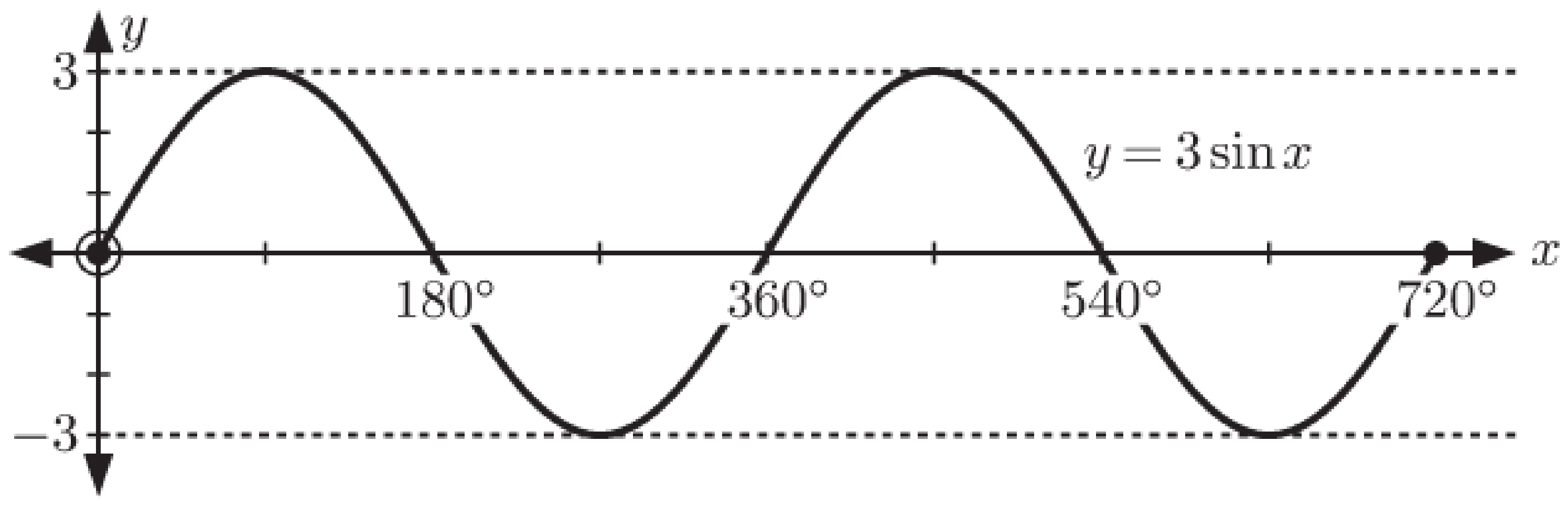
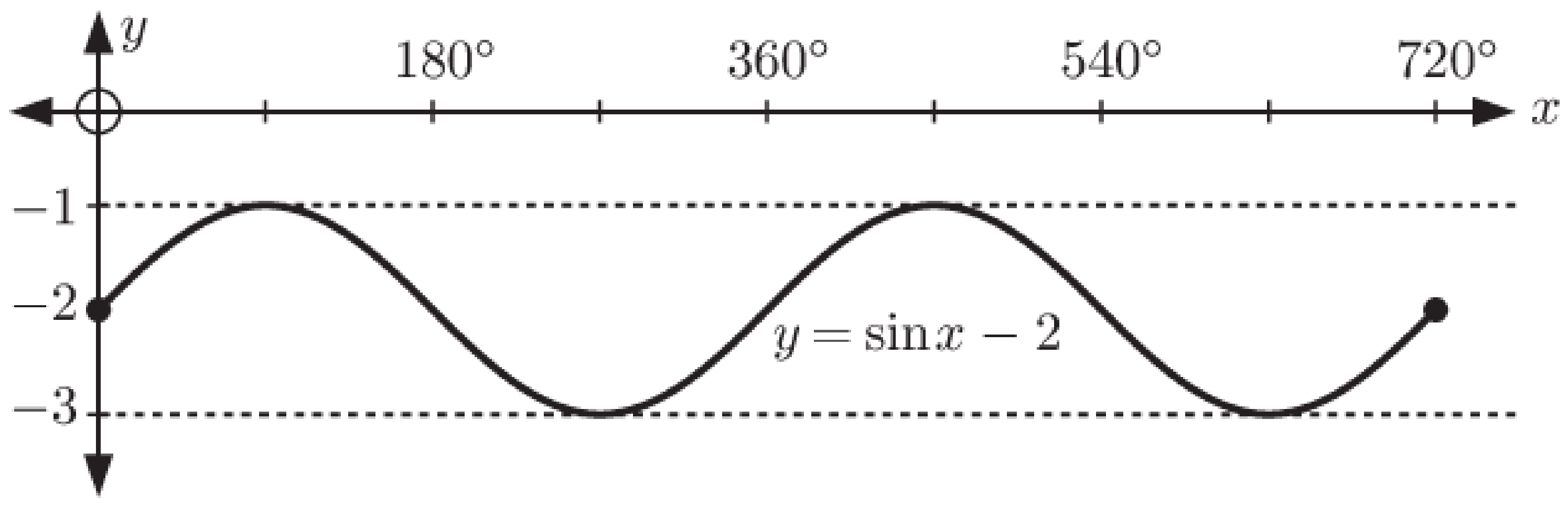


c



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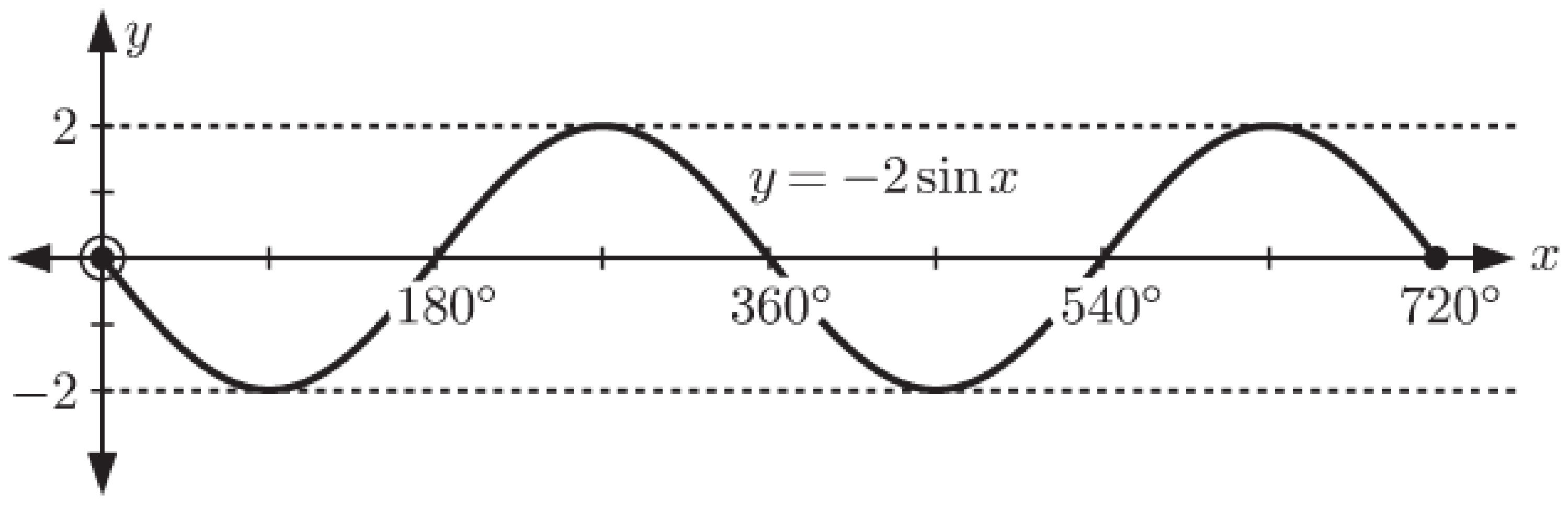

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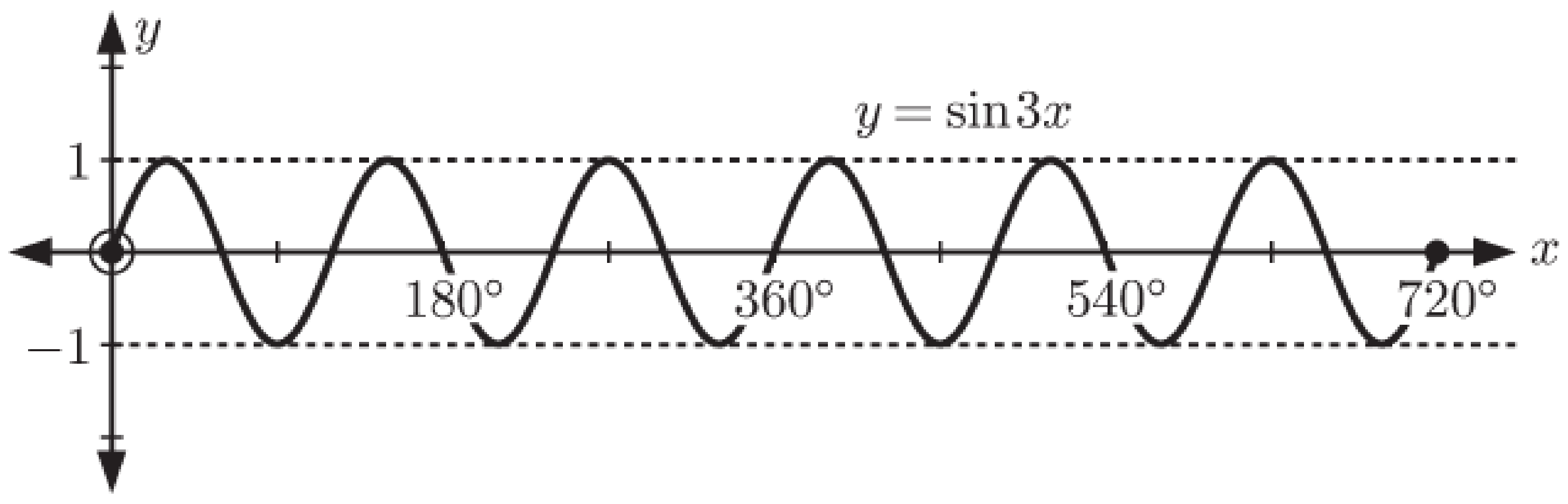

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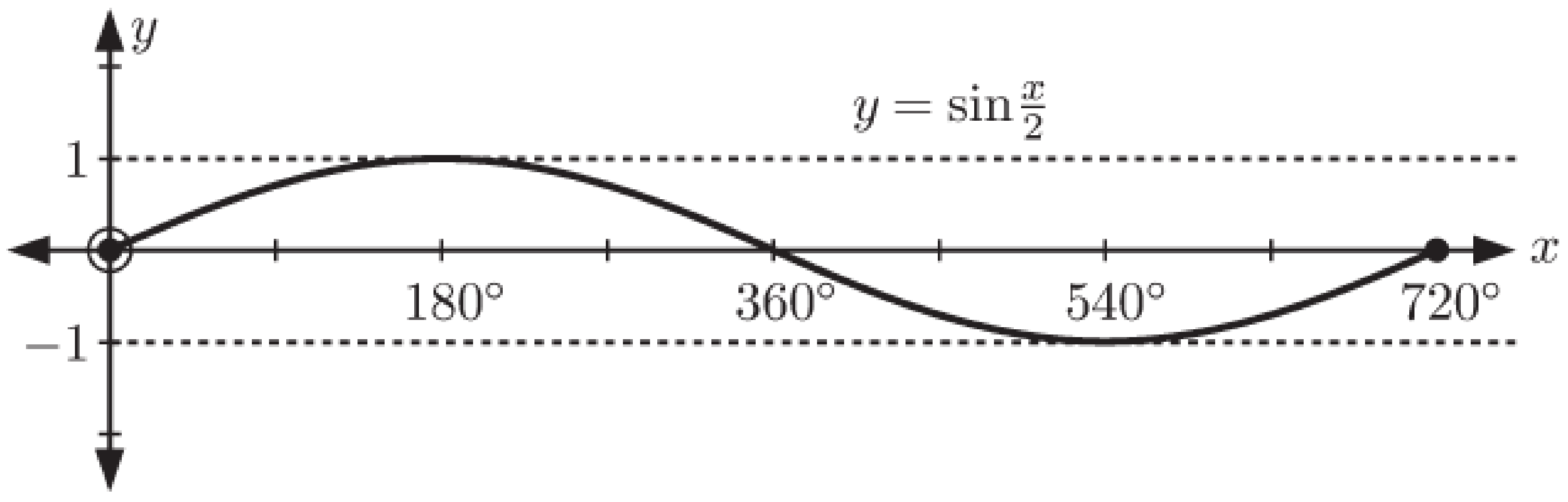
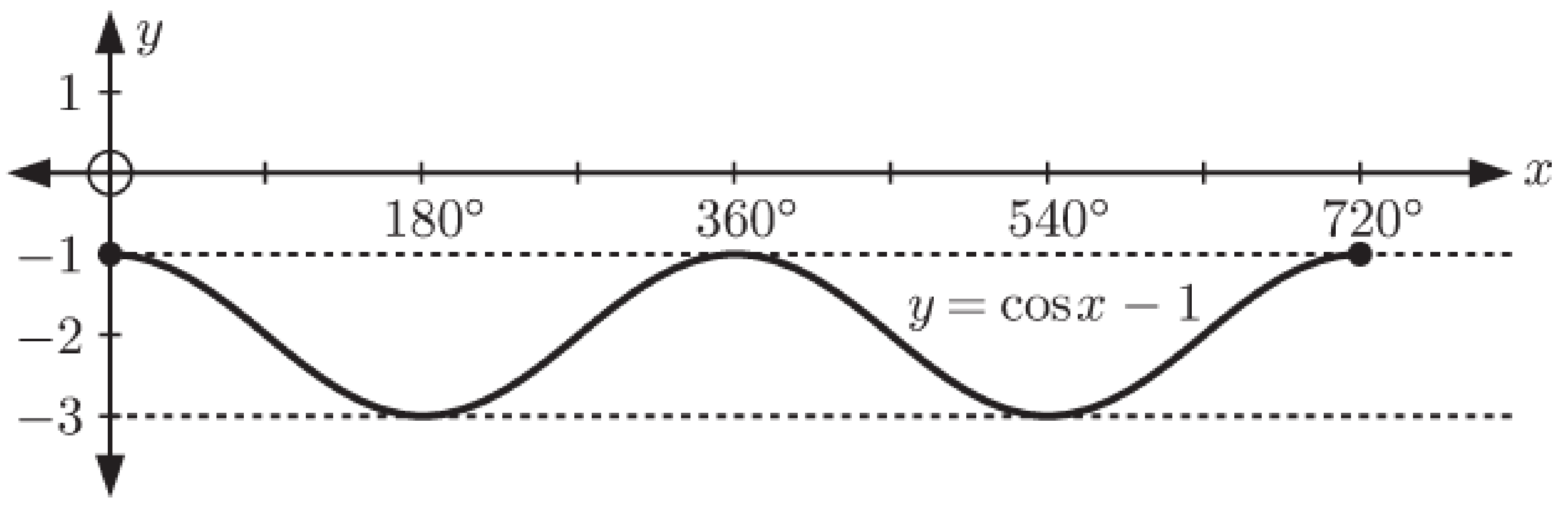


8 **a** A vertical stretch with scale factor $\frac{1}{2}$.

b A reflection in the x-axis.

c A translation 3 units upwards.

9 **a**



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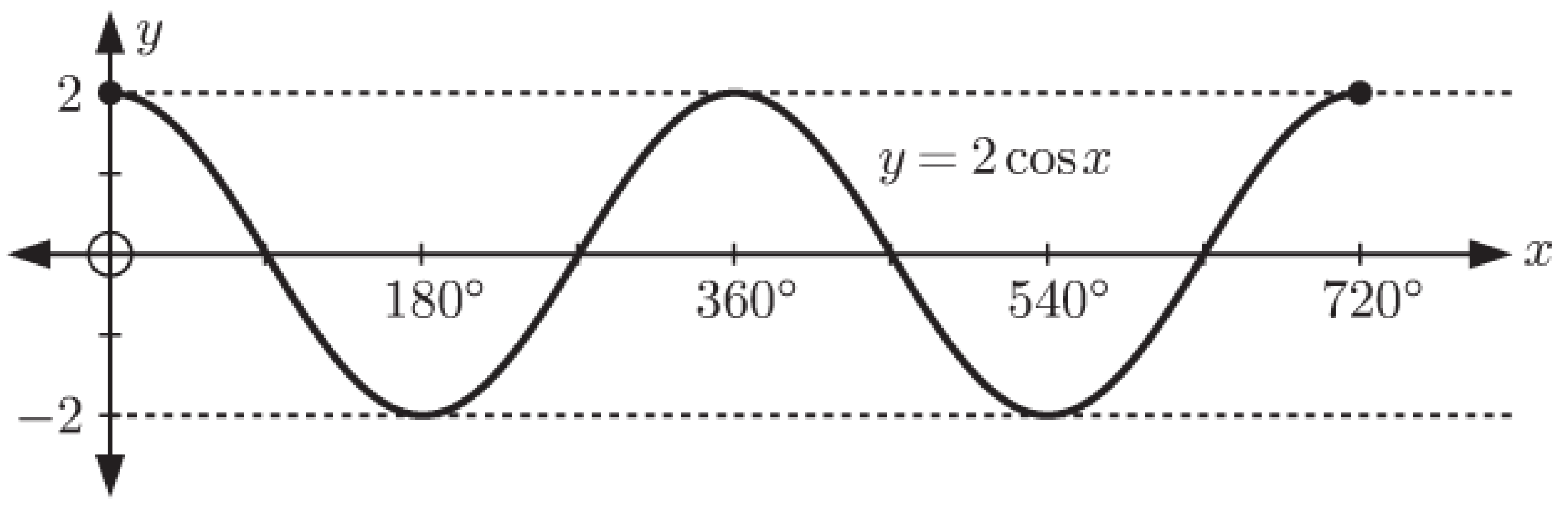

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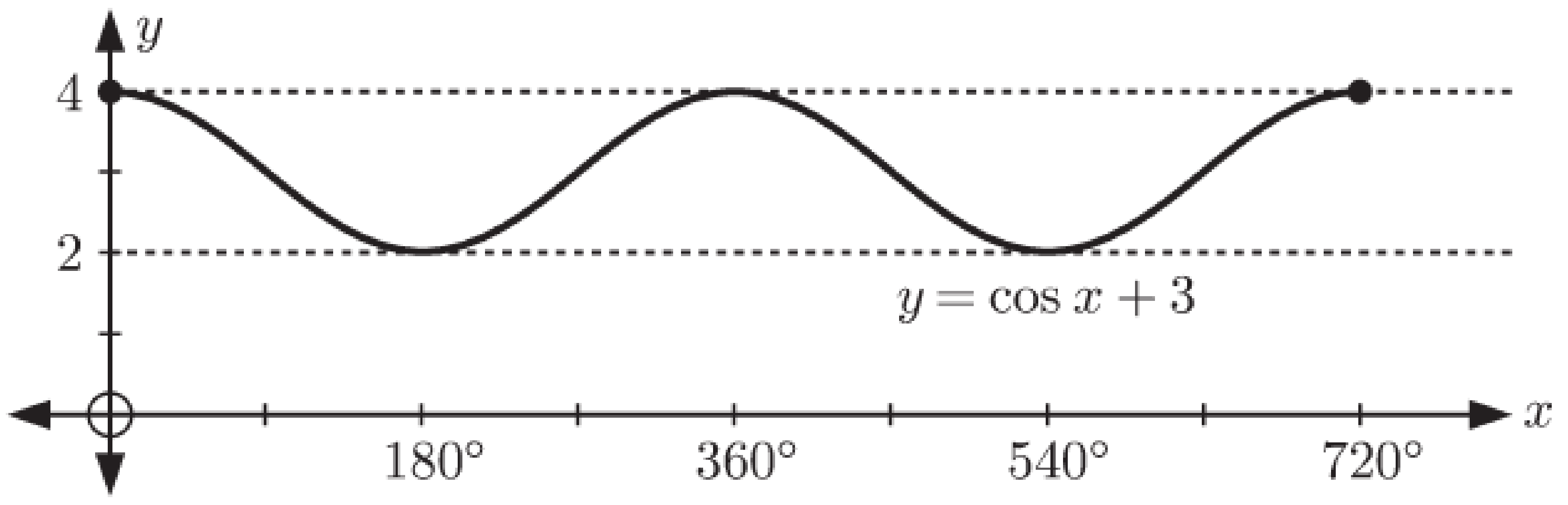

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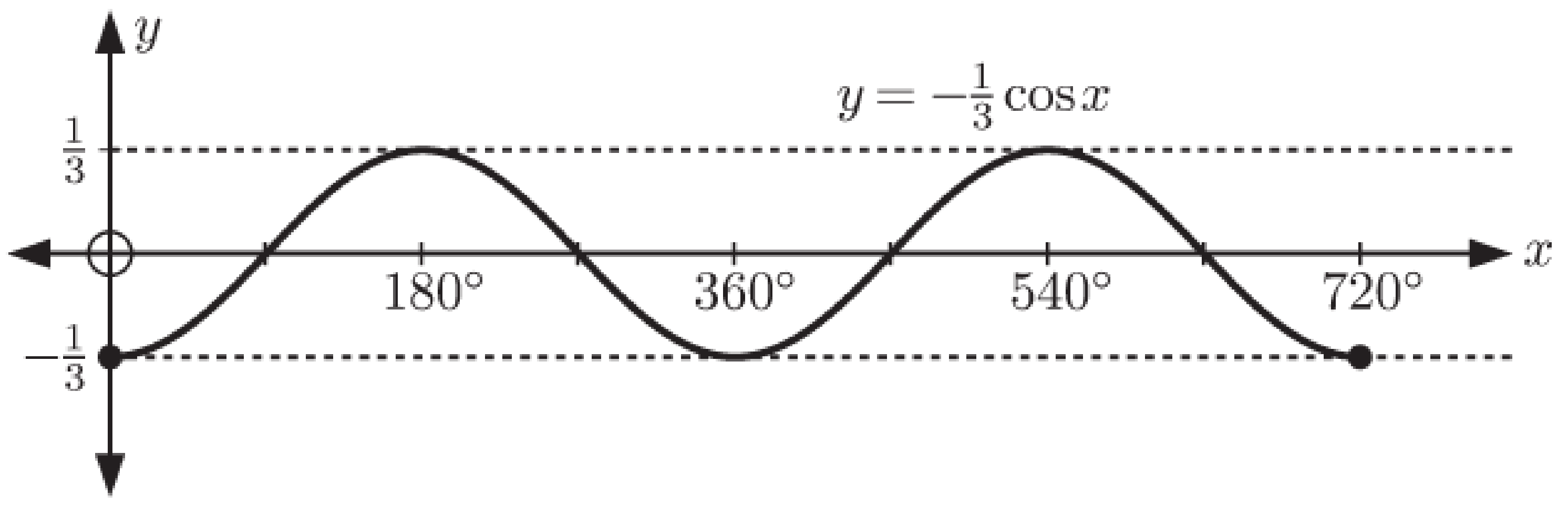

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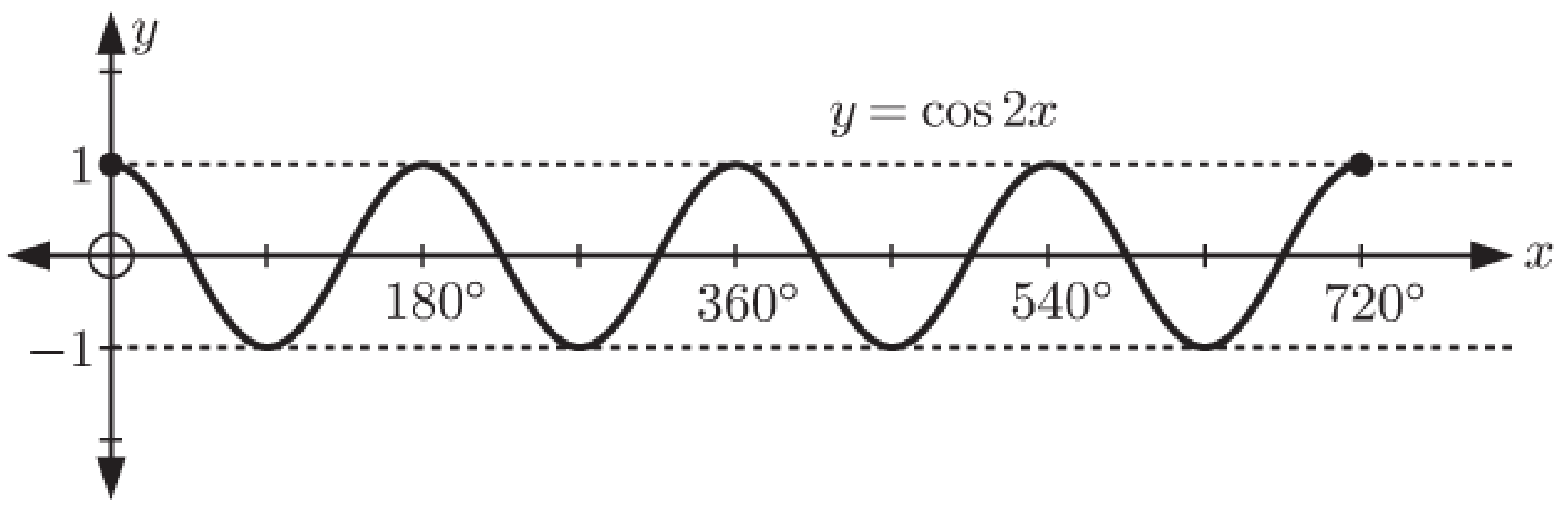

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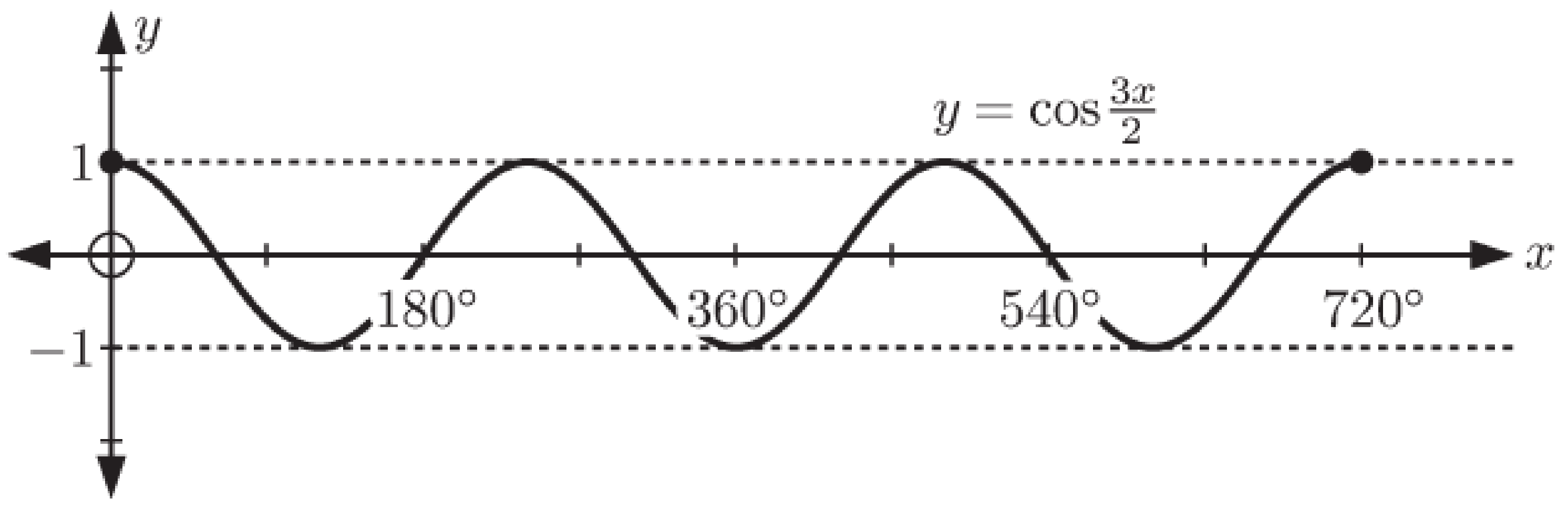
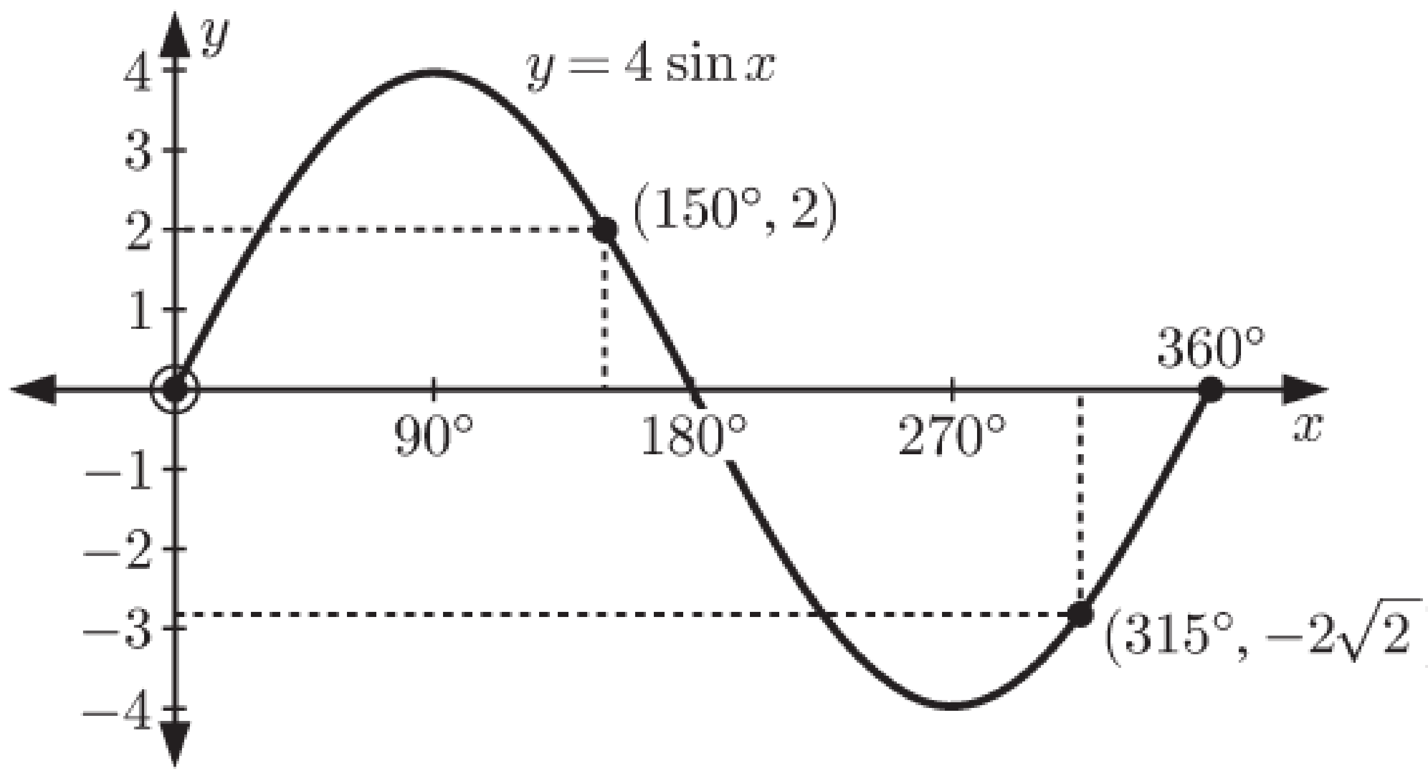


10 **a**



b **i** $y = 2$ **ii** $y = -2\sqrt{2} \approx -2.83$

11 **a** $d > 3$ **b** $d < -3$ **c** $-3 < d < 3$

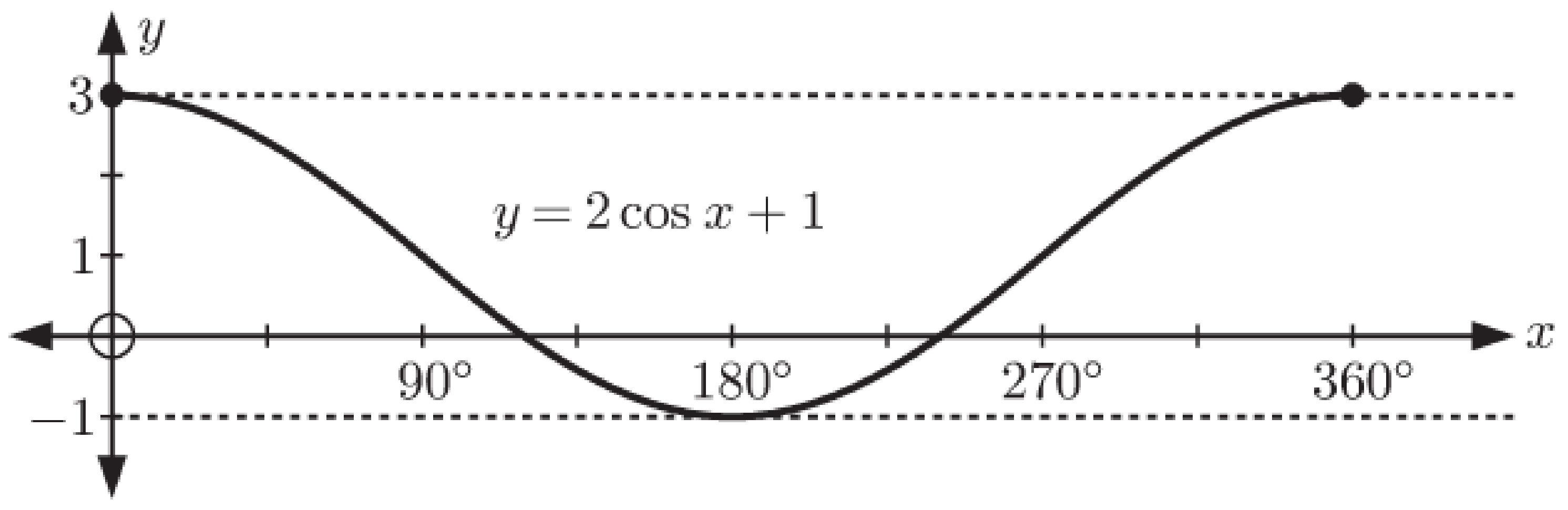
12 **a** 4 **b** $\frac{2\pi}{3}$ **c** $\{y \mid -2 \leqslant y \leqslant 6\}$

13 **a** A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2.

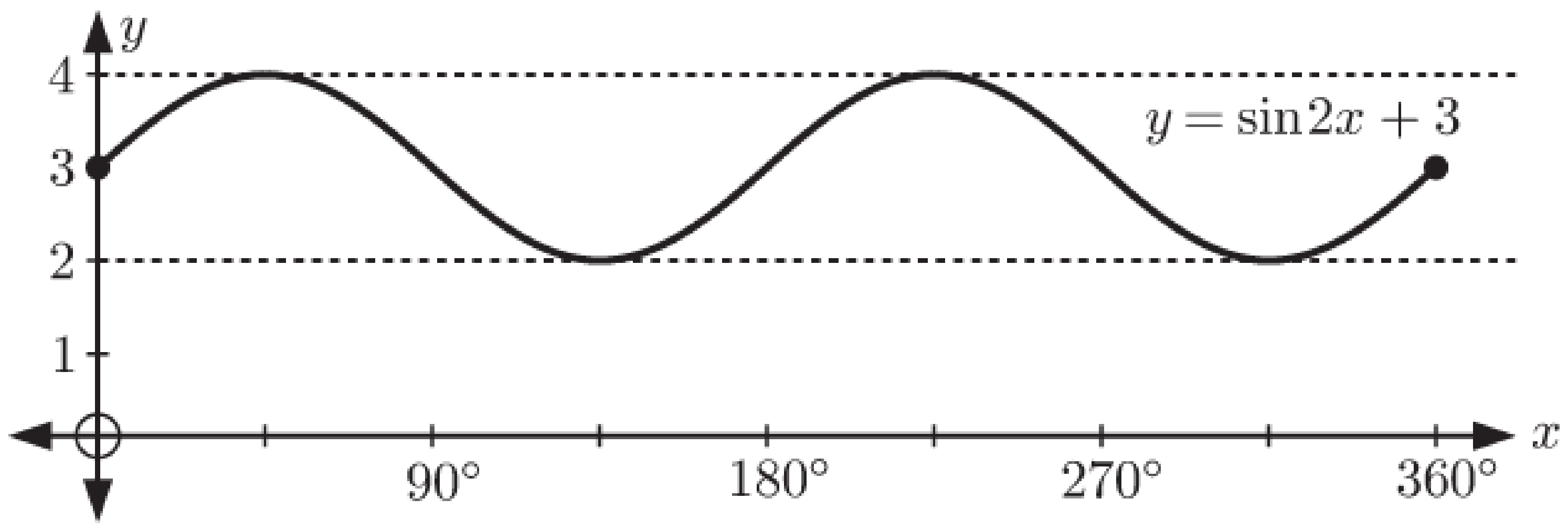
b A vertical stretch with scale factor 3, then a translation 5 units downwards.

c A reflection in the x-axis, then a vertical stretch with scale factor 2.

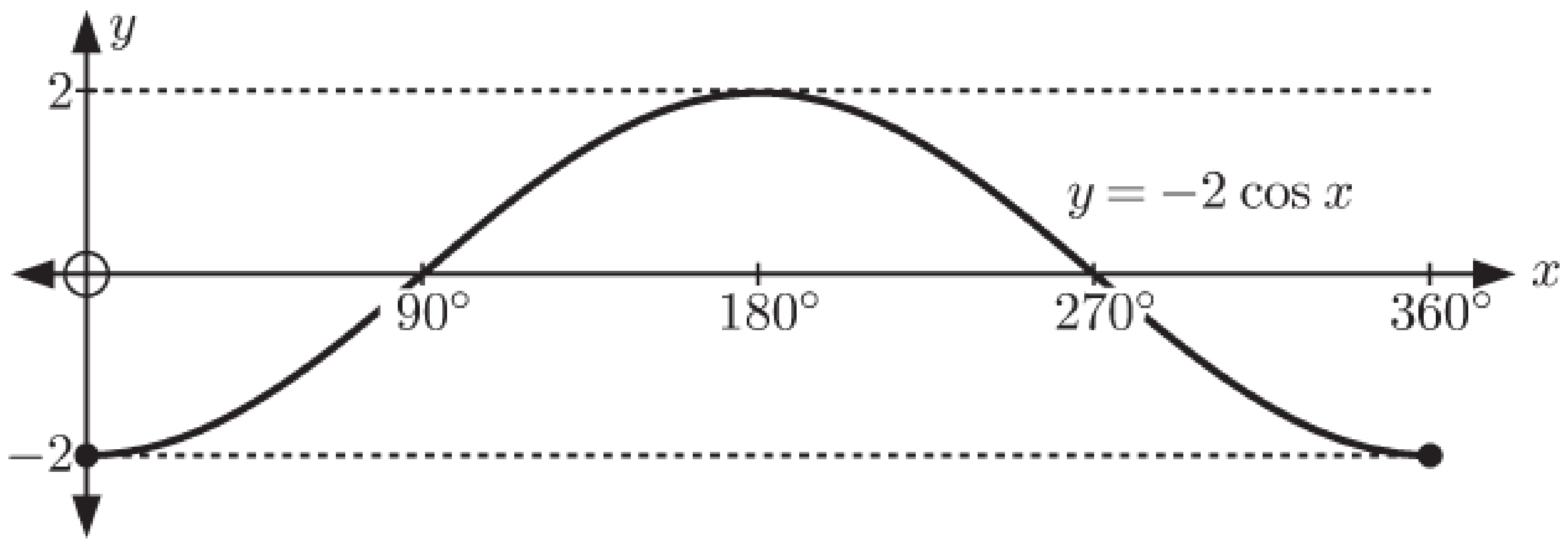
14 **a**



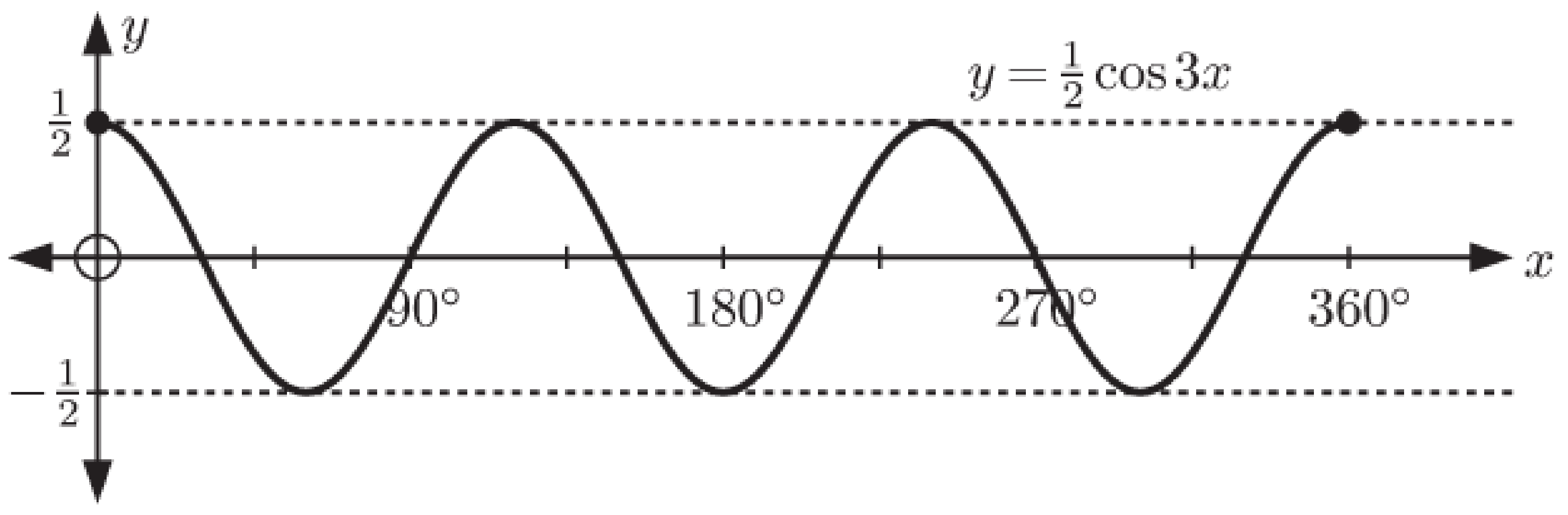
b



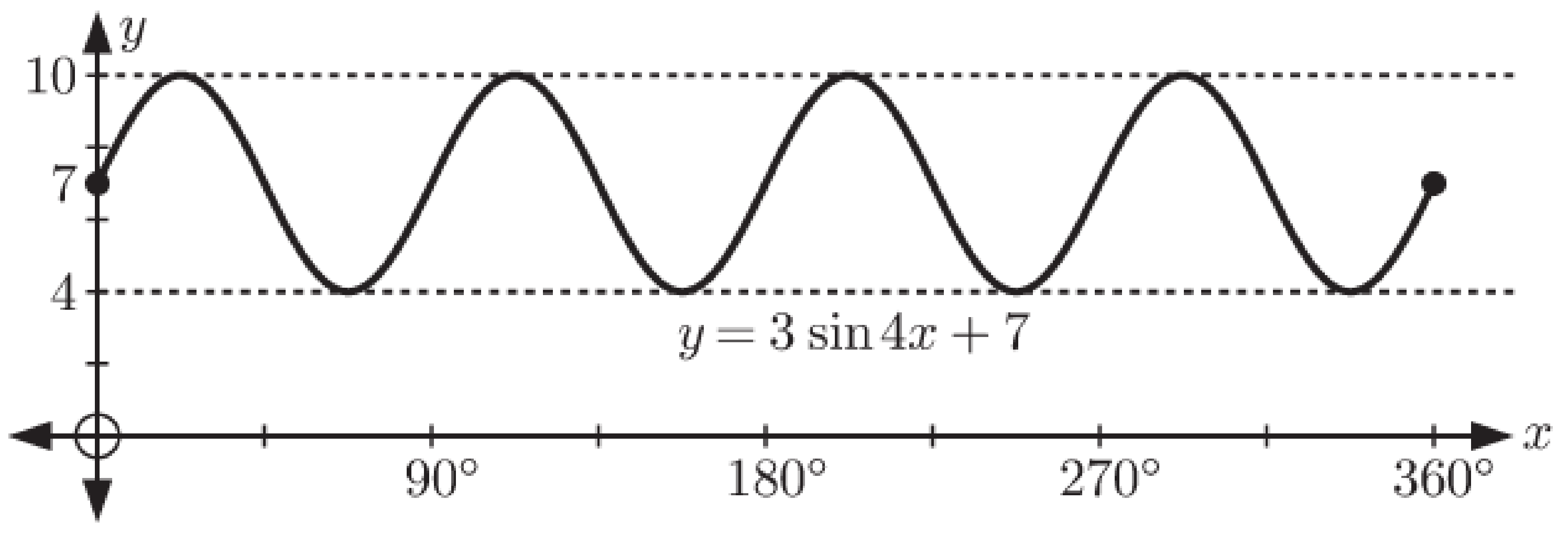
c



d



e



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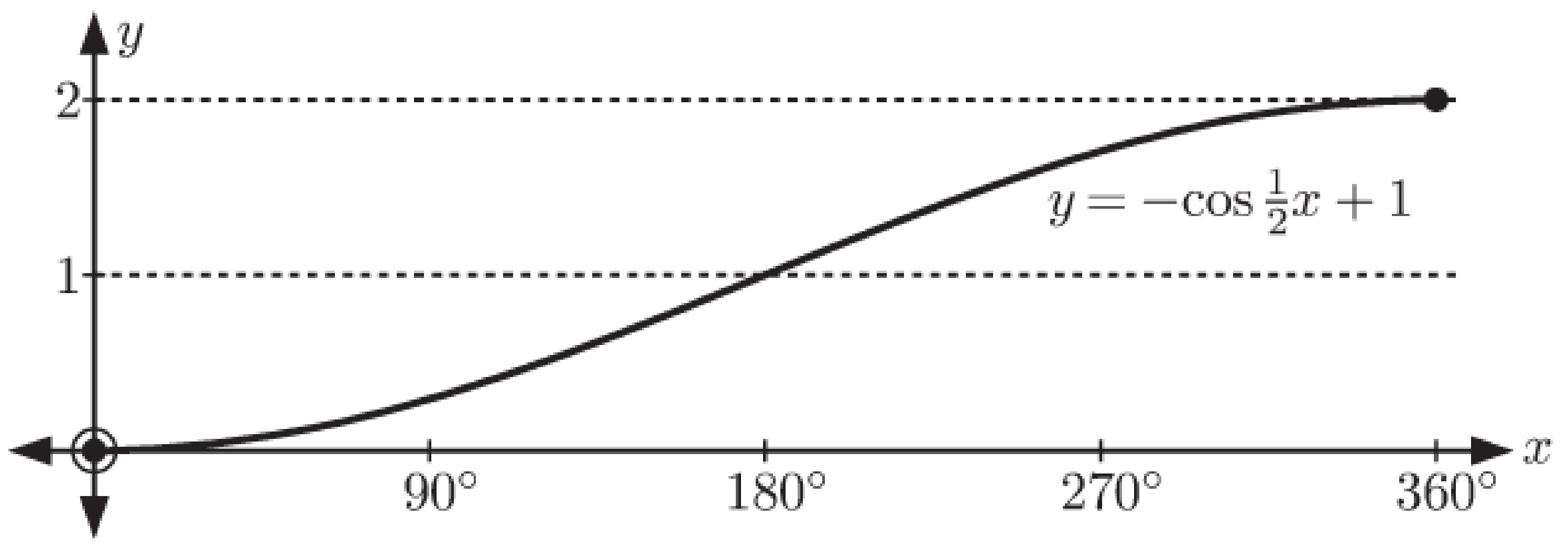
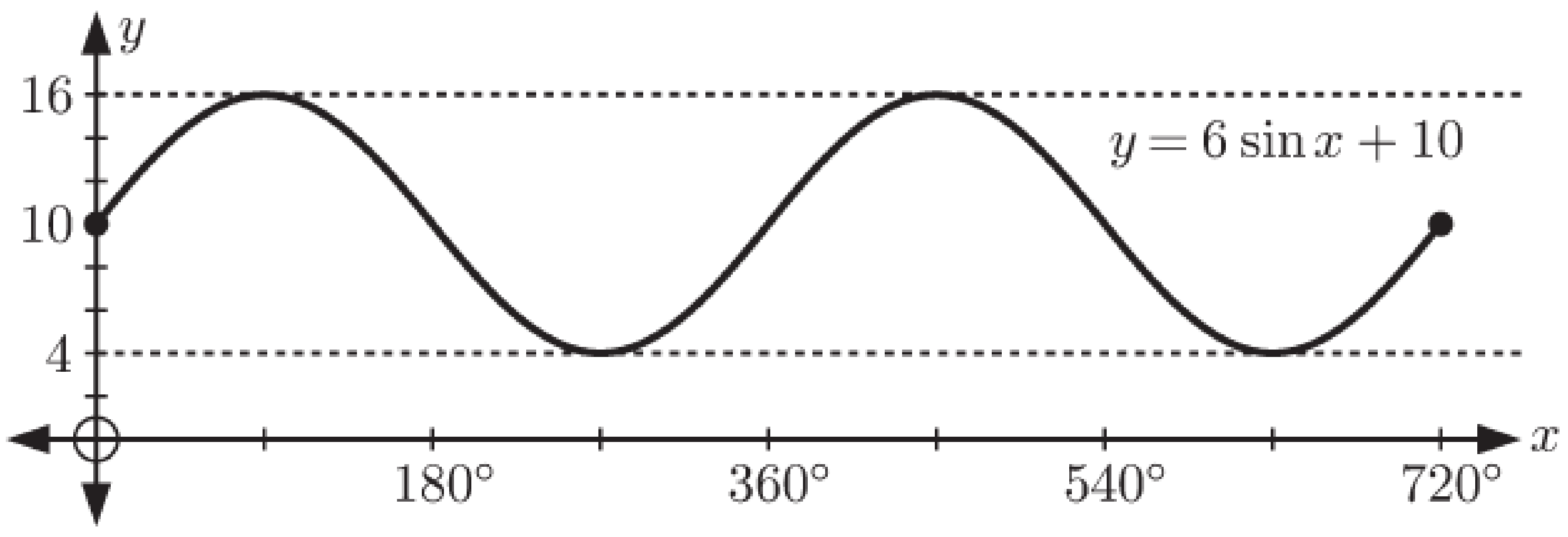


15 **a**



b $y = 13$ **c** max. value 16, when $x = 90°$ and $450°$

d minimum value = 4, when $x = 270°$ and $630°$

16 **a** $a = 4, \ d = 1$ **b** $a = 3, \ d = -1$

c $a = -2, \ d = 3$ **d** $a = \frac{1}{3}, \ d = \frac{4}{3}$

17 **a** $y = \sin x - 2$ **b** $y = \sin 3x$ **c** $y = -2\sin 2x$

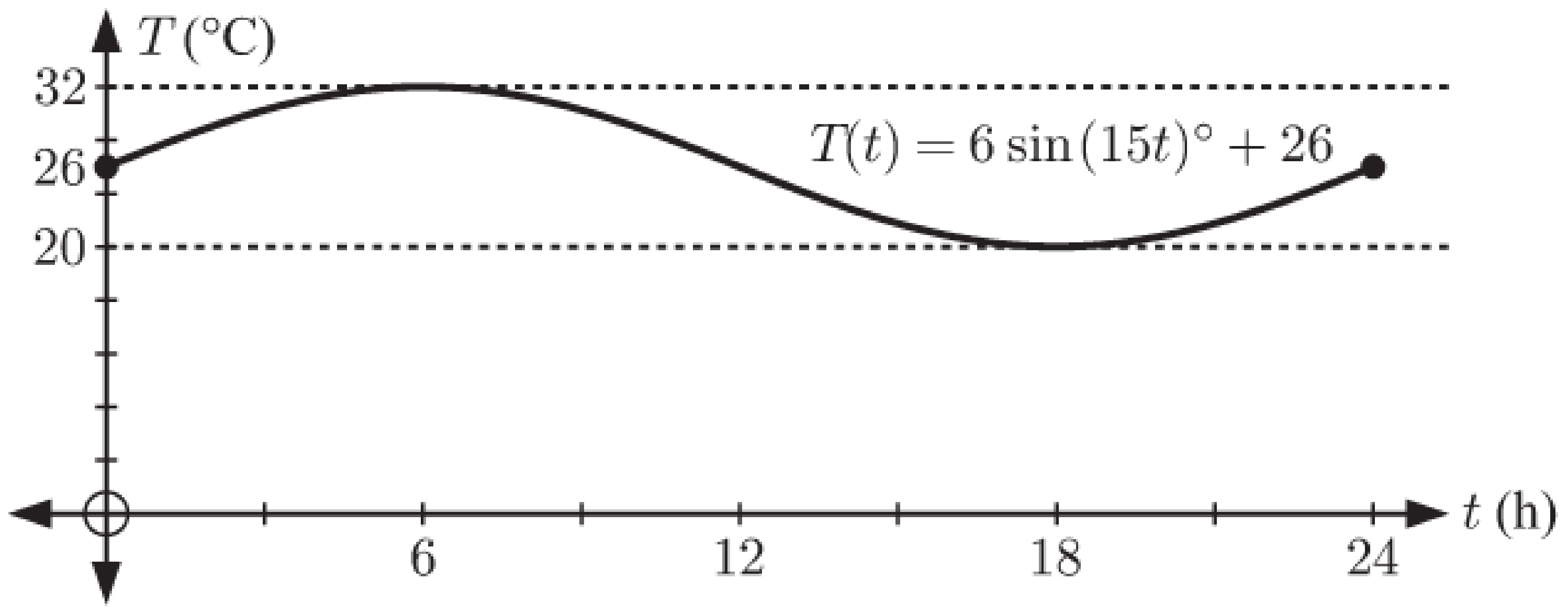
d $y = 2\sin x + 1$ **e** $y = 4\sin\frac{x}{2} - 1$ **f** $y = 6\sin 3x$

18 **a** $y = 2\cos 2x$ **b** $y = \cos\frac{x}{2} + 2$ **c** $y = -5\cos 4x$

d $y = -3\cos 3x + 3$

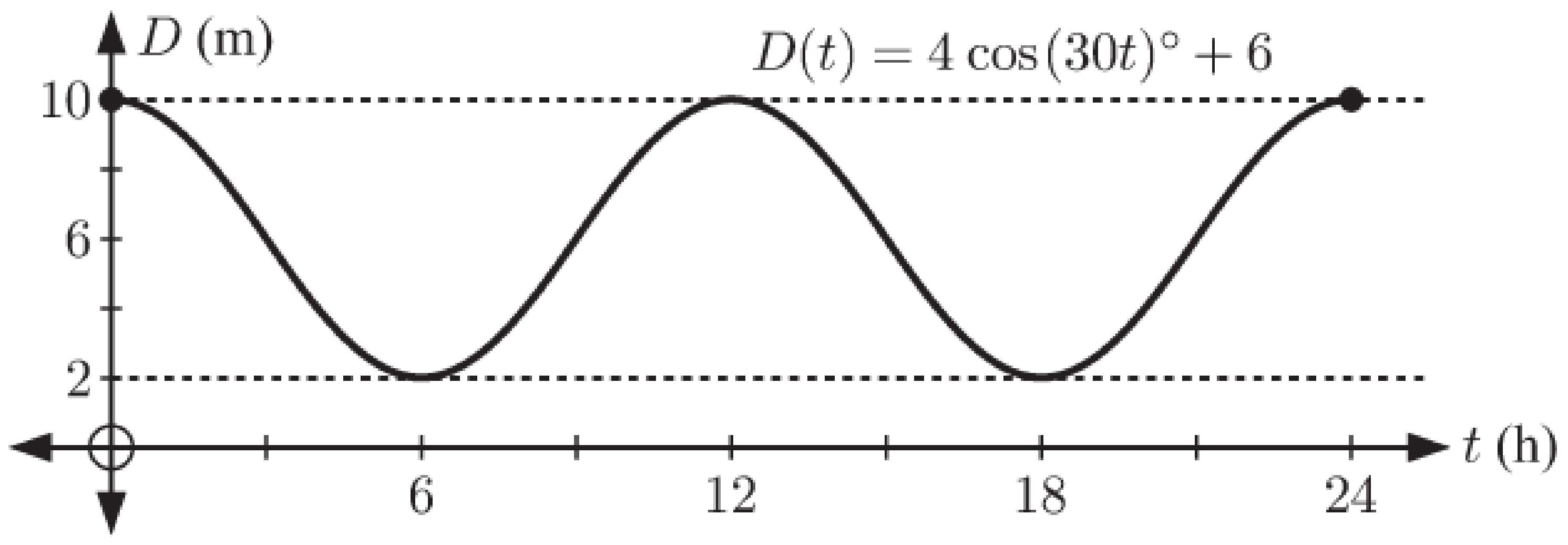
EXERCISE 9E

1 **a**



b **i** 26°C **ii** 29°C **c** 32°C, at 6 pm

2 **a**

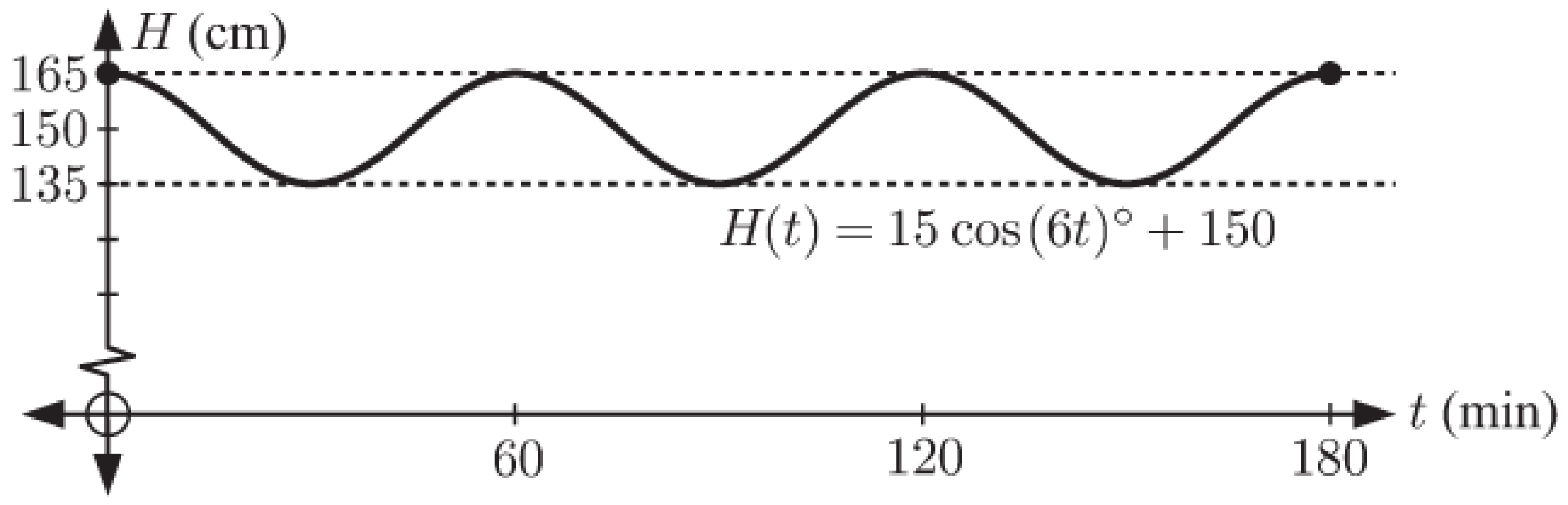


b highest = 10 m, at midnight, midday, and midnight the next day

lowest = 2 m, at 6 am and 6 pm

c no (water height is 4 m)

3 **a**

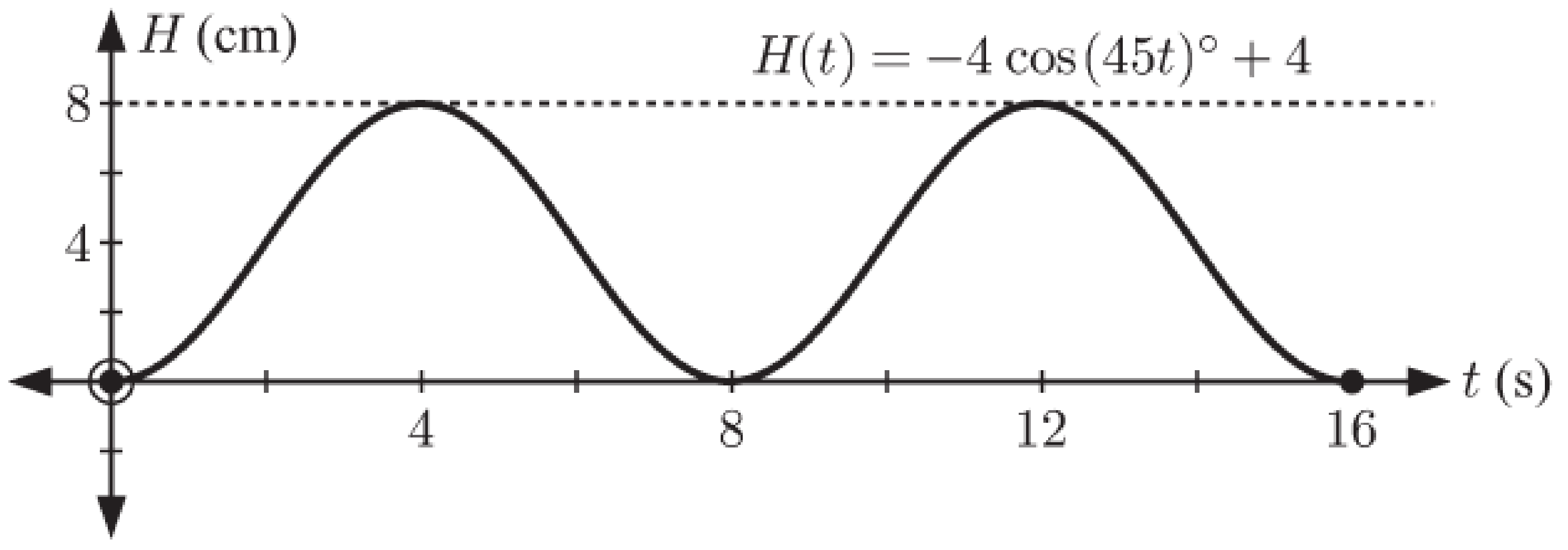


b 15 cm

c **i** ≈ 160.0 cm **ii** ≈ 138.9 cm **iii** ≈ 158.8 cm

iv ≈ 138.9 cm

4 **a**



b 4 cm

c no (ball diameter is 4.28 cm, gate height ≈ 3.07 cm)

5 **a** period $= \dfrac{360}{b} = 12$, $\therefore$ $b = \frac{360}{12} = 30$

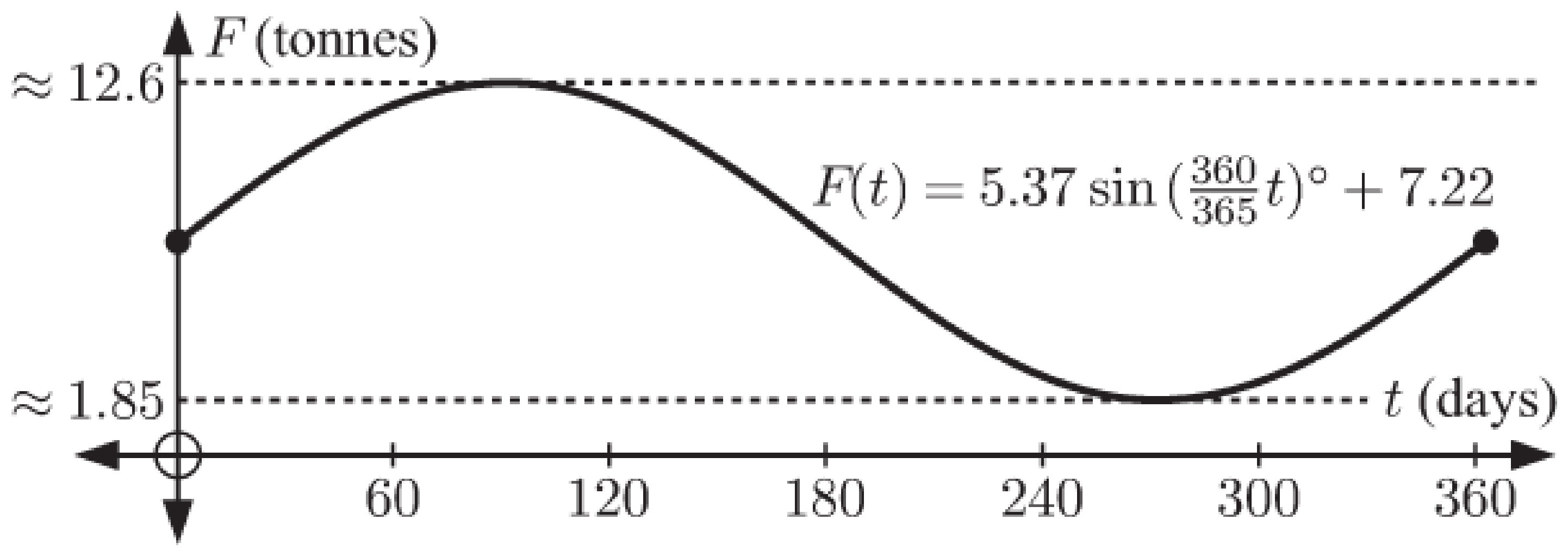
b $a = 4.5$, $d = 7.5$ **c** 5 units

6 **a** $d = 120$ **b** every 24 days **c** $a = 18$

d 102 cents per litre **e** ≈ 103 cents per litre

7 **a** $b = \frac{360}{365} \approx 0.986$ **b** $a \approx 5.37$, $d \approx 7.22$

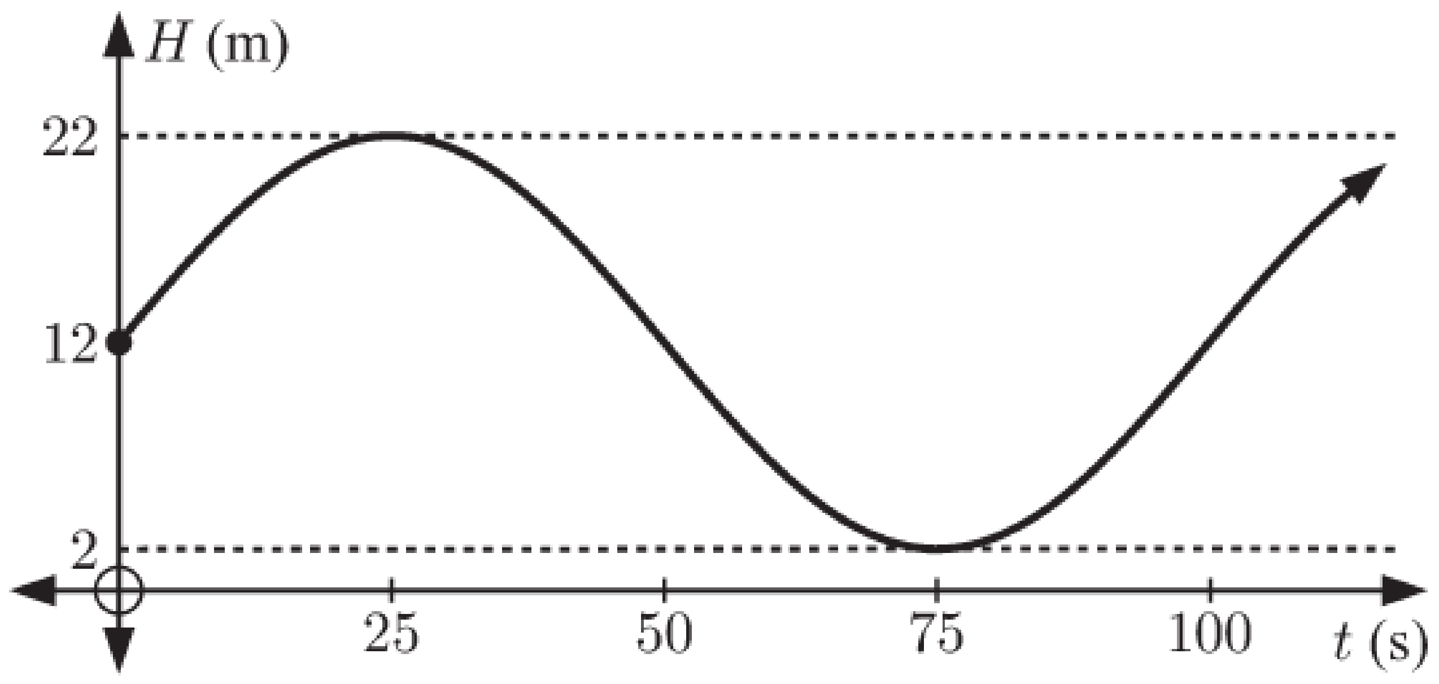
c



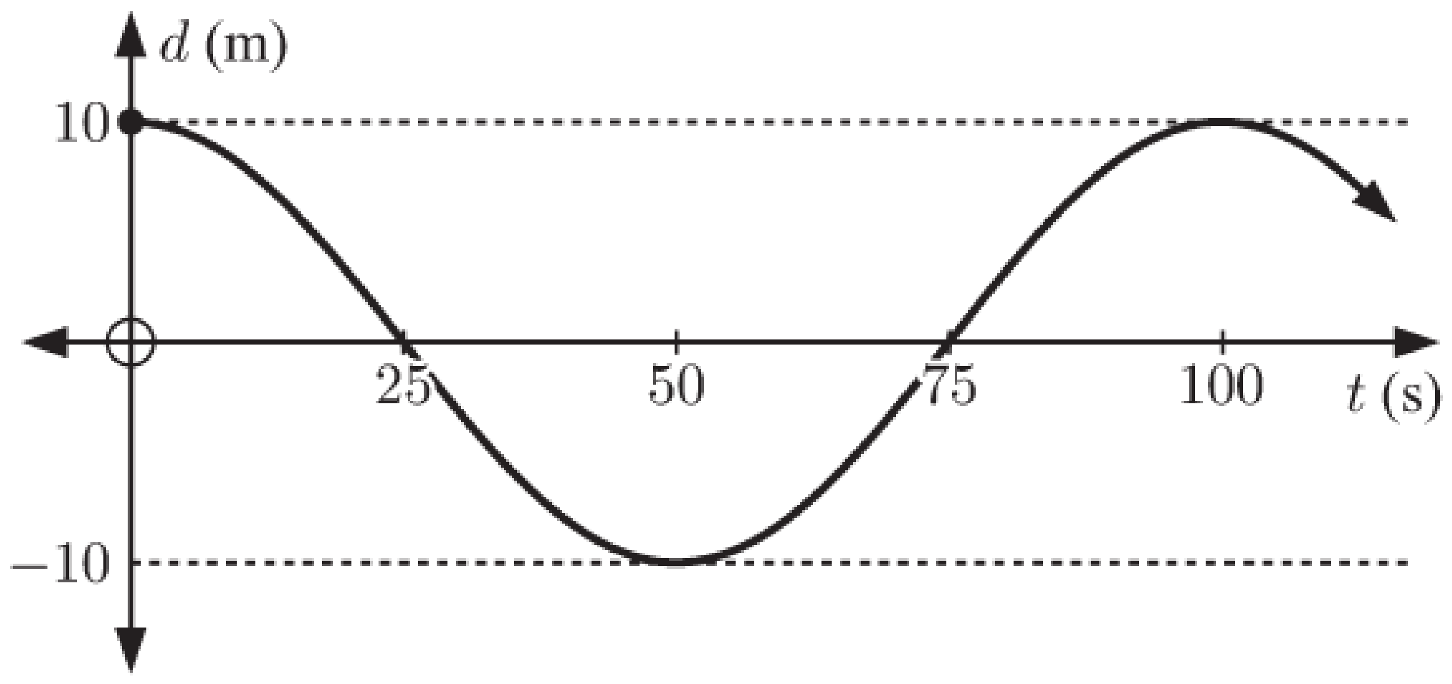
d ≈ 2.22 tonnes

e maximum ≈ 12.6 tonnes, minimum ≈ 1.85 tonnes

8 **a**



b



c Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graphs differ by a horizontal translation of 25 s and the principal axis is also translated by 12 m.

d **i** $H = 10\sin(\frac{18}{5}t)^\circ + 12$ m **ii** $D = 10\cos(\frac{18}{5}t)^\circ$ m

Note: The function of horizontal displacement of the light will be different depending on how the coordinate system is defined.

9 **a** $H(t) = 3\sin(90t)^\circ + 4$ m **b** below the water

10 **a** $h = 60\cos(720t)^\circ + 70$ cm **b** ≈ 88.5 cm

REVIEW SET 9A

1 M(0.292, 0.956), N(−0.985, −0.174), P(0.602, −0.799)

2 **a** not periodic **b** periodic **3** **a** 5 **b** $\frac{1}{4}$

4 **a** 1800° **b** 90° **c** 720° **d** 120°

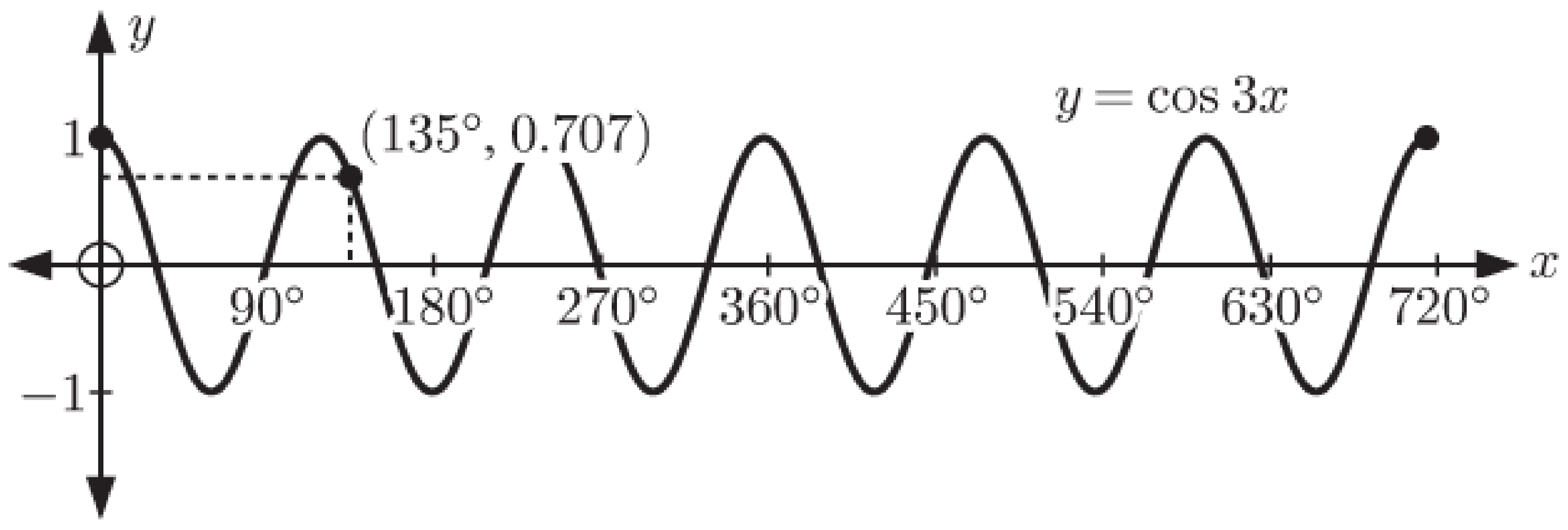
5 **a** $x \approx 115°, 245°, 475°$, or $605°$

b $x \approx 25°, 335°, 385°, 695°$, or $745°$

6

Function	*Period*	*Amplitude*	*Range*
$y = -3\sin\frac{x}{4} + 1$	1440°	3	$-2 \leqslant y \leqslant 4$
$y = 2\cos 5x - 7$	72°	2	$-9 \leqslant y \leqslant -5$

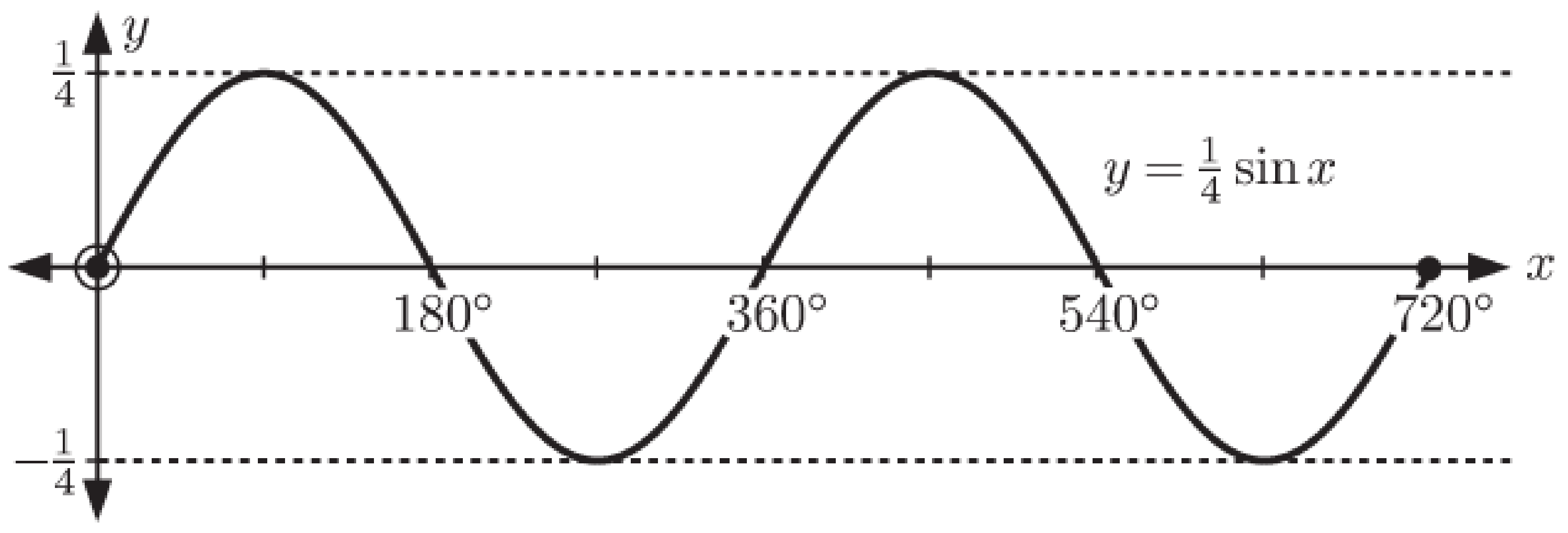
7 **a**



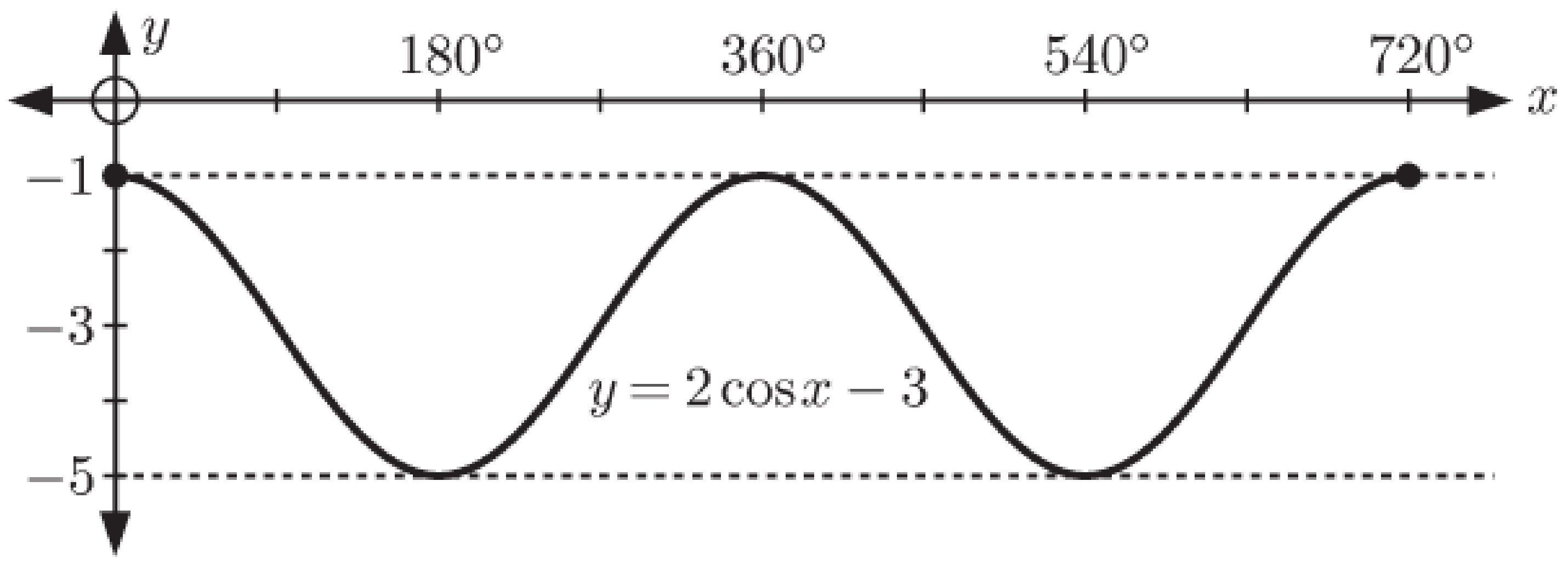
b $y = \frac{1}{\sqrt{2}} \approx 0.707$

8 A horizontal stretch with scale factor $\frac{1}{2}$, then a vertical stretch with scale factor 3.

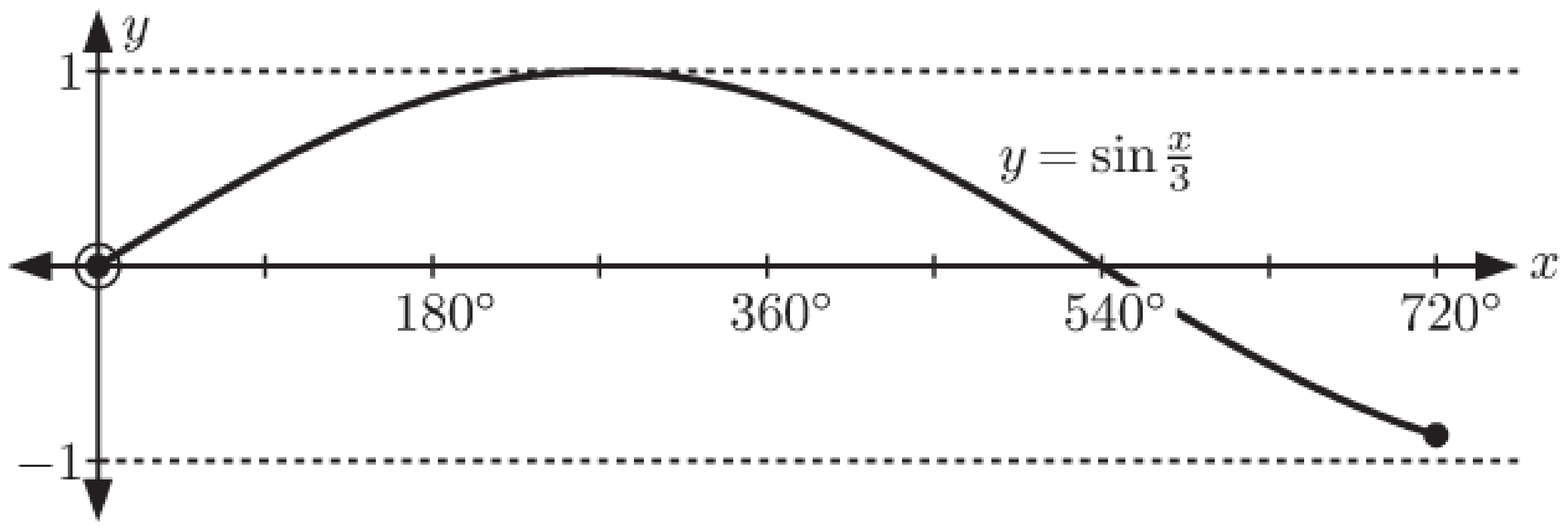
9 **a**



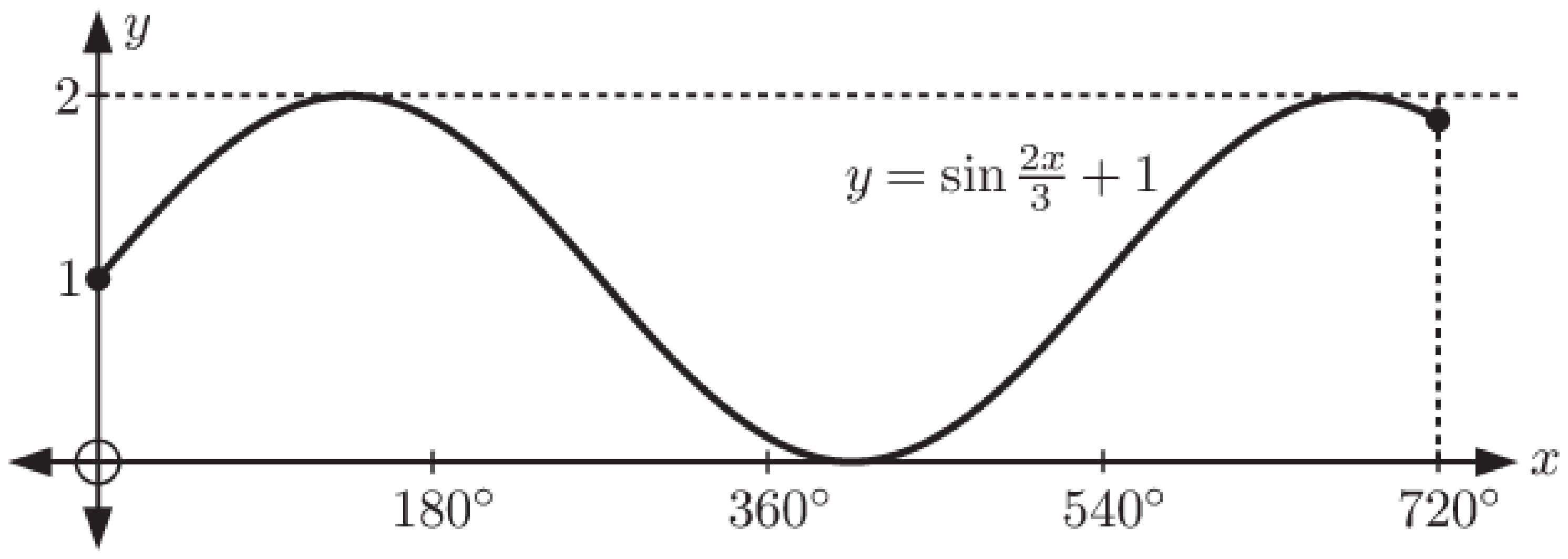
b



c



d



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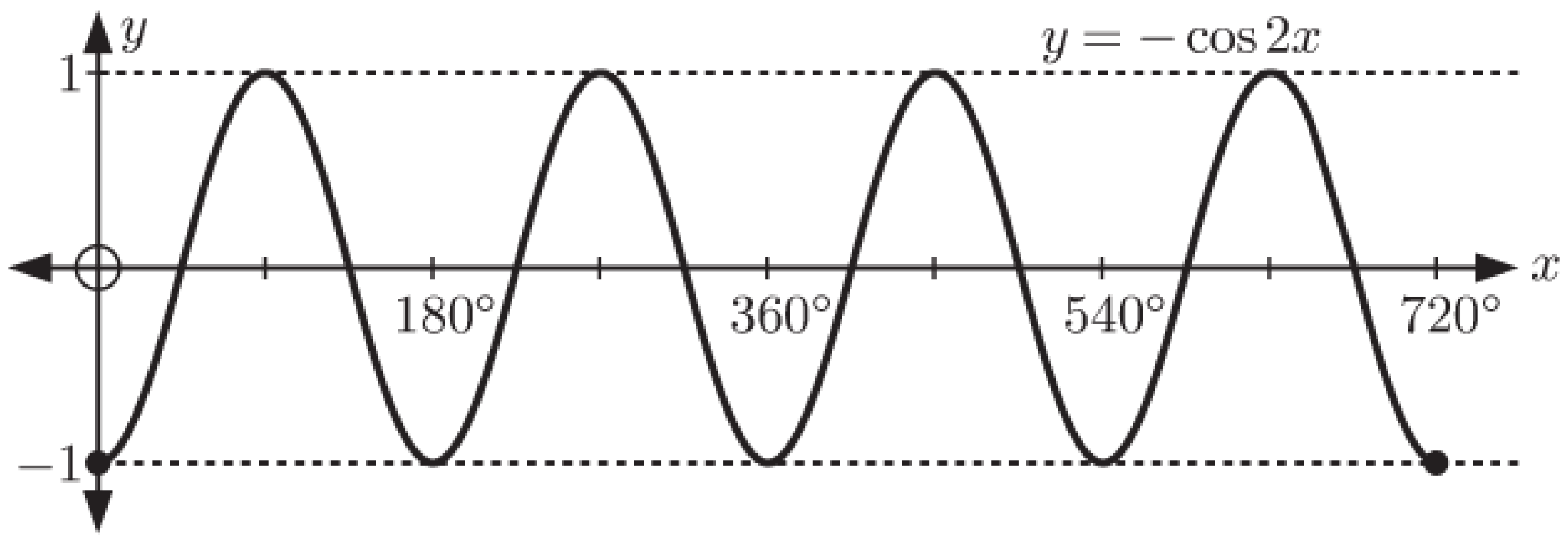

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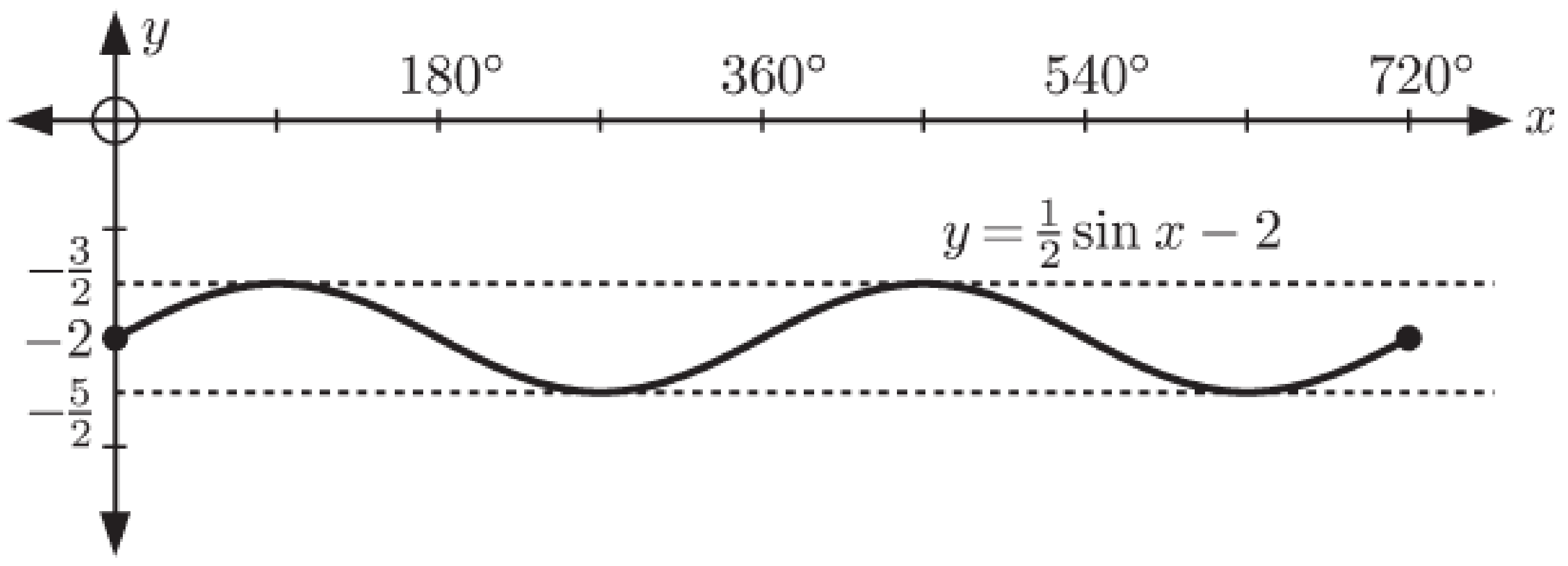
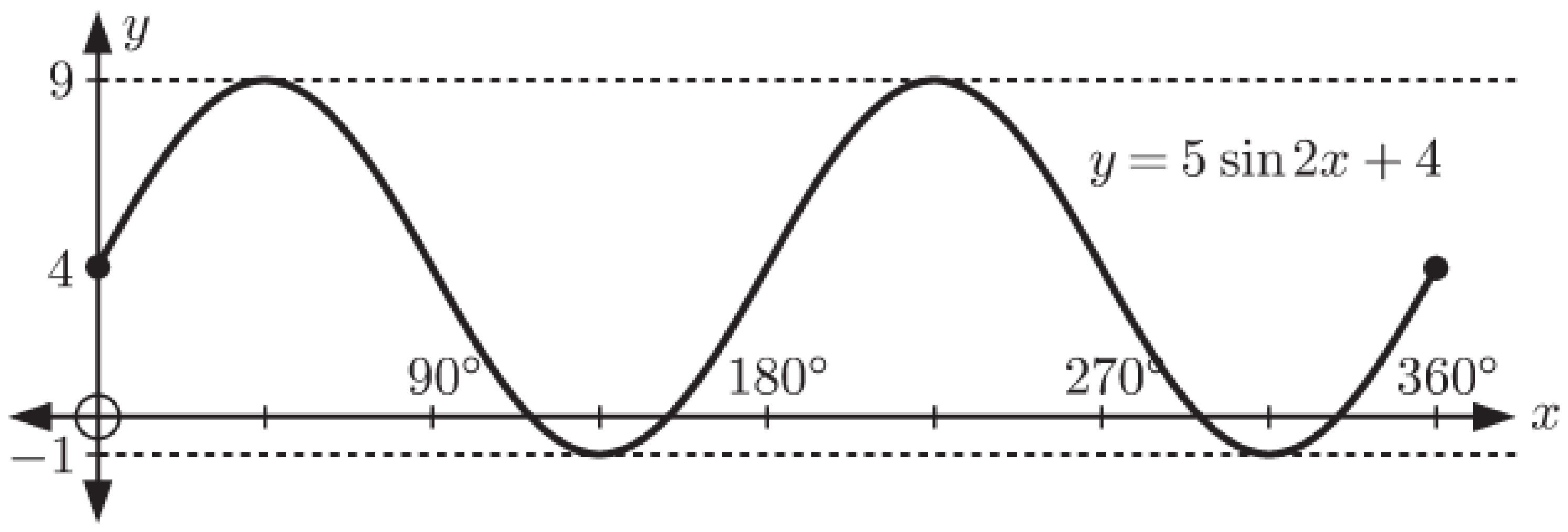


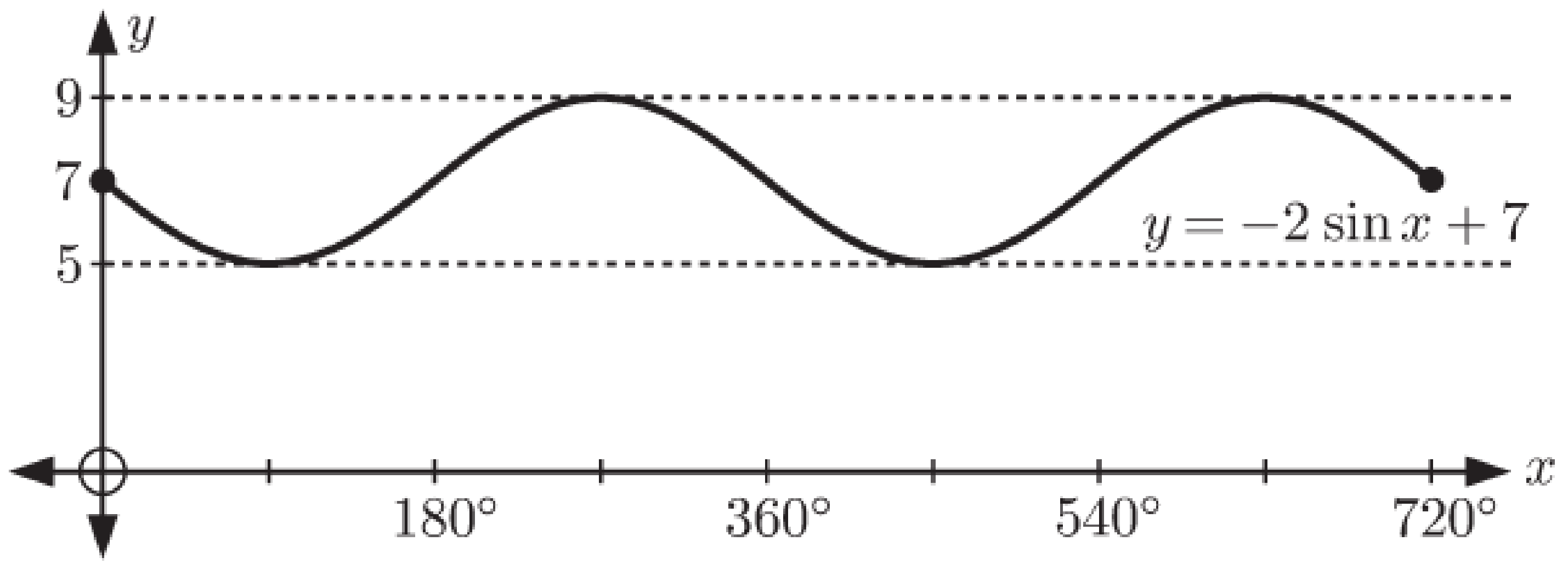
10 **a** 5 **b** $y = 4$ **c** 180°

d



e $y = 9$

11 **a**



b $y = 6$

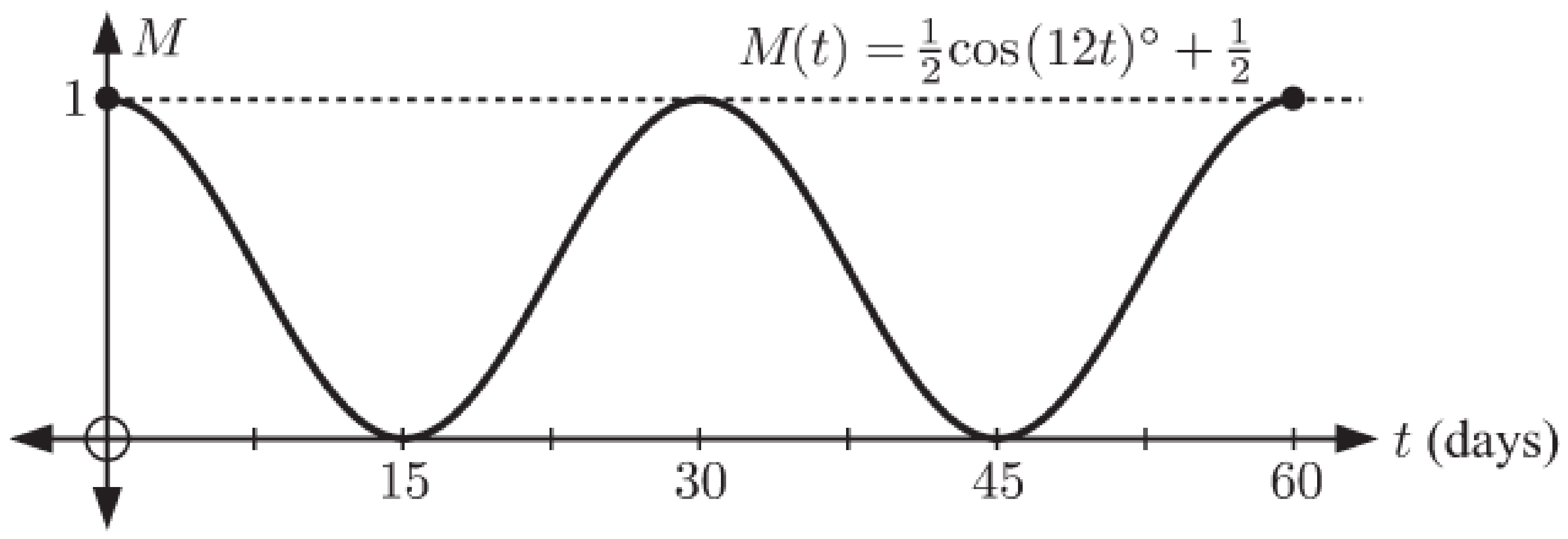
c maximum $= 9$, when $x = 270°$ and $630°$

d minimum $= 5$, when $x = 90°$ and $450°$

12 **a** $y = -4\cos 2x$ **b** $y = \cos\frac{3x}{2} + 2$

13 **a** $a = 8$, $d = 3$ **b** $\{y \mid -5 \leqslant y \leqslant 11\}$

14 **a**

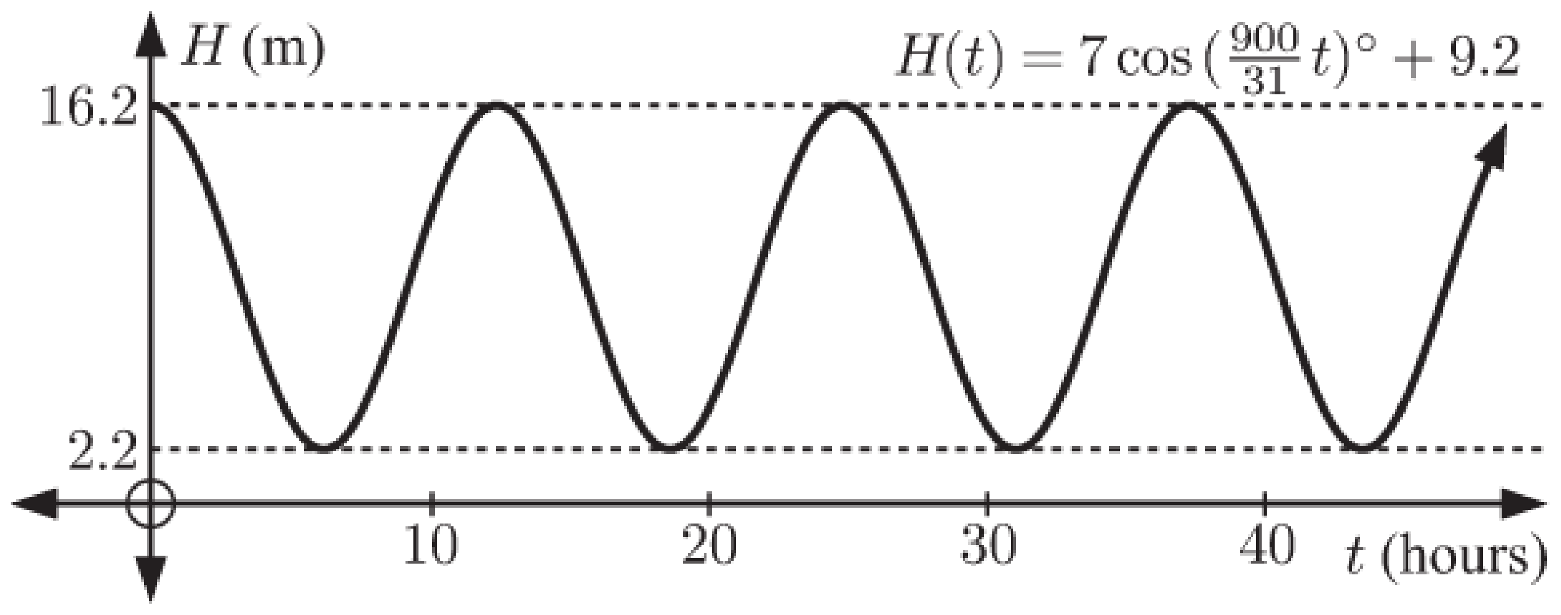


b **i** 0.75 **ii** 0.25 **iii** ≈ 0.835 **iv** ≈ 0.165

c once every 30 days **d** 16th January, 15th February

15 **a** $a = 7$, $b = \frac{360}{12.4} = \frac{900}{31}$ **b** $d = 9.2$

c



d ≈ 4.92 m

16 **a** $H = 8\sin(360t)° + 11$ cm **b** ≈ 3.39 cm

REVIEW SET 9B

1 **a** $\cos 500° = \cos(360° + 140°) = \cos 140°$

b $\sin(-100°) = \sin(620° - 720°) = \sin 620°$

2 **a** The function repeats itself over and over in a horizontal direction, in intervals of length 8 units.

b **i** 8 **ii** 5 **iii** -1

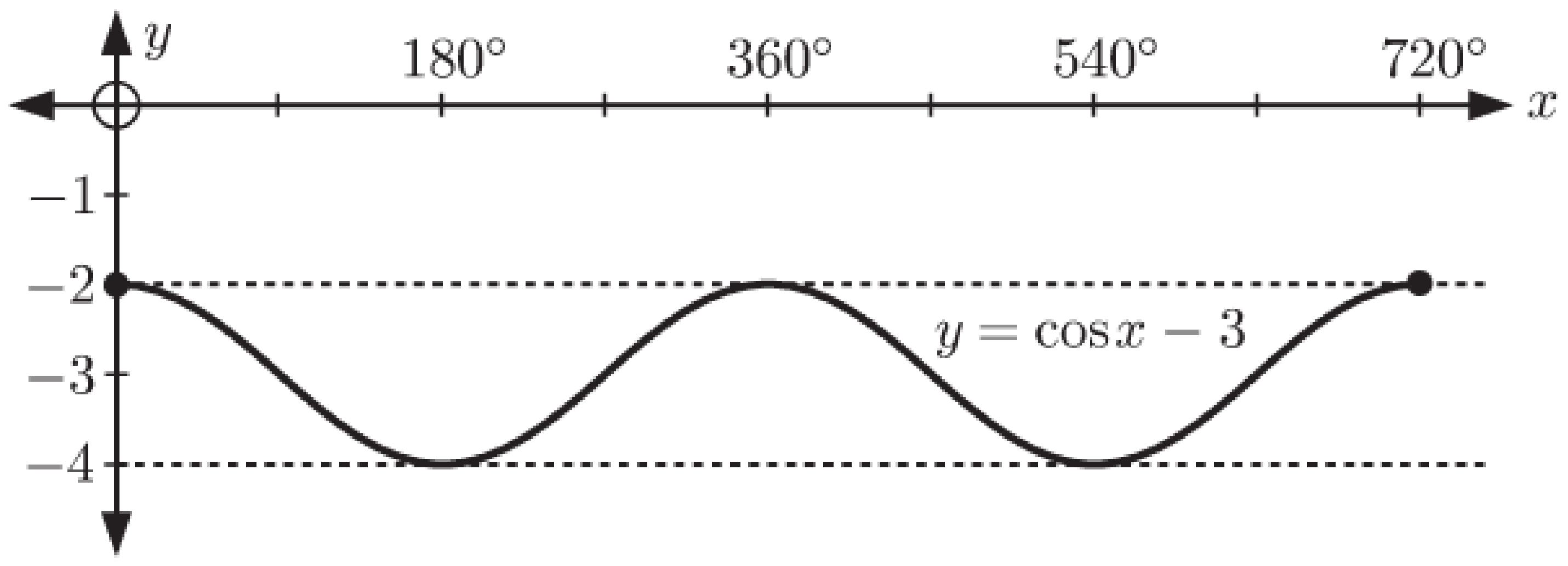
3 **a** $120°$ **b** $720°$ **4** **a** $y = 5$ **b** $y = -4$

5 **a** $b = \frac{1}{3}$ **b** $b = 24$ **c** $b = 40$

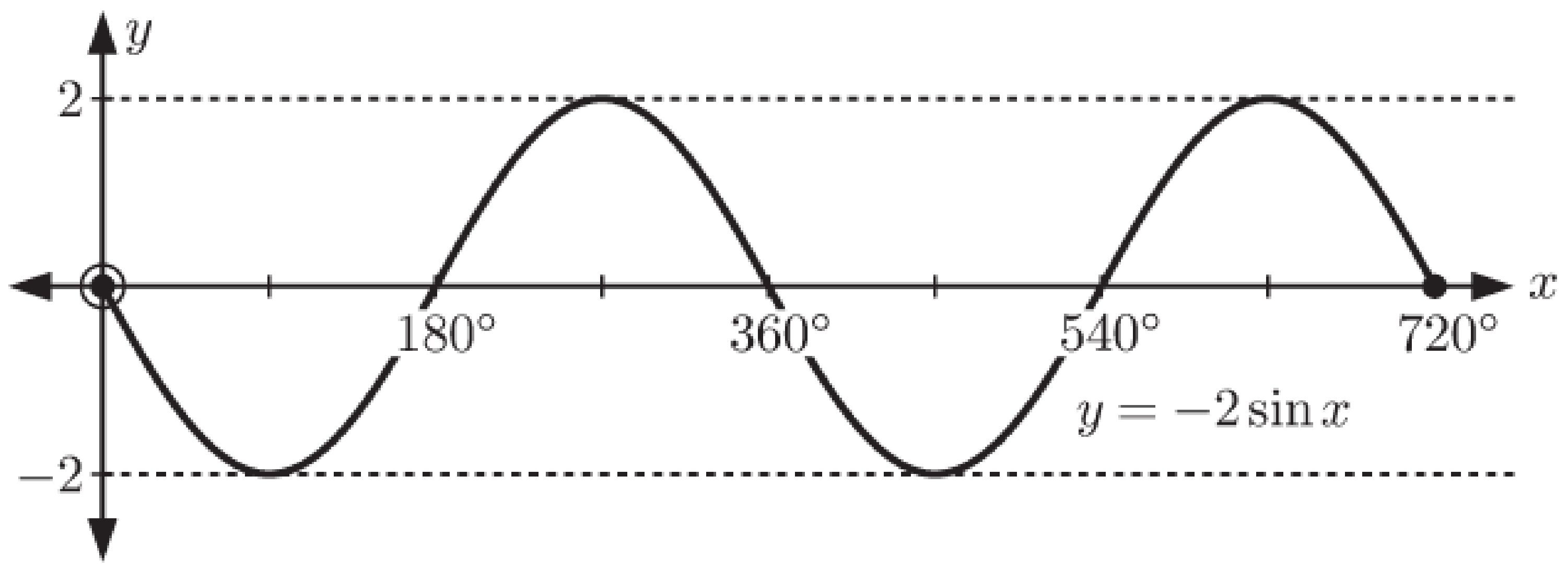
6 **a** maximum $= 2$, minimum $= -8$

b maximum $= \frac{4}{3}$, minimum $= \frac{2}{3}$

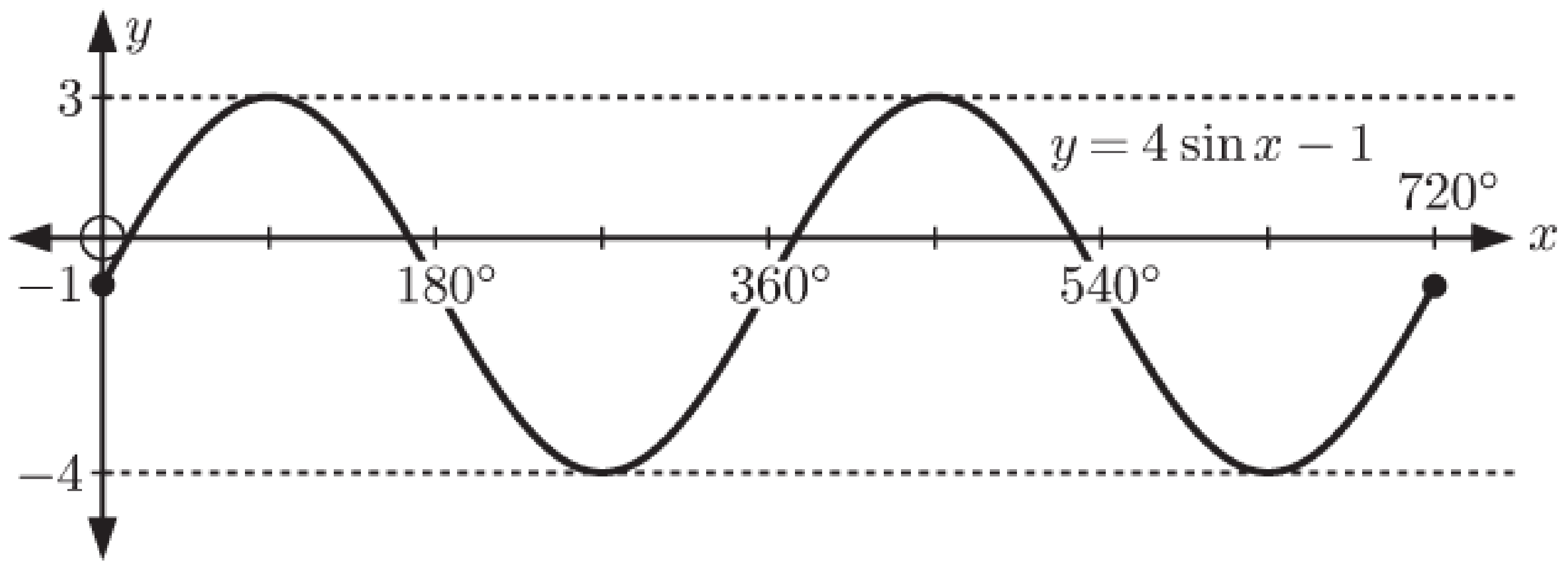
7 **a**



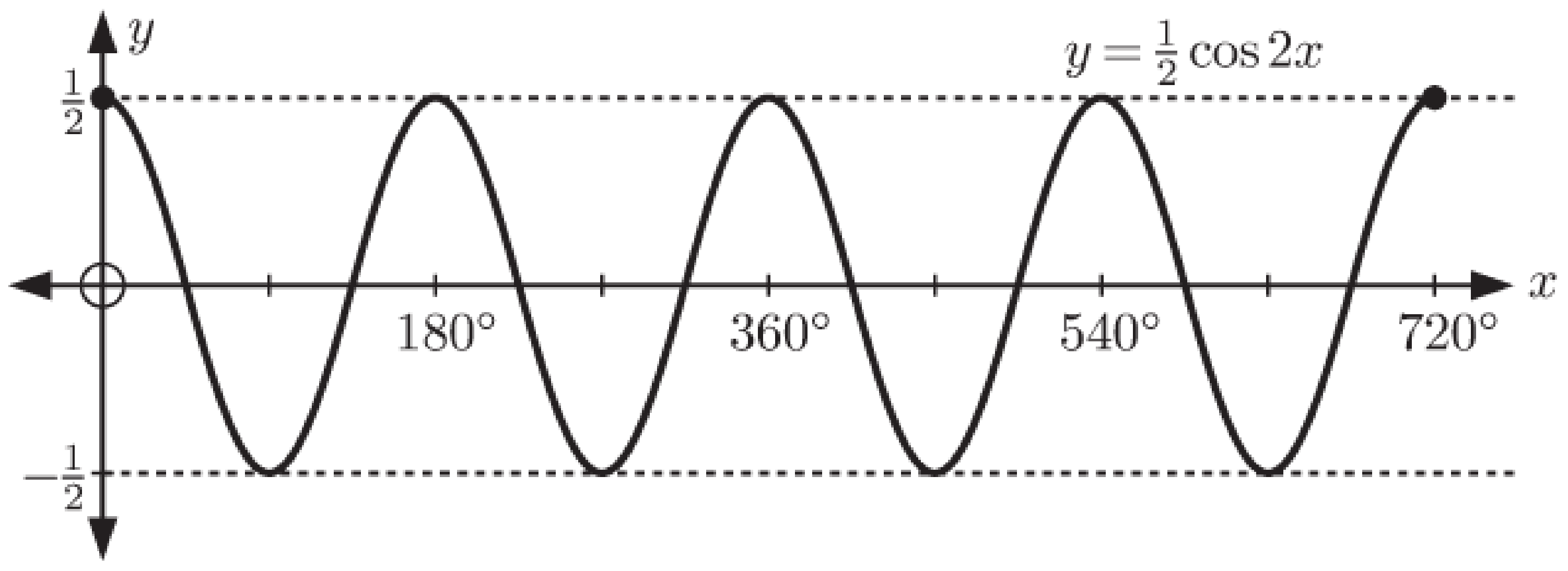
b



c



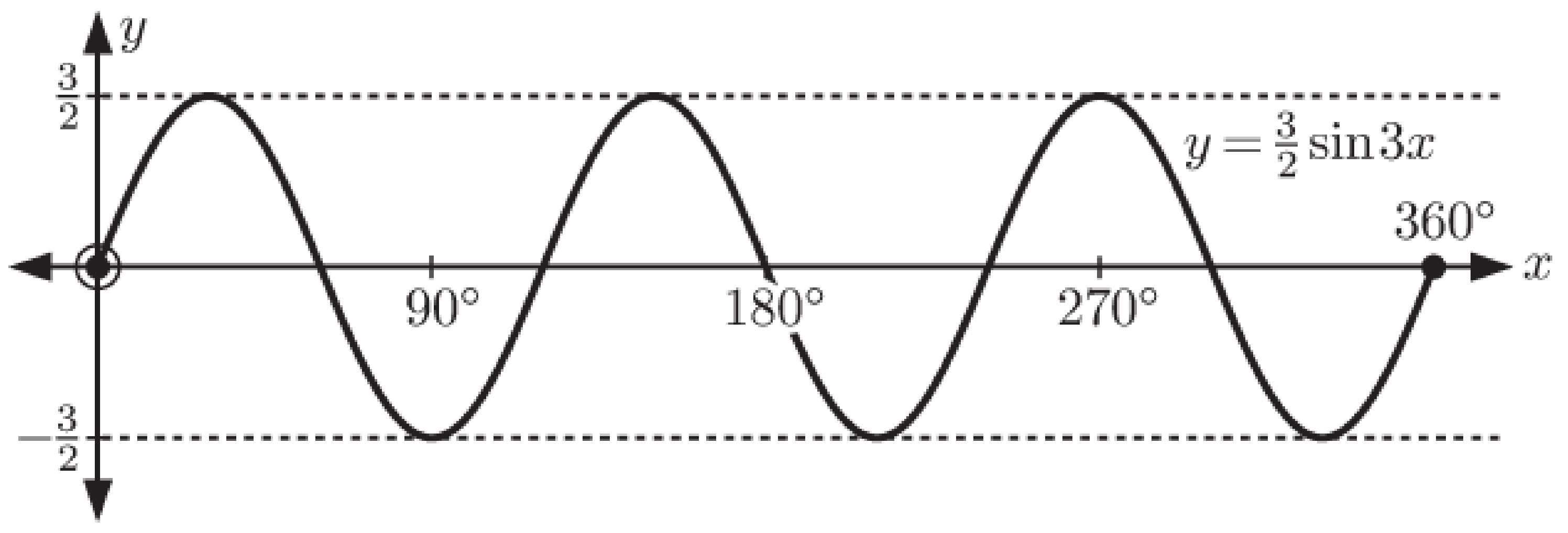
d



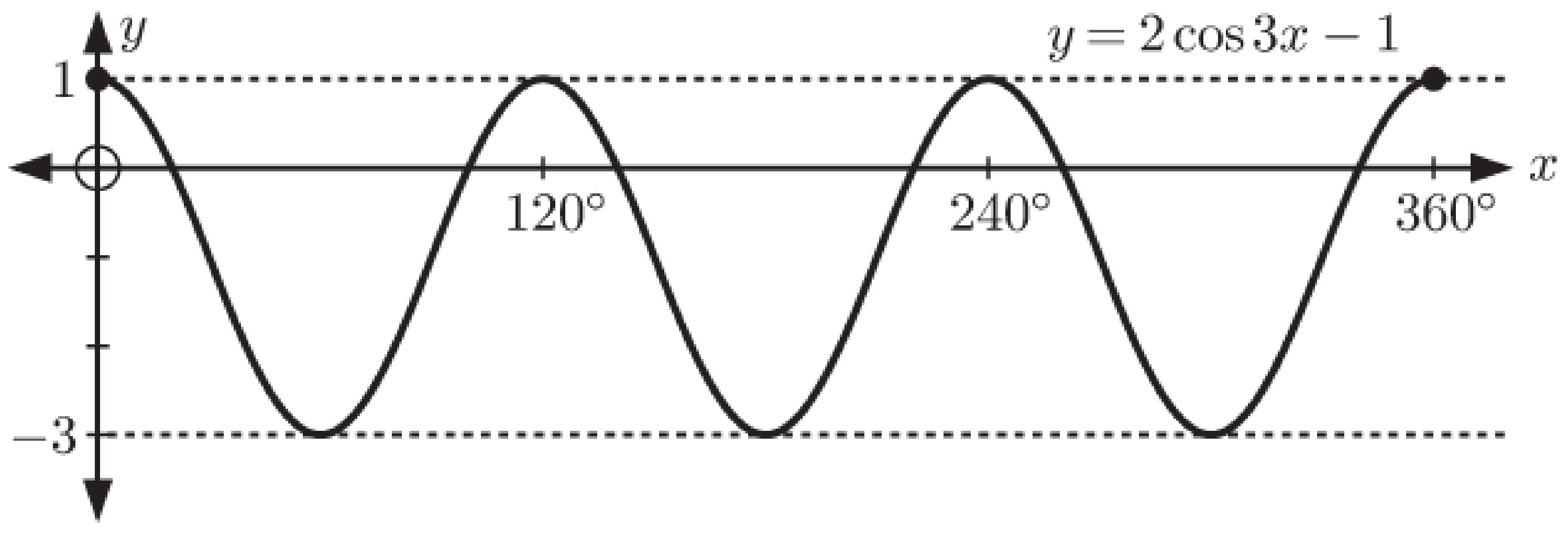
8 **a** A vertical stretch with scale factor $\frac{1}{3}$, then a translation 1 unit upwards.

b A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{2}{3}$.

9 **a**



b



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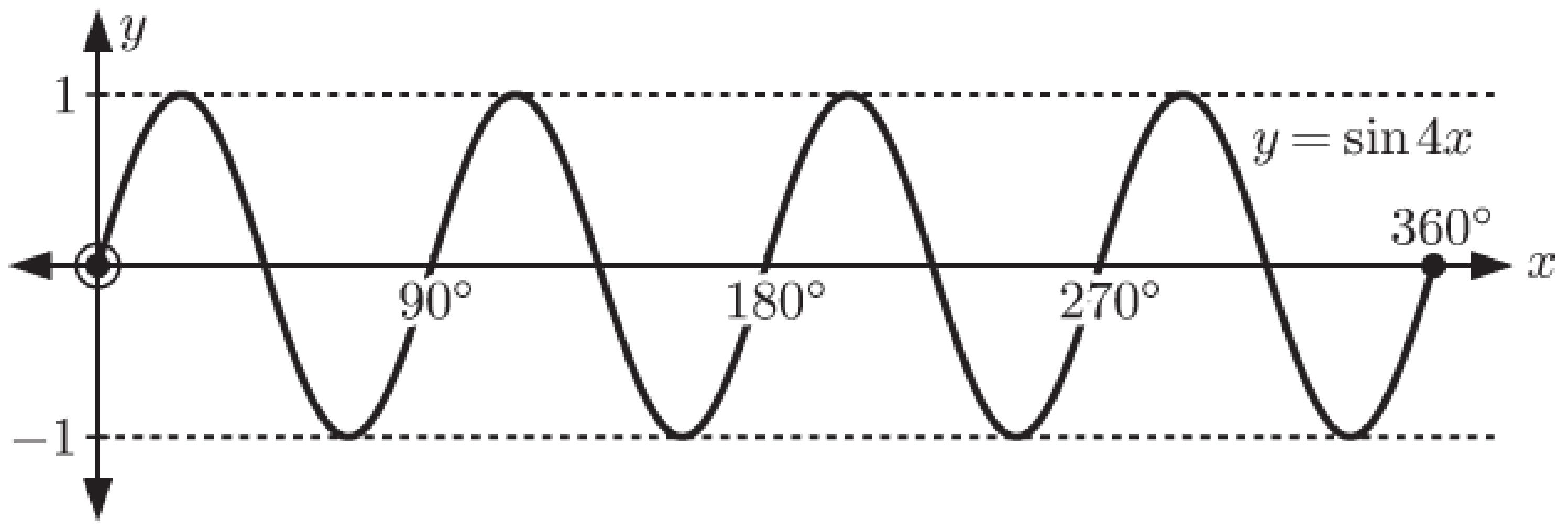

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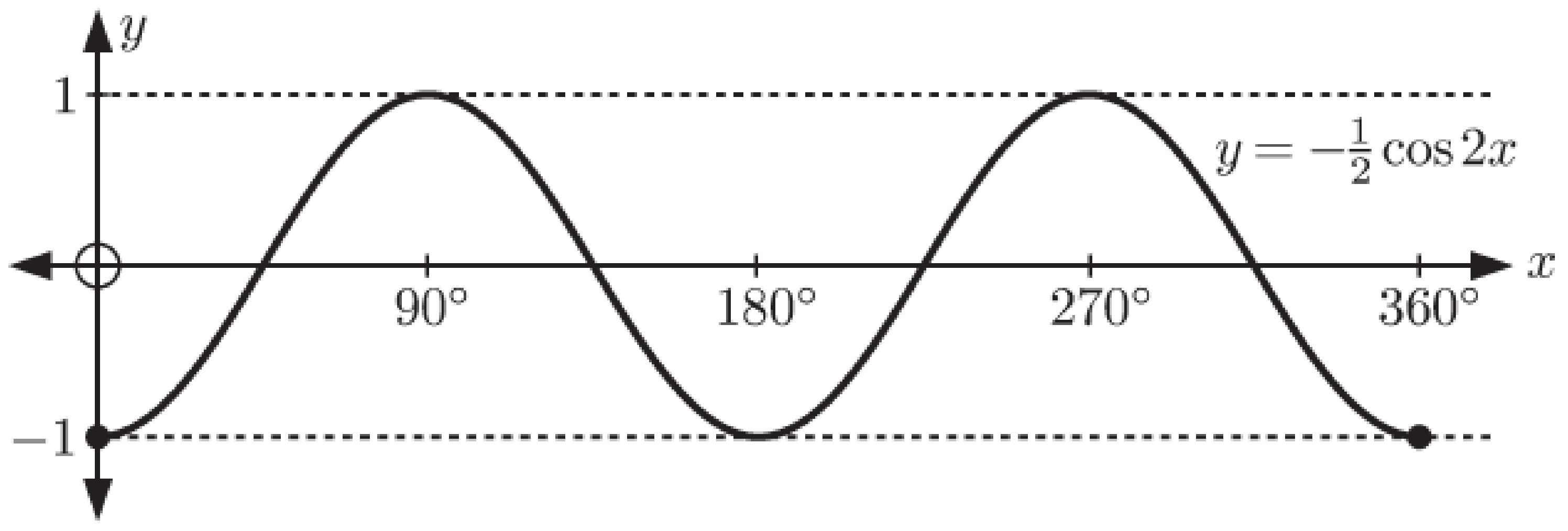
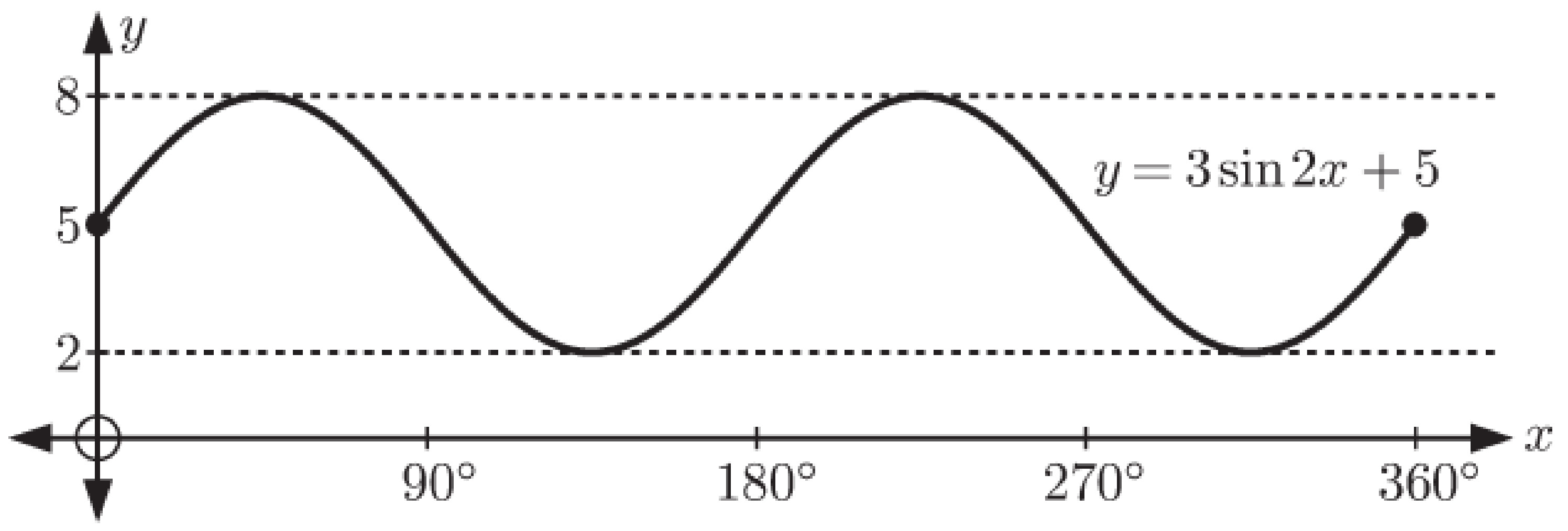


10 **a**

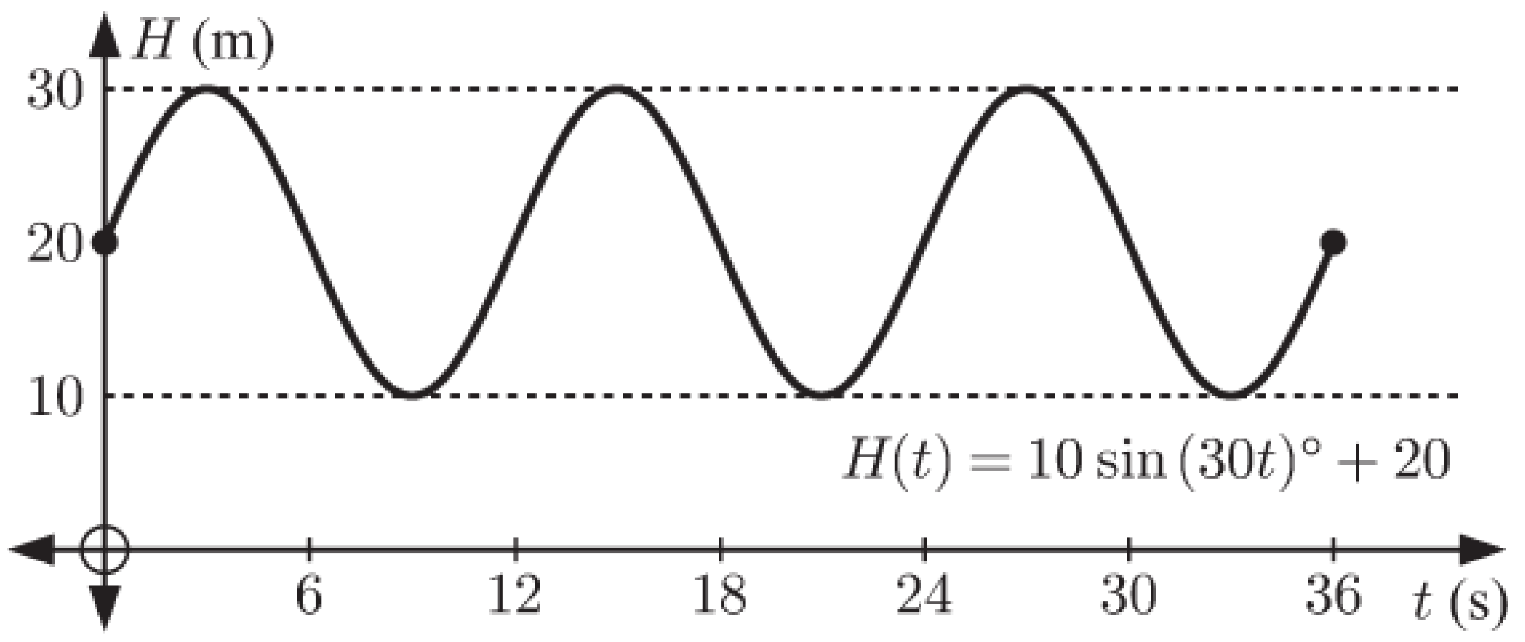


b $y = \frac{3\sqrt{3}}{2} + 5 \approx 7.60$

11 $y = 4\sin x + 6$

12 **a** $a = \frac{7}{2}$, $b = 2$, $d = -\frac{1}{2}$ **b** $a = \frac{1}{2}$, $d = \frac{3}{2}$

13 **a**

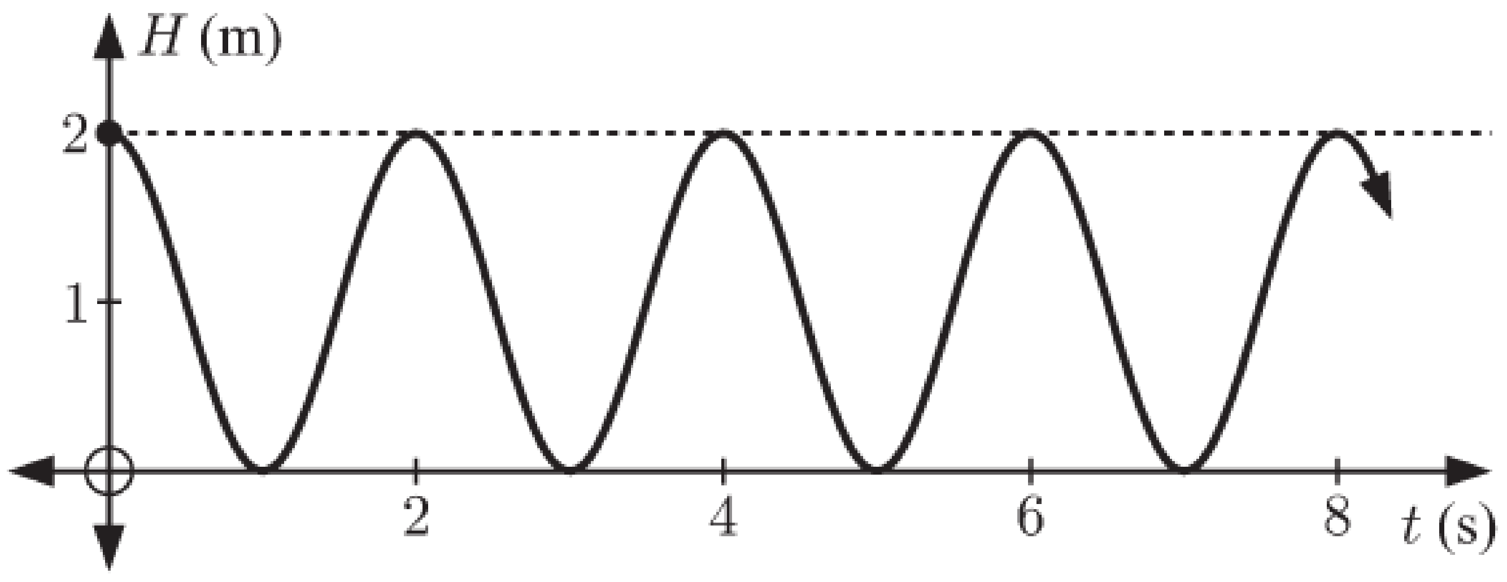


b 10 m **c** 10 m **d** 12 seconds

14 **a** $b = 30$ **b** $a = -\frac{5}{2}$, $d = 12$ **c** 13.25 hours

d The parameters b and d will stay the same since the number of daylight hours should be periodic over 12 months, with a mean of 12 daylight hours per day. The parameter a will change depending on the latitude of the city.

15 **a**



b $H = \cos(180t)^\circ + 1$ m **c** ≈ 1.31 m

EXERCISE 10A.1

1 **a** In one minute, Kirsten's heart is expected to beat 62 times.

b 3720 beats per hour

2 **a** $\approx 0.001\,50$ errors per word

b ≈ 0.150 errors per 100 words

3 Niko, £12.35 per hour

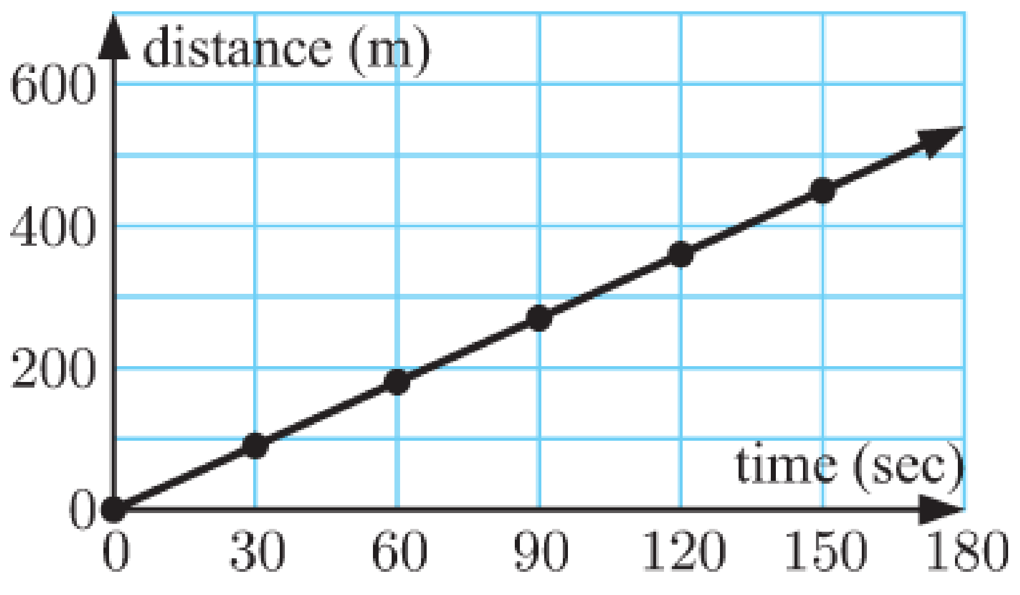
4 **a** $\approx 0.000\,177$ mm per km **b** ≈ 1.77 mm per 10 000 km

5 **a** ≈ 89.0 km h^{-1} **b** ≈ 24.7 m s^{-1}

EXERCISE 10A.2

1 **a** Yes. The distance increases by the same amount each time interval.

b



c 3 m s^{-1}

2 **a** Yes. The height increases by the same amount each time interval.

b 5 cm per week

3 **a** 3 **b** -2 **c** $-\frac{1}{4}$

EXERCISE 10A.3

1 **a** No, as the graph is not a straight line.

b **i** 60 km h^{-1} **ii** 100 km h^{-1}

2 **a** 100 m per hour **b** 100 m per hour (downwards)

3 **a** $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** $-\frac{5}{4}$ **d** -2

4 **a** **i** 3 **ii** 2.5 **iii** 2.1 **iv** 2.01 **v** 2.001

b The average rate of change approaches 2.

EXERCISE 10B

1 **a** 0.5 m s^{-1} **b** 2 m s^{-1} **2** **a** 1 **b** 4

3 **a** ≈ 1 m s^{-1} **b** ≈ 3 km h^{-1}

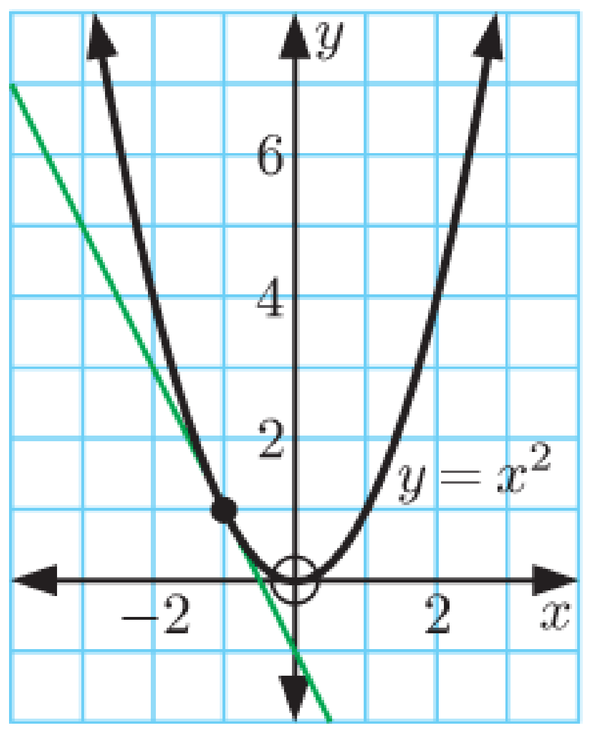
c $\approx$ \$44.40 per item sold **d** ≈ -4.3 bats per week

4 **a** 8000 L **b** 3000 L **c** 10 667 L per hour

d 3000 L per hour

e The rate at which water is leaking is decreasing.

5 **a**, **b**



c -2

EXERCISE 10C

1 **a** 7 **b** 7 **c** 11 **d** 16 **e** 0 **f** 5

2 **a** 5 **b** 7 **c** c

3 **a** -2 **b** 7 **c** -1

4 **a** 1 **b** -3 **c** 5 **d** -1 **e** 6 **f** -4

g -8 **h** 1 **i** 2

EXERCISE 10D

1 **a** $(3+h)^2$ **b** $\dfrac{(3+h)^2 - 9}{(3+h) - 3} = 6 + h$ for $h \neq 0$

c **i** 7 **ii** 6.5 **iii** 6.1 **iv** 6.01 **d** 6

2 **a** **i** 2

ii 8

c $2a$

b

x-coordinate	*Gradient of tangent*
1	2
2	4
3	6
4	8

3 **a** 4 **b** 5 **c** 0 **d** -3

4 **a** $(1+h)^3 = 1 + 3h + 3h^2 + h^3$ **b** **ii** 3

5 **b** **i** -1 **ii** $-\frac{1}{9}$

EXERCISE 10E

1 **a** $f(0) = 4$ **b** $f'(0) = -1$ **2** $f'(2) = 1$

3 **a** positive **b** negative **c** negative **d** positive

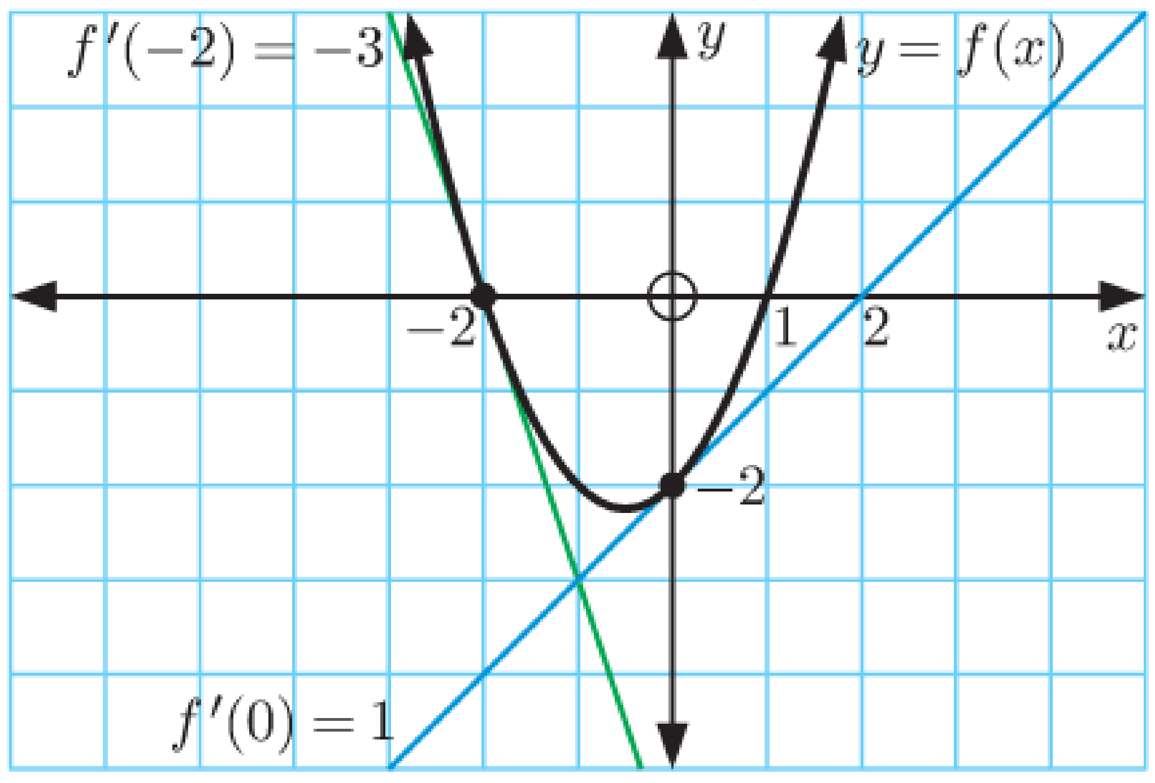
4 **a** **i** $f'(-2) = -3$

At $x = -2$, the derivative function is -3, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3.

ii $f'(0) = 1$

At $x = 0$, the derivative function is 1, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1.

b



EXERCISE 10F

1 **a** $f'(x) = 0$ **b** $f'(x) = 1$ **c** $f'(x) = 2$

d $f'(x) = -1$

2 **a** $\dfrac{dy}{dx} = 2x$ **b** $\dfrac{dy}{dx} = -2x$ **c** $\dfrac{dy}{dx} = 4x + 1$

d $\dfrac{dy}{dx} = -2x + 5$

3 **a** $f'(x) = 6x$

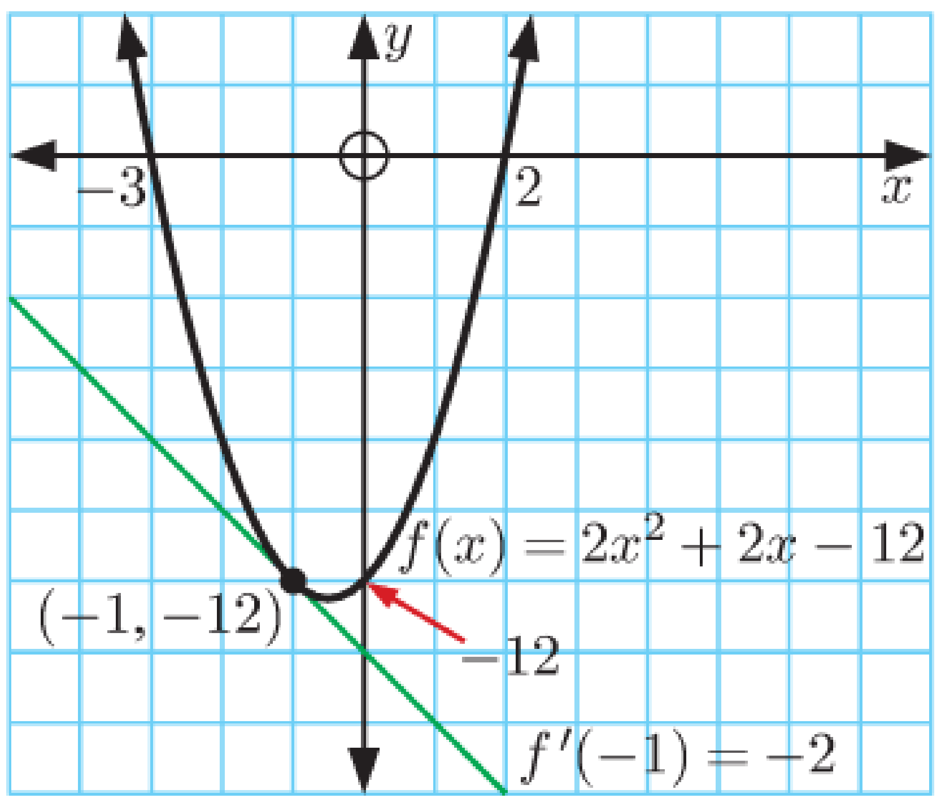
b $f'(2) = 12$. The gradient of the tangent to $y = f(x)$ at the point where $x = 2$ is 12.

4 **a** **i** 3 **ii** -1 **b** $f'(x) = -2x + 3$

c $f'(0) = 3$, $f'(2) = -1$

5 **a** $f'(x) = 4x + 2$ **b** $(-1, -12)$

c



6 **a** $f'(x) = x - 1$ **b** **i** -3 **ii** $(5, \frac{11}{2})$

7 **a**

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	27	12	3	0	3	12	27

c $\frac{dy}{dx} = 3x^2$

8 **a**

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$-\frac{1}{9}$	$-\frac{1}{4}$	-1	undefined	-1	$-\frac{1}{4}$	$-\frac{1}{9}$

b $\frac{dy}{dx} = -\frac{1}{x^2}$

9 **a**

$f(x)$	$f'(x)$
x^1	1
x^2	$2x$
x^3	$3x^2$
x^{-1}	$-\frac{1}{x^2}$
x^0	0

b If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

EXERCISE 10G

1 **a** $3x^2$ **b** $8x^7$ **c** $11x^{10}$ **d** 6 **e** $6x^2$
f $14x$ **g** $15x^4$ **h** $30x^5$ **i** $2x + 1$ **j** $2x + 3$
k 5 **l** $2x$ **m** $4x + 1$ **n** $6x - 7$ **o** $-4x$
p $2x^3 - 12x$ **q** $3x^2 - 8x + 6$ **r** $-1 - 12x^2$
s $\frac{3}{5}x^2 - 7x$ **t** $8x - 4$

2 **a** $-\frac{2}{x^3}$ **b** $-\frac{5}{x^6}$ **c** $-\frac{8}{x^9}$ **d** $-\frac{3}{x^2}$
e $-\frac{12}{x^4}$ **f** $\frac{28}{x^5}$ **g** $2 - \frac{6}{x^3}$ **h** $2x + \frac{6}{x^2}$
i $\frac{6}{x^4}$ **j** $-\frac{1}{x^2} + \frac{15}{x^4}$ **k** $-\frac{4}{x^3} - \frac{36}{x^5}$ **l** $3 + \frac{1}{x^2} - \frac{4}{x^3}$
m $\frac{16}{x^3} - \frac{12}{x^4}$ **n** $-\frac{2}{5x^3}$ **o** $4 + \frac{1}{4x^2}$
p $1 + \frac{3}{x^2}$ **q** $2x - \frac{4}{x^2}$ **r** $-\frac{2}{x^2} + \frac{10}{x^3}$

3 **a** $f'(x) = 12x^2 - 1$ **b** 47 **c** -1

4 **a** $g'(x) = 1 - \frac{1}{x^2}$ **b** $\frac{8}{9}$ **c** $\frac{3}{4}$

5 **a** $\frac{dy}{dx} = 100$ **b** $\frac{dy}{dx} = 2\pi x$ **c** $\frac{dy}{dx} = 6 - \frac{5}{x^2}$
d $\frac{dy}{dx} = 7.5x^2 - 2.8x$ **e** $\frac{dy}{dx} = 10$ **f** $\frac{dy}{dx} = 12\pi x^2$
g $\frac{dy}{dx} = 2x - 1$ **h** $\frac{dy}{dx} = 2x - 10$ **i** $\frac{dy}{dx} = -1 - 2x$

6 **a** $\frac{dy}{dt} = 2t^3 - \frac{1}{3}$ **b** $\frac{dy}{dt} = \frac{1}{2t^2}$ **c** $\frac{dV}{dr} = 4\pi r^2$

7 **a** 4 **b** 22 **c** $-\frac{16}{729}$ **d** -7 **e** $\frac{13}{4}$ **f** -11

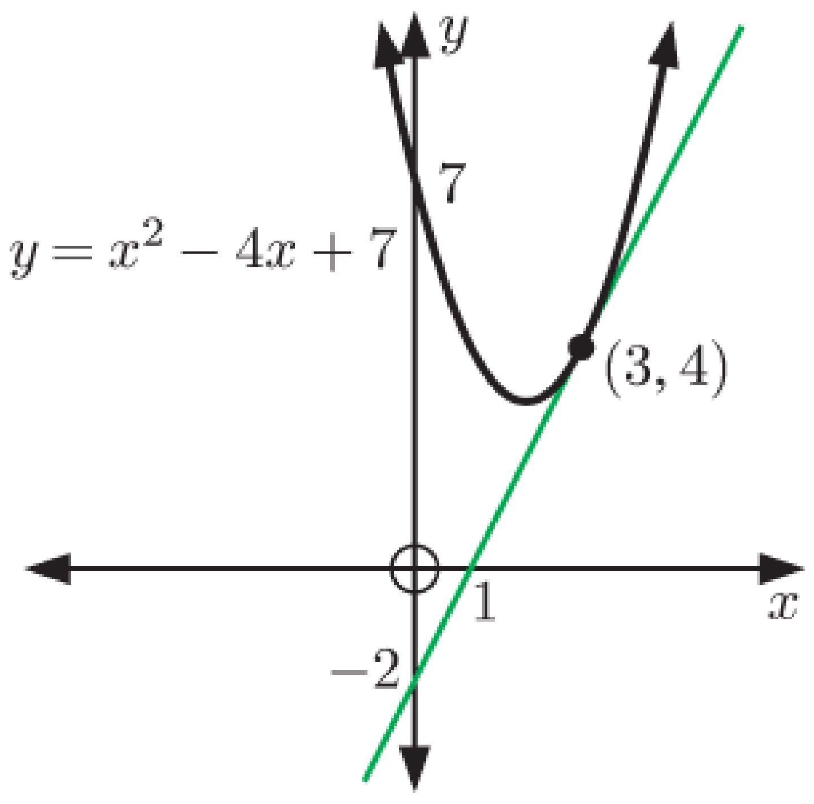
8 $b = 3,\ c = -4$

9 $\frac{dy}{dx} = 4 + \frac{3}{x^2}$, $\frac{dy}{dx}$ is the gradient function of $y = 4x - \frac{3}{x}$ from which the gradient of the tangent at any point can be found.

10 **a** $\frac{dS}{dt} = 4t + 4$ This gives the speed of the car at time t, in metres per second.
b $\frac{dS}{dt} = 16\ \text{m s}^{-1}$ at $t = 3$. This is the speed of the car after 3 seconds.

11 $\frac{dC}{dx} = 7$ when $x = 1000$. It costs £7 to produce each toaster when 1000 toasters are produced each week.

12



The tangent has gradient 2 at the point $(3, 4)$.

13 **a** $(-1, -3)$ **b** $(-3, -1)$ **c** $(2, \frac{5}{2})$
d $(-1, 4)$ or $(1, 0)$ **e** $\left(-\frac{b}{2a}, \frac{b^2}{4a} - \frac{b^2}{2a} + c\right)$

14 $\text{P}(-4, 4)$ **15** $a = 7$ **16** $a = -9,\ b = 8$

17 $a = 3,\ b = 6$ **18** **a** 1 **b** $a = -7,\ b = -15$

REVIEW SET 10A

1 $\frac{4}{3}$

2 **a** Yes. The height increases by the same amount each time interval.
b $1.6\ \text{m s}^{-1}$

3 **a** -1 **b** -1 **4** $-\frac{1}{2}$

5 **a** $f'(x) = 2x + 2$ **b** $\frac{dy}{dx} = -6x$

6 **a** $\frac{dy}{dx} = 4x$ **b** 16 **c** $x = -3$

7 **a** $f'(x) = 3x^2 - 6x$
b $f'(-1) = 9,\ f'(3) = 9$
Gradients of tangents are both 9.
$\therefore$ the tangents are parallel.

8 **a** $f'(x) = 15x^2$ **b** $f'(x) = 6x^5 - 5$
c $f'(x) = 14x + \frac{3}{x^2}$ **d** $f'(x) = 3 + \frac{8}{x^3}$

9 10 **10** **a** -17 **b** -17 **11** $\text{P}(6, 5)$ **12** $a = 2$

13 **a** -5 **b** -12 **c** $\frac{7}{9}$ **d** -1

14 $S'(t) = 0.9t^2 - 36t + 550$
This gives the instantaneous rate of change in weight, in grams per second, for a given value of t.

15 **a** $f(3) = 2,\ f'(3) = -1$ **b** $f(x) = x^2 - 7x + 14$

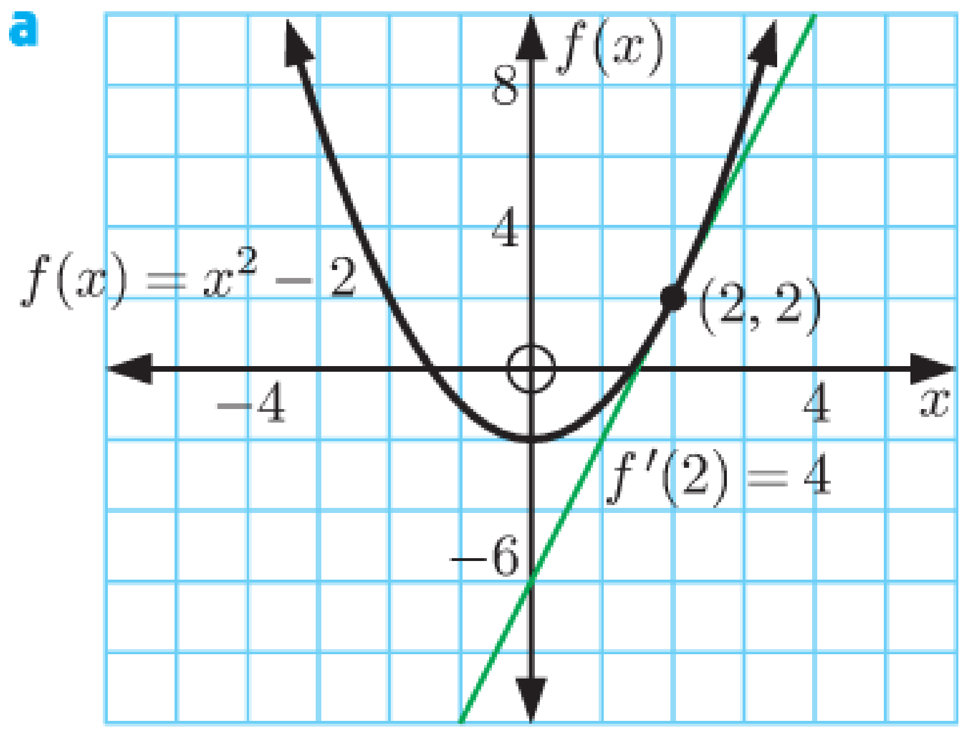
REVIEW SET 10B

1 $\frac{1}{2}$ **2** **a** 2°C per hour **b** −2.5°C per hour

3 **a** -3 **b** 3

4 **a** negative **b** positive **c** positive **d** negative

5 **a**



b 4

6 **a** $f'(x)=4x^3-2$

b $f'(-2)=-34$. The gradient of $f'(x)$ at the point where $x=-2$ is -34.

7 **a** $\dfrac{dy}{dx}=2x+5$ **b** $(-4,\ -6)$

8 **a** **i** 447.2 m **ii** 432.8 m **b** $f'(t)=-9.6t\ \ \text{m s}^{-1}$

c **i** $-9.6\ \text{m s}^{-1}$ **ii** $-19.2\ \text{m s}^{-1}$

9 **a** $f'(x)=21x^2$ **b** $f'(x)=6x-3x^2$

c $f'(x)=8x-12$ **d** $f'(x)=7+4x$

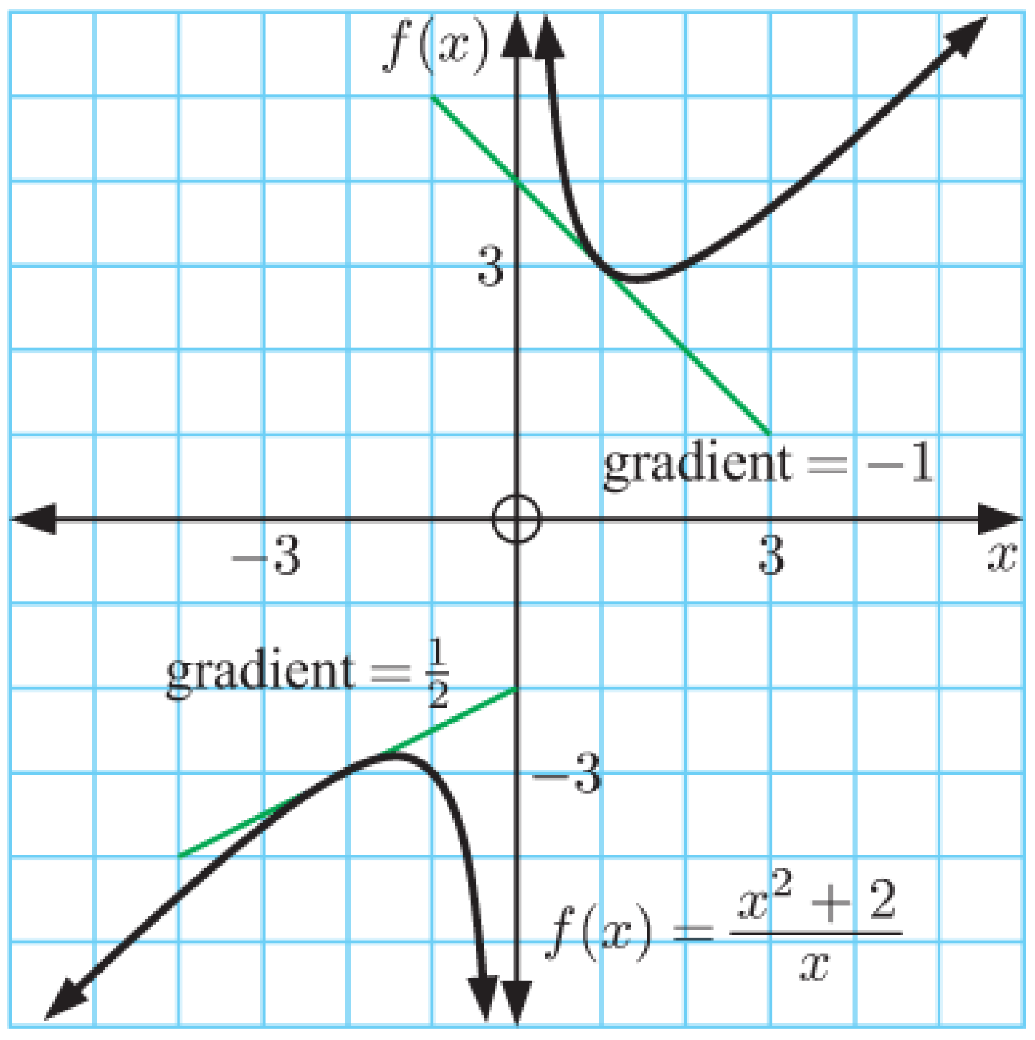
10 **a** $f'(x)=4x^3-3$ **b** 29 **c** -3 **11** **a** -7 **b** -3

12 **a** $\dfrac{dy}{dx}=6x-7$ **b** $\dfrac{dy}{dx}=6x^2-12x+7$

c $\dfrac{dy}{dx}=-\dfrac{3}{x^2}+\dfrac{15}{x^4}$

13 **a** $f'(x)=1-\dfrac{2}{x^2}$ **b** **i** -1 **ii** $\frac{1}{2}$

c



14 $a=-14,\ b=21$

EXERCISE 11A

1 **a** $f'(x)=2x-4$ **b** $y=-2x-1$

2 **a** $y=8x-16$ **b** $y=12x+16$ **c** $y=-3x-6$

d $y=-\frac{3}{4}x+2$ **e** $y=7x-5$ **f** $y=-3x-5$

g $y=2x$ **h** $y=-3x-8$ **i** $y=-7x+11$

j $y=-2x-2$ **k** $y=-5x-9$ **l** $y=-5x-1$

3 **a** $f'(x)=-2x+6$ **b** $x=3$

c The tangent at the point where $x=3$ is horizontal.

4 **a** $k=-5$ **b** $y=4x-15$ **5** $y=-3x+1$

6 $a=-4,\ b=7$

7 **a** $f'(x)=2x-\dfrac{8}{x^3}$ **b** $x=\pm\sqrt{2}$

c When $x=\sqrt{2}$, $y=4$ and when $x=-\sqrt{2}$, $y=4$.
$\therefore$ tangents are $y=4$.

8 **a** $f'(x)=2x$ **b** $y=2x$ **c** When $x=0,\ y=0$

9 **a** **i** $y=x+5$ **ii** $(2,\ 7)$

b **i** $y=\frac{17}{3}x-5$ **ii** $(-\frac{1}{3},\ -\frac{62}{9})\approx(-0.333,\ -6.89)$

c **i** $y=6.75x-1.75$ **ii** $(-3,\ -22)$

d **i** $y=2x$ **ii** $(1,\ 2)$

e **i** $y=3.25x+0.75$ **ii** $(-1.67,\ -4.67)$

10 **a** $(-4,\ -64)$ **b** $(4,\ -31)$ **c** $(-1,\ -2)$

11 **a** $(5,\ 0)$ **b** $(-1.58,\ 2)$ **c** $(0.8,\ -1.4)$ **d** $(0,\ -5)$

12 R$(0.5,\ -6)$ **13** **a** $y=2x+2$ **b** $(-1,\ 0)$

EXERCISE 11B

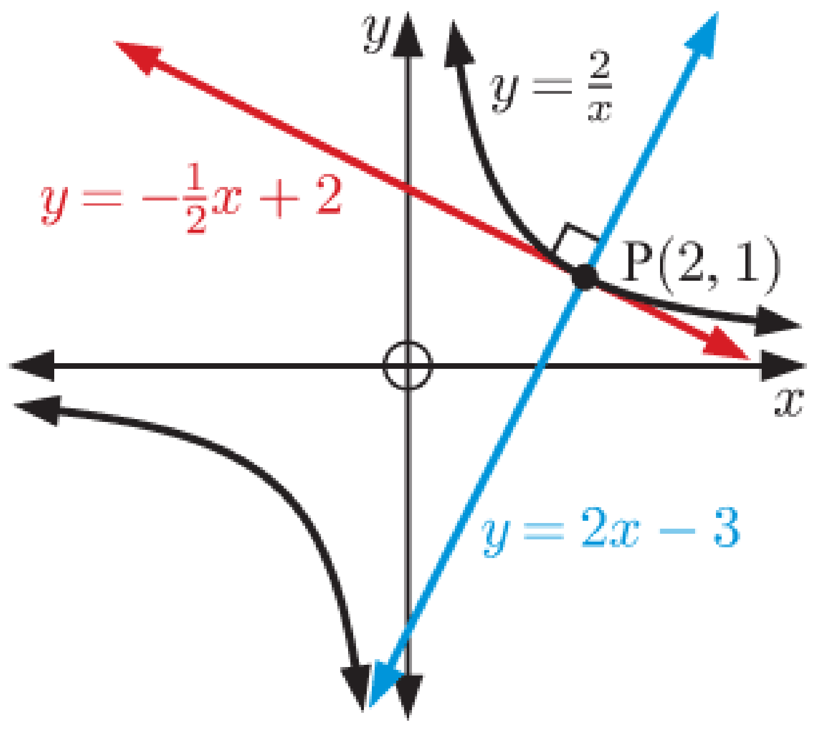
1 **a** $y=-\frac{1}{6}x+\frac{19}{2}$ **b** $y=-\frac{1}{7}x+\frac{26}{7}$ **c** $y=x+2$

d $y=-\frac{1}{3}x+\frac{1}{3}$ **e** $y=-\frac{1}{3}x+3$ **f** $y=-\frac{1}{2}x+\frac{1}{2}$

2 **a** **i** $y=-\frac{1}{2}x+2$

ii $y=2x-3$

b



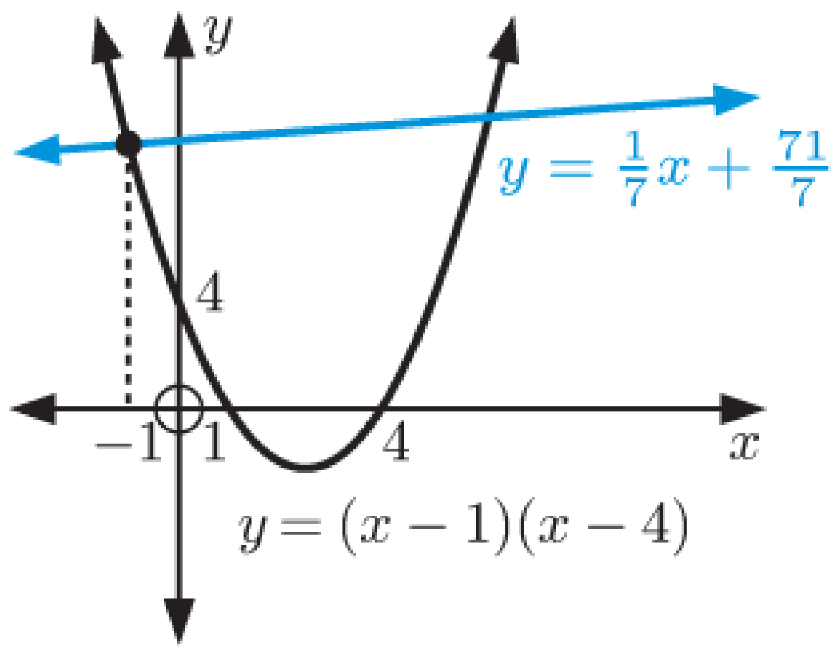
3 **a** $y=-2x+4$ **b** $y=-\frac{9}{62}x+\frac{1259}{186}$

4 **a** x-intercepts 1 and 4, y-intercept 4

b $f'(x)=2x-5$

c $y=\frac{1}{7}x+\frac{71}{7}$

d



5 **a** $(-2.25,\ 5.0625)$ **b** $(1,\ 3)$

c The normal does not meet the curve again.

d $\approx(-3.78,\ -6.55)$ and $\approx(0.777,\ -6.85)$

6 **a** $(-2,\ 0)$ **b** $(-13,\ 3)$ **c** $\approx(1.33,\ -1.67)$

d $\approx(0,\ 0.333)$

7 $a=-4$

8 **a** **i** $a=-1$ **ii** P$(2,\ 2)$ **b** $y=-x+4$ **c** Q$(0,\ 4)$

d $y=-x+4$. This is the same line as the normal to the curve at P.

EXERCISE 11C

1 **a** **i** $x\geqslant 0$ **ii** never **b** **i** never **ii** $-2<x\leqslant 3$

c **i** $x\leqslant 2$ **ii** $x\geqslant 2$ **d** **i** $x\in\mathbb{R}$ **ii** never

e **i** $1\leqslant x\leqslant 5$ **ii** $x\leqslant 1,\ x\geqslant 5$

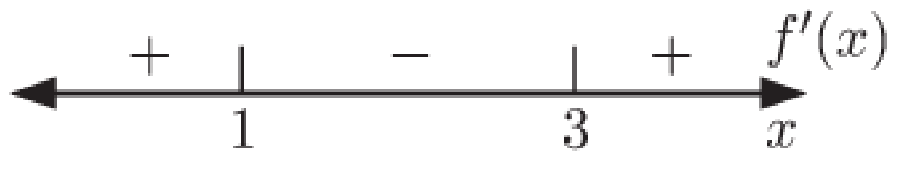
f **i** $2\leqslant x<4,\ x>4$ **ii** $x<0,\ 0<x\leqslant 2$

g **i** $-2<x\leqslant 0$ **ii** $0\leqslant x<2$

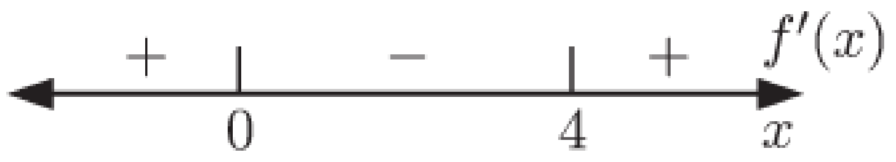
h **i** $x\leqslant 0,\ 2\leqslant x\leqslant 6$ **ii** $0\leqslant x\leqslant 2,\ x\geqslant 6$

2 **a** **i** $x\leqslant 1,\ x\geqslant 3$ **ii** $1\leqslant x\leqslant 3$

b $f'(x)=3x^2-12x+9$
$=3(x-3)(x-1)$



3 **a** $f'(x)=3x^2-12x$
$=3x(x-4)$



b increasing for $x\leqslant 0$ and $x\geqslant 4$
decreasing for $0\leqslant x\leqslant 4$

4 **a** increasing for all $x\in\mathbb{R}$ **b** decreasing for all $x\in\mathbb{R}$

c increasing for $x\geqslant 0$, decreasing for $x\leqslant 0$

d decreasing for all $x\in\mathbb{R}$

e increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$

f decreasing for all $x \neq 0$

g increasing for all $x > 0$, decreasing for all $x < 0$

h increasing for $x \leqslant 0$ and $x \geqslant 4$,
decreasing for $0 \leqslant x \leqslant 4$

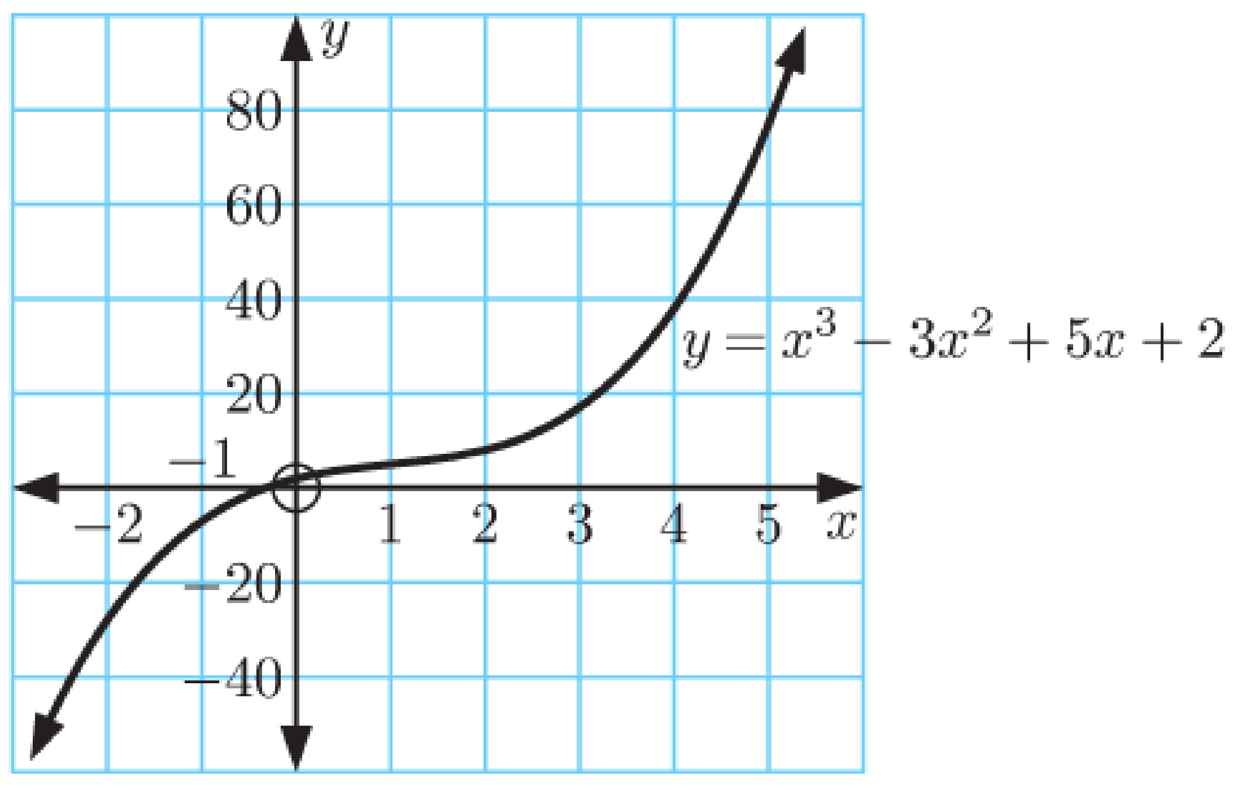
i increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$

5 **a** **i** $f'(x) = -12x^2 + 30x + 18$
ii $x = 3$ or $-\frac{1}{2}$
iii increasing for $-\frac{1}{2} \leqslant x \leqslant 3$,
decreasing for $x \leqslant -\frac{1}{2}$ and $x \geqslant 3$

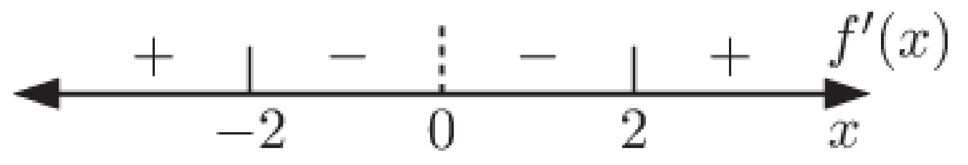
b **i** $f'(x) = 3x^2 - 12x + 3$
ii $x = 2 \pm \sqrt{3}$
iii increasing for $x \leqslant 2 - \sqrt{3}$ and $x \geqslant 2 + \sqrt{3}$,
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

6 **a** $f'(x) = 3x^2 - 6x + 5$
b $\Delta = -24$ which is < 0 and $a > 0$
$\therefore$ $f'(x)$ lies entirely above the x-axis.
$\therefore$ $f'(x) > 0$ for all x.
$\therefore$ $f(x)$ is increasing for all x.

c



7 **b**

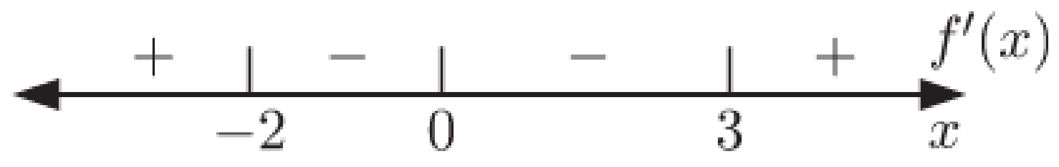


c increasing for $x \leqslant -2$ and $x \geqslant 2$,
decreasing for $-2 \leqslant x < 0$ and $0 < x \leqslant 2$

EXERCISE 11D

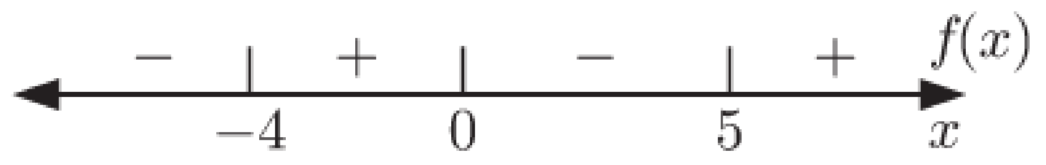
1 **a** A is a local maximum. O is a stationary inflection.
B is a local minimum.

b



c **i** $x \leqslant -2$ and $x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

d

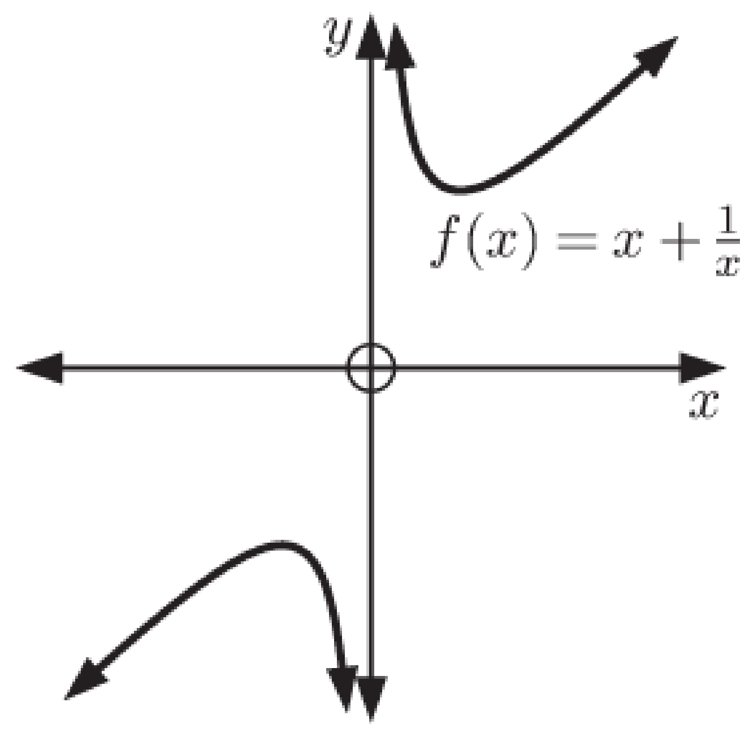


2 **a** 4 **b** **i** $(-2, 9)$ **ii** $(2, -6)$
c **i** 11, when $x = 6$ **ii** -10, when $x = 4$
d 9, when $x = -2$ **e** -6, when $x = 2$

3 **a** $x = \frac{5}{4}$
b $f'(x) = 4x - 5$. $f'(x) = 0$ when $x = \frac{5}{4}$
The vertex of the quadratic (a local minimum in this case) is always on the axis of symmetry.

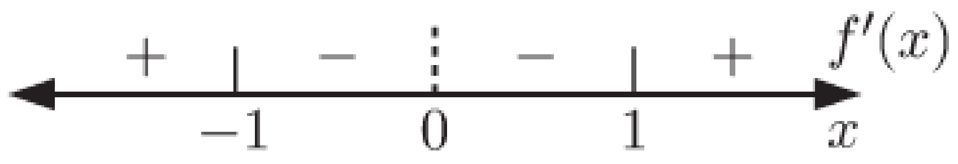
4 **a** P is a local maximum, Q is a local minimum.
b $f'(x) = 3x^2 + 12x - 15 = 3(x + 5)(x - 1)$
c P$(-5, 60)$, Q$(1, -48)$

5 **a**



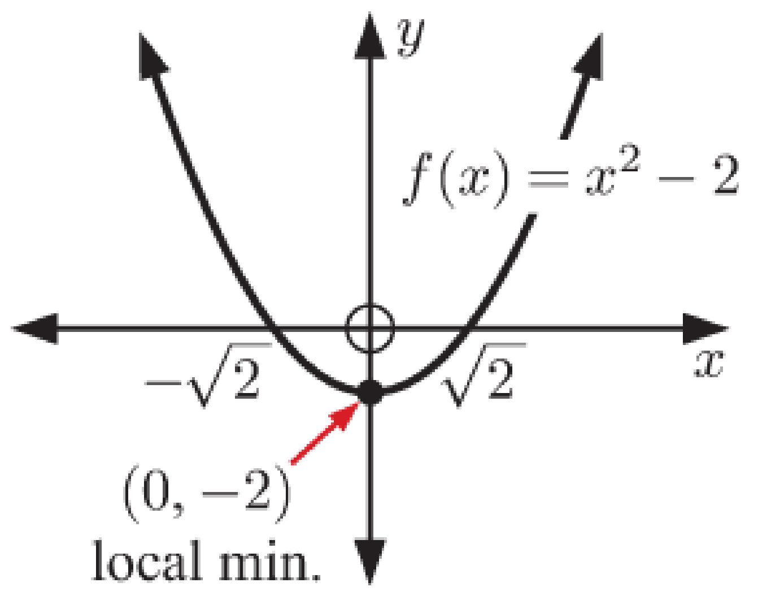
b $f'(x) = 1 - \dfrac{1}{x^2}$

c

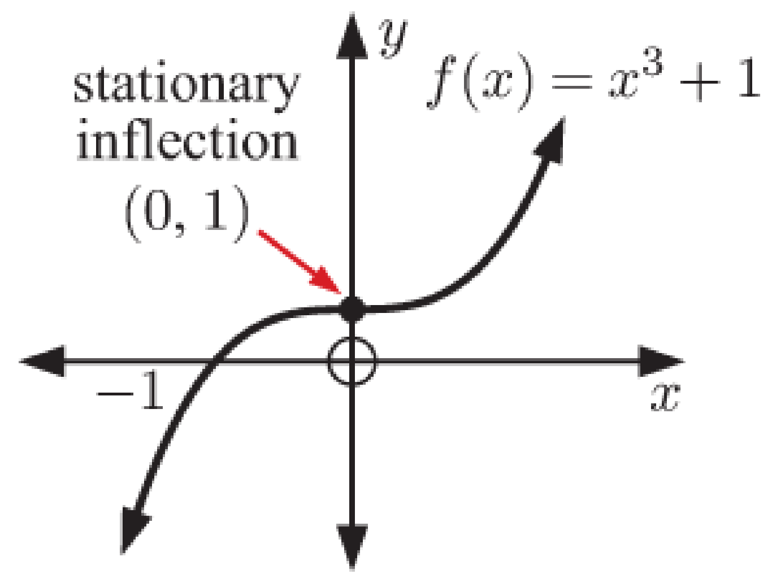


d local maximum at $(-1, -2)$, local minimum at $(1, 2)$

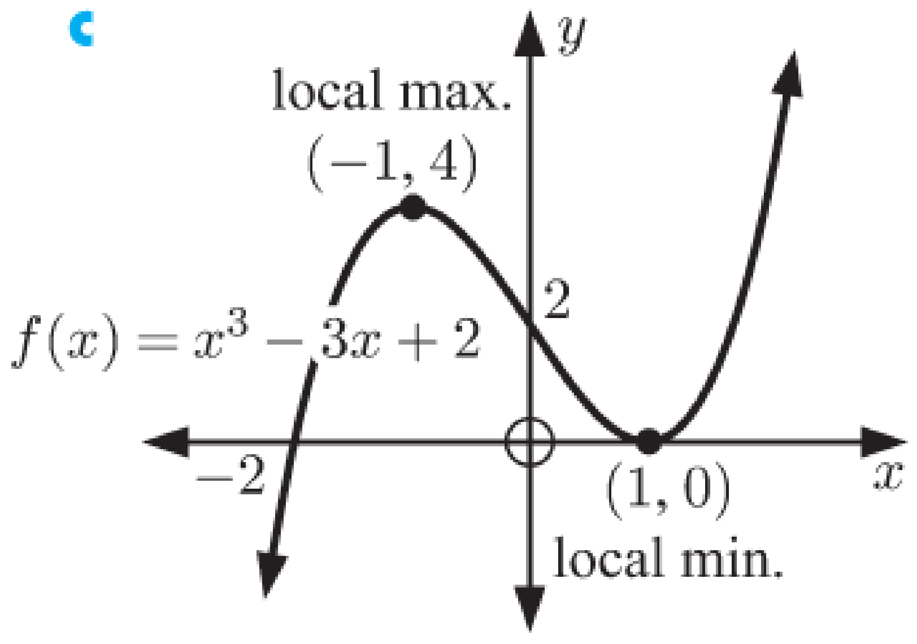
6 **a**



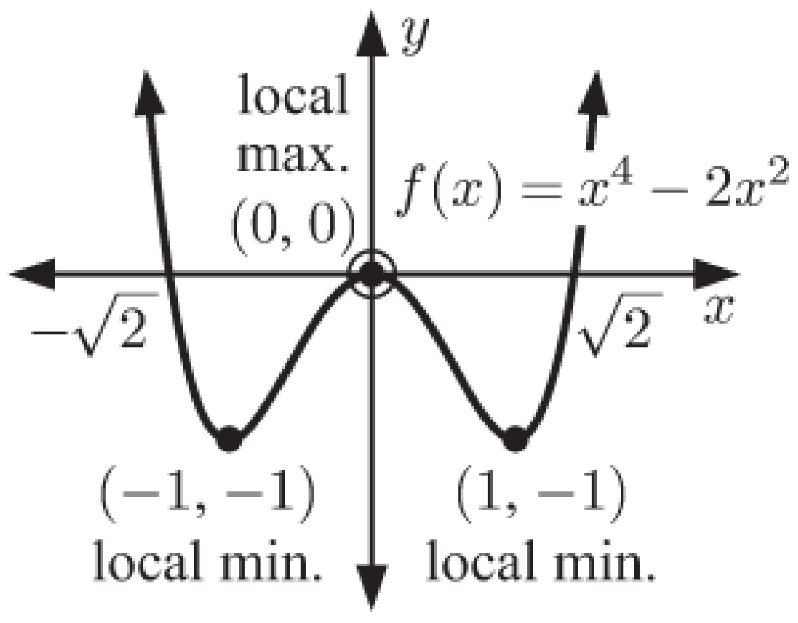
b



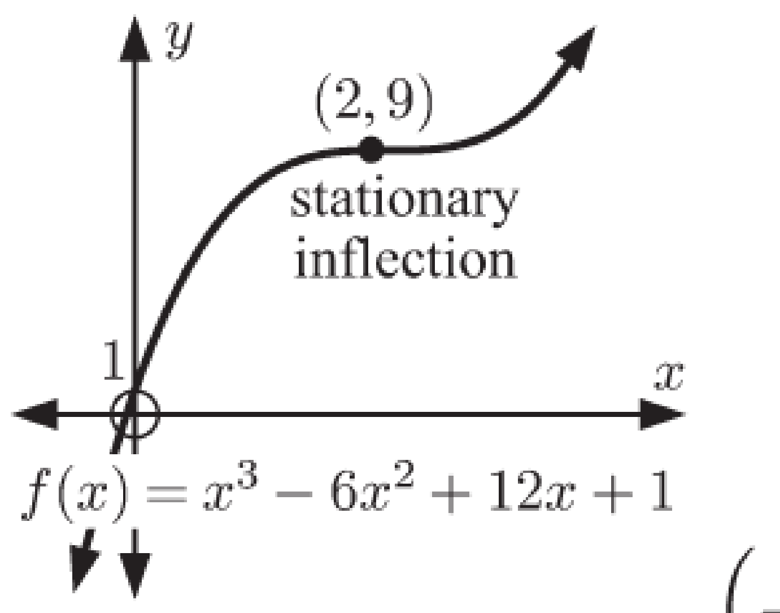
c



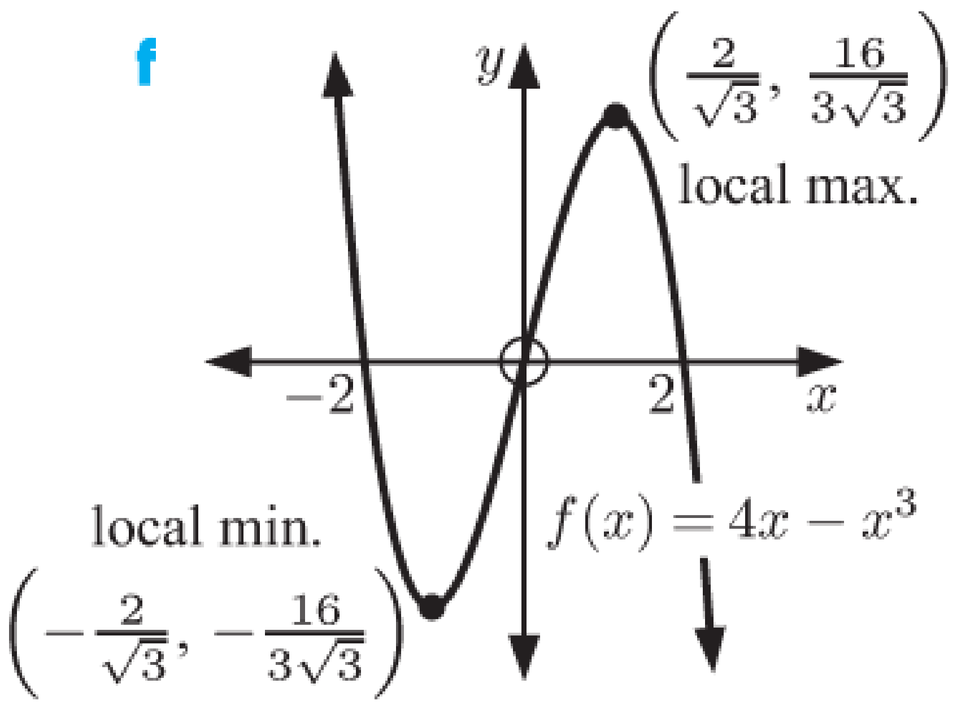
d



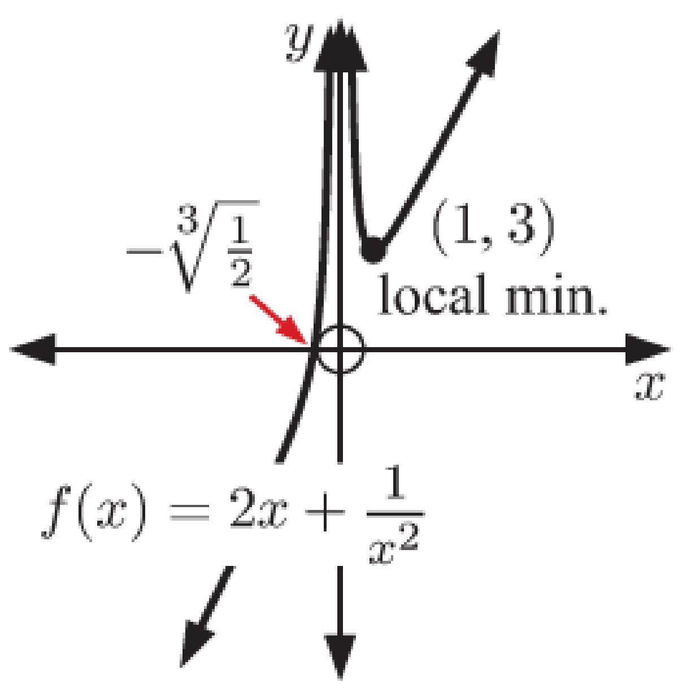
e



f



g



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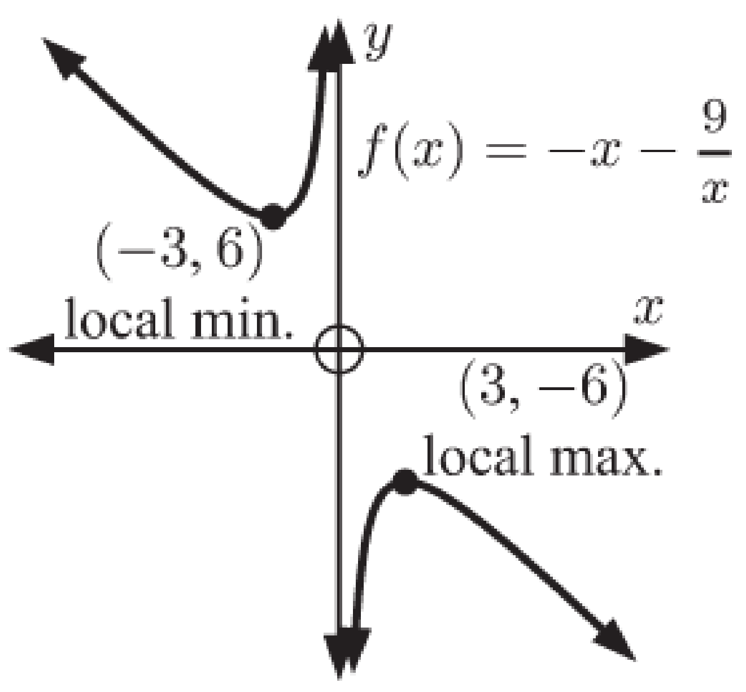
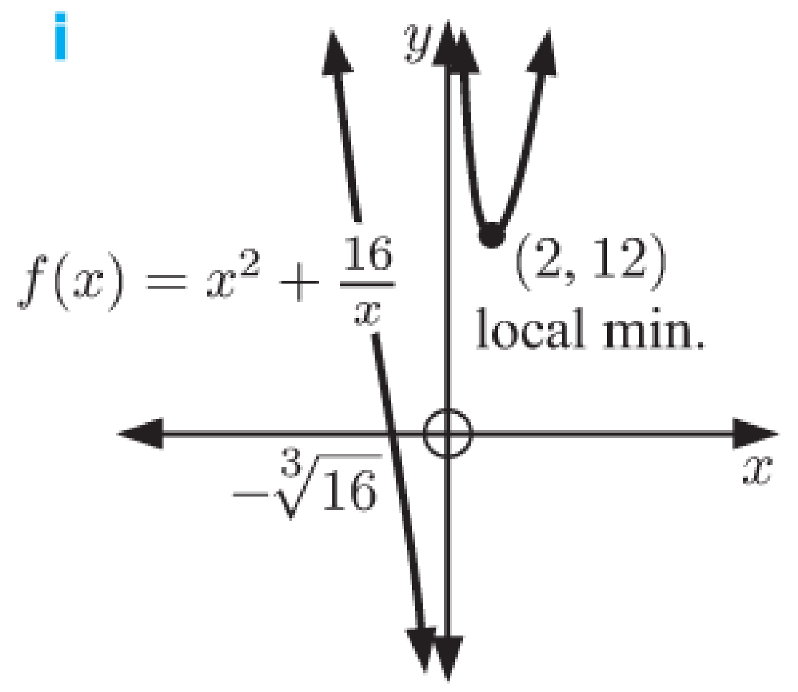


i



7 **a** $x = -\dfrac{b}{2a}$ **b** local min. if $a > 0$, local max. if $a < 0$

8 **a** $a = 9$ **b** $(-4, 113)$

9 **a** $a = -12$, $b = -13$ **b** $f'(x) = 3x^2 - 12$
c local maximum at $(-2, 3)$, local minimum at $(2, -29)$

10 $P(x) = -9x^3 - 9x^2 + 9x + 2$

11 **a** greatest value 63 when $x = 5$,
least value -18 when $x = 2$

b greatest value 4 when $x = 3$ and $x = 0$,
least value -16 when $x = -2$

c greatest value 20 when $x = 4$,
least value 12 when $x = 2$

d greatest value ≈ 7.13 when $x \approx 0.869$,
least value ≈ -6.76 when $x \approx -1.54$

REVIEW SET 11A

1 **a** $y = 4x + 2$ **b** $y = 7x - 14$ **c** $y = -1$

2 $a = 2,\ b = 3$ **3** $(-2, -25)$

4 **a** $y = -\frac{1}{15}x + \frac{182}{15}$ **b** $y = -\frac{1}{2}x + \frac{13}{2}$

5 $(10.1, -13.0)$

6 **a** P(2, 5) **b** $y = x + 3$ **c** $(-3, 0)$ **d** $y = -x + 7$

7 **a** increasing for $x \leqslant -1$ and $x \geqslant 4$,
decreasing for $-1 \leqslant x \leqslant 4$

b increasing for $x > -3$, decreasing for $x < -3$

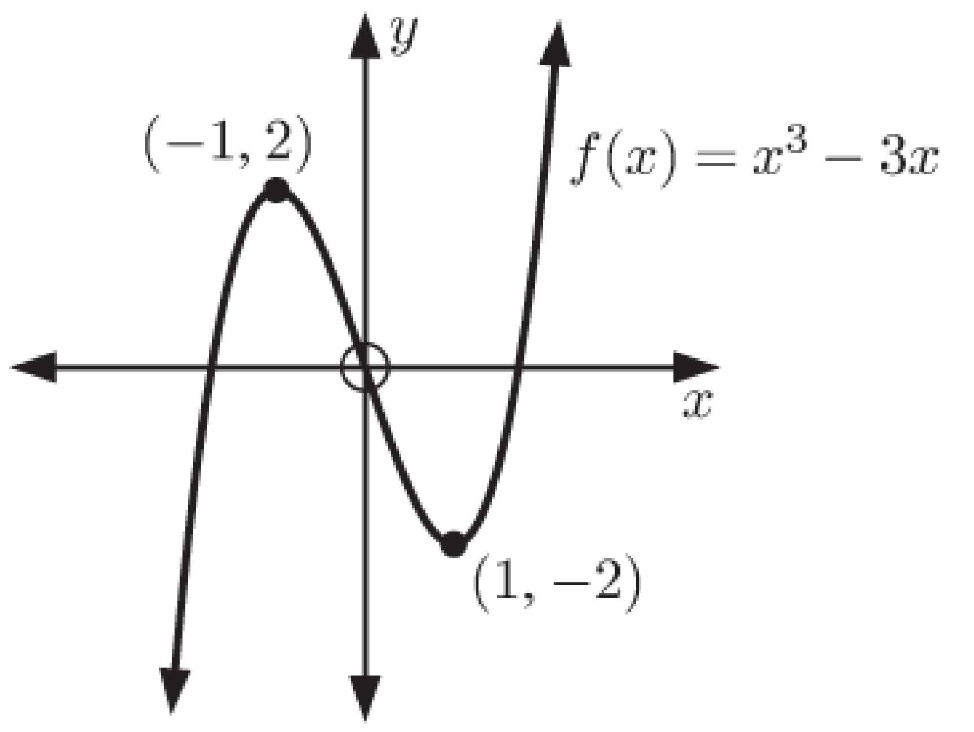
c increasing for $x \leqslant 6$, never decreasing

8 **a** 0

b $f'(x) = 3x^2 - 3$

c local maximum at $(-1, 2)$, local minimum at $(1, -2)$

d



9 **a** $f'(x) = 3x^2 - 12$ **b** 15 **c** Q(2, −12)

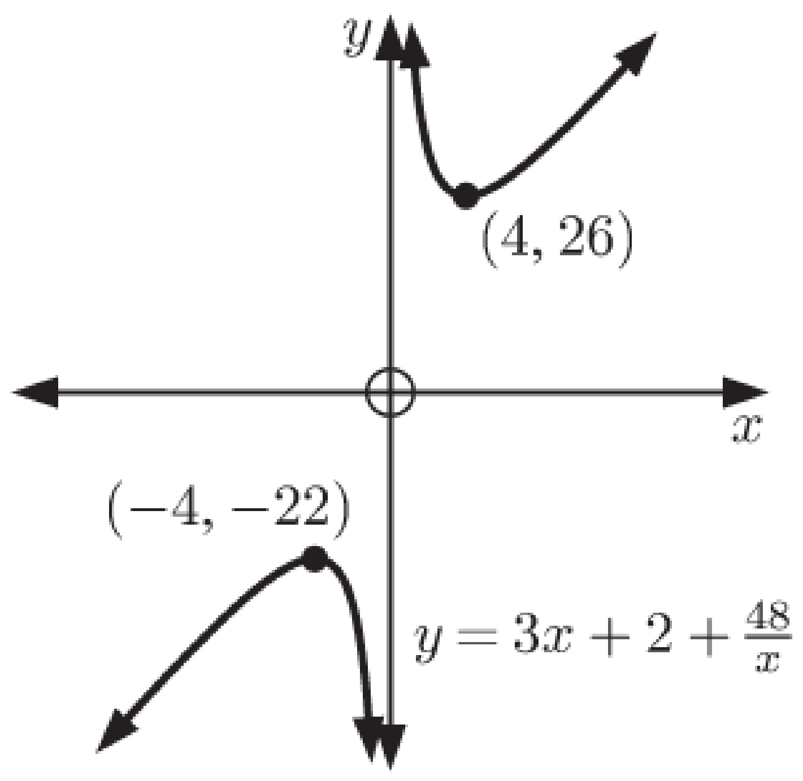
10 **a** $-6 \leqslant x \leqslant 2$ **b** $x \leqslant -6$ and $x \geqslant 2$

11 local maximum at $(1, 3)$, local minimum at $(\frac{1}{3}, \frac{77}{27})$

12 **a** $f'(x) = 3 - \dfrac{48}{x^2}$

b local maximum at $(-4, -22)$, local minimum at $(4, 26)$

c



13 **a** maximum value $25\frac{1}{3}$, minimum value $-1\frac{2}{3}$

b maximum value $10\frac{8}{25}$, minimum value 6

REVIEW SET 11B

1 **a** $y = 9x - 11$ **b** $y = 31x - 43$ **c** $y = -24x + 36$

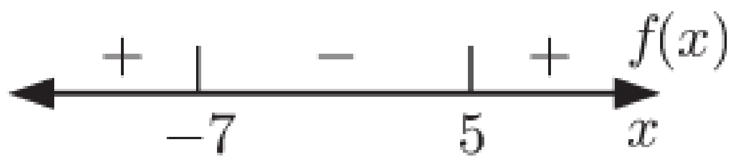
2 **a** $y = \frac{1}{2}x + 2$ **b** $x = 1$

3 **a** $a = 2$ **b** $y = 3x - 1$ **c** $(-4, -13)$

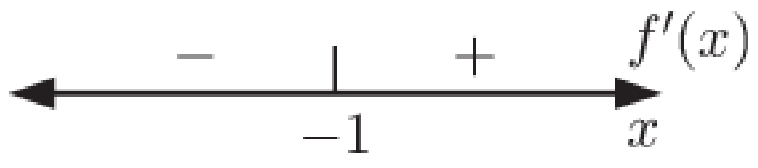
4 $(\frac{1}{2}, \frac{1}{4})$ **5** $a = 3,\ b = 7$

6 $(-1.32, -0.737)$ and $(1.32, -1.26)$

7 **a**



b



8 **a** $-1 \leqslant x \leqslant 0$ and $x \geqslant 4$ **b** $x \leqslant -1$ and $0 \leqslant x \leqslant 4$

9 **a** $a = -9$

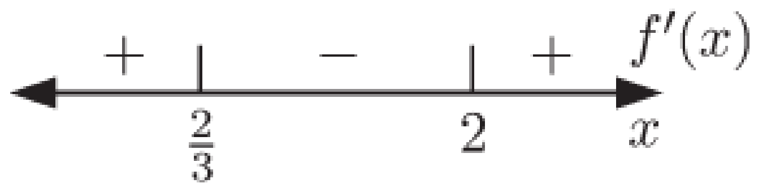
b local maximum at $(-1, 55)$, local minimum at $(3, 23)$

10 **a** $A = -3,\ B = 7$

b local maximum at $(-1, 9)$, local minimum at $(1, 5)$

11 **a** y-intercept is 0

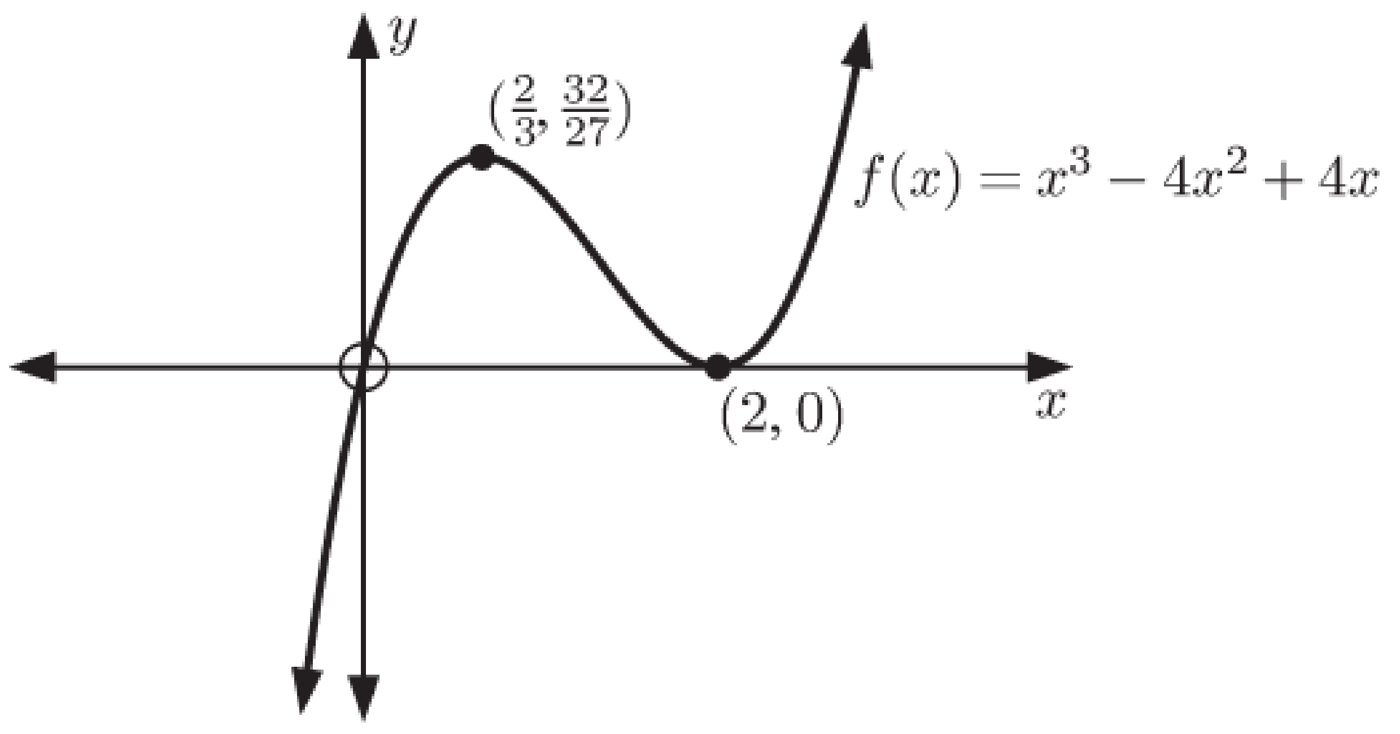
b $f'(x) = 3x^2 - 8x + 4$



c increasing for $x \leqslant \frac{2}{3}$ and $x \geqslant 2$,
decreasing for $\frac{2}{3} \leqslant x \leqslant 2$

d local maximum at $(\frac{2}{3}, \frac{32}{27})$, local minimum at $(2, 0)$

e



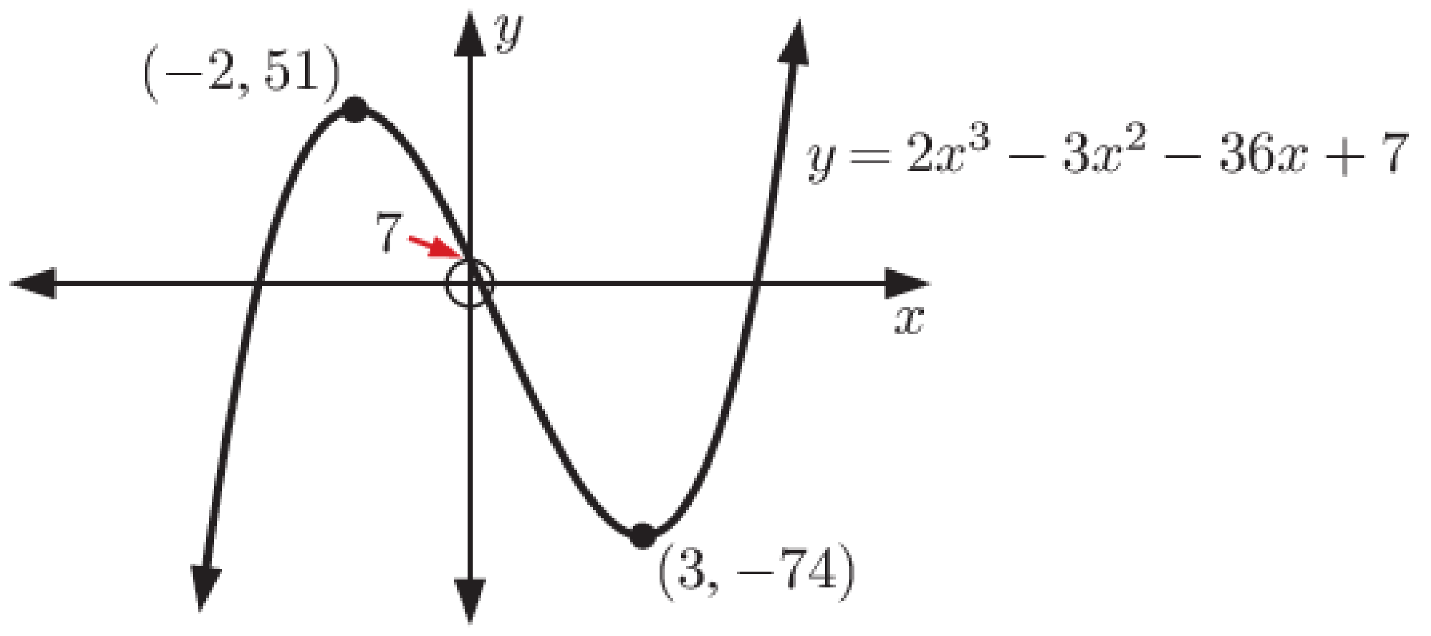
12 maximum value 21, minimum value 1

13 **a** $f'(x) = 6x^2 - 6x - 36$

b local maximum at $(-2, 51)$, local minimum at $(3, -74)$

c increasing for $x \leqslant -2$ and $x \geqslant 3$,
decreasing for $-2 \leqslant x \leqslant 3$

d



EXERCISE 12A.1

1 **a** $\dfrac{dM}{dt} = 3t^2 - 6t$ **b** $\dfrac{dR}{dt} = 8t + 4$

2 **a** **i** $\text{cm}^2\,\text{s}^{-1}$

ii $\dfrac{dA}{dt}$ tells us the rate at which the area is changing after t seconds.

b **i** $\text{m}^3\,\text{min}^{-1}$

ii $\dfrac{dV}{dt}$ tells us the rate at which the volume is changing after t minutes.

3 **a** \$118 000 **b** $\dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

c 20, which means that in 8 years from now, the business' profits will be increasing at a rate of \$20 000 per year.

4 **a** 190 m^3 per day **b** 180 m^3 per day

5 **a** $B'(t) = 0.9t^2 + 30$ million per hour
$B'(t)$ is the instantaneous rate of growth of the bacteria.

b $B'(3) = 38.1$ million per hour
After 3 hours, the bacteria are increasing at a rate of 38.1 million per hour.

c $B'(t)$ is always positive for $0 \leqslant t \leqslant 10$
$\therefore$ $B(t)$ is increasing for $0 \leqslant t \leqslant 10$.

6 **a** 1.2 m

b $s'(t) = 28.1 - 9.8t$; this is the speed of the ball (in $\text{m}\,\text{s}^{-1}$).

c $t \approx 2.87$ s; the ball has reached its maximum height.

d ≈ 41.5 m

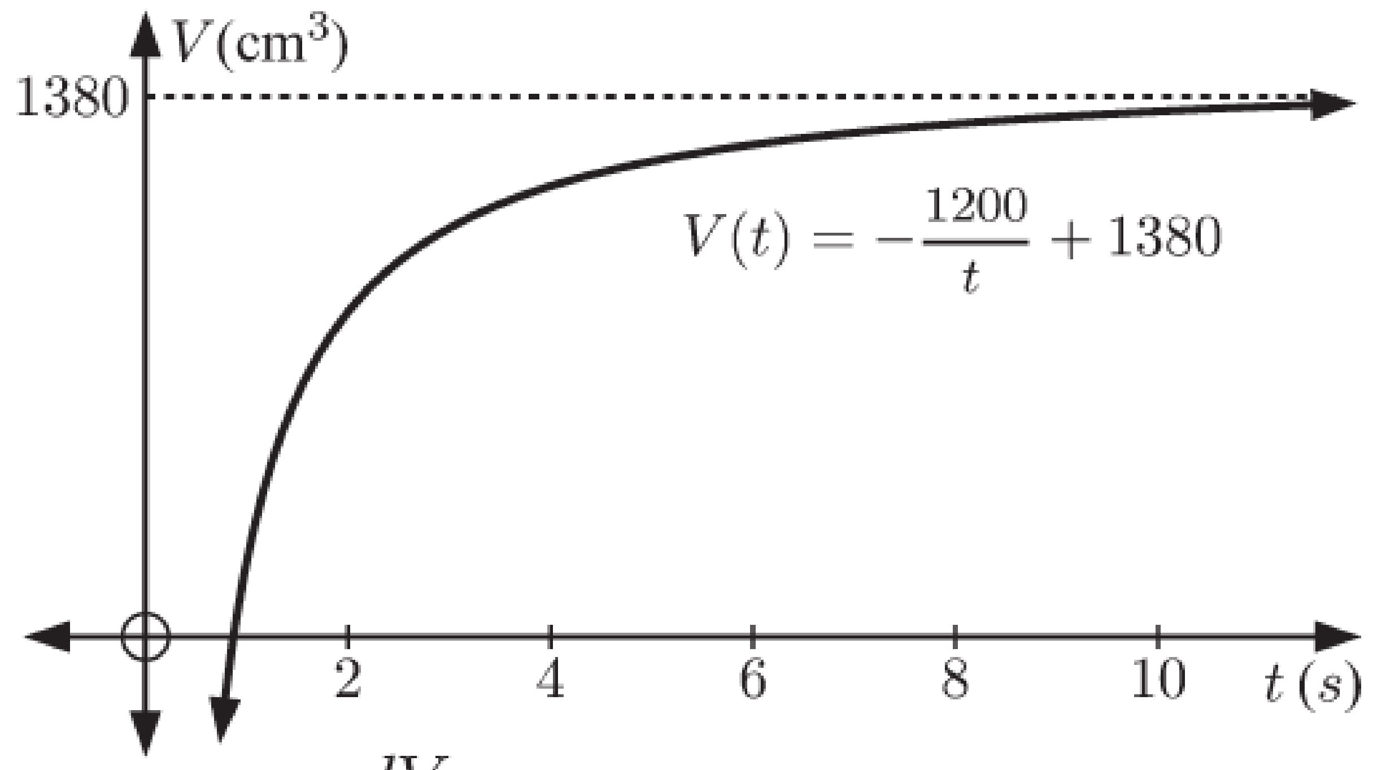
e **i** 28.1 m s^{-1} **ii** 8.5 m s^{-1} **iii** -20.9 m s^{-1}

The sign tells us whether the ball is travelling upwards $(+)$ or downwards $(-)$.

f ≈ 5.78 s

7 **a** $\dfrac{1200}{x^2}$ cm^3/s **b** **i** 300 cm^3/s **ii** $33\frac{1}{3}$ cm^3/s

c



d as $t \to \infty$, $\dfrac{dV}{dt} \to 0$, the amount of air in the tyre reaches its maximum

8 **a** **i** $\dfrac{dh}{dt} = 0.2t + 0.15$ metres per year

ii 0.35 m per year **iii** 2.2 m

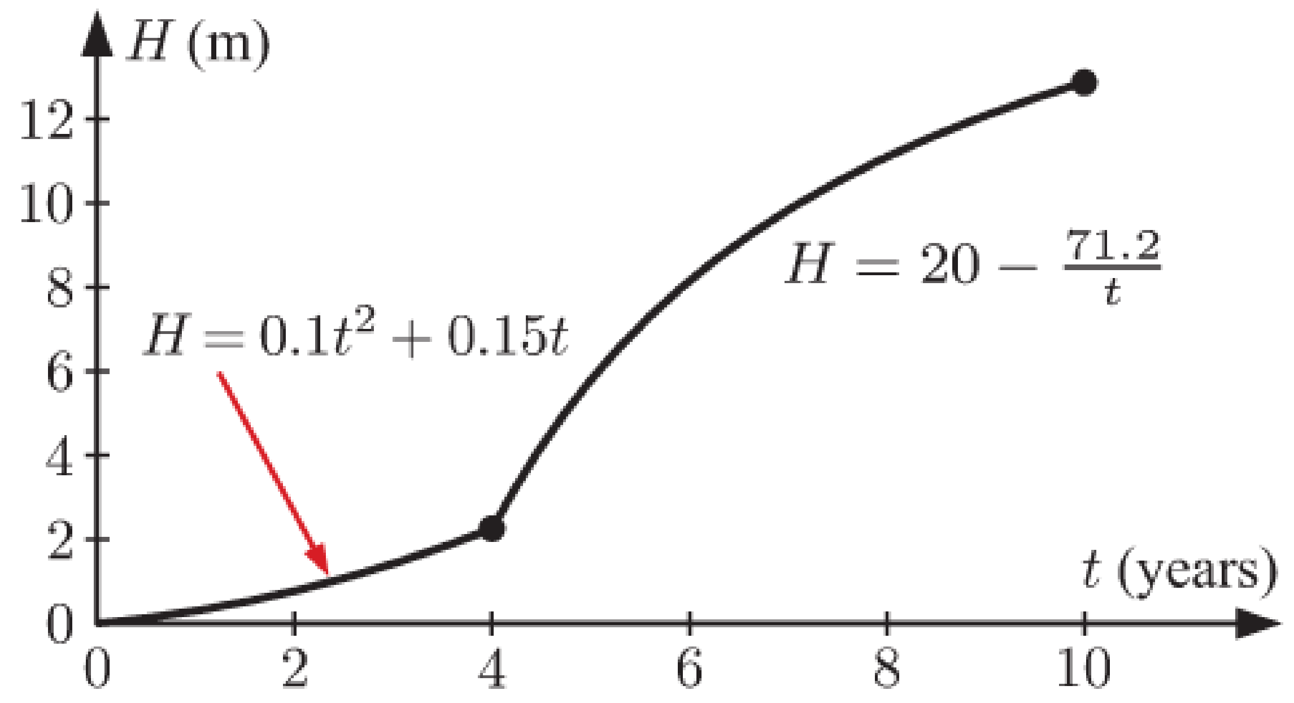
b **i** $k = 71.2$ **ii** $\dfrac{dH}{dt} = \dfrac{71.2}{t^2}$

iii Yes, since $\dfrac{dh}{dt} \neq \dfrac{dH}{dt}$.

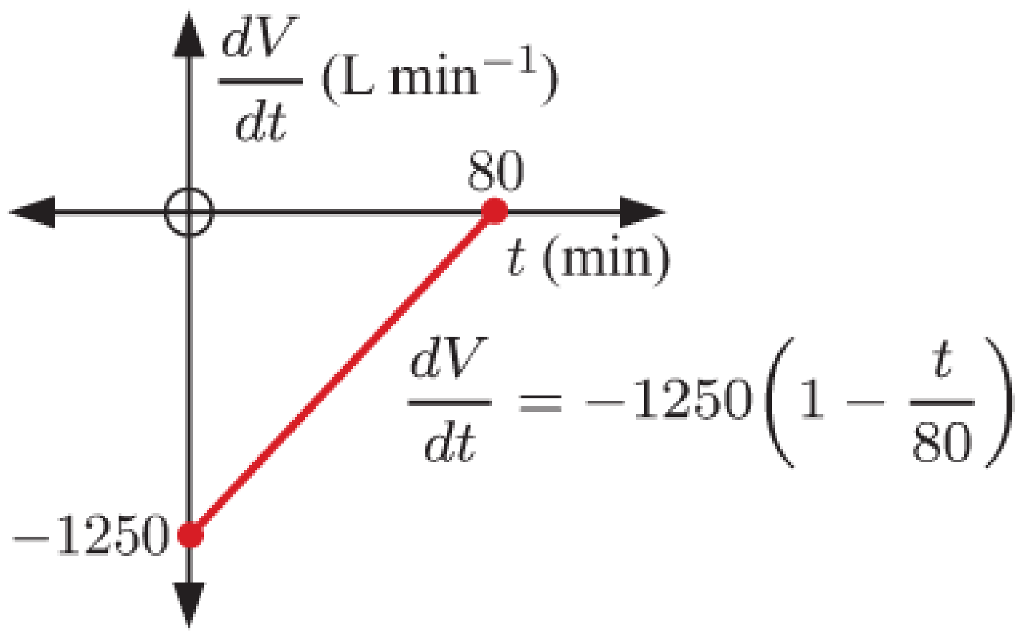
iv It means that according to the model the tree will never stop growing.

v After 10 years, height $= 12.88$ m
After 20 years, height $= 16.44$ m
After 50 years, height $= 18.576$ m
The rate of growth is slowing as the years progress.

c



9 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$ L min^{-1}



b At $t = 40$ min, $V = -625$ L min^{-1} which means the tank is draining water at a rate of 625 L min^{-1}.

c At time $t = 0$ when the tap was first opened.

EXERCISE 12A.2

1 **a** $\dfrac{dT}{dr} = 2r + \dfrac{100}{r^2}$ **b** $\dfrac{dA}{dh} = 2\pi + \frac{1}{2}h$

2 pounds per item produced

3 **a** $C(0) = \$14\,230$, which is the fixed operation cost without producing any items.

b $C'(x) = -0.000\,021\,6x^2 + 0.0122x + 18$
This is the rate at which the production cost (in dollars) is increasing per item when x items are made. It gives an estimate of the cost of making the $(x+1)$th item each day.

c $C'(300) = 19.716$; the cost of producing the 301st item each day is $\approx \$19.72$.

d $C(301) - C(300) \approx \$19.72$

4 **a** $C'(x) = 7 - 0.0002x$

b $C'(220) \approx \$6.96$; this estimates the cost of making the 221st pair of jeans each day.

c $C(221) - C(220) \approx \$6.96$; this is the actual cost of making the 221st pair of jeans each day.

5 **a** **i** 4500 euros **ii** 4000 euros

b **i** decreasing at ≈ 210.22 euros per km h^{-1}
ii increasing at ≈ 11.31 euros per km h^{-1}

c ≈ 79.4 km h^{-1}

6 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

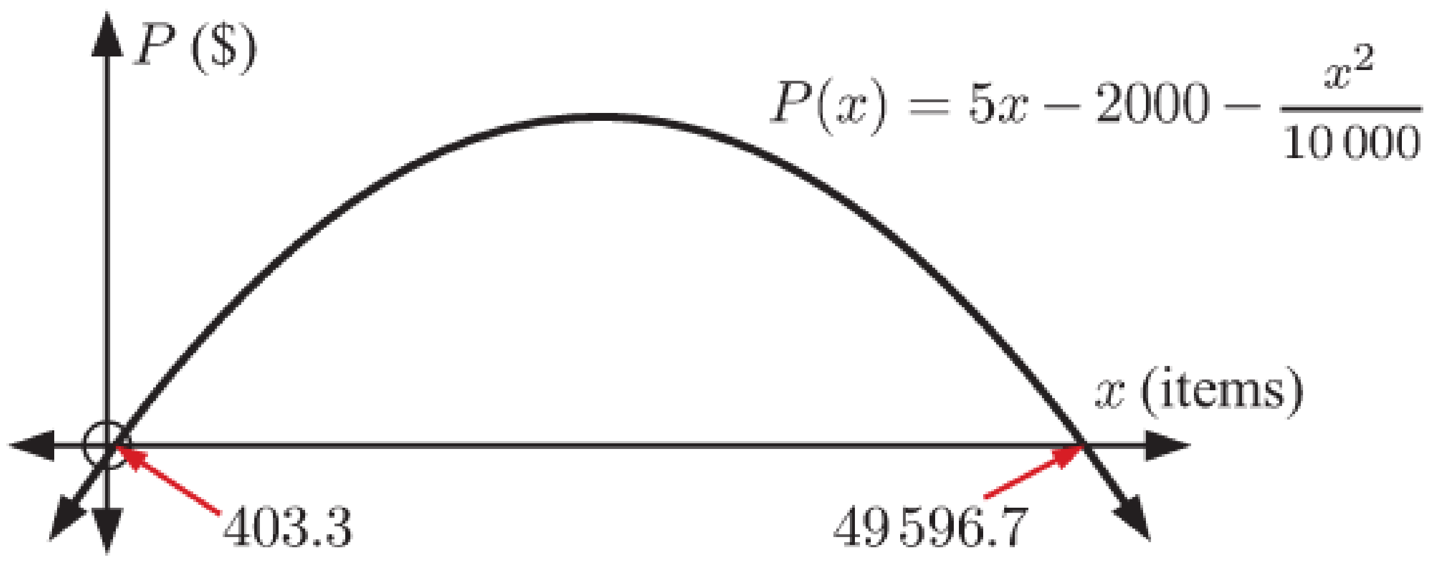
b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}$, $\dfrac{dy}{dx} = 0.175$, the height of the hill is increasing as the gradient is positive.

When $x = 1\frac{1}{2}$, $\dfrac{dy}{dx} = -0.225$, the height of the hill is decreasing as the gradient is negative.

c ≈ 2.55 km from the sea, ≈ 63.1 m deep

7 **a**



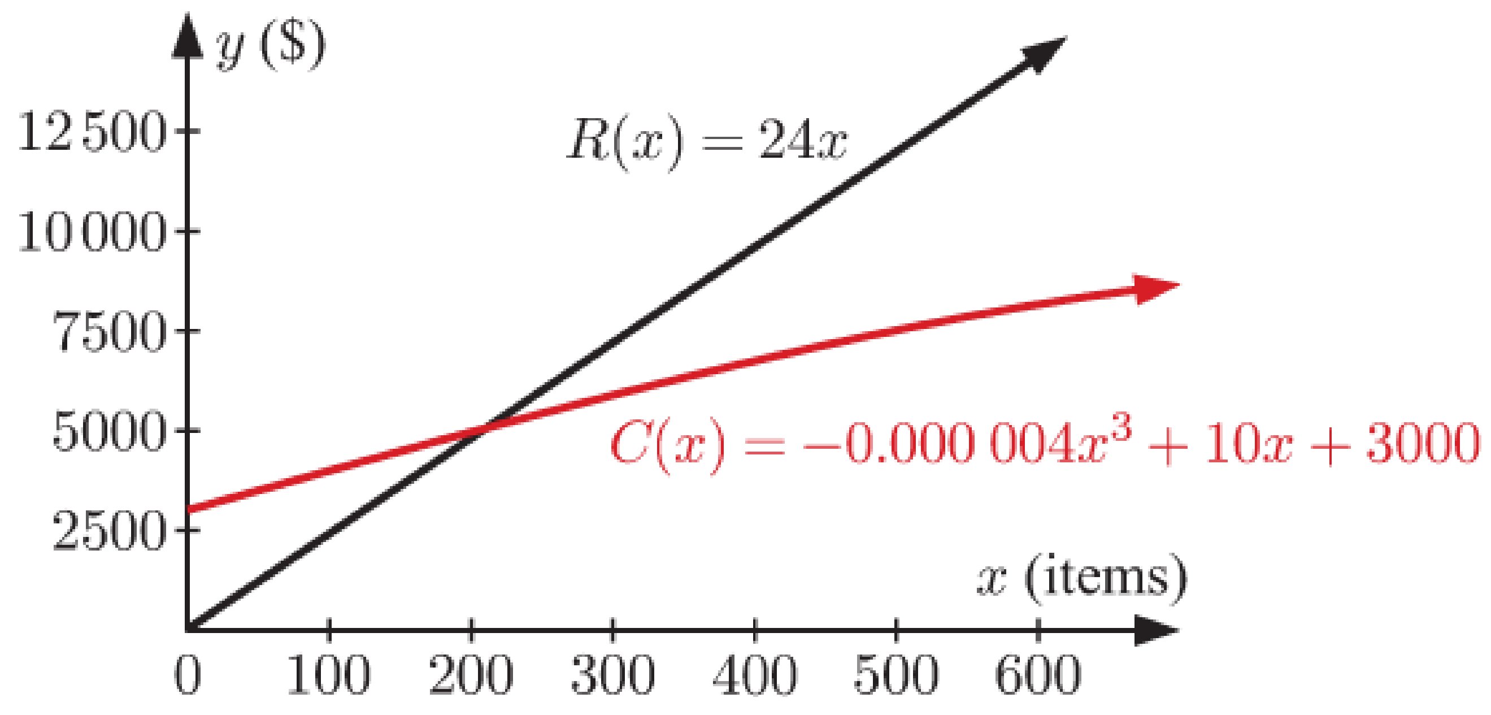
b $404 \leqslant x \leqslant 49\,596$

c $P'(x) = 5 - \dfrac{x}{5000}$; this is the rate at which the profit earned is increasing or decreasing when producing x items per year.

d $0 \leqslant x \leqslant 25\,000$

8 **a** $R(x) = 24x$

b



c $C(x) = R(x)$ when $x \approx 212$. This is the breakeven point, when the revenue is equal to the cost of producing the items.

d $P(x) = R(x) - C(x) = 0.000\,004x^3 + 14x - 3000$ dollars

e $P'(x) = 0.000\,012x^2 + 14$

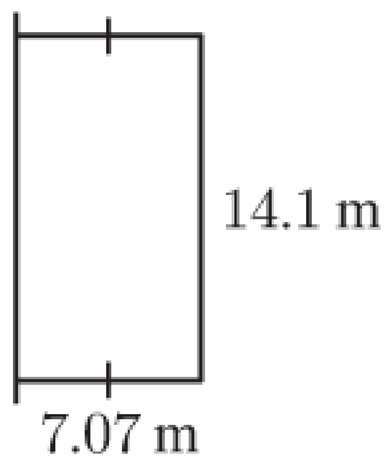
f $P'(120) \approx 14.17$, which means that the total profit is increasing at a rate of $\approx \$14.17$ per item when 120 items are produced.

EXERCISE 12B.1

1 250 items per day **2** ≈ 53.8 m **3** **b** 15 m × 30 m

4 **b** $L_{\min} \approx 28.3$ m, when $x \approx 7.07$

c



5 **c** $C = 27x - \dfrac{x^3}{4}$ **d** $\dfrac{dC}{dx} = 27 - \frac{3}{4}x^2$

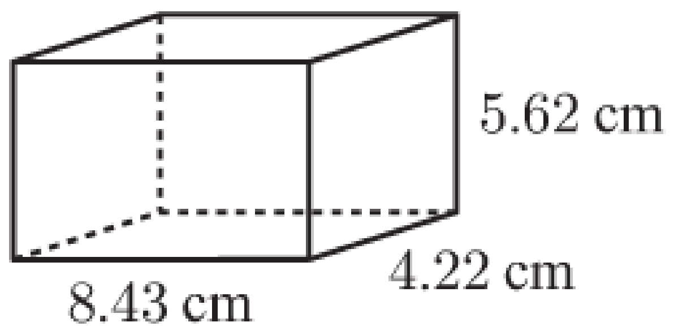
e $\dfrac{dC}{dx} = 0$ when $x = 6$ **f** 6 cm by 6 cm

6 **a** **Hint:** $V = 200 = 2x \times x \times h$

b **Hint:** Show $h = \dfrac{100}{x^2}$ and substitute into the surface area equation.

c $A_{\min} \approx 213$ cm^2, when $x \approx 4.22$

d

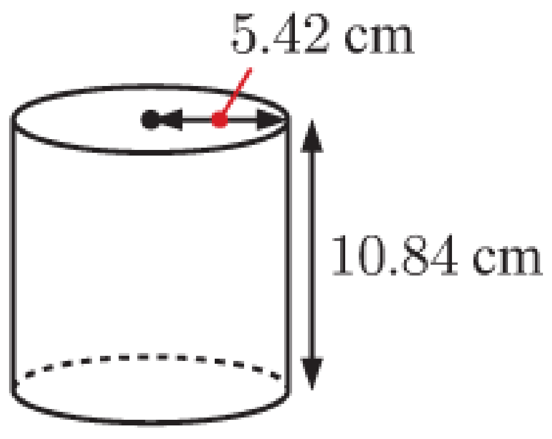


7 **b** 6 cm × 6 cm **8** 10 blankets per day

9 **a** **Hint:** Recall that $V_{\text{cylinder}} = \pi r^2 h$ and that 1 L = 1000 cm^3.

b **Hint:** Recall that $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$.

c



10 **a** $0 \leqslant x \leqslant \frac{200}{\pi} \approx 63.7$

b $l = 100$, $x = \frac{100}{\pi} \approx 31.8$, $A = \frac{20\,000}{\pi} \approx 6370$ m^2

11 **a** $y = 30 - x$ **b** $A(x) = x(30 - x)$ cm^2

c $A'(x) = 30 - 2x$ **d** $x = 15$, 15 cm × 15 cm

12 **a** 2.225 m **b** yes, at $x = 3.5$

c $h'(0) = -2.471$, $h'(7) = 2.471$
The units are metres per metre.

13 **a** $(40 - 2x)$ cm × $(33 - \frac{3}{2}x)$ cm × x cm

c $V'(x) = 9x^2 - 252x + 1320$

d maximum volume ≈ 4100 cm^3 when dimensions are ≈ 26.0 cm × 22.5 cm × 6.98 cm.

14 **b** $\dfrac{d}{dx}(A^2) = 200x - 4x^3$, A^2 is maximised when $x = \sqrt{50}$.

c $\sqrt{50}$ cm × $\sqrt{50}$ cm

15 **a** $D = \sqrt{(x-1)^2 + (x^2-4)^2}$ {distance formula}

$\therefore\ D^2 = (x-1)^2 + (x^2-4)^2$

b $\dfrac{d}{dx}(D^2) = 4x^3 - 14x - 2$; D^2 is minimised when $x \approx 1.94$.

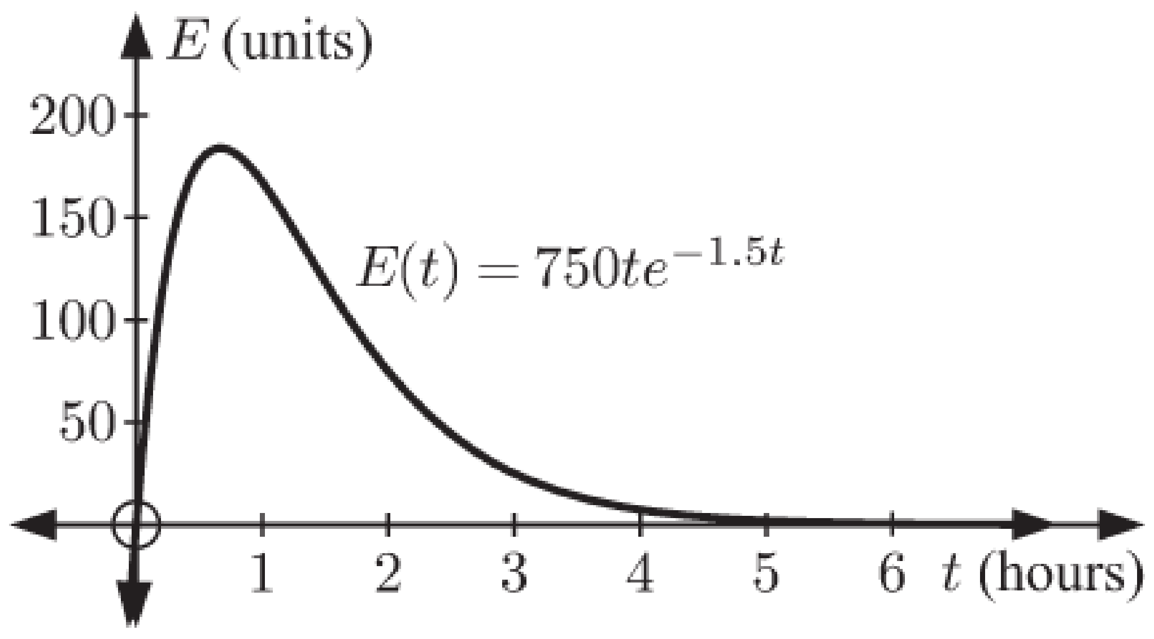
c (1.94, 3.76), $D \approx 0.969$ units from A

16 **a** **Hint:** Use the cosine rule. **b** ≈ 3550 **c** ≈ 5:36 pm

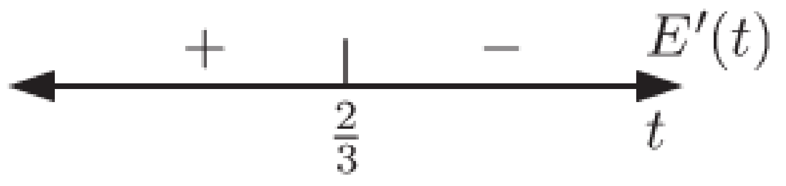
17 **b** ≈ 3.37 m

EXERCISE 12B.2

1 **a**



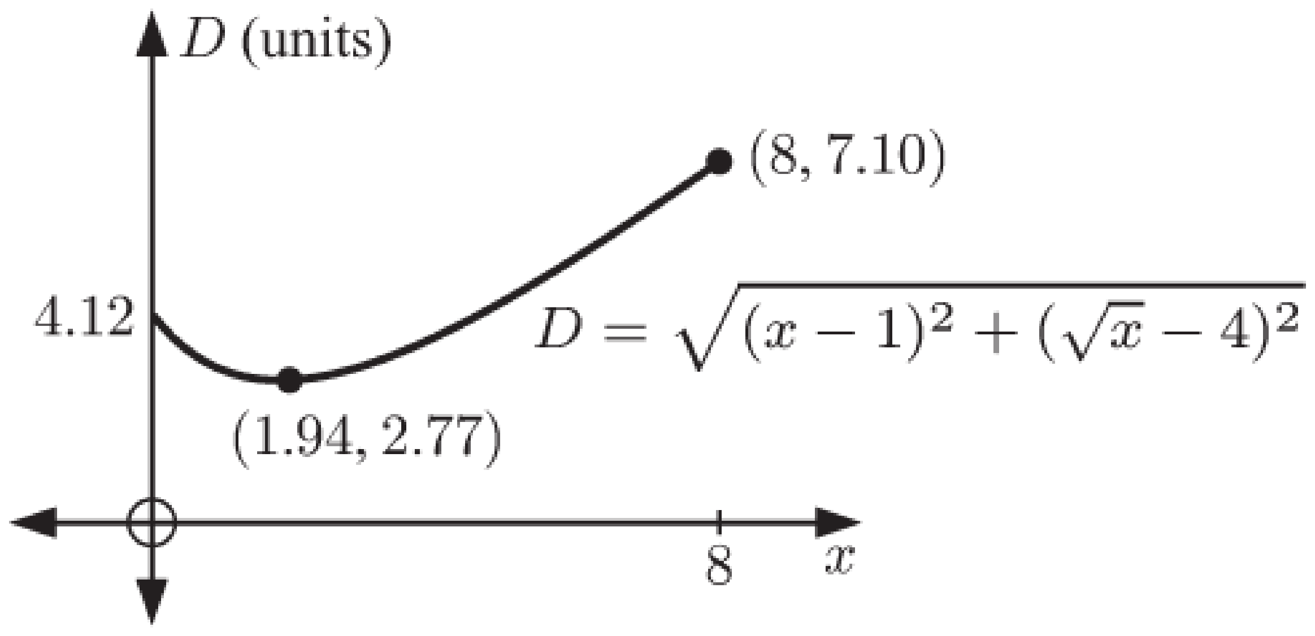
b



c 40 minutes after being administered

2 **a** $D = \sqrt{(x-1)^2 + (\sqrt{x}-4)^2}$ {distance formula}

b



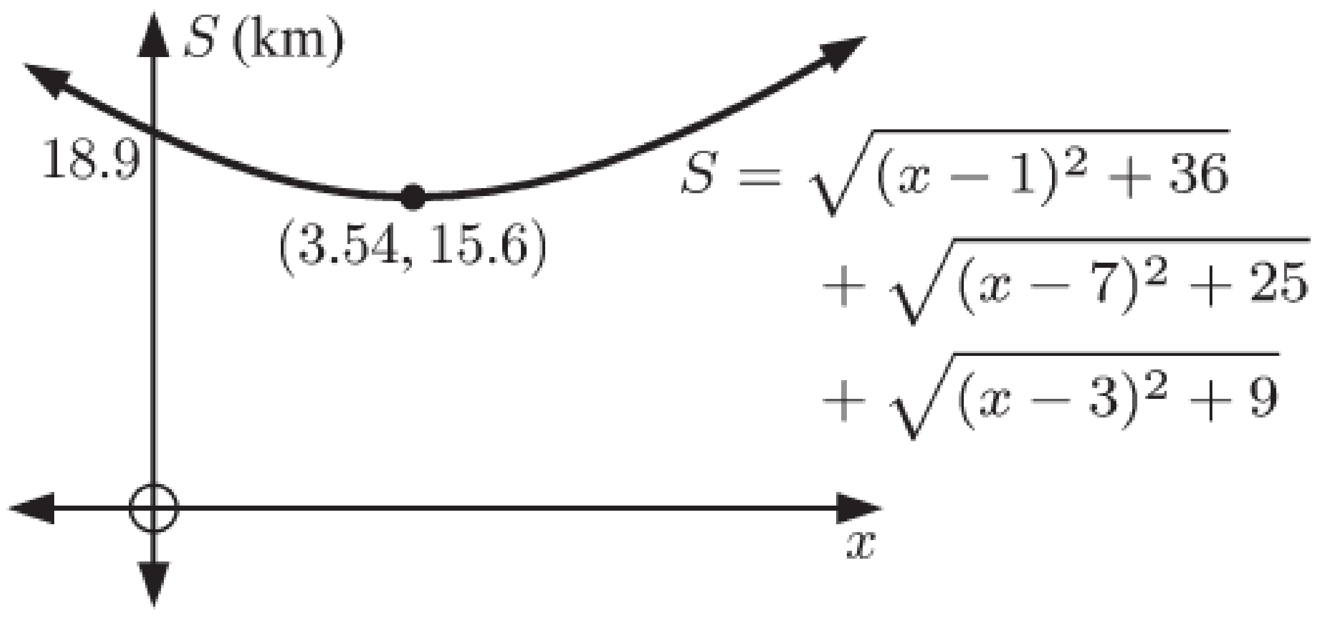
c $D \approx 2.77$ units, when $x \approx 1.94$

d $\dfrac{dD}{dx} = 0$ when $x \approx 1.94$

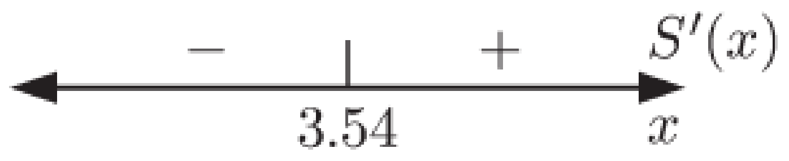
3 **b** PA $= \sqrt{(x-1)^2 + 36}$, PB $= \sqrt{(x-7)^2 + 25}$

c $S = \sqrt{(x-1)^2 + 36} + \sqrt{(x-7)^2 + 25}$
$+ \sqrt{(x-3)^2 + 9}$

d



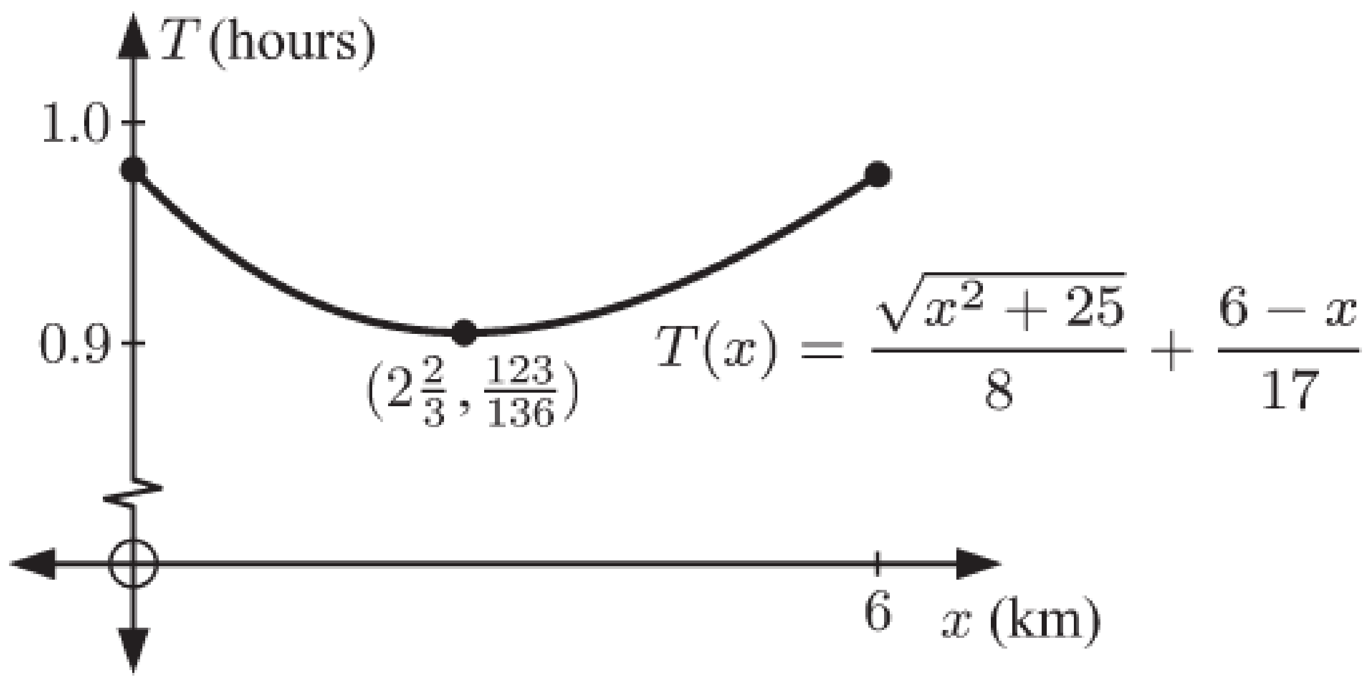
e



f at (3.54, 8)

4 **a** X is between A and C.

c

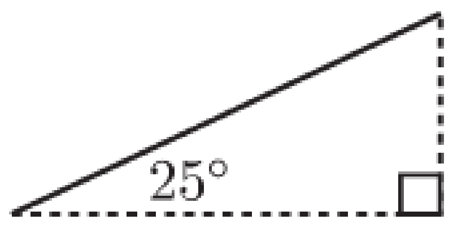


d $x = 2\frac{2}{3}$; Peter should swim to the point $2\frac{2}{3}$ km from A, then run to C, to minimise his total time.

EXERCISE 12C

1 **a** $h(0) = 3$ $\therefore$ $c = 3$ **b** $h'(x) = 2ax + b$

c



$h'(0)$ = gradient = tan 25°
$\therefore$ b = tan 25°

d $x \approx 14.5$

2 **a** $C(x) \approx -0.178x^3 + 15.3x^2 + 3280x + 24\,500$

b €331 700

3 **a** $P(0) = 21.5$ $\therefore$ $d = 21.5$

b $P'(t) = 3at^2 + 2bt + c$ **c** $t = 15$

d
$$3375a + 225b + 15c = 5$$
$$125\,000a + 2500b + 50c = -13.9$$
$$675a + 30b + c = 0$$
$\therefore$ $P(t) \approx 0.000\,136t^3 - 0.0263t^2 + 0.697t + 21.5$

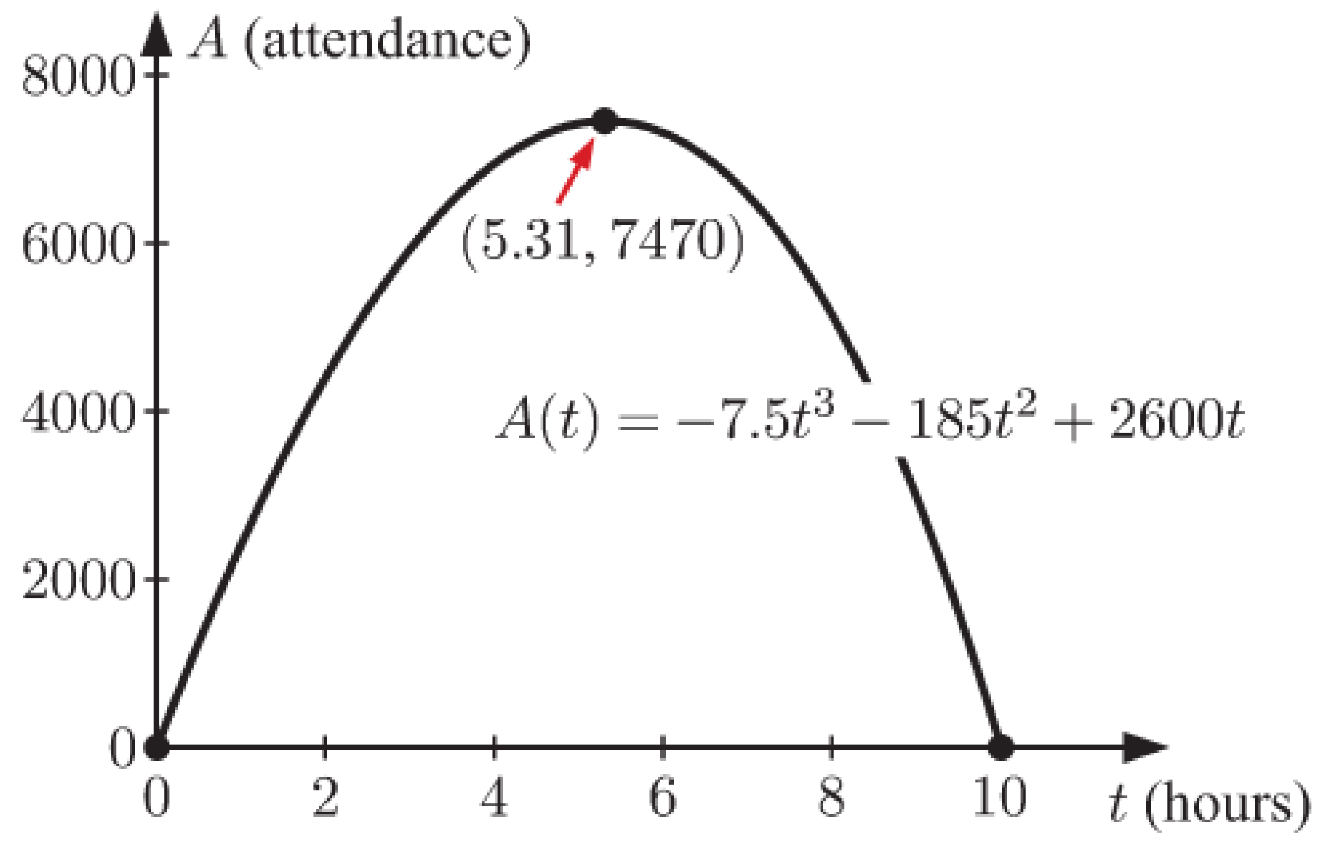
e **i** ≈ 22.4 million **ii** ≈ −1.98 million

f The model provided a reasonable estimate when interpolating between data points. The extrapolation in **e ii** however predicted a negative population, which is not possible.

4 **a** $A(0)=0,\ \ A(10)=0,\ \ A'(2)=1770,\ \ A'(8)=-1800$

b $A(t)=-7.5t^3-185t^2+2600t,\quad 0\leqslant t\leqslant 10$

c



The model seems reasonable; the attendance was always increasing until shortly after 1 pm, then it began to decrease.

d $\approx$ 7470 people at 1:19 pm

5 **a** **B**, Ir^2 is approximately constant

b $I\approx\dfrac{180\,266}{r^2}$ **c** $\dfrac{dI}{dr}\approx-\dfrac{360\,532}{r^3}$

d decreasing by $\approx$ 0.361 lux per cm

e The surface area of a sphere is proportional to the square of its radius. The total illuminance from a point is a sphere. Since $SA\propto r^2$ and $I\propto\dfrac{1}{r^2}$ the total illuminance is a constant.

REVIEW SET 12A

1 **a** $C(0)=5400$ dollars, which is the fixed operating cost without producing any items.

b $C'(x)=-0.0006x^2+24$; this is the rate at which the production cost (in dollars) is increasing per item when x thousand pairs of chopsticks are produced. It estimates the cost of producing the $(x+1)$ thousandth pair of chopsticks.

c $C'(100)=18$; the cost of producing the 101st lot of one thousand pairs of chopsticks is approximately \$18.

d $C(101)-C(100)\approx 17.94$, so the actual cost is approximately \$17.94 .

2 **a** **i** €312 **ii** €1218.75

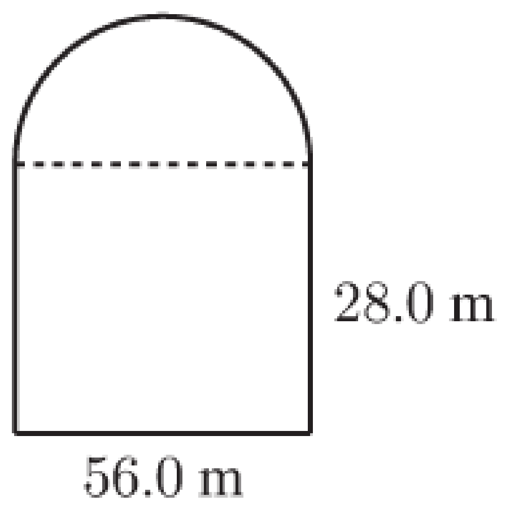
b **i** €9.10 per hour per km h^{-1}

ii €7.50 per hour per km h^{-1}

c 3 km h^{-1}

3 **b** $\mathrm{C}(\sqrt{3},\,6)$

4 **c**



5 **a** $y=\dfrac{1}{x^2}$ **c** $C'(x)=48x-\dfrac{96}{x^2}$

d 1.26 m × 1.26 m × 0.630 m, total cost $\approx$ \$114.29

6 **a** $\{t\mid 0\leqslant t\leqslant 5.9\}$ **b** $\approx$ 3.33 s **c** $\approx$ 49.8 m

d $\approx$ 36.0 m

7 **a** $h(0)=1.6\ \ \therefore\ \ c=1.6$

b $h'(0)=16.4\ \ \therefore\ \ b=16.4$ **c** $a=-4.9$

d $\approx$ 15.3 m at $t\approx 1.67$

REVIEW SET 12B

1 **a** $H'(t)=19-1.6t$ m s^{-1}

b $H'(0)=19$ m s^{-1}, $H'(10)=3$ m s^{-1}, $H'(20)=-13$ m s^{-1}

These are the instantaneous speeds at $t=0$, 10, and 20 s.
A positive sign means the ball is travelling upwards.
A negative sign means the ball is travelling downwards.

c $\approx$ 23.8 s

2 **a** \$4930.25

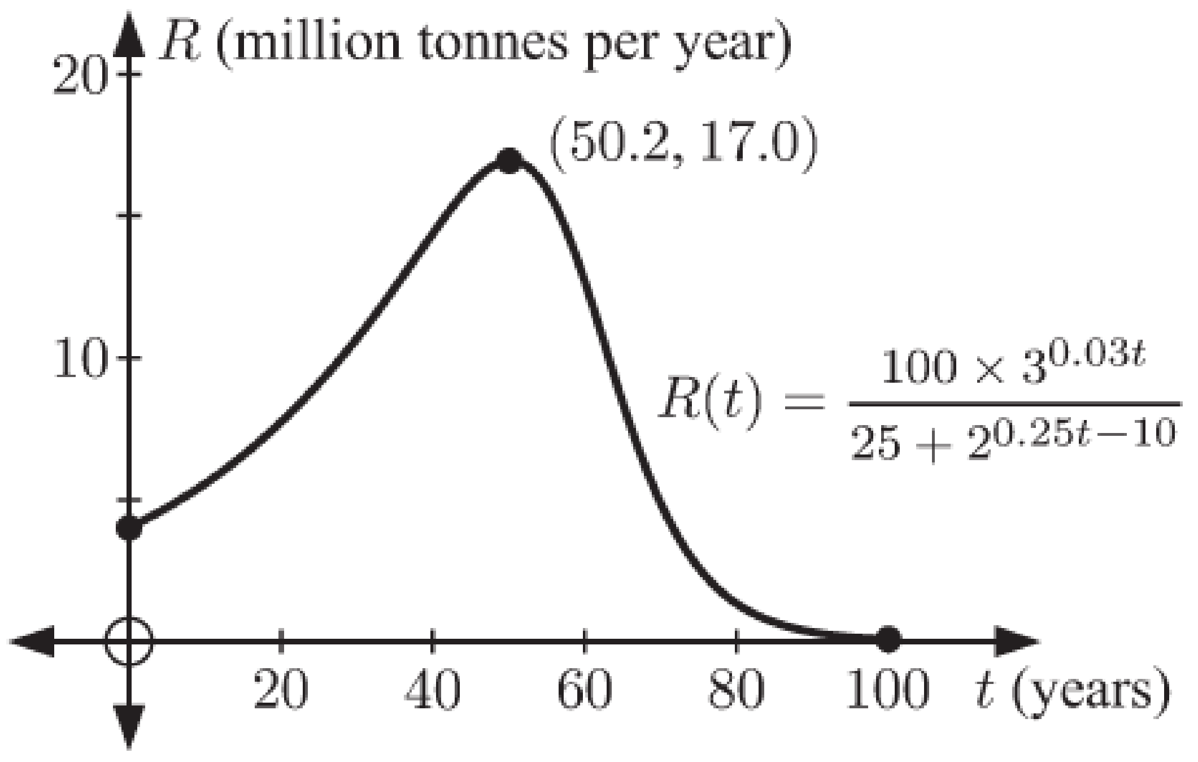
b **i** decreasing at $\approx$ \$1.39 per km h^{-1}

ii increasing at $\approx$ \$2.83 per km h^{-1}

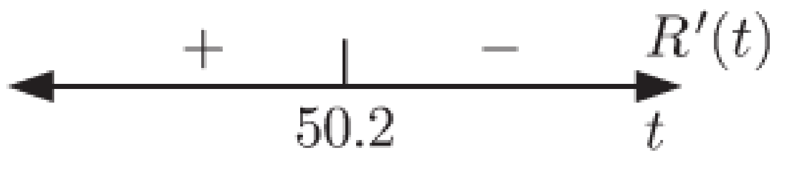
c $\approx$ 79.4 km h^{-1}

3 6 cm from each end **4** $x\approx 2.11$

5 **a**



b



c $\approx$ 7.72 million tonnes per year

d $\approx$ 28.0 years and $\approx$ 63.1 years after mining begins

e $\approx$ 17.0 million tonnes per year, $\approx$ 50.2 years after mining begins

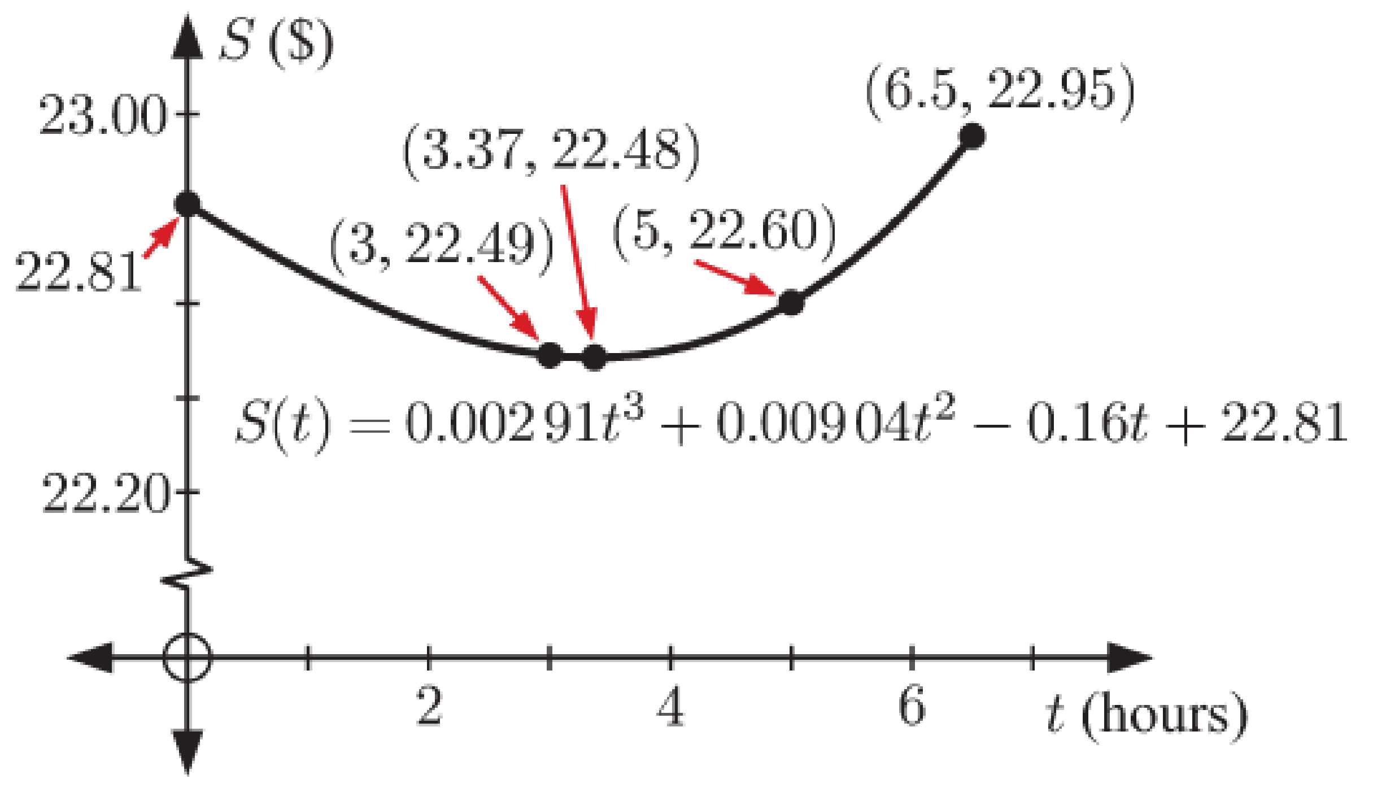
6 **a** $P(x)=133\,100-\dfrac{300\,000}{x}-950x$ **b** $\approx$ 18 days

7 **a** $S(0)=22.81\ \ \therefore\ \ d=22.81$

b $S'(0)=-0.16\ \ \therefore\ \ c=-0.16$

c $a\approx 0.002\,91,\ \ b\approx 0.009\,04$

d



e $\approx$ \$22.95

f minimum $\approx$ \$22.48 at 12:52 pm,
maximum $\approx$ \$22.95 at 4 pm

EXERCISE 13A.1

1 **a** **i** 0.4 units^2 **ii** 0.6 units^2 **b** 0.5 units^2

2 **a** $\approx$ 0.653 units^2 **b** $\approx$ 0.737 units^2

3

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3}$.

4 **a** **i**

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

ii

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

iii

n	A_L	A_U
5	0.113 28	0.313 28
10	0.153 33	0.253 33
50	0.190 13	0.210 13
100	0.195 03	0.205 03
500	0.199 00	0.201 00
1000	0.199 50	0.200 50
10 000	0.199 95	0.200 05

iv

n	A_L	A_U
5	0.083 20	0.283 20
10	0.120 83	0.220 83
50	0.156 83	0.176 83
100	0.161 71	0.171 71
500	0.165 67	0.167 67
1000	0.166 17	0.167 17
10 000	0.166 62	0.166 72

b **i** $\frac{1}{2}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{5}$ **iv** $\frac{1}{6}$ **c** $\dfrac{1}{a+1}$ units^2

5 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $n = 10\,000$

EXERCISE 13A.2

1 **a** ≈ 2.3479 units^2 **b** ≈ 7.25 units^2

2 **a** $4\frac{1}{2}$ units^2

b $y = 3 - x$ is a straight line.
We are finding the area of a triangle with base and height each 3 units.

3 **a** ≈ 5.2650 units^2 **b** ≈ 0.1346 units^2
c ≈ 0.9920 units^2

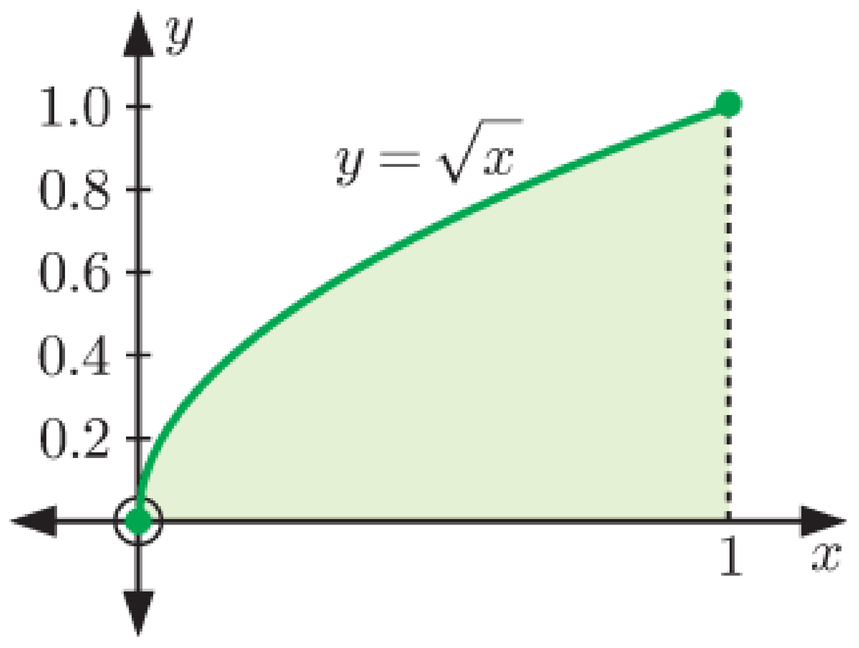
4 **a**

n	*Area estimate*
8	3.0898
40	3.1369
100	3.1404
1000	3.1416

b $n = 1000$

EXERCISE 13B

1 **a**



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\displaystyle\int_0^1 \sqrt{x}\, dx \approx 0.67$

d Using the trapezoidal method with 8 subintervals,

$$\int_0^1 \sqrt{x}\, dx \approx 0.6581\,.$$

With just 8 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 50$.

2 **a** $A_L = \dfrac{2}{n}\displaystyle\sum_{i=0}^{n-1}\sqrt{1+{x_i}^3}, \quad A_U = \dfrac{2}{n}\displaystyle\sum_{i=1}^{n}\sqrt{1+{x_i}^3},$

where $x_i = \dfrac{2i}{n}$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

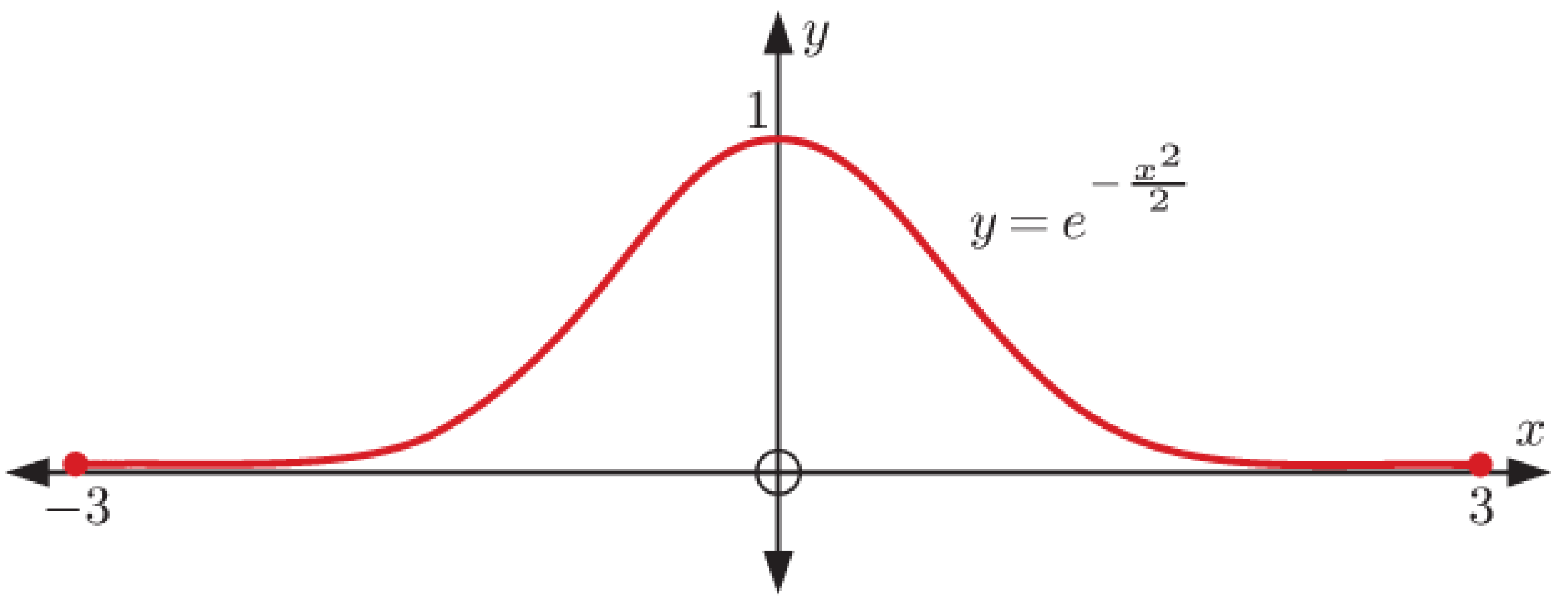
c $\displaystyle\int_0^2 \sqrt{1+x^3}\, dx \approx 3.24$

d Using the trapezoidal method with 10 subintervals,

$$\int_0^2 \sqrt{1+x^3}\, dx \approx 3.2480\,.$$

With just 10 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 100$.

3 **a**



b lower ≈ 1.2493, upper ≈ 1.2506

c $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx \approx 2.4999, \quad \sqrt{2\pi} \approx 2.5066$

d 7 subintervals

4 **a** 18 **b** 4.5 **c** 2π **5** **b** **i** 0 **ii** 22

EXERCISE 13C

1 **a** $F'(x) = 2x$
$\therefore$ the antiderivative of $2x$ is x^2.
b 8 units^2

2 **a** $F'(x) = 3x^2$
$\therefore$ the antiderivative of $3x^2$ is x^3.
b 1 unit^2

3 **a** $F'(x) = x^3$
$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b **i** 4 units^2 **ii** $16\tfrac{1}{4}$ units^2 **iii** $20\tfrac{1}{4}$ units^2

d $\displaystyle\int_0^3 x^3\,dx = \int_0^2 x^3\,dx + \int_2^3 x^3\,dx$

EXERCISE 13D

1 **a** $\dfrac{d}{dx}(x^2) = 2x$

$\therefore$ the antiderivative of x is $\tfrac{1}{2}x^2$.

b $\tfrac{1}{2}x^2 + c$

2 **a** $\dfrac{d}{dx}(x^3) = 3x^2$

$\therefore$ the antiderivative of x^2 is $\tfrac{1}{3}x^3$.

b $\tfrac{1}{3}x^3 + c$

3 **a** $\dfrac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$.

b $-\dfrac{1}{x} + c$

4 **a** $\dfrac{d}{dx}(x^{-2}) = -2x^{-3}$

$\therefore$ the antiderivative of x^{-3} is $-\tfrac{1}{2}x^{-2}$.

b $-\dfrac{1}{2x^2} + c$

5 **a** The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

b $\tfrac{1}{6}x^6$

6 $\dfrac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\therefore \displaystyle\int (3x^2 + 2x)\,dx = x^3 + x^2 + c$

7 $\dfrac{d}{dx}(3x^4 - 2x^2) = 12x^3 - 4x$

$\therefore \displaystyle\int (3x^3 - x)\,dx = \tfrac{3}{4}x^4 - \tfrac{1}{2}x^2 + c$

8 $\dfrac{d}{dx}\left(\dfrac{1}{x} + 2x\right) = -\dfrac{1}{x^2} + 2$

$\therefore \displaystyle\int \left(\frac{1}{x^2} - 2\right) dx = -\frac{1}{x} - 2x + c$

EXERCISE 13E

1 **a** $3x + c$ **b** $-2x + c$ **c** $x^2 + c$

d $x^3 + c$ **e** $x^5 + c$ **f** $-\tfrac{1}{8}x^4 + c$

g $\tfrac{2}{15}x^5 + c$ **h** $-\dfrac{2}{x} + c$ **i** $-\dfrac{1}{4x^2} + c$

2 **a** $x^2 - x + c$ **b** $\tfrac{1}{2}x^2 + 3x + c$ **c** $4x - \tfrac{1}{2}x^2 + c$

d $\tfrac{3}{4}x^2 + \tfrac{1}{2}x + c$ **e** $\tfrac{1}{3}x^3 - 2x + c$ **f** $5x - \tfrac{1}{3}x^3 + c$

g $\tfrac{1}{3}x^2 + x^3 + c$ **h** $\tfrac{1}{6}x^4 - \tfrac{4}{3}x + c$ **i** $\dfrac{x^2}{2} - \dfrac{1}{x} + c$

3 **a** $y = 6x + c$ **b** $y = \tfrac{4}{3}x^3 + c$

c $y = -\dfrac{1}{x} + c$ **d** $y = \tfrac{1}{2}x^4 - 4x + c$

e $y = x^4 + x^3 + c$ **f** $y = 2x + \dfrac{1}{x} + c$

4 **a** $\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 - 2x + c$ **b** $\tfrac{2}{3}x^3 - \tfrac{3}{2}x^2 + x + c$

c $-\tfrac{1}{4}x^4 + \tfrac{4}{3}x^3 - 3x + c$ **d** $\tfrac{1}{4}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + c$

e $\tfrac{1}{5}x^5 - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x + c$ **f** $-\dfrac{1}{6x^2} + \dfrac{2}{x} + c$

g $\tfrac{1}{8}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{2}x^2 + c$ **h** $-\dfrac{4}{x} + \tfrac{1}{3}x^3 - \tfrac{1}{16}x^4 + c$

5 **a** $\tfrac{4}{3}x^3 + 2x^2 + x + c$ **b** $\tfrac{1}{3}x^3 + 2x - \dfrac{1}{x} + c$

c $9x - 2x^3 + \tfrac{1}{5}x^5 + c$ **d** $-\dfrac{4}{3x^3} - \dfrac{4}{x} + x + c$

e $\tfrac{1}{4}x^4 + x^3 + \tfrac{3}{2}x^2 + x + c$

f $\tfrac{1}{5}x^5 - x^4 + 2x^3 - 2x^2 + x + c$

6 **a** $f(x) + g(x)$

EXERCISE 13F

1 **a** $f(x) = x^2 - x + 3$ **b** $f(x) = x^3 + x^2 - 7$

c $f(x) = 3x - \tfrac{2}{3}x^3 - \tfrac{4}{3}$ **d** $f(x) = \tfrac{1}{2}x^2 + \dfrac{2}{x} - \tfrac{1}{2}$

e $f(x) = \tfrac{1}{4}x^4 - 2x - 56$ **f** $f(x) = x^3 - 2x^2 + x + 12$

2 $y = \dfrac{x^2}{2} - \dfrac{2x^3}{3} + \tfrac{22}{3}$ **3** $y = x + \dfrac{3}{x} - 4$

4 $f(x) = -x^2 + x + 3$ **5** $f(x) = 3x^3 + 1$

EXERCISE 13G

1 **a** $\displaystyle\int_1^2 x^3\,dx = \tfrac{15}{4}$ and $\displaystyle\int_1^2 (-x^3)\,dx = -\tfrac{15}{4}$

b $\displaystyle\int_0^1 x^7\,dx = \tfrac{1}{8}$ and $\displaystyle\int_0^1 (-x^7)\,dx = -\tfrac{1}{8}$

2 **a** $\tfrac{1}{3}$ **b** $\tfrac{7}{3}$ **c** $\tfrac{8}{3}$ **d** 1

3 **a** -4 **b** $\tfrac{25}{4}$ **c** $\tfrac{9}{4}$

4 **a** $\tfrac{1}{3}$ **b** $-\tfrac{3}{2}$ **c** $-\tfrac{7}{6}$

5 **a** $\tfrac{1}{4}$ **b** $\tfrac{2}{3}$ **c** 18 **d** 3 **e** $\tfrac{2}{3}$

f $\tfrac{61}{3}$ **g** $\tfrac{17}{6}$ **h** $\tfrac{27}{4}$ **i** $\tfrac{11}{108}$

6 **a** $m = 4$ **b** $m = -1$ or $\tfrac{4}{3}$

7 **a** 2 **b** ≈ 4.667 **c** ≈ 1.718 **d** ≈ 0.7183

e ≈ 1.296 **f** ≈ 1.494

8 **a** $\displaystyle\int_2^7 f(x)\,dx$ **b** $\displaystyle\int_4^6 f(x)\,dx$ **c** $\displaystyle\int_1^9 g(x)\,dx$

9 **a** -5 **b** 4

10 **a** 4 **b** 0 **c** -8 **d** $k = -\tfrac{7}{4}$

EXERCISE 13H

1 **a** 6 units^2 **b** 6 units^2

2 **a** 30 units^2 **b** $4\tfrac{1}{2}$ units^2 **c** $13\tfrac{1}{2}$ units^2

3 **a** $5\tfrac{1}{3}$ units^2 **b** $12\tfrac{2}{3}$ units^2

4 **a** $\tfrac{1}{3}$ units^2 **b** $63\tfrac{3}{4}$ units^2 **c** $2\tfrac{1}{6}$ units^2

5 **a** A$(-2, 0)$, B$(3, 0)$ **b** $20\tfrac{5}{6}$ units^2

6 9 units^2 **7** **a** $4\tfrac{1}{2}$ units^2 **b** $4\sqrt{3}$ units^2 **8** $\tfrac{1}{2}$ units^2

9 **a** $a = 16$ **b** $a = \sqrt{3}$ **c** $a = 3$ **d** $a = 2$

10 **a** **i** $4\tfrac{1}{2}$ units^2 **ii** 9 units^2 **b** $4\tfrac{1}{2}$ units^2

REVIEW SET 13A

1 **a** $A = 4.25$, $B = 6.25$ **b** ≈ 5.3125

c $\displaystyle\int_0^2 (4-x^2)\,dx = 5\tfrac{1}{3}$

In **a**, $\dfrac{A+B}{2} = 5.25$

Both answers provide good approximations of the integral.

2 **a** 2π **b** 4 **3** **a** 10 **b** $\frac{\pi}{2}$

4 $\dfrac{d}{dx}(x^4 - x^2) = 4x^3 - 2x$

$\therefore \displaystyle\int (2x^3 - x)\,dx = \frac{x^4}{2} - \frac{x^2}{2} + c$

5 **a** $5x + c$ **b** $2x^3 + c$ **c** $3x - x^2 + c$

6 **a** $2x^2 + \dfrac{2}{x} + c$ **b** $\frac{1}{12}x^4 + x^2 + c$ **c** $-\dfrac{1}{2x^2} + \dfrac{2}{x} + c$

7 **a** $-\frac{3}{5}x^5 + 2x^3 + c$ **b** $\frac{3}{2}x^2 - x + \dfrac{1}{x} + c$

c $\frac{4}{3}x^3 - 6x^2 + 9x + c$

8 $f(x) = x^3 - 2x^2 + x + 2$ **9** $y = -2x^2 + 3x + 2$

10 **a** 8 **b** $-\frac{1}{12}$ **c** $11\frac{13}{15}$ **11** **a** $b = 3$ **b** $b \approx 1.86$

12 **a** $\approx 1.236\,17$ **b** $\approx 1.952\,49$

13 **a** 9 units^2 **b** 36 units^2 **14** **a** 39 units^2 **b** 12 units^2

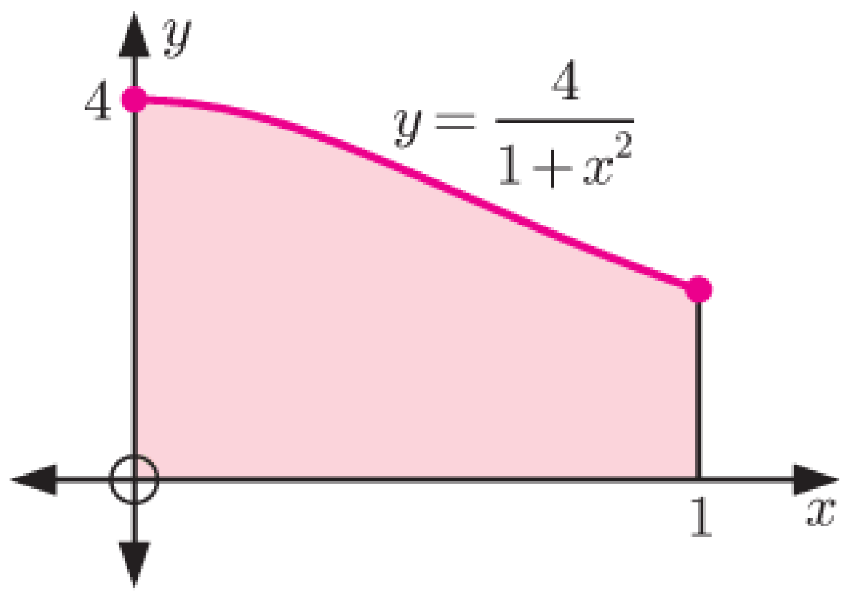
15 **a** ≈ 6.2365

b It is an over estimate. Joining the interval endpoints with straight line segments would give a larger area than the shaded area.

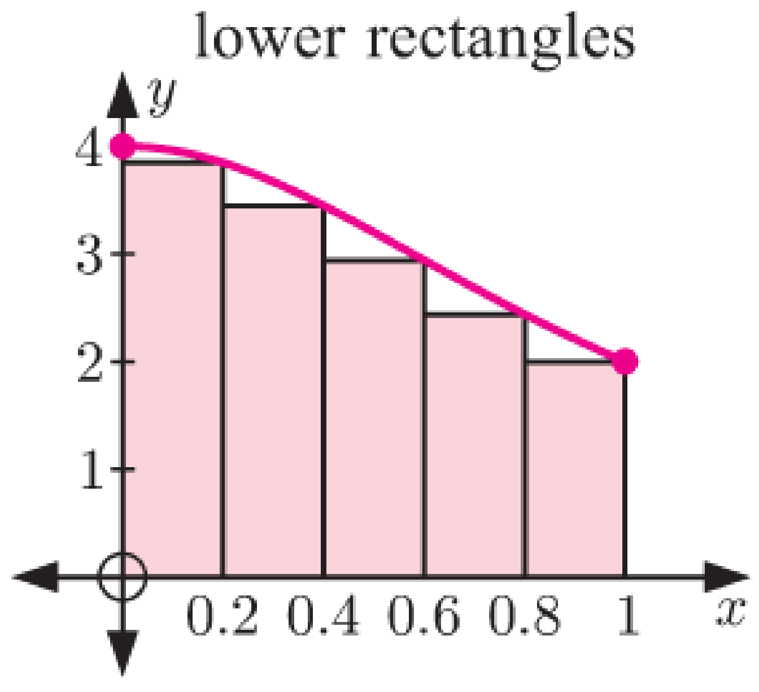
c $\displaystyle\int_{-2}^{-1} x^4\,dx = \tfrac{31}{5} = 6.2$

REVIEW SET 13B

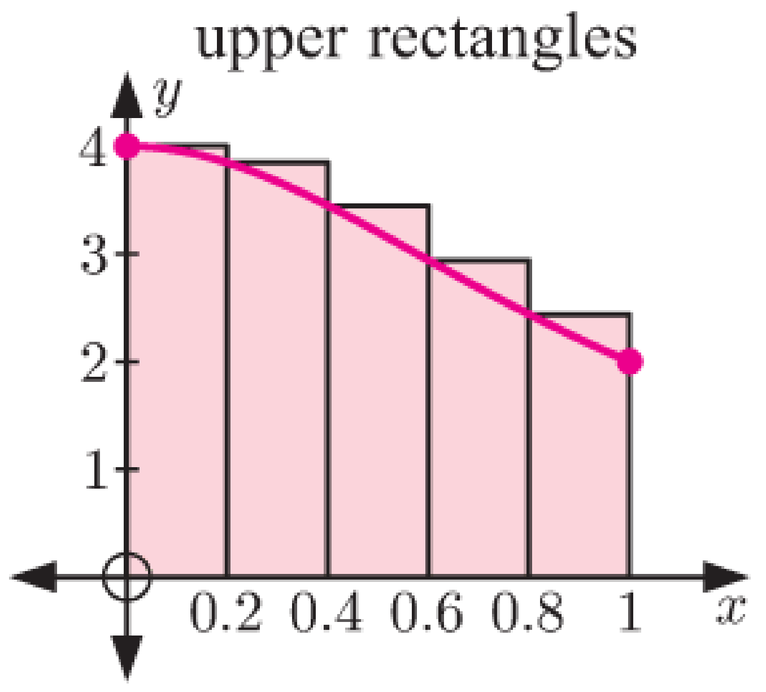
1 **a**



lower rectangles



upper rectangles



b

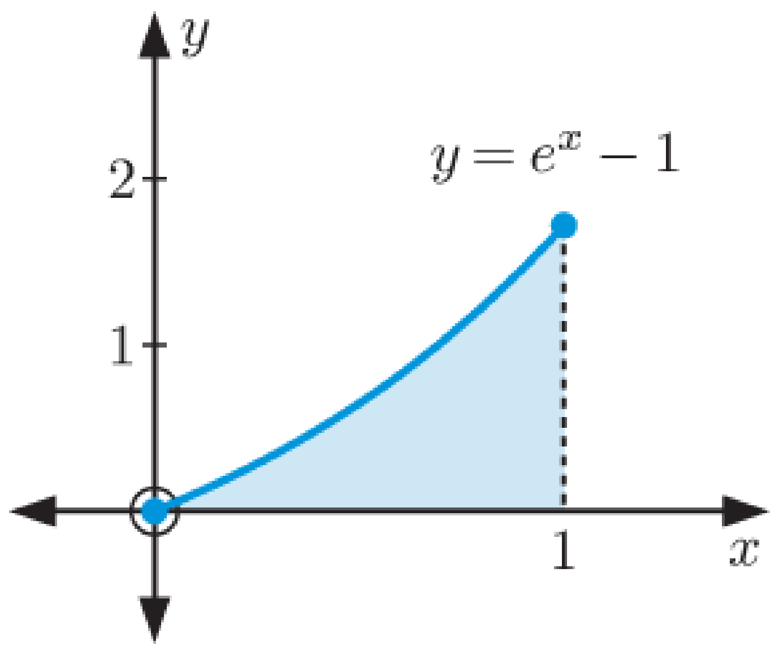
n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\dfrac{3.1396 + 3.1436}{2} = 3.1416$ **d** ≈ 3.1399

e $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \pi \approx 3.1416$

Both answers in **c** and **d** are good approximations of the integral, but the trapezoidal method required fewer subintervals and is hence more efficient.

2 **a**



b **i** ≈ 0.6338
ii ≈ 0.8056
iii ≈ 0.7197

c $\displaystyle\int_0^1 (e^x - 1)\,dx = e - 2 \approx 0.7183$

$\dfrac{0.6338 + 0.8056}{2} = 0.7197$

Our approximations of the integral using the rectangle and trapezoidal methods with 10 subintervals are almost equal.

3 $\dfrac{d}{dx}\left(2x - \dfrac{3}{x}\right) = 2 + \dfrac{3}{x^2}$

$\therefore \displaystyle\int \left(-\frac{3}{x^2} - 2\right) dx = \frac{3}{x} - 2x + c$

4 **a** $-\dfrac{5x^3}{3}$ **b** $-\dfrac{6}{x}$ **c** $x + \dfrac{x^3}{18}$

5 **a** $x + \dfrac{2}{x} + c$ **b** $3x^3 - 12x^2 + 16x + c$ **c** $4x - \frac{2}{3}x^3 + c$

6 **a** $\dfrac{x^2}{6} + x + c$ **b** $x^3 - 2x + c$ **c** $9x + 6x^2 + \dfrac{4x^3}{3} + c$

7 $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + \frac{13}{6}$

8 We could include any arbitrary constant c in the function y and obtain the same expression for $\dfrac{dy}{dx}$ by differentiating.

Integration is the reverse process of differentiation, so when we integrate $\dfrac{dy}{dx}$ to obtain y, we include an arbitrary constant c.

The constant will always cancel when we find a definite integral, so we do not include it in this case.

9 **a** 1 **b** -3 **c** $\frac{8}{15}$ **10** $a = \frac{3}{2}$

11 **a** ≈ 3.528 **b** ≈ 2.963

12 **a** 6 **b** 3 **c** $k = -\frac{5}{3}$ **13** **a** 22 units^2 **b** $\frac{2}{9}$ units^2

14 **a** $50\frac{5}{12}$ units^2 **b** 4 units^2 **15** $k = \frac{4}{3}$

EXERCISE 14A

1 **a** continuous **b** discrete **c** continuous **d** continuous
e discrete **f** discrete **g** continuous **h** continuous

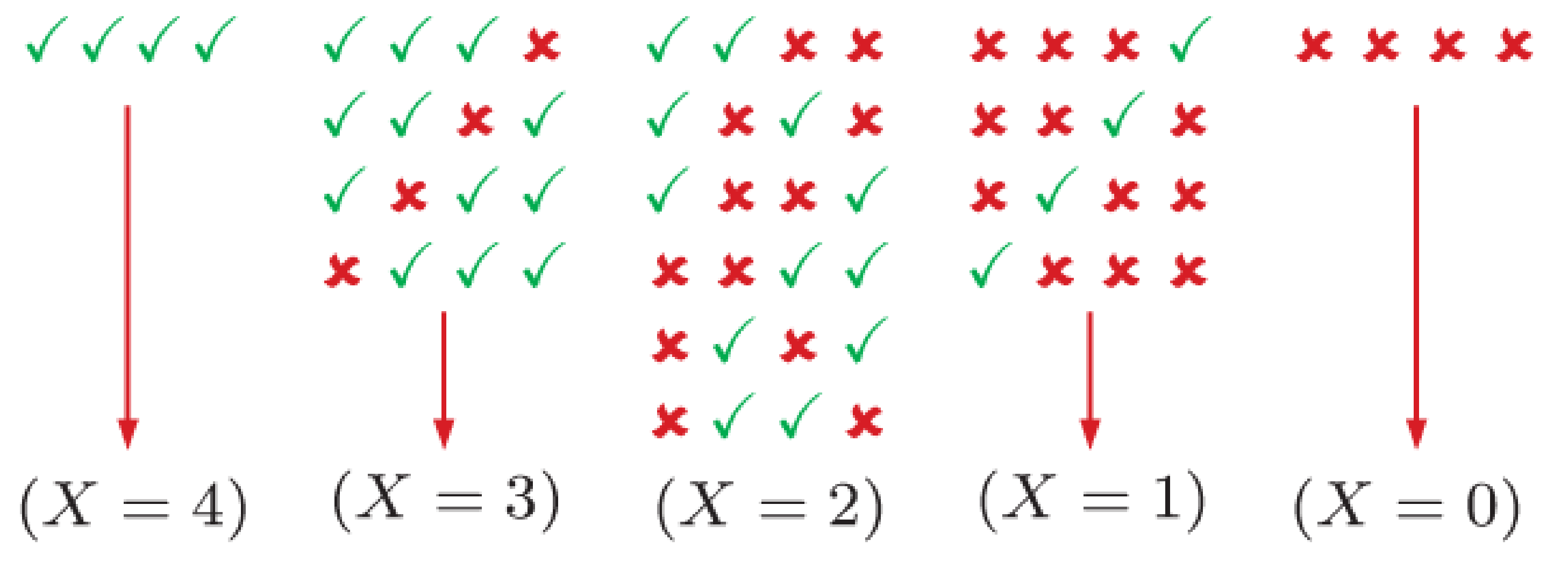
2 **a** **i** X = the height of water in the rain gauge
ii continuous **iii** $0 \leqslant X \leqslant 400$ mm
b **i** X = stopping distance **ii** continuous
iii $0 \leqslant X \leqslant 50$ m
c **i** number of switches until failure
ii discrete **iii** any integer $\geqslant 1$

3 **a** X has a set of distinct possible values.
b $X = 2, 3, 4, 5, 6, 7, 8, 9,$ or 10

4 **a** $X = 4, 5, 6,$ or 7 **b** **i** $X = 5$ **ii** $X = 6$ or 7

5 **a** $X = 0, 1, 2, 3,$ or 4

b



c **i** $X = 2$ **ii** $X = 2, 3$, or 4

6 **a** $X = 0, 1, 2$, or 3

b

HHH	HHT HTH THH	TTH THT HTT	TTT
$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c No, for example there is probability $\frac{1}{8}$ that $X = 3$, and probability $\frac{3}{8}$ that $X = 2$.

EXERCISE 14B

1 **a** **i** yes **ii** no **iii** yes **iv** no

b For **a iii**, X is a uniform random variable.

2 **a** $k = 0.2$ **b** $k = \frac{1}{7}$

3 **a** $a = 0.2$

b No, as the probabilities of each outcome are not all equal.

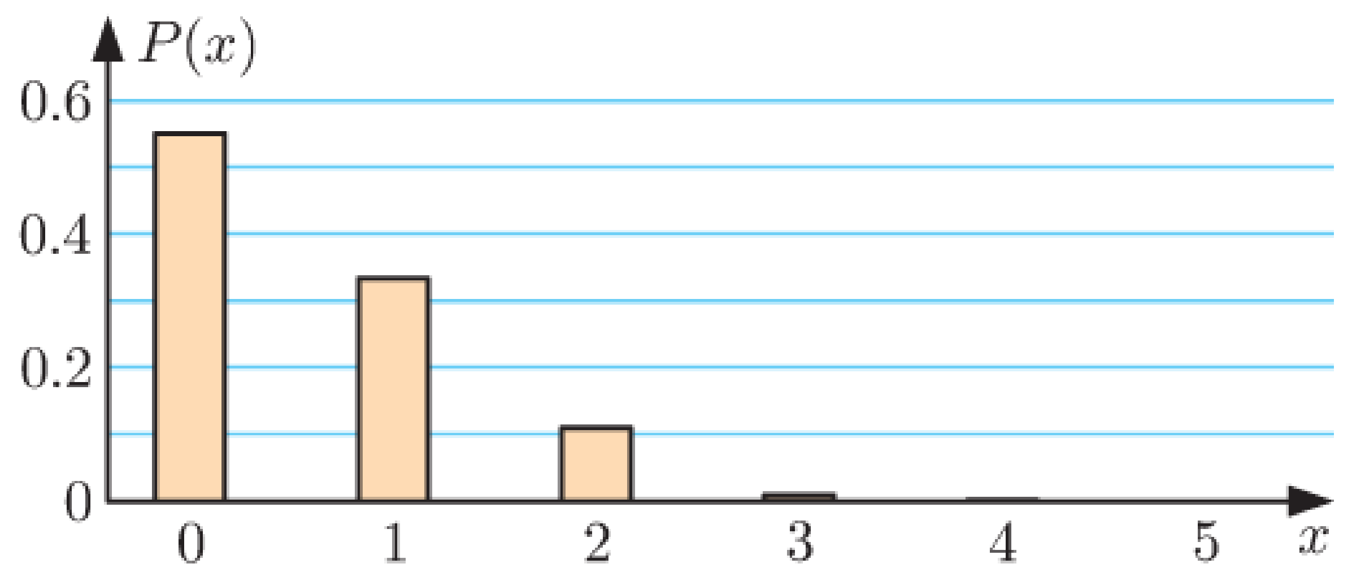
c 2 **d** $\text{P}(X \geqslant 2) = 0.65$

4 **a** $P(2) = 0.1088$

b $a = 0.5488$ is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.

d



e mode = 0 home runs, median = 0 home runs

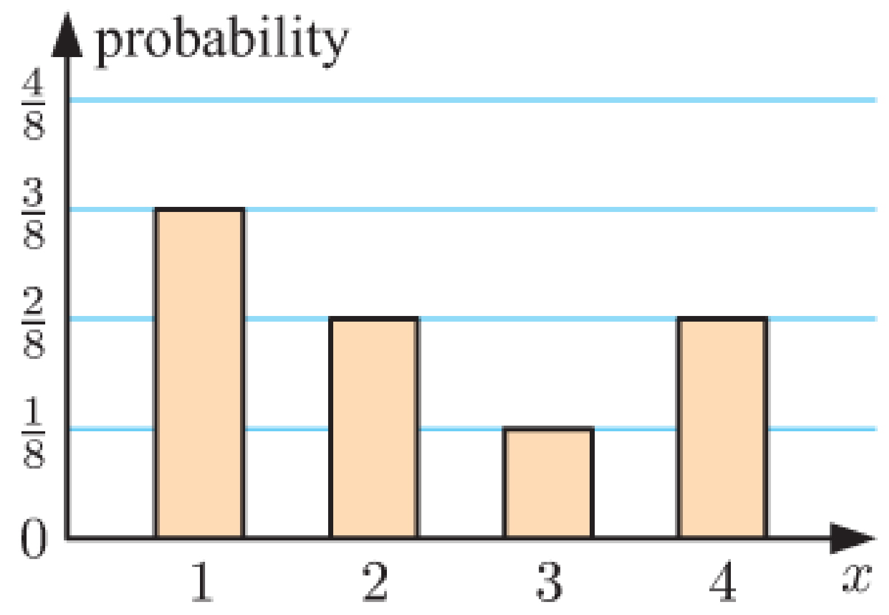
5 **a** $k = 0.04$ **b** 0 tyres

c $\text{P}(X > 1) = 0.12$ which is the probability that more than 1 tyre will need replacing on a car being inspected.

6 **a**

x	1	2	3	4
$\text{P}(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

b



c mode = 1
median = 2

d $\text{P}(X \leqslant 3) = \frac{3}{4}$

7 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$\text{P}(X = x)$	0.24	0.35	0.27	0.14

c mode = 2 bedrooms, median = 2 bedrooms

8 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$\text{P}(X = x)$	0.48	0.28	0.08	0.16

c mode = 1 shot, median = 2 shots

9 **a** $P(0) = \frac{1}{10}$, $P(1) = \frac{2}{10}$, $P(2) = \frac{3}{10}$, $P(3) = \frac{4}{10}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and

$$\sum_{i=1}^{n} P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$$

$\therefore$ $P(x)$ is a valid probability function.

b $P(1) = \frac{6}{11}$, $P(2) = \frac{6}{22}$, $P(3) = \frac{6}{33}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and

$$\sum_{i=1}^{n} P(x_i) = \frac{6}{11} + \frac{6}{22} + \frac{6}{33} = 1$$

$\therefore$ $P(x)$ is a valid probability function.

10 **a** $k = \frac{1}{12}$ **b** $k = \frac{12}{25}$

11 **a** $a = 10$ **b** $\text{P}(X = 1) = \frac{3}{10}$ **c** 2

EXERCISE 14C.1

1 **a** $\text{E}(X) = 1.7$ **b** $\text{E}(X) = 2.5$ **c** $\text{E}(X) = 3.85$

d $\text{E}(X) = 30$

2 **a** $a = \frac{1}{2}$ **b** 3 **c** $\mu = 2\frac{2}{5}$

3 ≈ 11.7 points **4** 1.57 fish

5 **a** $a = 0.25$ **b** 4 books **c** 3.13 books

6 5.25 lollies **7** **a** $\frac{4}{15}$ **b** ≈ 8.93 pins

8 $a = 0.1$, $b = 0.4$

9 **a** offensive strategy: P(draw) = 0.15
defensive strategy: P(draw) = 0.5

b offensive strategy: 1.05 points per game
defensive strategy: 1.1 points per game

c defensive strategy

d Yes, an offensive strategy would then be better.

10 **a** **i** car park B **ii** car park A **iii** car park B

b Zoe should choose car park A as the expected cost for car park A is \$14.80 whereas the expected cost for car park B is \$15.25.

11 \$390

EXERCISE 14C.2

1 fair **2** **a** \$3.50 **b** $-\$0.50$ **c** no

3 **a** $\approx -\$0.05$ **b** lose $\approx$ \$5.41 **4** $-\$0.75$

5 **a** **i** 0.3 **ii** 0.1 **b** $\text{E}(X) = 2.5$ tokens

c No, as the player can expect to lose half a token on average per game.

6 **a** Expected gain $\approx -\$0.67 \neq \$0$ **b** \$30 **7** \$4.75

EXERCISE 14D

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (H or T) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

2

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

3 **a** $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

b $4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}$

c $6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$

4 **a** $5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{5}{32}$ **b** $10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{16}$

c $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$

5 **a** $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ **b** $6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$ **c** $\frac{8}{9}$

6 **a** $15\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2 = \frac{1215}{4096}$ **b** $\frac{347}{2048}$ **c** $\frac{347}{2048}$

EXERCISE 14E

1 **a** ≈ 0.0305 **b** ≈ 0.265

2 **a** ≈ 0.476 **b** ≈ 0.840 **c** ≈ 0.160 **d** ≈ 0.996

3 **a** ≈ 0.0280 **b** $\approx 0.002\,46$ **c** ≈ 0.131 **d** ≈ 0.710

4 $\approx 0.000\,864$ **5** **a** ≈ 0.998 **b** ≈ 0.807

6 **a** ≈ 0.0388 **b** ≈ 0.405 **c** ≈ 0.573 **7** ≈ 0.0341

8 **a** ≈ 0.863 **b** ≈ 0.475 **9** **a** $\frac{1}{36}$ **b** ≈ 0.846

10 **a** ≈ 0.0905 **b** ≈ 0.622

c Yes, the probability that Shelley is on time for work each day of a 5 day week is now $\approx 87.2\%$.

11 **a** ≈ 0.0388 **b** 25 solar components

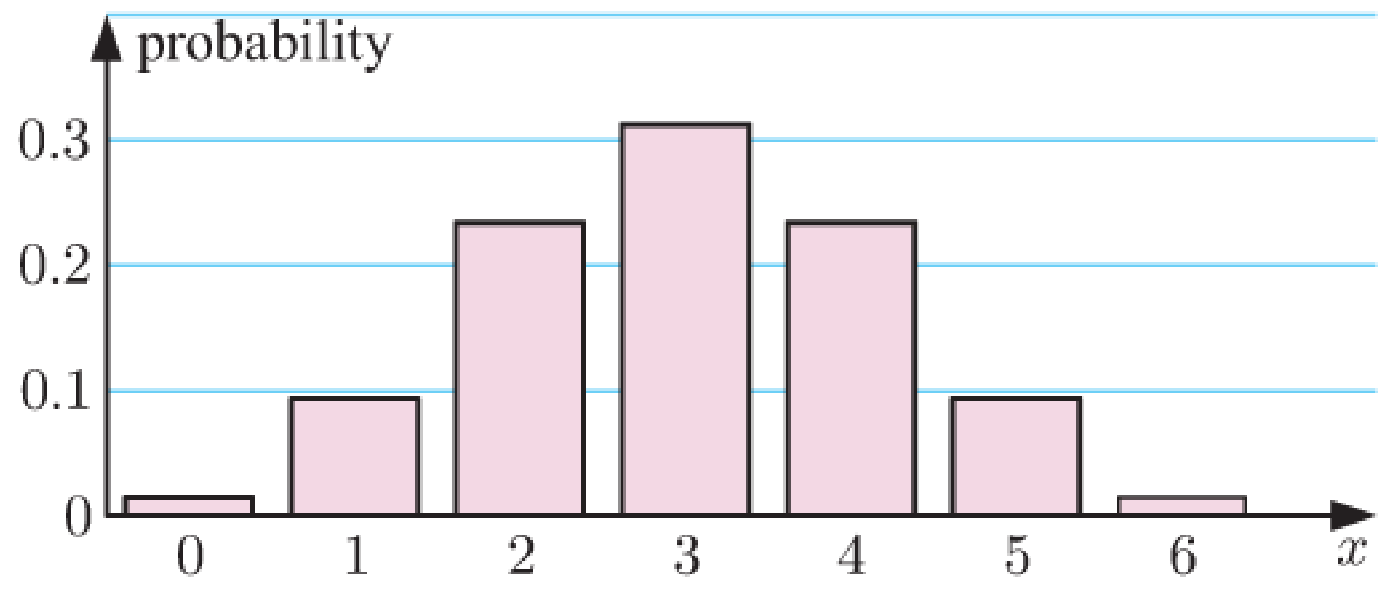
EXERCISE 14F

1 **a** **i** $\mu = 3,\ \sigma \approx 1.22$

ii

x_i	0	1	2	3
$P(x_i)$	0.0156	0.0938	0.2344	0.3125

x_i	4	5	6
$P(x_i)$	0.2344	0.0938	0.0156



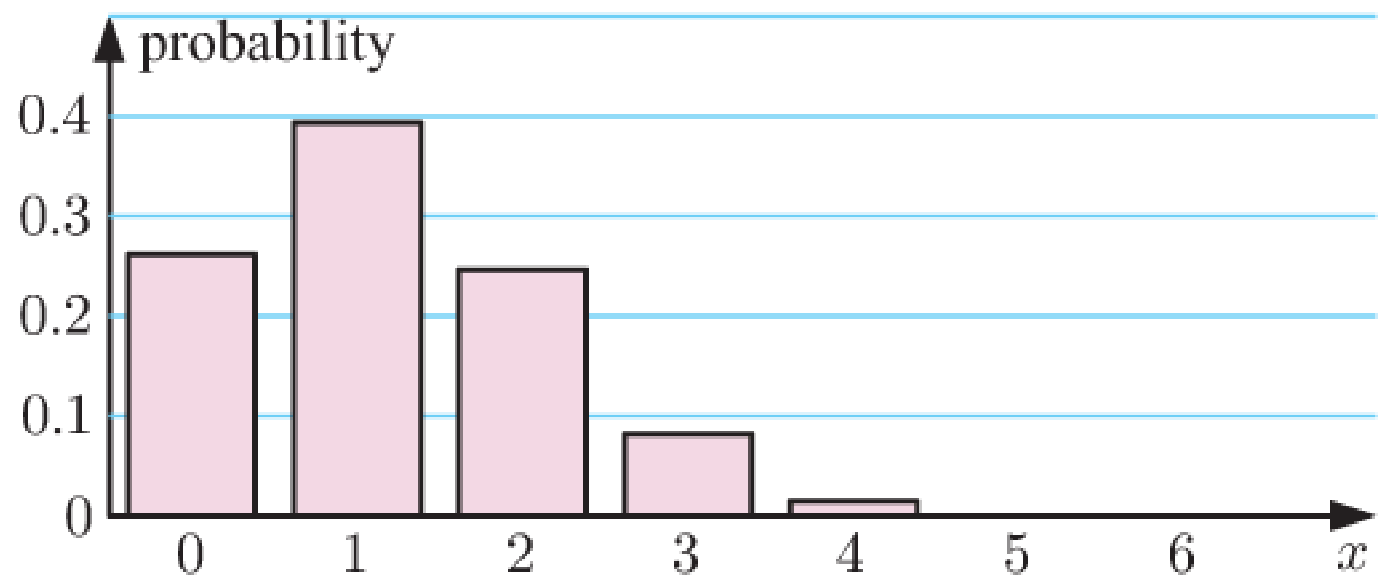
iii The distribution is symmetric.

b **i** $\mu = 1.2,\ \sigma \approx 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.2621	0.3932	0.2458	0.0819

x_i	4	5	6
$P(x_i)$	0.0154	0.0015	0.0001



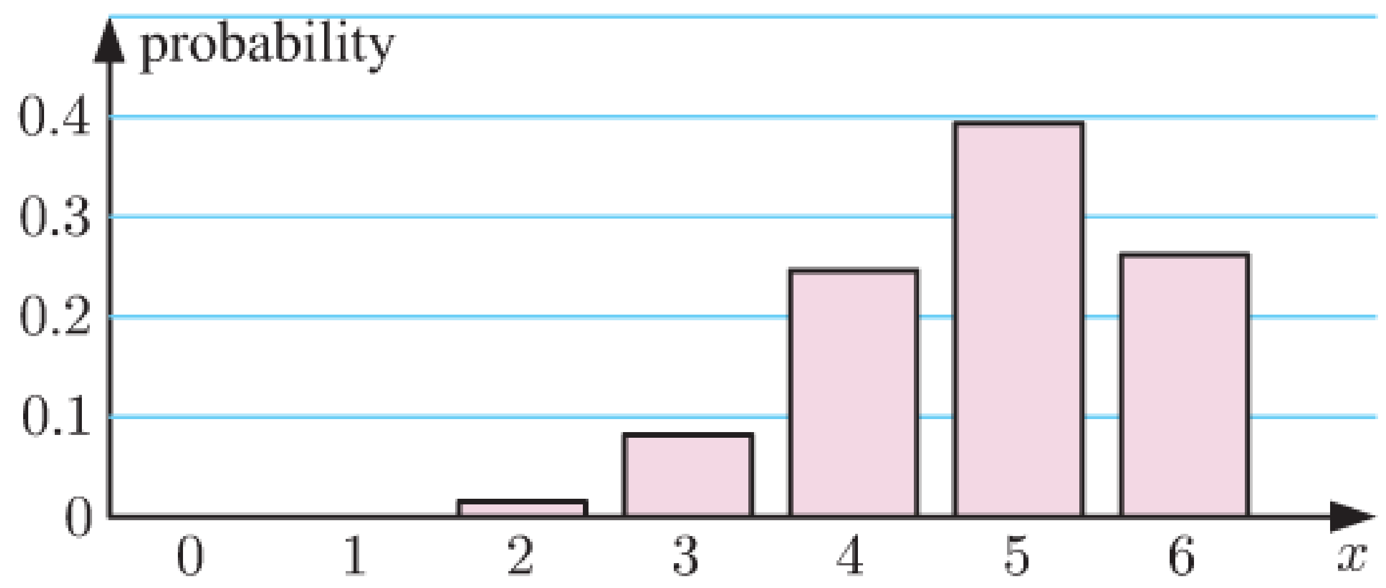
iii The distribution is positively skewed.

c **i** $\mu = 4.8,\ \sigma = 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.0001	0.0015	0.0154	0.0819

x_i	4	5	6
$P(x_i)$	0.2458	0.3932	0.2621



iii The distribution is negatively skewed and is the exact reflection of **b**.

2 $\mu = 5,\ \sigma^2 = 2.5$

3 **a** $\mu = 1.2,\ \sigma \approx 1.07$ **b** $\mu = 28.8,\ \sigma \approx 1.07$

4 $\mu = 3.9,\ \sigma \approx 1.84$

5 **a** $\mu = 28.5,\ \sigma \approx 2.67$ **b** ≈ 0.740

6 **a**
$$\begin{aligned}\mu_X &= np\\ &= 100 \times \tfrac{1}{2}\\ &= 50\end{aligned} \qquad \begin{aligned}\mu_Y &= np\\ &= 300 \times \tfrac{1}{6}\\ &= 50\end{aligned}$$

b $\sigma_X = 5,\ \sigma_Y \approx 6.45$

c X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y, which means there are more values of X which lie close to the mean.

d **i** ≈ 0.729 **ii** ≈ 0.606

REVIEW SET 14A

1 **a** discrete **b** continuous **c** discrete

2 **a** **i** yes **ii** no **iii** no **iv** yes **v** yes **vi** yes

b the distribution in **a iv**

3 **a** $a = \frac{5}{9}$ **b** $\frac{4}{9}$

4 **a** $k = 0.05$ **b** 0.15 **c** 2 **d** $\mathrm{E}(X) = 1.7$

5 **a** X has a set of distinct possible values.

b $X = 0,\ 1,$ or 2

c

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d 1.2 green balls

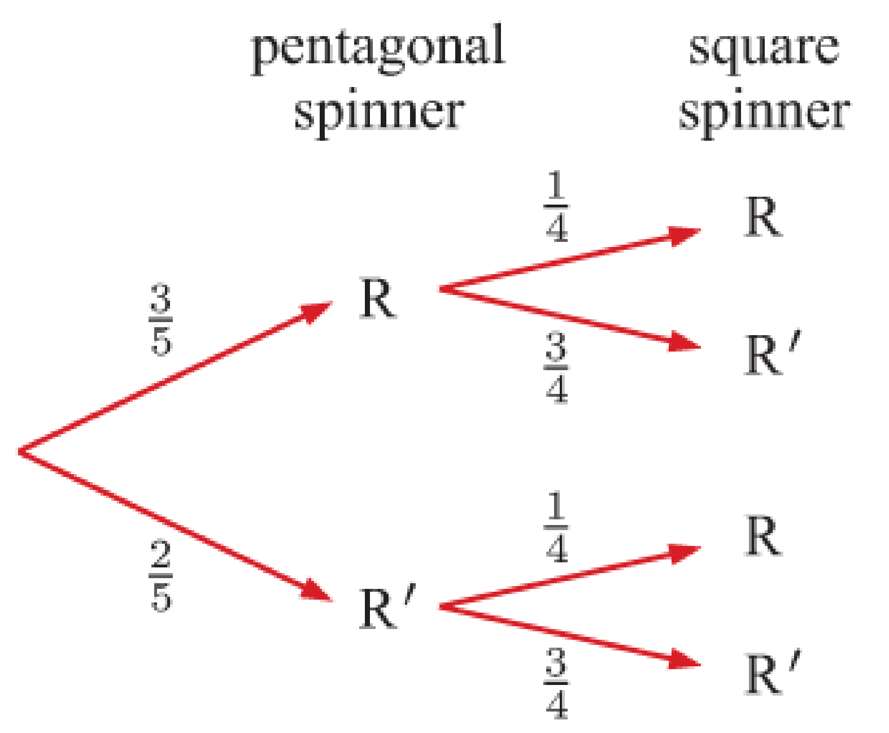
6 ≈ 3.83

7 **a** \$7 **b** No, she would lose \$1 per game in the long run.

8 **a** $\frac{64}{3125} = 0.020\,48$ **b** $\frac{128}{625} = 0.2048$

9 **a** $a = -\frac{1}{84}$ **b** 4 marsupials

10 **a**



b $\frac{11}{20}$

c **i** $X \sim \mathrm{B}(10,\ \frac{11}{20})$

ii $\mathrm{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9 \approx 0.004\,16$

$\mathrm{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1 \approx 0.0207$

It is more likely that exactly one red will occur 9 times.

11 **a** **i** ≈ 0.0751 **ii** ≈ 0.166 **b** ≈ 4.97 games

12 **a** ≈ 0.544 **b** ≈ 0.456

REVIEW SET 14B

1 **a** X is the number of hits that Sally has in each match.
$X = 0,\ 1,\ 2,\ 3,\ 4,$ or 5

b **i** $k = 0.23$ **ii** $\mathrm{P}(X \geqslant 2) = 0.79$

iii $\mathrm{P}(1 \leqslant X \leqslant 3) = 0.83$

c mode = 3 hits, median = 3 hits

3 **a** 2 **b** 3 **c** 2.7

4 **a** **i** Naomi **ii** Rosslyn **b** Rosslyn

5 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{1}{10}$ **b** \$2.70 per game

6 **a** The probability of rolling a two is not the same for each die. So X is not a binomial random variable.

b

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c $\frac{5}{12}$

7 $a = 0.15,\ \ b = 0.35$

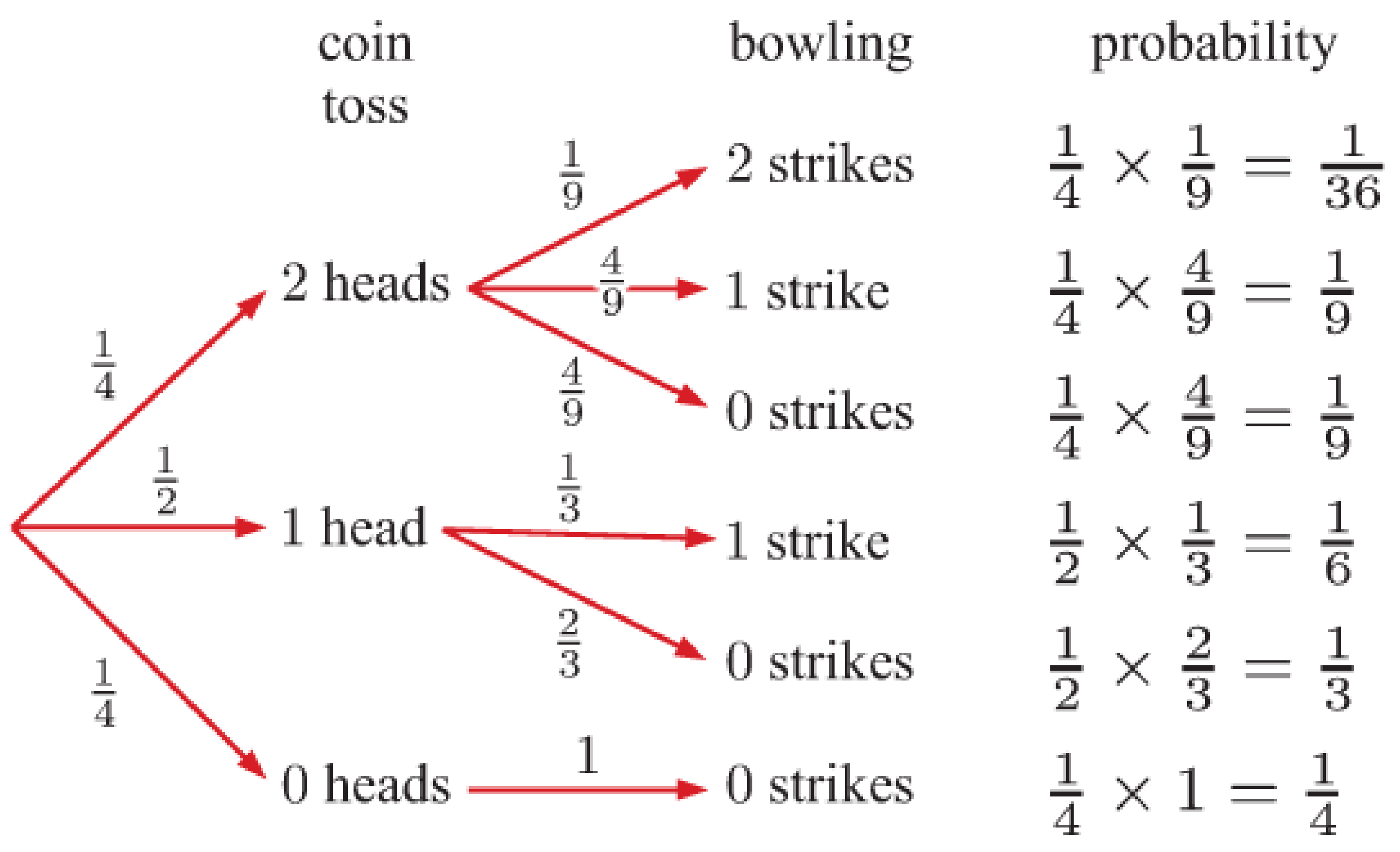
8 **a** $\mathrm{E}(X) = 2.1$ **b** $\mathrm{E}(Y) = 1.9$

9 **a** The probability of spinning a 3 is the same for each spin.

b $\mu = 4,\ \ \sigma \approx 1.79$

10 **a** 42 donations **b** **i** ≈ 0.334 **ii** ≈ 0.0931

11 **a**


coin toss
bowling
probability
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
2 heads
1 head
0 heads
$\frac{1}{9}$
$\frac{4}{9}$
$\frac{4}{9}$
$\frac{1}{3}$
$\frac{2}{3}$
1
2 strikes
1 strike
0 strikes
1 strike
0 strikes
0 strikes
$\frac{1}{4} \times \frac{1}{9} = \frac{1}{36}$
$\frac{1}{4} \times \frac{4}{9} = \frac{1}{9}$
$\frac{1}{4} \times \frac{4}{9} = \frac{1}{9}$
$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
$\frac{1}{4} \times 1 = \frac{1}{4}$

b

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c $\approx \$3.33$

d $\approx -\$1.67$, Suvi should not play the game many times.

EXERCISE 15A.1

1 **B**, **D**, and **F**

2 **a** The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

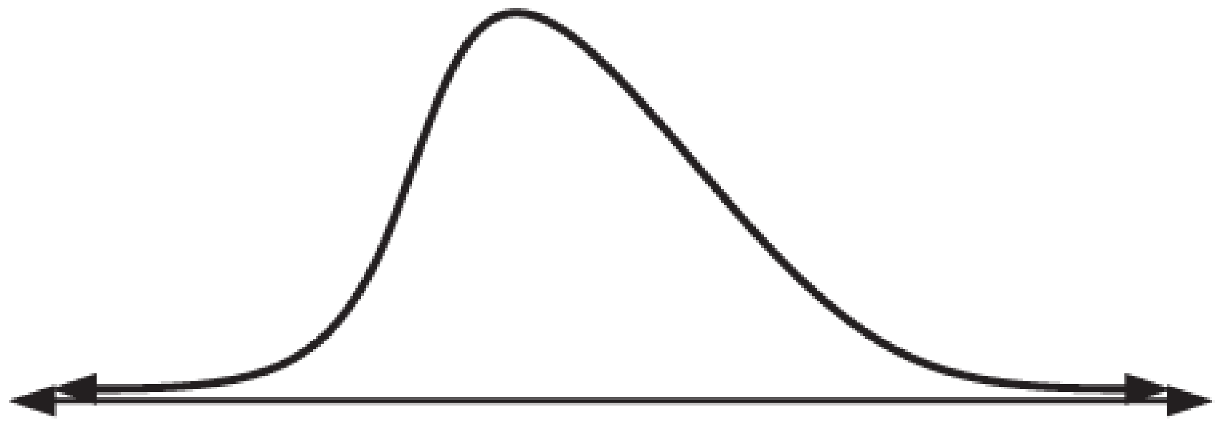
b The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

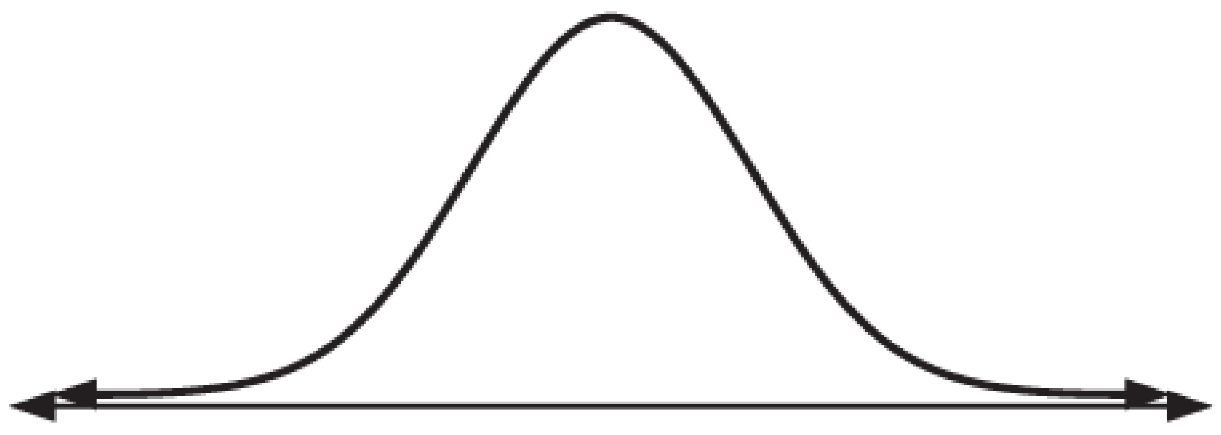
c The times may be affected by:

- the distance that the students live from their school
- walking speed
- physical fitness
- the terrain.

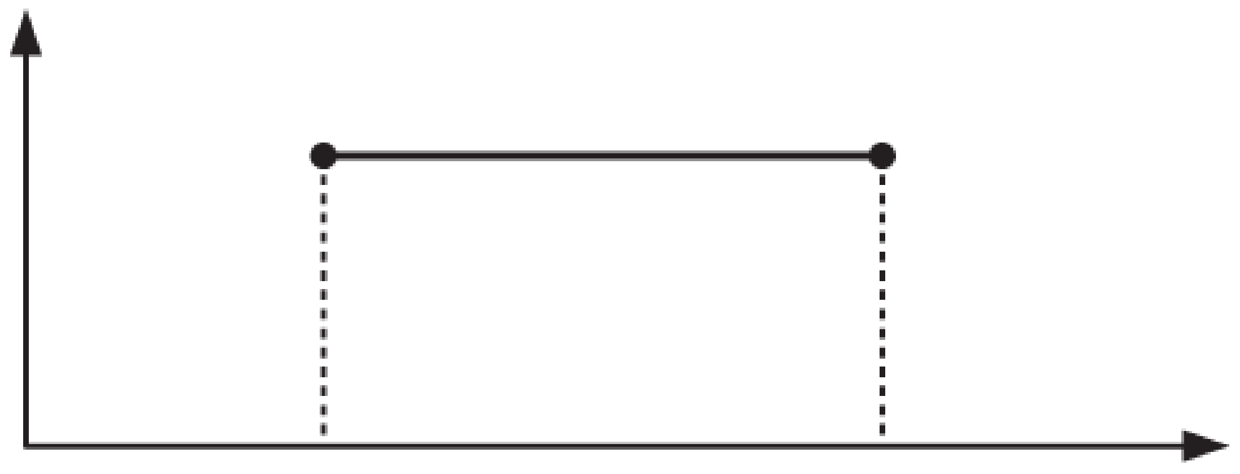
3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



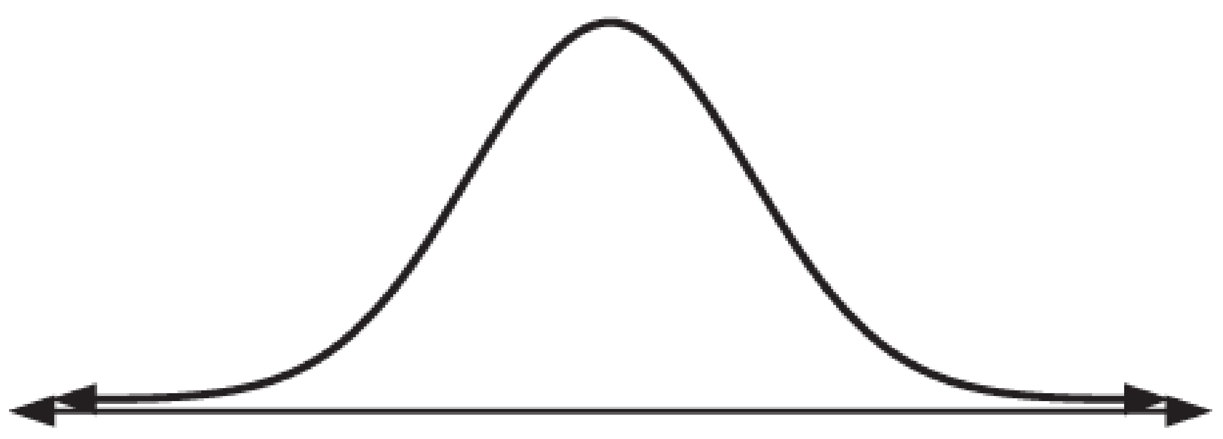
b The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



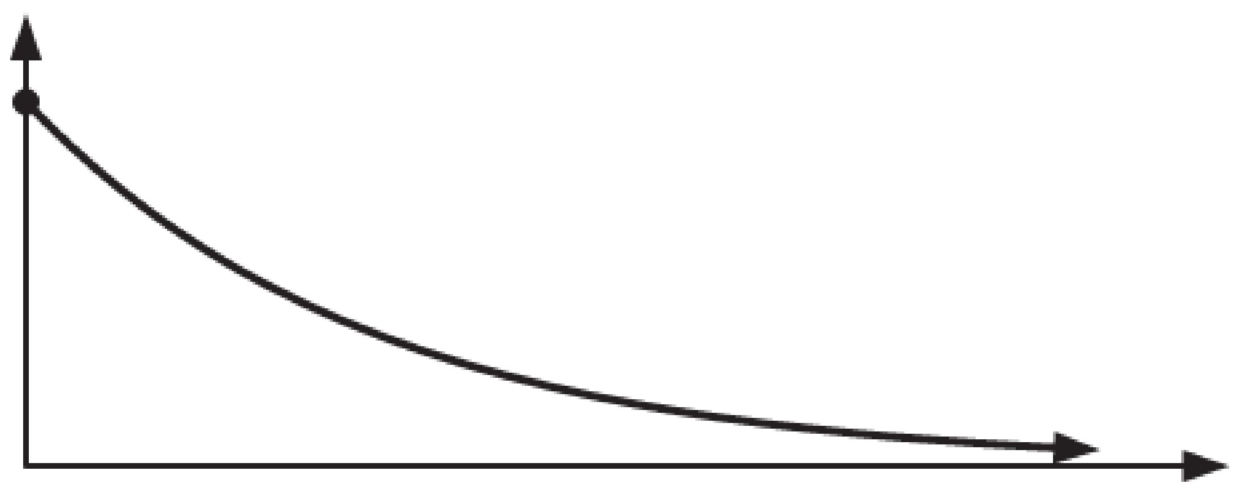
c The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



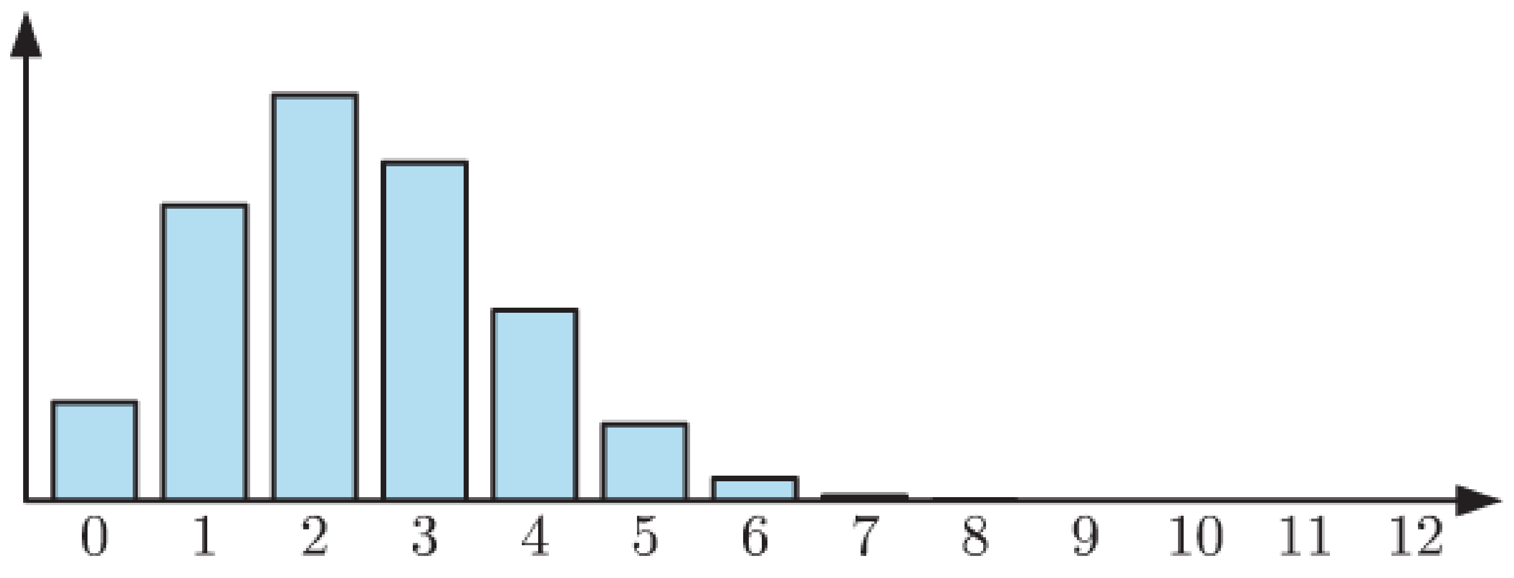
d The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



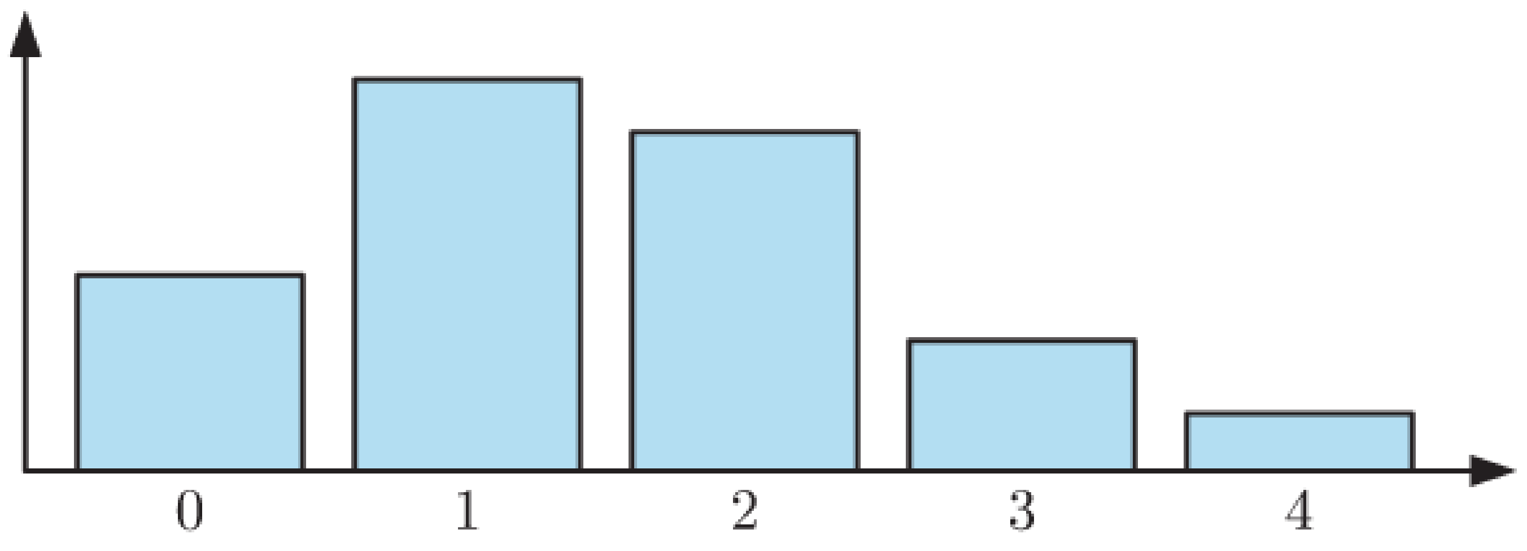
e The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



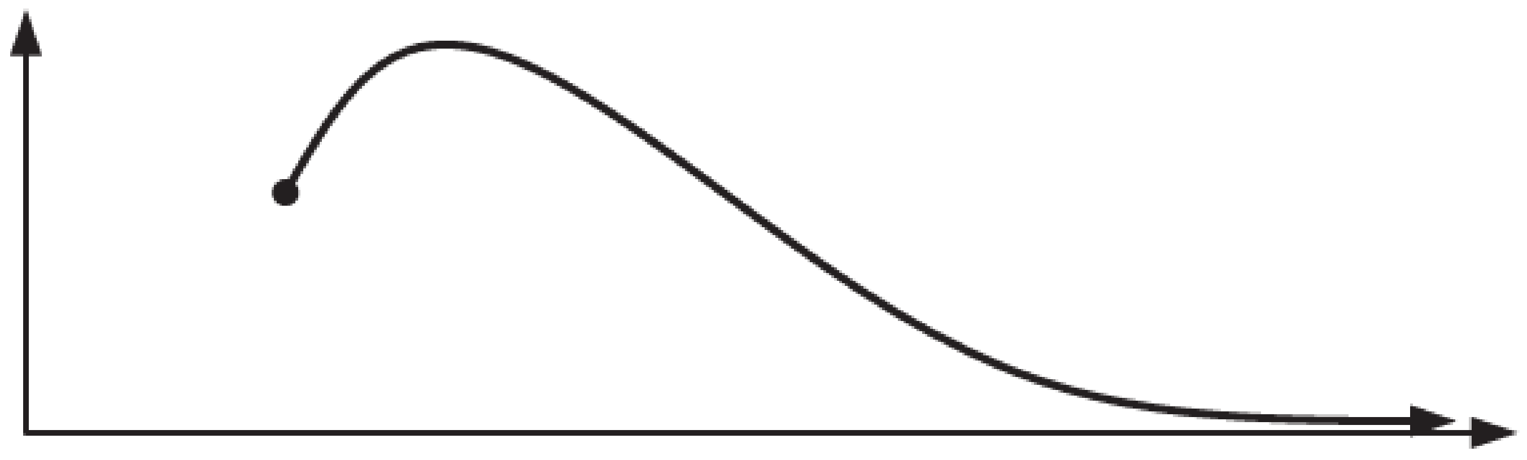
f The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



g The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



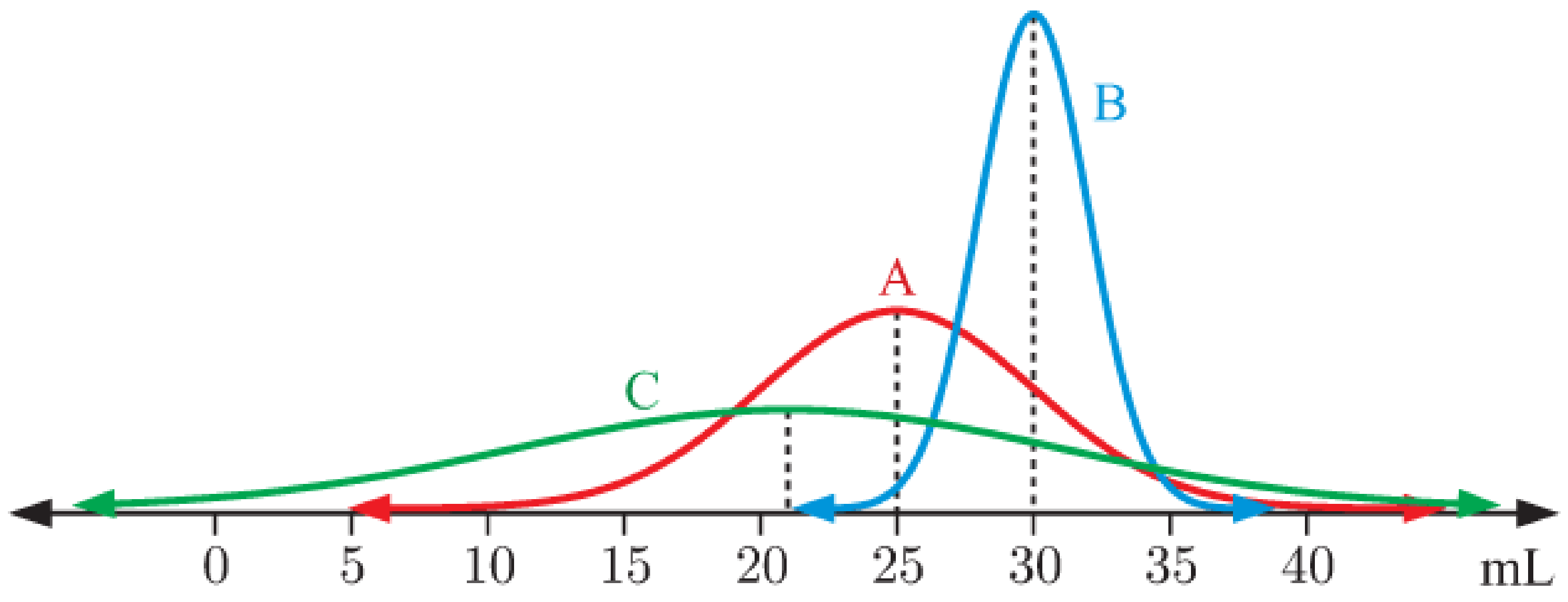
h The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



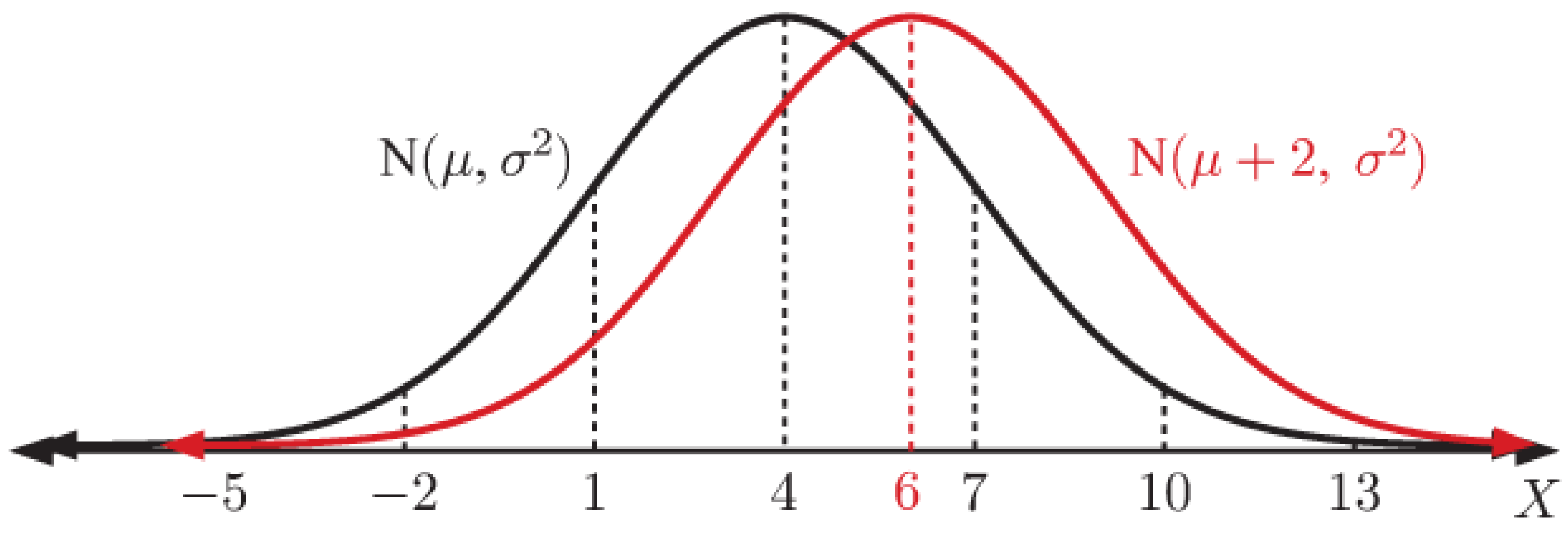
EXERCISE 15A.2

1 **a** B **b** D **c** A **d** C

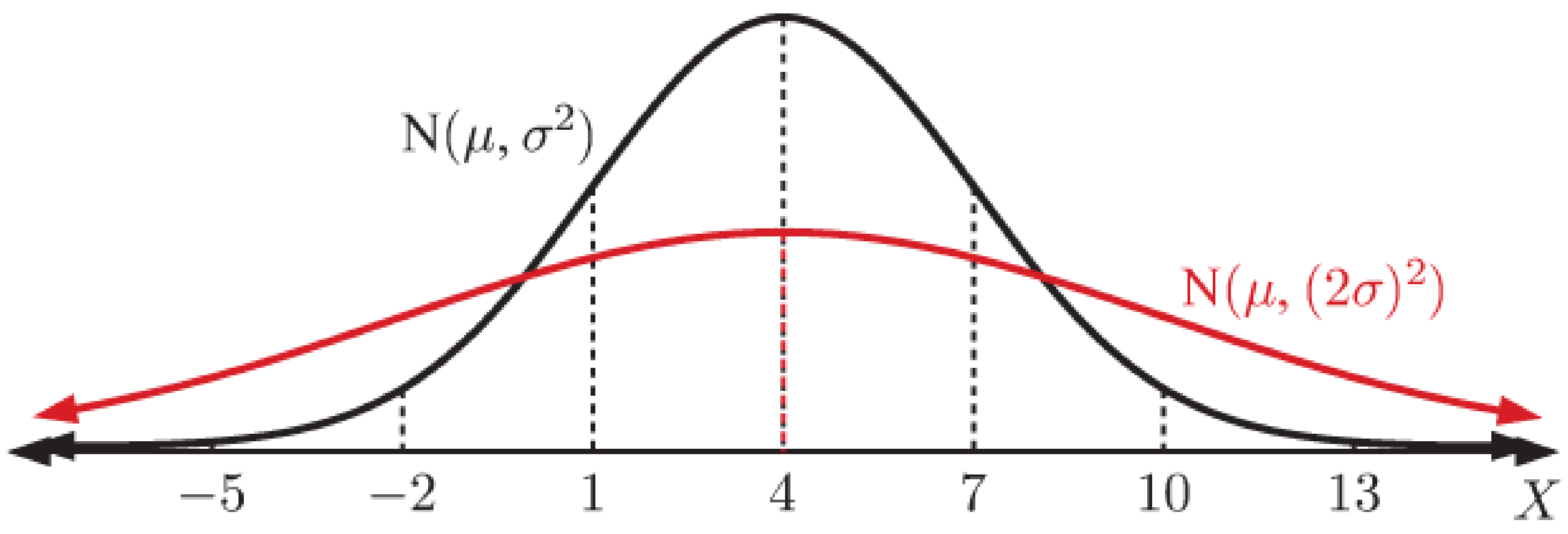
2



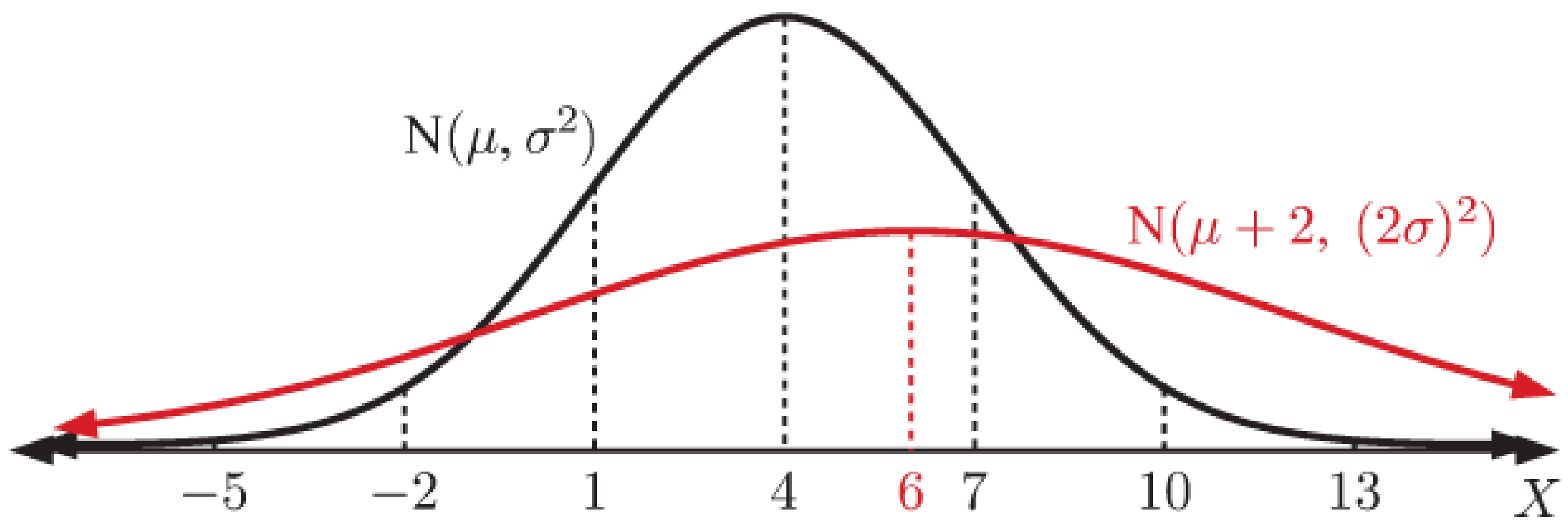
3 **a**



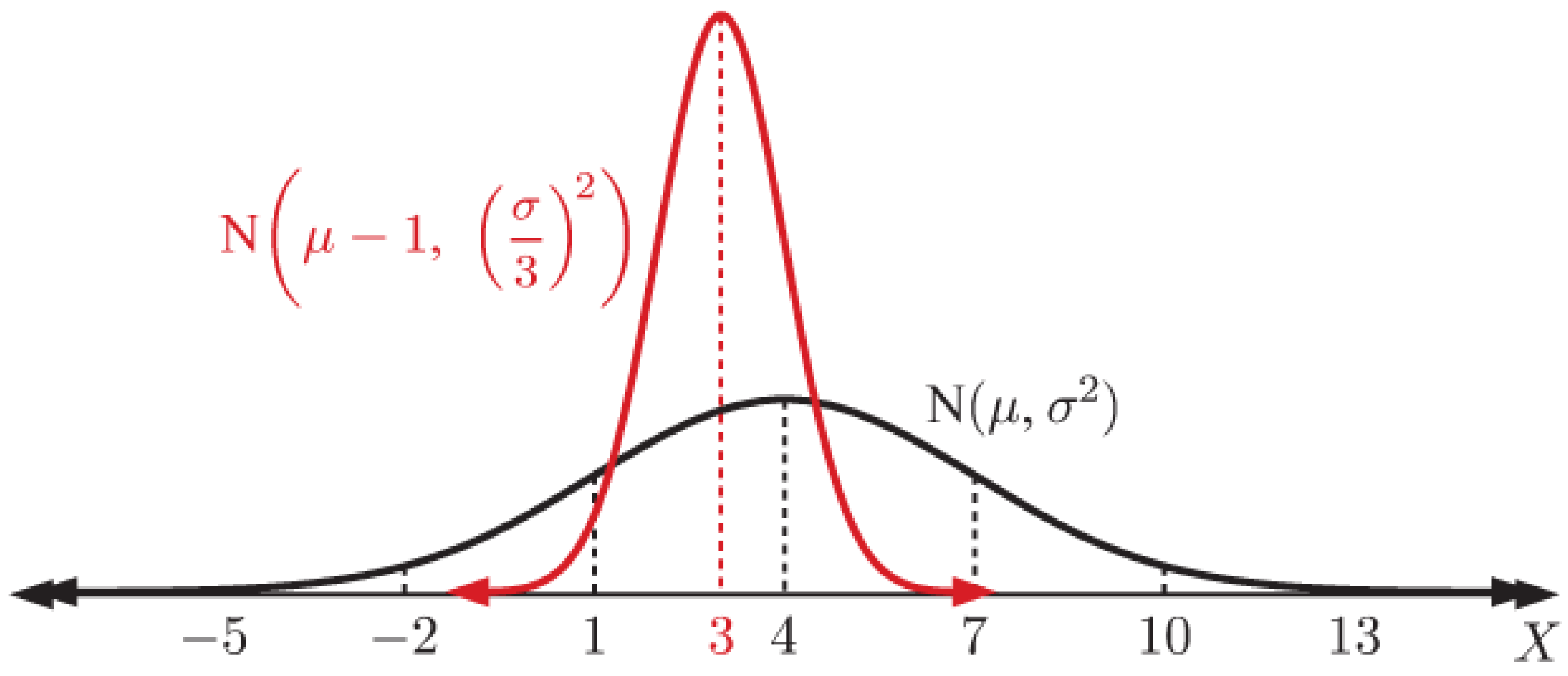
b



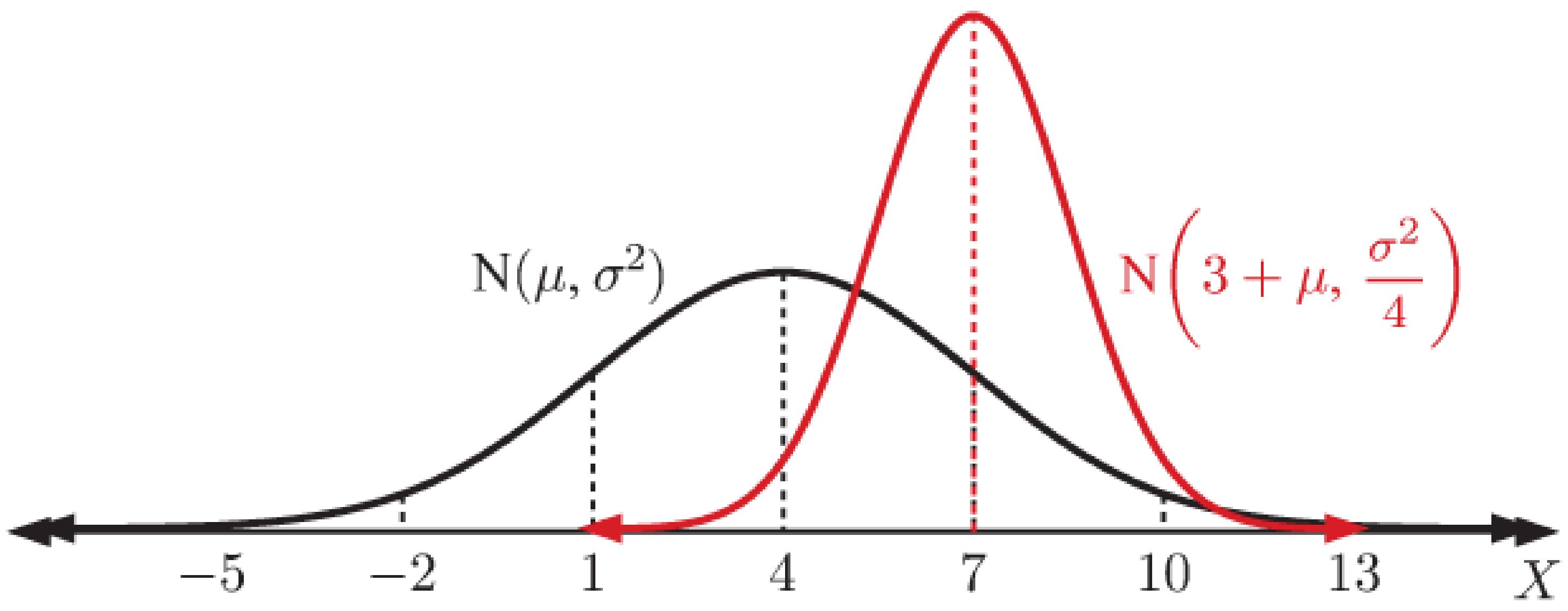
c



d



e



EXERCISE 15B.1

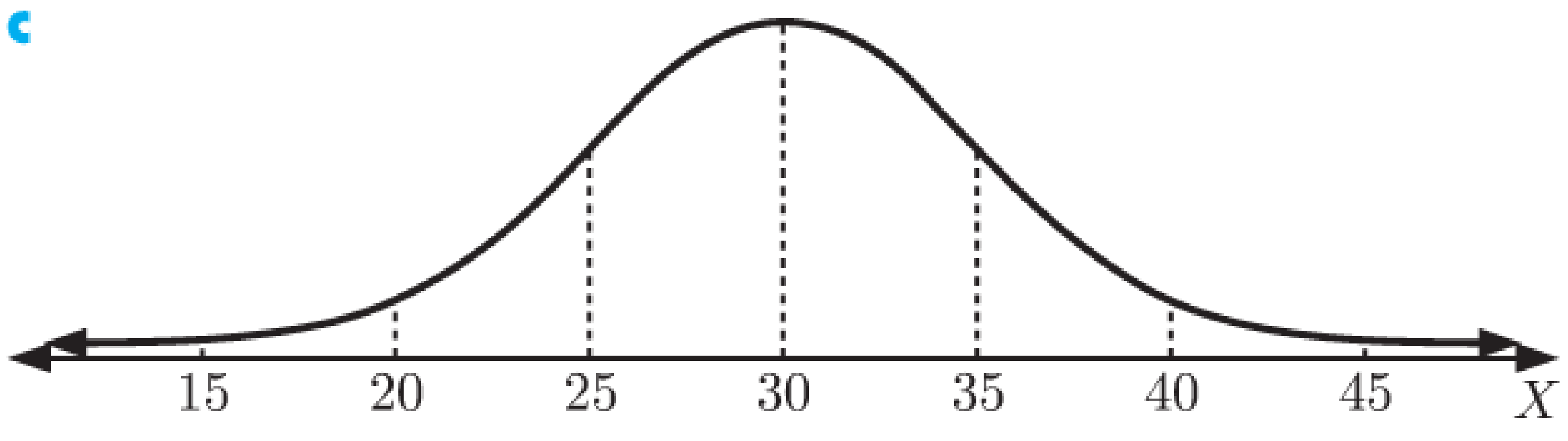
1 **a** **i** 40 **ii** 25

b **i** 1 standard deviation above the mean

ii 2 standard deviations below the mean

iii 3 standard deviations above the mean

c

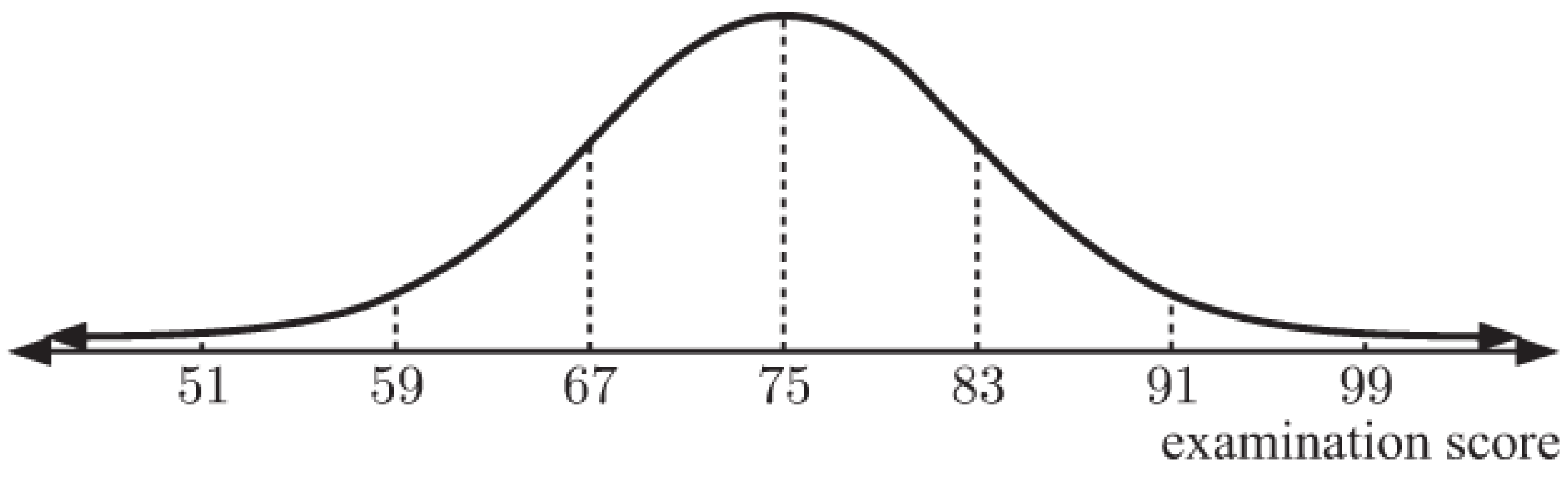


d $\approx 34.13\%$ **e** ≈ 0.1359

2 **a** $\mu = 20, \quad \sigma = 4$

b **i** $\approx 34.13\%$ **ii** $\approx 13.59\%$ **iii** $\approx 2.28\%$

3 **a**



b **i** $\approx 15.87\%$ **ii** $\approx 2.28\%$ **iii** $\approx 81.85\%$

4 **a** ≈ 0.6826 **b** ≈ 0.0228

5 **a** **i** $\approx 34.13\%$ **ii** $\approx 47.72\%$

b **i** ≈ 0.0228 **ii** ≈ 0.8413

c ≈ 68 students **d** $k \approx 178$

6 **a** ≈ 0.8413 **b** ≈ 0.8185

7 **a** ≈ 459 babies **b** ≈ 446 babies

8 **a** ≈ 41 days **b** ≈ 254 days **c** ≈ 213 days

9 **a** ≈ 5 competitors **b** ≈ 32 competitors

c ≈ 137 competitors

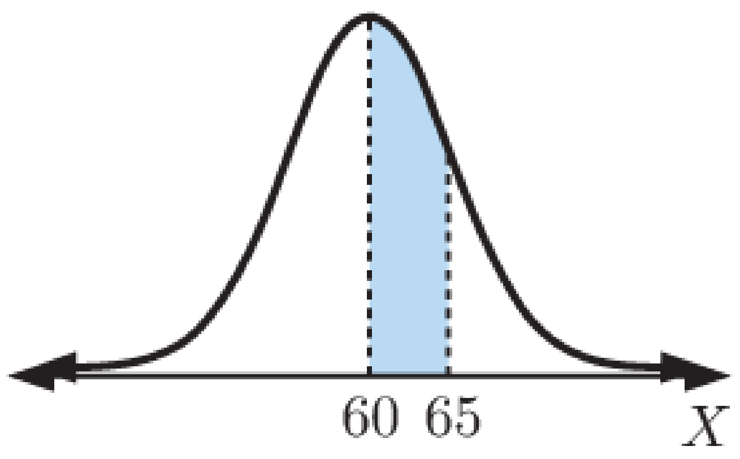
10 **a** $\mu = 176$ g, $\sigma = 24$ g **b** $\approx 81.85\%$

11 **a** **i** $\approx 84.13\%$ **ii** $\approx 2.28\%$

b **i** ≈ 0.0215 **ii** ≈ 0.9544 **c** ≈ 0.0223

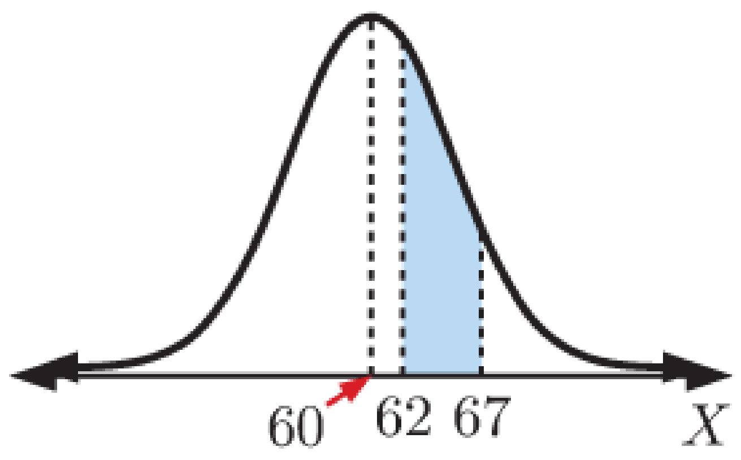
EXERCISE 15B.2

1 **a**



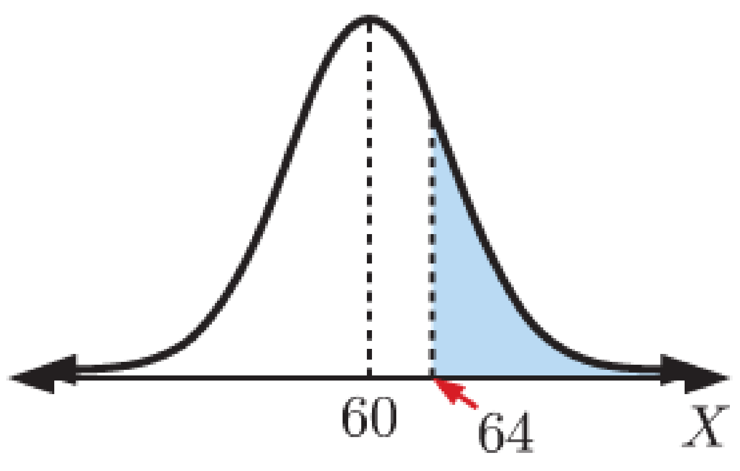
$P(60 \leqslant X \leqslant 65) \approx 0.341$

b



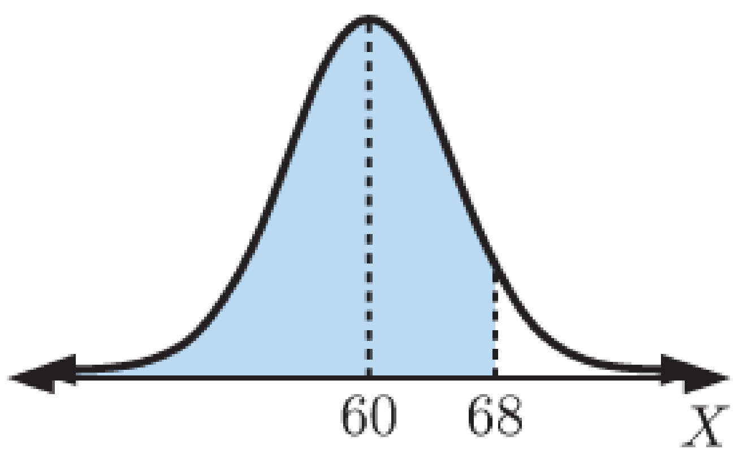
$P(62 \leqslant X \leqslant 67) \approx 0.264$

c



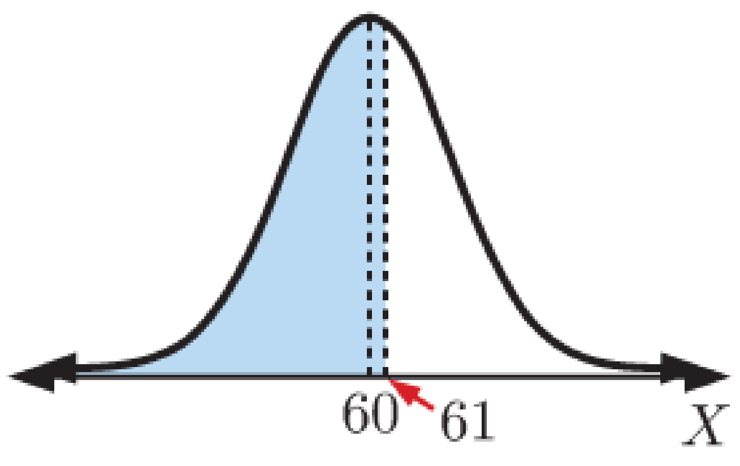
$P(X \geqslant 64) \approx 0.212$

d



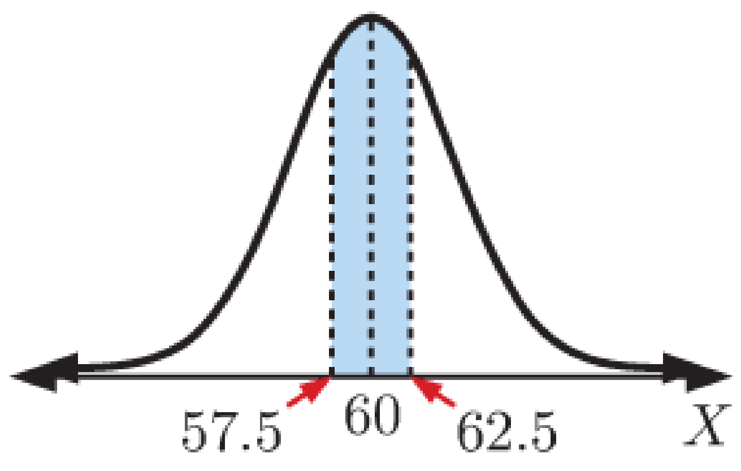
$P(X \leqslant 68) \approx 0.945$

e



$P(X \leqslant 61) \approx 0.579$

f



$P(57.5 \leqslant X \leqslant 62.5)$
≈ 0.383

2 **a** ≈ 0.334 **b** ≈ 0.166 **3** ≈ 0.378

4 **a** ≈ 0.303 **b** ≈ 0.968 **c** ≈ 0.309

5 **a** ≈ 0.0509 **b** $\approx 52.1\%$ **c** ≈ 47 eels

6 **a** **i** $\approx 90.4\%$ **ii** $\approx 4.78\%$ **b** \$4160

7 **a** **i** $\approx 12.7\%$ **ii** $\approx 52.0\%$

b **i** 21.6 kL **ii** ≈ 76 customers

8 **a** **i** $\approx 21.5\%$ **ii** $\approx 95.2\%$

b **i** Enrique **ii** Damien

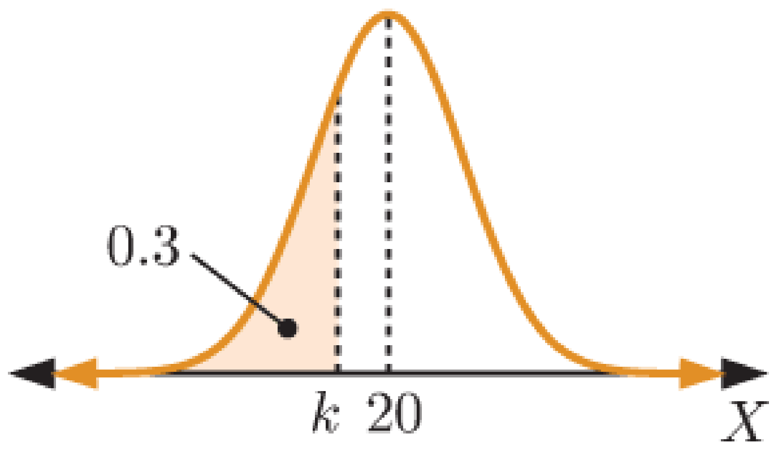
9 **a** $\approx 10.3\%$ **b** ≈ 0.456

10 **a** $\approx 84.1\%$ **b** ≈ 0.880

11 **a** ≈ 0.133 **b** ≈ 0.971

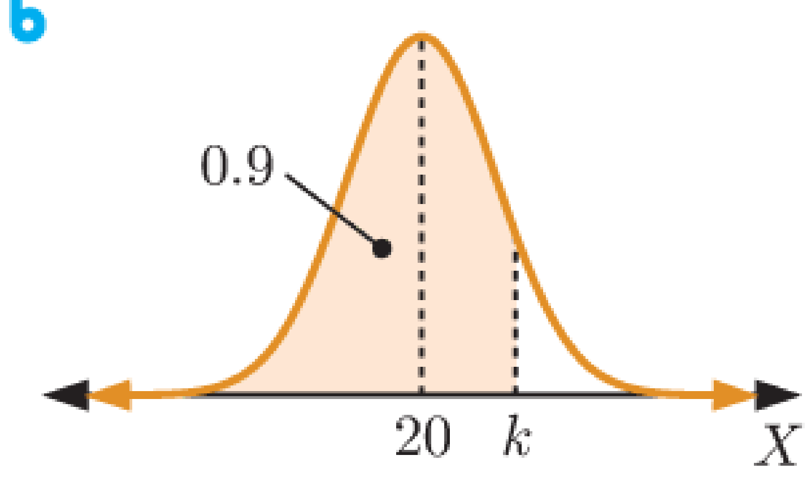
EXERCISE 15C

1 **a**



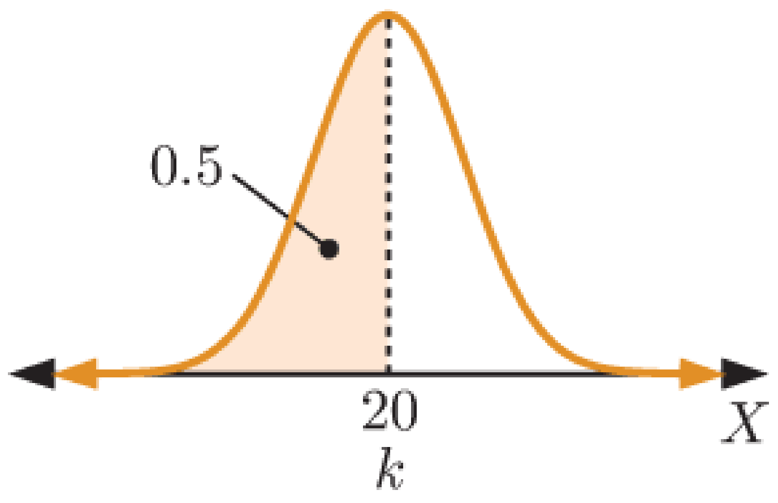
$k \approx 18.4$

b



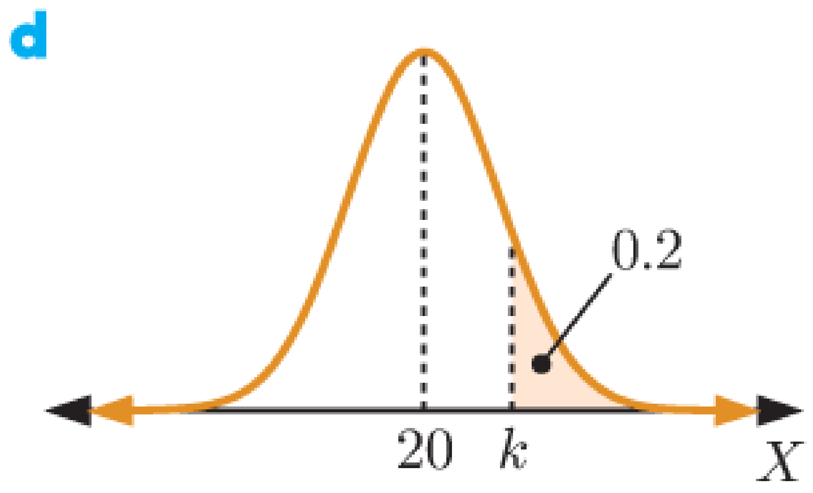
$k \approx 23.8$

c



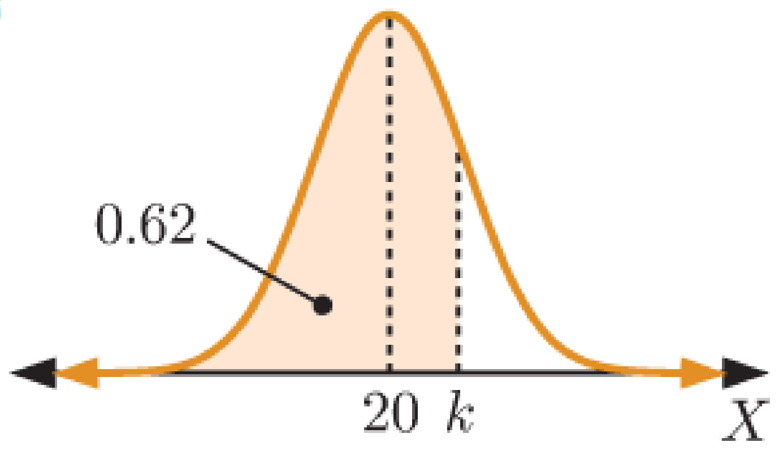
$k = 20$

d



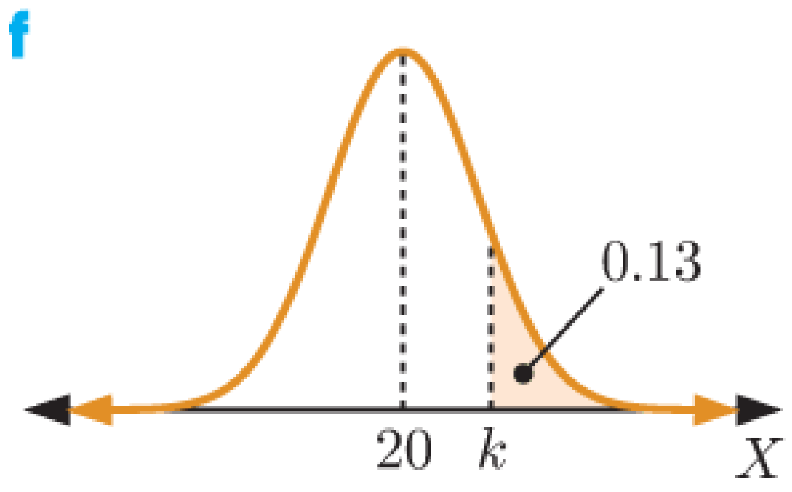
$k \approx 22.5$

e



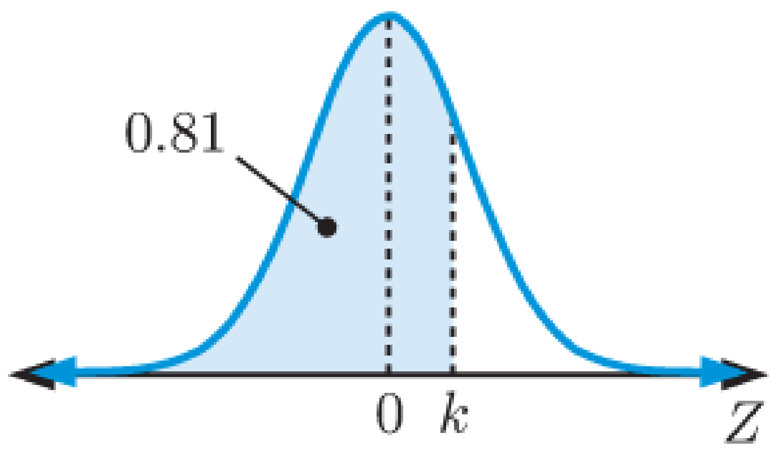
$k \approx 20.9$

f



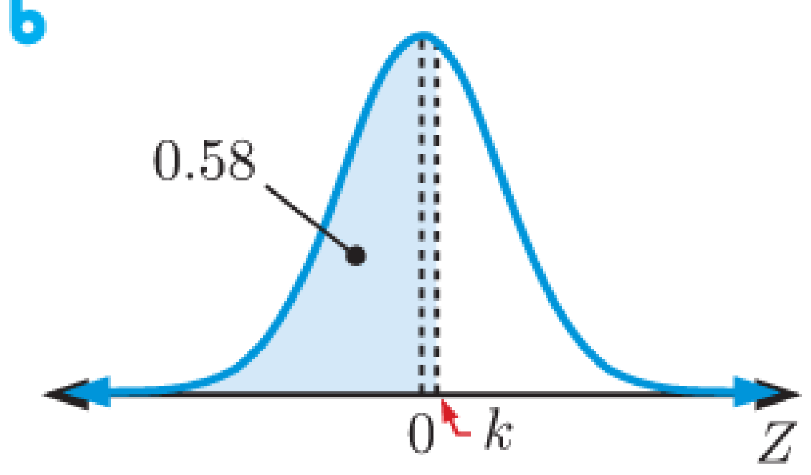
$k \approx 23.4$

2 **a**



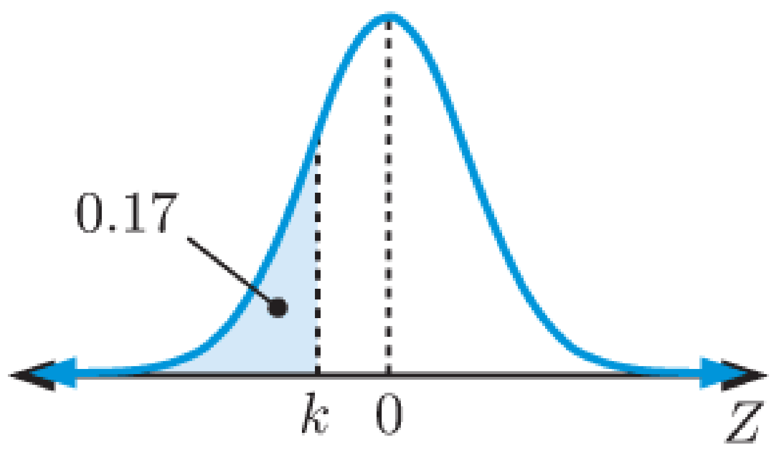
$k \approx 0.878$

b



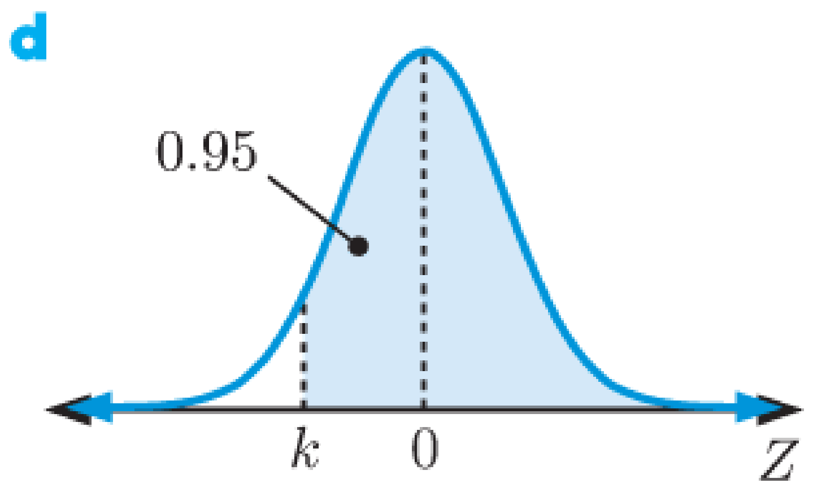
$k \approx 0.202$

c



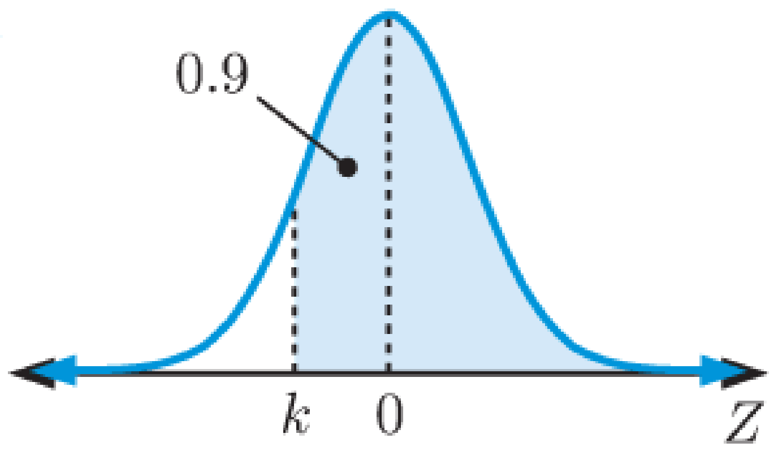
$k \approx -0.954$

d



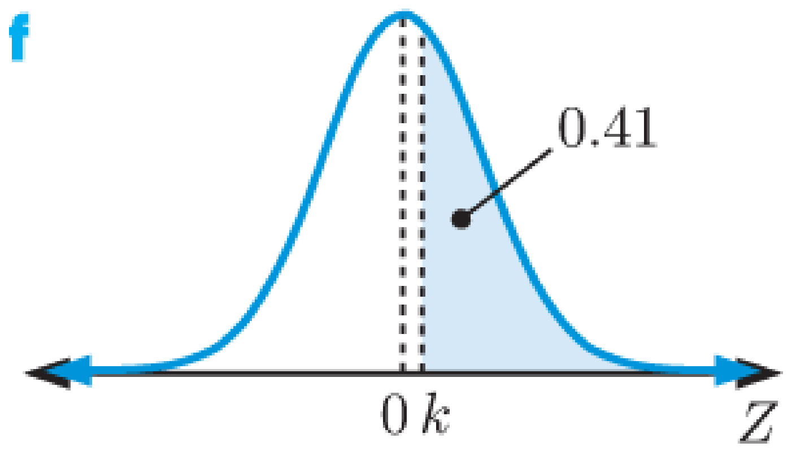
$k \approx -1.64$

e



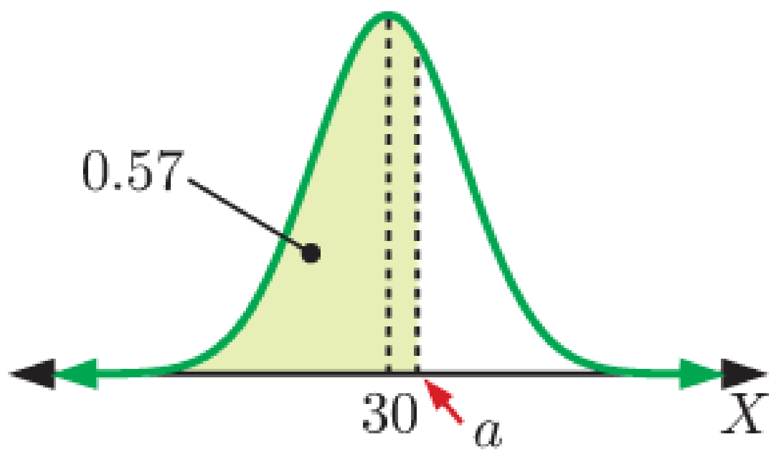
$k \approx -1.28$

f



$k \approx 0.228$

3 **a**



$\therefore \quad a > 30$

b $a \approx 30.9$

c **i** 0.43 **ii** 0.07

4 **a** $k \approx 12.5$ **b** $k \approx 18.8$ **c** $k \approx 4.93$

5 **a** ≈ 0.212 **b** $k \approx 75.1$

6 **a** $a \approx 42.0$ **b** $a \approx 46.7$ **c** $a \approx 40.1$

7 ≈ 24.7 cm **8** ≈ 75.2 mm

9 **a** **i** $k \approx 54.5$ **ii** $l \approx 69.8$

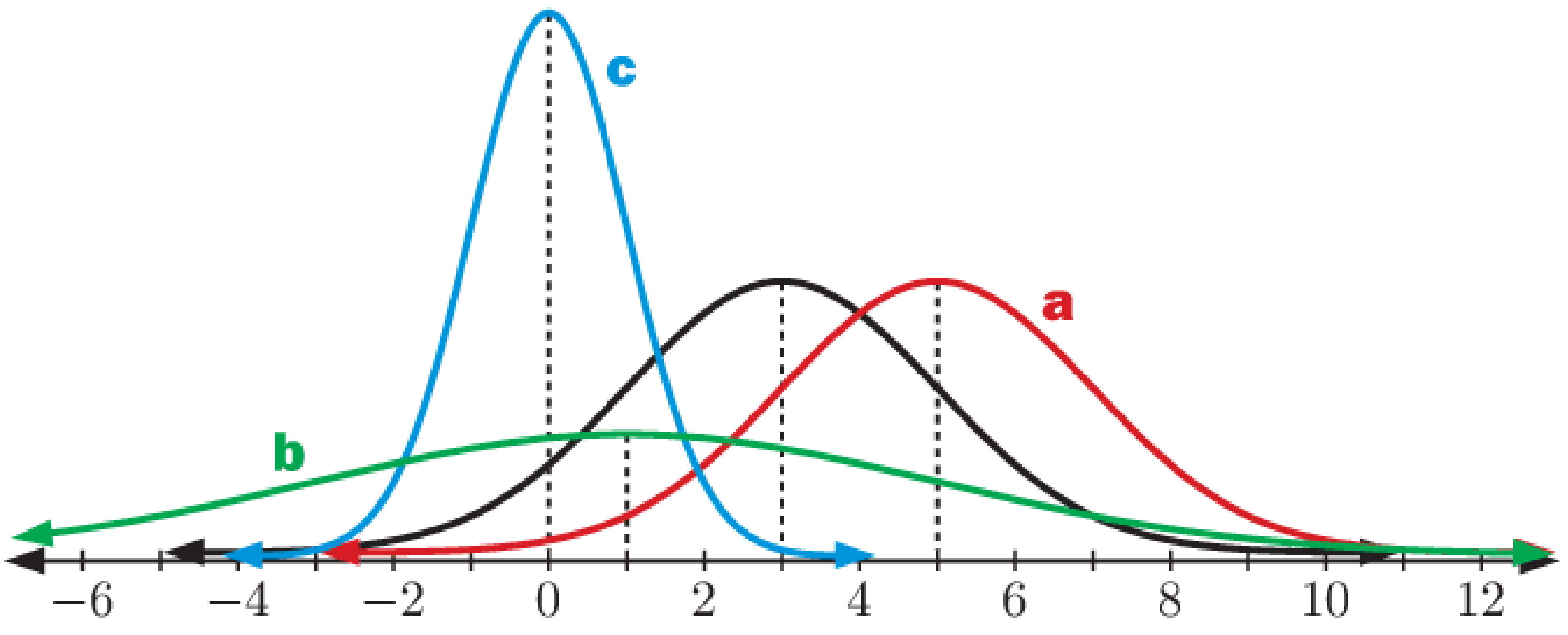
10 ≈ 501.8 mL to 504.0 mL **11** ≈ 31.0°C

REVIEW SET 15A

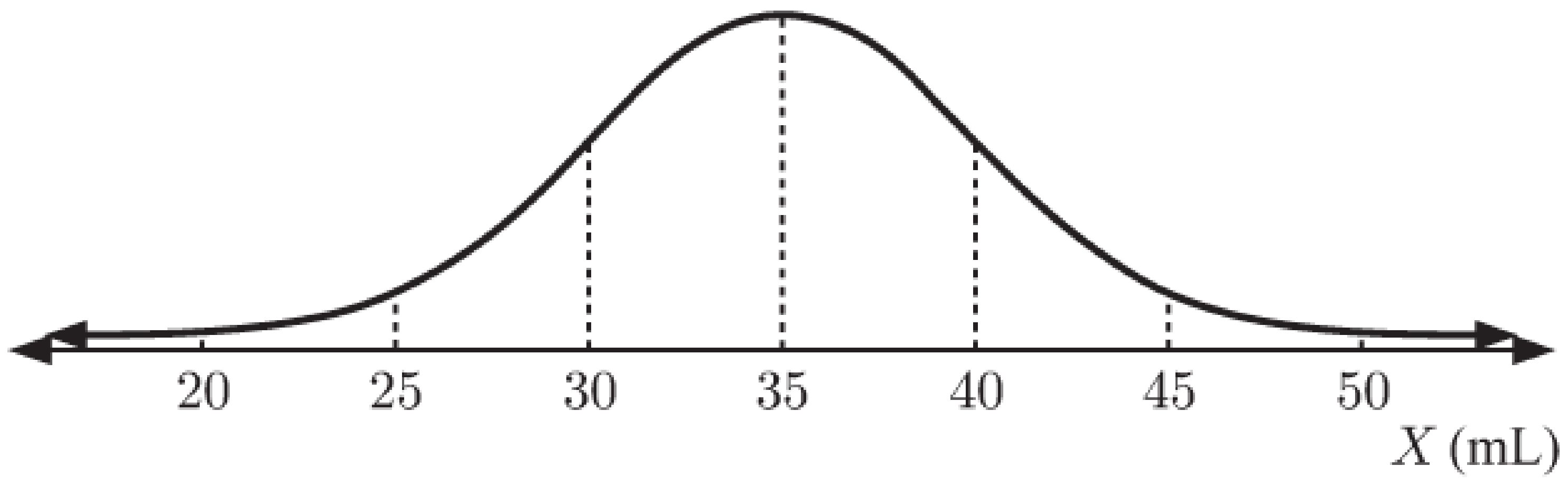
1 **a** The distribution of times taken for students to read a novel is likely to be positively skewed, and hence not normal.

b The mean amount spent on groceries at a supermarket is likely to occur most often, with variations around the mean occurring symmetrically as a result of random variation in the prices of items bought and/or the quantities of items bought (for example weights of fruits and vegetables). So the distribution is likely to be normal.

2



3 **a**



b **i** $\approx 47.7\%$ **ii** $\approx 2.28\%$

4 **a** $\approx 2.28\%$ **b** $\approx 68.26\%$ **c** $\approx 95.44\%$

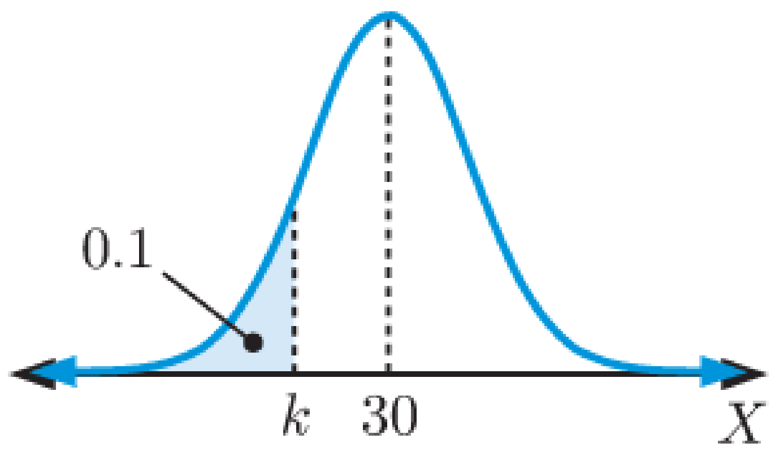
5 **a** $\approx 50.2\%$ **b** ≈ 7 oysters

6 **a** ≈ 0.364 **b** ≈ 0.356 **c** $k \approx 18.2$

7 **a** $\approx 6.68\%$ **b** ≈ 0.854

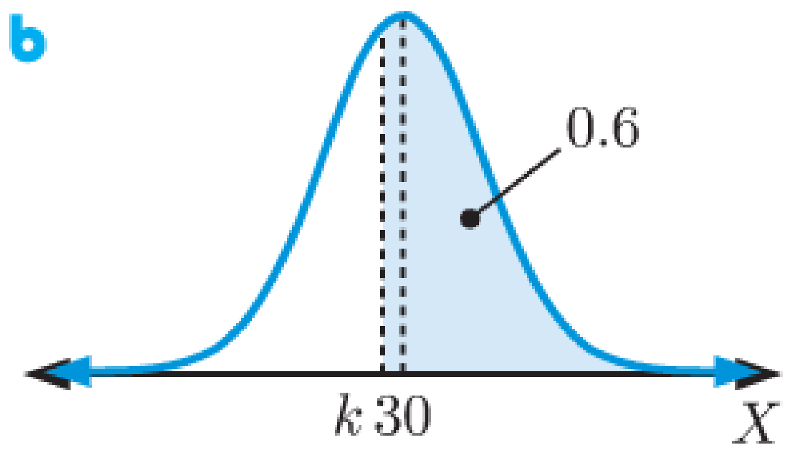
8 **a** ≈ 0.758 **b** ≈ 0.115 **c** ≈ 0.285

9 **a**



$k \approx 19.7$

b



$k \approx 28.0$

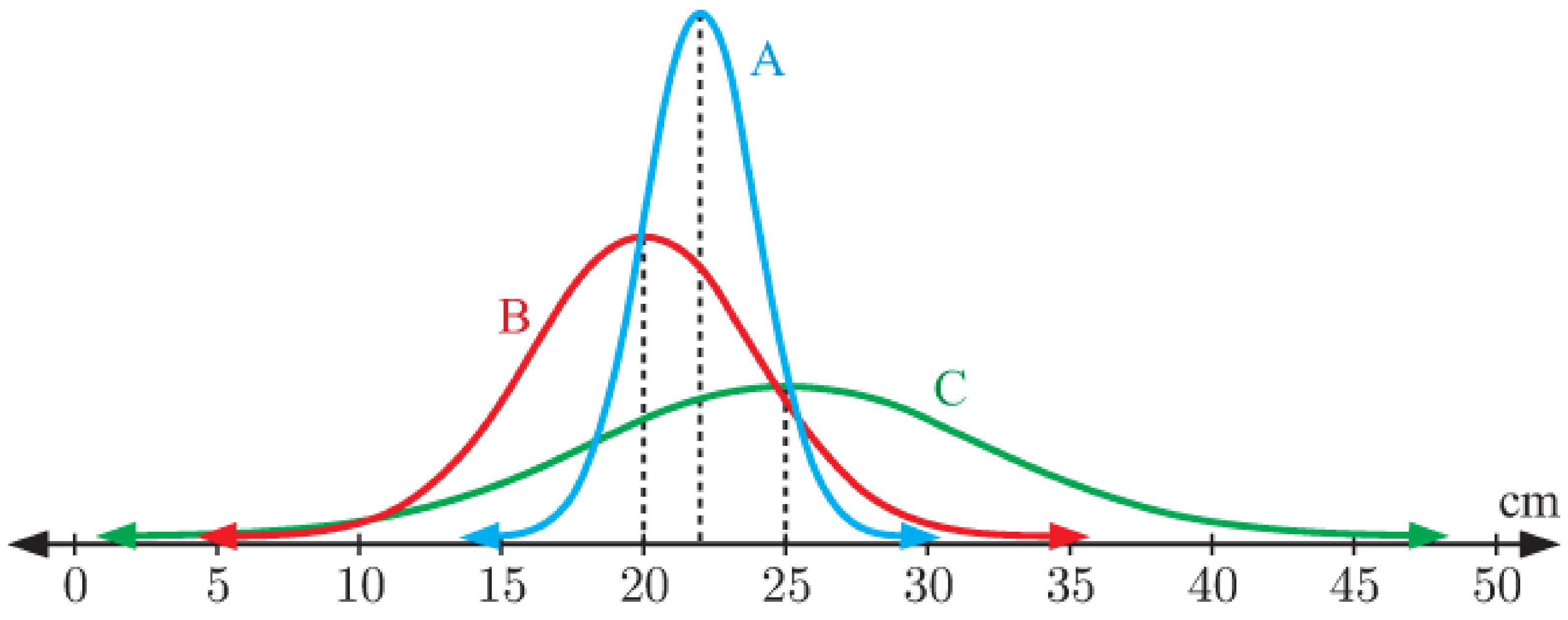
10 **a** ≈ 0.260 **b** ≈ 29.3 weeks

11 **a** $k \approx 28.1$ **b** $k \approx 26.5$ **c** $k \approx 25.0$

12 **a** **i** ≈ 0.0736 **ii** ≈ 0.0406 **b** ≈ 0.644

REVIEW SET 15B

1



2 **a** $\mu = 32, \ \sigma = 5$

b **i** $\approx 34.13\%$ **ii** $\approx 84.13\%$ **iii** $\approx 2.28\%$

3 $k \approx 1.96$

4 **a** **i** $\approx 2.28\%$ **ii** $\approx 84.0\%$ **b** ≈ 0.3413

5 **a** **i** $\approx 76.1\%$ **ii** $\approx 96.0\%$ **b** ≈ 0.598

c $x \approx 61.9$

6 **a** ≈ 0.479 **b** ≈ 0.0766 **c** $k \approx 55.2$

7 ≈ 162 seconds

8 **a** $a \approx 9.05$ **b** $a \approx 13.7$ **c** $a \approx 10.4$

9 **a** ≈ 0.258 **b** ≈ 243 suitcases **c** 23 kg

10 **a** **i** ≈ 0.0362 **ii** ≈ 0.610 **iii** ≈ 0.566

b $k \approx 74.4$ **c** $a \approx 81.0,\ b \approx 102$ **d** ≈ 0.506

11 **a** $\approx 68.3\%$ **b** ≈ 0.0884

12 **a** **i** ≈ 0.722 **ii** ≈ 0.798 **b** ≈ 0.0563

EXERCISE 16A

1 **a** $H_0\colon\ \mu = 80$ {new globe lasts as long as old globe}

$H_1\colon\ \mu > 80$ {new globe lasts longer than old globe}

b $H_0\colon\ \mu = 80$ {new globe lasts as long as old globe}

$H_1\colon\ \mu < 80$ {new globe does not last as long as old globe}

2 $H_0\colon\ \mu = 26.3$ {new top speed is the same as old top speed}

$H_1\colon\ \mu > 26.3$ {new top speed is greater than old top speed}

3 $H_0\colon\ \mu = 250$ {mean weight of chips per bag is 250 g}

$H_1\colon\ \mu \neq 250$ {mean weight of chips per bag is *not* 250 g}

4 $H_0\colon\ \mu = 80$ {mean weight of paper is 80 g per m^2}

$H_1\colon\ \mu \neq 80$ {mean weight of paper is *not* 80 g per m^2}

5 $H_0\colon\ \mu = 27$ {mean travel time is the same as before}

$H_1\colon\ \mu < 27$ {mean travel time is lower than before}

6 $H_0\colon\ \mu = 2.7$ {fat content of Brand B's muesli bars is 2.7 g}

$H_1\colon\ \mu > 2.7$ {fat content of Brand B's muesli bars is *greater* than 2.7 g}

EXERCISE 16B

1 **a** **i** ≈ -1.89 **ii** ≈ 0.0336

b As the p-value < 0.05, we reject H_0 in favour of H_1.

2 **a** $H_0\colon\ \mu = 80$ and $H_1\colon\ \mu > 80$ **b** ≈ 3.40

c $\approx 0.000\,409$ **d** As the p-value < 0.01, we reject H_0.

e We conclude that $\mu > 80$ at the 1% significance level.

3 The customer's claim is valid at the 5% level of significance.

4 We conclude that the herd fineness has changed between 2015 and 2019, at the 5% level of significance.

5 It is not justified to adjust the machine.

6 There is insufficient evidence to support the underfilling claim at a 1% level of significance.

7 **a** Since the growth of carrots depends on many factors such as genetic makeup and environment, it is reasonable to assume the weight of carrots is normally distributed.

b The buyer will purchase the crop.

EXERCISE 16C

1 **a** Let μ_1 be the population mean weight of the tomatoes from last year, and

μ_2 be the population mean weight of the tomatoes from this year.

$H_0\colon\ \mu_1 = \mu_2$ and $H_1\colon\ \mu_1 < \mu_2$

b There is insufficient evidence to support the claim that the weight of the crop increased at a 1% level of significance.

2 There is insufficient evidence to support the claim that high school students sleep less than middle school students at a 5% significance level.

3 There is insufficient evidence to support the claim that *Brand B* is better than *Brand A* at a 5% significance level.

4 There is insufficient evidence to support the claim that student's test results increase with a week of tutoring at a 5% significance level.

5 **a** There is sufficient evidence to support the claim that there is a significant difference between the runners' times at a 5% level.

b The mean time for Jesiah ≈ 12.6 s and the mean time for Billy ≈ 11.7 s. So, Billy is faster.

EXERCISE 16D.1

1 **a** Let p_1 be the population proportion of "heads", and p_2 be the population proportion of "tails".

$H_0\colon\ p_1 = 0.5,\ \ p_2 = 0.5$

$H_1\colon\ p_1 \neq 0.5$ and $p_2 \neq 0.5$

b 42 **c** 48 heads, 48 tails

d

Side	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
heads	54	48	0.75
tails	42	48	0.75
		Total	1.5

So, $\chi^2_{\text{calc}} = 1.5$

e 1 **f** ≈ 0.221

g Since $p > 0.05 = \alpha$, there is insufficient evidence to suggest that the coin is biased at a 5% significance level.

2 There is insufficient evidence to suggest that there has been a change in the proportions of voters supporting each party at a 1% significance level.

3 There is sufficient evidence that the proportions of each ice cream flavour sold are not all the same at a 10% significance level. Brian should change the amounts of each ice cream flavour that he makes.

4 There is sufficient evidence to support the claim that there was a significant change in London's demographics between 2001 and 2011 at a 5% significance level.

5 **a**

Band	10	9	8	7	6	5 and below
Expected frequency	11.85	25.05	44.7	44.55	20.25	3.6

b There is sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level.

c Since the expected frequency for "Band 5 and below" is less than 5, the sample size is not large enough for χ^2 to be distributed appropriately. By combining "Band 6" and "Band 5 and below" we can obtain more reliable results.

d There is still sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level. Although we have obtained the same result both times, the result is more reliable now that each of the expected frequencies is sufficiently large.

EXERCISE 16D.2

1 **a** $\chi^2_{\text{crit}} = 9.49$

b As $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we reject H_0 in favour of H_1.

2 **a** Let p_1, p_2, p_3, and p_4 be the population proportions of red, yellow, green, and blue lollies respectively.

$H_0\colon\ p_1 = \frac{1}{4},\ p_2 = \frac{1}{4},\ \ldots,\ p_4 = \frac{1}{4}$

$H_1\colon$ at least one of $p_1, p_2, p_3, p_4 \neq \frac{1}{4}$

b $\chi^2_{\text{calc}} \approx 2.02$ **c** $\chi^2_{\text{crit}} = 6.25$

d As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to reject H_0.

e As $p \approx 0.569 > 0.1 = \alpha$, there is insufficient evidence to reject H_0 at a 10% significance level.

3 As $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$ $(\chi^2_{\text{calc}} \approx 28.8,\ \chi^2_{\text{crit}} = 13.28)$, there is sufficient evidence to support the claim that the ISP's changes were effective.

EXERCISE 16E.1

1 **a**

	Drove to work	*Cycled to work*	*Public transport*
Male	25.3	7.7	11
Female	20.7	6.3	9

b

	Junior school	*Middle school*	*High school*
Plays sport	38.28	56.76	69.96
Does not play sport	19.72	29.24	36.04

c

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*
Sunburnt	10.92	6.16	3.92
Not sunburnt	28.08	15.84	10.08

2 **a**

	Pass Maths test	*Fail Maths test*
Male	30	20
Female	30	20

b In a sample of 100 students, we would expect 30 to be male and pass the Maths test.

c

f_{obs}	f_{exp}	$f_{obs} - f_{exp}$	$(f_{obs} - f_{exp})^2$	$\frac{(f_{obs} - f_{exp})^2}{f_{exp}}$
24	30	-6	36	1.2
26	20	6	36	1.8
36	30	6	36	1.2
14	20	-6	36	1.8
			Total	6

$\chi^2_{calc} = 6$

3 $\chi^2_{calc} \approx 6.61$, $\text{df} = 2$, $p \approx 0.0368$
As $p < 0.05$, we reject H_0, and conclude that the variables *weight* and *suffering diabetes* are dependent.

4 **a** $\chi^2_{crit} = 4.61$

b $\chi^2_{calc} \approx 8.58$, $\text{df} = 2$, $p \approx 0.0137$
As $\chi^2_{calc} > 4.61$, we reject H_0. So at a 10% level, we conclude that *age* and the *party they wish to vote for* are dependent.

5 **a** $\chi^2_{calc} \approx 23.6$, $\text{df} = 3$, $p \approx 0.000\,029\,9$
As $p < 0.05$, we reject H_0. So at a 5% level, *reason for travelling* and *rating* are dependent.

b Guests travelling for a holiday are more likely to give a higher rating.

6 $\chi^2_{calc} \approx 7.94$, $\text{df} = 6$, $p \approx 0.242$
As $p > 0.1$, we do not reject H_0. So at a 10% level, *position* and *injury type* are independent.

7 **a**

		Own a pet?	
		Yes	*No*
Age	0 - 19	≈ 4.02	≈ 3.98
	20 - 29	≈ 27.1	≈ 26.9
	30 - 49	≈ 50.2	≈ 49.8
	50+	≈ 36.7	≈ 36.3

b Yes, 4.02 and 3.98.

c

		Own a pet?	
		Yes	*No*
Age	0 - 29	37	25
	30 - 49	42	58
	50+	39	34

d $\chi^2_{calc} \approx 5.22$, $\text{df} = 2$, $p \approx 0.0735$
As $p > 0.05$, we do not reject H_0. So at a 5% level, *age* and *owning a pet* are independent.

8 **a**

		Intelligence level			
		Low	*Average*	*High*	*Very high*
Smoking habits	*Non smoker*	≈ 262	≈ 383	≈ 114	≈ 4.69
	Medium level smoker	≈ 133	≈ 194	≈ 57.7	≈ 2.38
	Heavy smoker	≈ 107	≈ 157	≈ 46.6	≈ 1.93

b $\chi^2_{calc} \approx 16.9$, $\text{df} = 6$, $p = 0.009\,59$
As $\chi^2_{calc} > 16.81$, we reject H_0. So at a 1% level, we conclude that *intelligence level* and *cigarette smoking* are not independent.

c

		Intelligence level			
		Low	*Average*	*High/Very high*	*Sum*
Smoking habits	*Non smoker*	279	386	98	763
	Medium level smoker	123	201	63	387
	Heavy smoker	100	147	66	313
	Sum	502	734	227	1463

d $\chi^2_{calc} \approx 13.2$, $\text{df} = 4$, $p = 0.0104$
As $p > 0.01$, we do not reject H_0. So at a 1% level, *intelligence level* and *cigarette smoking* are independent. This is a different conclusion from the one in **b**.

EXERCISE 16E.2

1 **a**

		Result	
		Heads	*Tails*
Guess	*Heads*	49.4	54.6
	Tails	45.6	50.4

b $\chi^2_{calc} \approx 1.35$

c As $\chi^2_{calc} < 3.84$, we do not reject H_0. So at a 5% level, Horace's *guess* and *result* are independent.

d According to this test, Horace's claim is not valid.

2 **a**

		Result	
		Pass	*Fail*
Country	*France*	63.8	21.2
	Germany	168.2	55.8

b $\chi^2_{crit} = 2.71$

c $\chi^2_{calc} \approx 4.62$

d As $\chi^2_{calc} > 2.71$, we reject H_0. So at a 10% level, *motorbike test result* and *country* are dependent.

REVIEW SET 16A

1 H_0: $\mu = 0$ {the bus arrives on time}
H_1: $\mu > 0$ {the bus is late}

2 **a** There is 7.94% chance of observing this result if the null hypothesis is true.

b For a 10% significance level, we reject H_0 if there is less than 10% chance of observing this result.

c As the p-value $< 0.1 = \alpha$, we reject H_0 in favour of H_1.

3 There is insufficient evidence to reject the manufacturer's claim at the 5% level of significance.

4 **a** H_0: $\mu = 90$ and H_1: $\mu < 90$

b Rosario's concerns are justified.

5 **a** Let μ_1 be the population mean number of fish caught by Joe per fishing trip, and
μ_2 be the population mean number of fish caught by Ruben per fishing trip.
H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 > \mu_2$

b Joe's claim is not justified at a 5% level of significance.

6 There is sufficient evidence to support the claim that there is a significant difference between the time spent shopping by customers at the supermarkets at a 10% level.

7 **a** Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of shirts which are small, medium, large, X-large, and XX-large respectively.

i H_0: $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.35$, $p_4 = 0.25$, $p_5 = 0.1$

ii df $= 4$

b ≈ 0.0134

c Yes, the p-value < 0.05, so we reject H_0 in favour of H_1 on a 5% level of significance. Since we accept H_1, we conclude that the store should change its distribution.

8 **a**

Item rarity	super rare	rare	uncommon	common
Expected frequency	12.5	25	62.5	150

b $\chi^2_{\text{crit}} = 11.34$, df $= 3$, $\chi^2_{\text{calc}} \approx 10.0$

As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to justify Emmanuel's suspicions at a 1% significance level.

9 $\chi^2_{\text{calc}} \approx 42.1$, df $= 2$, $p = 7.37 \times 10^{-10}$

As $\chi^2_{\text{calc}} > 4.61$, we reject H_0 at a 10% significance level.

The variables *age of a driver* and their *opinion* are dependent.

REVIEW SET 16B

1 H_0: $\mu = 1.2$ {the mean minimum weight of Quickchick chickens is 1.2 kg}

H_1: $\mu < 1.2$ {the mean minimum weight of Quickchick chickens is less than 1.2 kg}

2 **a** 12.59

b As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to reject H_0.

3 There is sufficient evidence to justify the company's concerns that the systolic blood pressure of its employees is too high on a 5% level of significance.

4 There is insufficient evidence at the 5% significance level to support the claim that Arthur has improved.

5 There is insufficient evidence to support the claim that the average points total of the two suburbs is significantly different at a 10% significance level.

6 **a**

	Mean	*Standard deviation*
Revision course	$\overline{x}_1 \approx 34.6$	$s_1 \approx 2.82$
No revision course	$\overline{x}_2 = 32.1$	$s_2 \approx 4.48$

b There is sufficient evidence to support the claim that the revision course improved results at a 10% level of significance.

7 $\chi^2_{\text{calc}} = 5.215$, df $= 3$, $p \approx 0.157$

As $p > 0.05$, there is insufficient evidence to reject the manufacturer's claim at a 5% significance level.

8 $\chi^2_{\text{calc}} \approx 13.0$, df $= 6$, $p \approx 0.0433$

a As $p < 0.05$, we reject H_0. So, at a 5% level, P and Q are dependent.

b As $p > 0.01$, we do not reject H_0. So, at a 1% level, P and Q are independent.

9 $\chi^2_{\text{calc}} \approx 25.6$, df $= 9$, $p \approx 0.002\,41$

As $\chi^2_{\text{calc}} > 21.67$, we reject H_0. So at a 1% level, *education level* and *business success* are dependent.

EXERCISE 17A

1 **a** **i** 3 **ii** 3 **iii** 1

b **i** Site B as P lies in cell B.

ii PB $= \sqrt{10} \approx 3.16$ units, PA $= 3\sqrt{2} \approx 4.24$ units, PC $= \sqrt{26} \approx 5.10$ units

c **i** Q lies on the edge between cells A and C. Q is equally closest to sites A and C.

ii QA $= \sqrt{10} \approx 3.16$ units, QC $= \sqrt{10} \approx 3.16$ units, QB $= 3\sqrt{2} \approx 4.24$ units

2 **a** All points in the green cell are closest to site A than any other site.

b All points on the blue edge are equally closest to sites D and C.

c The red vertex is equally closest to sites A, B, and C.

3 **a** **i** site B **ii** site D **iii** site C **iv** site A

b **i** distance to site A $= \sqrt{5}$ units

distance to site D $= \sqrt{5}$ units

ii distance to site A $= \sqrt{13}$ units

distance to site D $= \sqrt{13}$ units

c $(-3, 0)$ is equally closest to sites A and D, hence it lies on an edge. $(-3, 2)$ is closest to site B and does not lie on an edge.

d 32 units^2

4 **a** true **b** true **c** not necessarily true

d not necessarily true

5 If the circle passed through another site then P would lie on an edge. If another site was contained in the circle, P would be closer to that site than X. P would not lie in cell X.

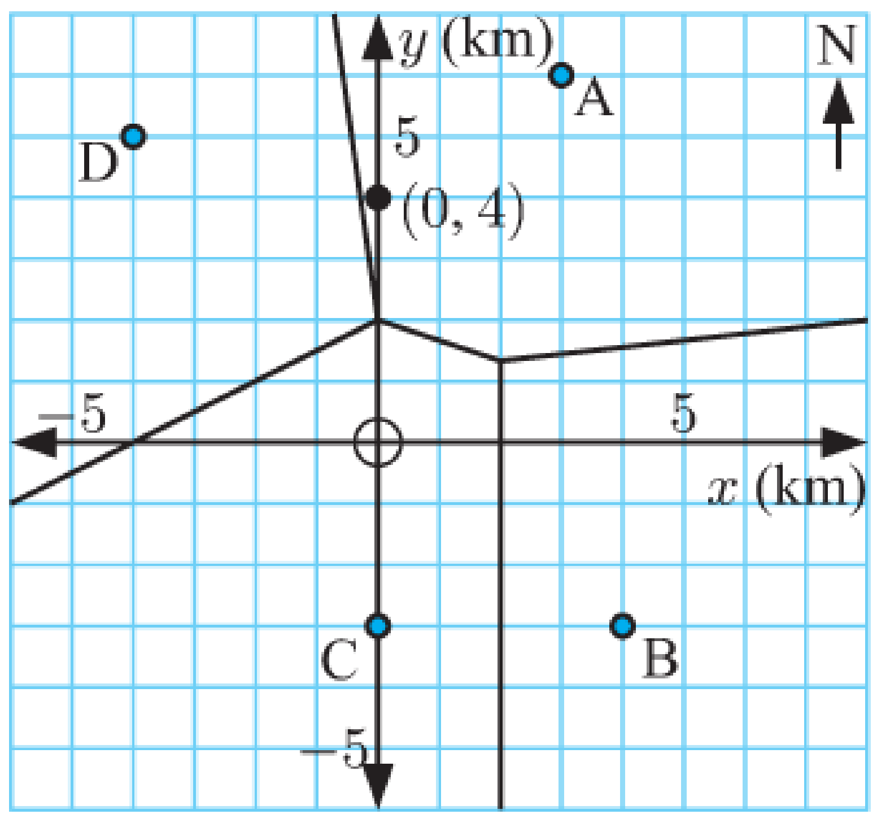
6 **a** **i** post office E **ii** post office B **iii** post office A

iv post office C

b $(-1, \frac{1}{2})$ and $(-1, -2)$

7 **a** **i** school C **ii** school A **iii** school D **iv** school B

b **i** school D **ii**



iii distance to site A $= \sqrt{13}$ km

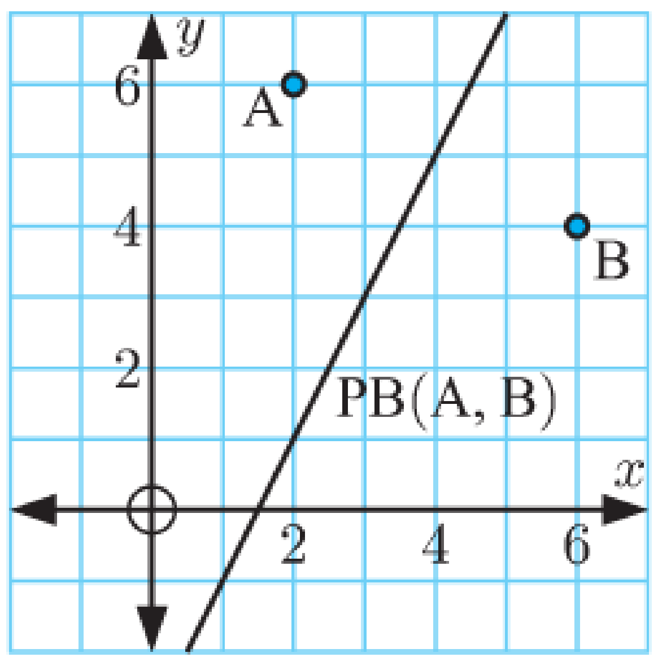
distance to site D $= \sqrt{17}$ km

c **i** $(0, 2)$ **ii** 5 km

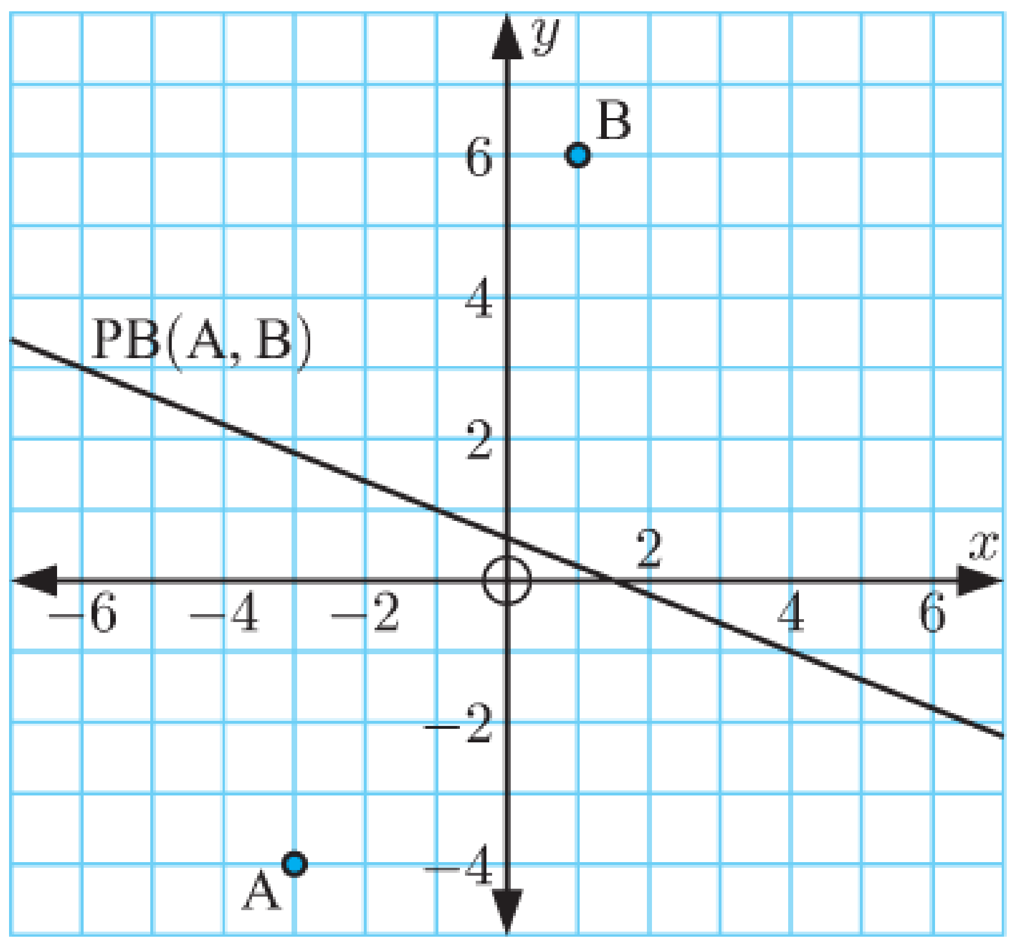
8 A vertex of a Voronoi diagram is equally closest to at least 3 sites (whose cells meet at that vertex). The circle's edge passes through site X, which is one of the sites V is closest to, and since every point on the edge is equidistant from the centre, the other closest sites must also lie on the edge of the circle.

EXERCISE 17B

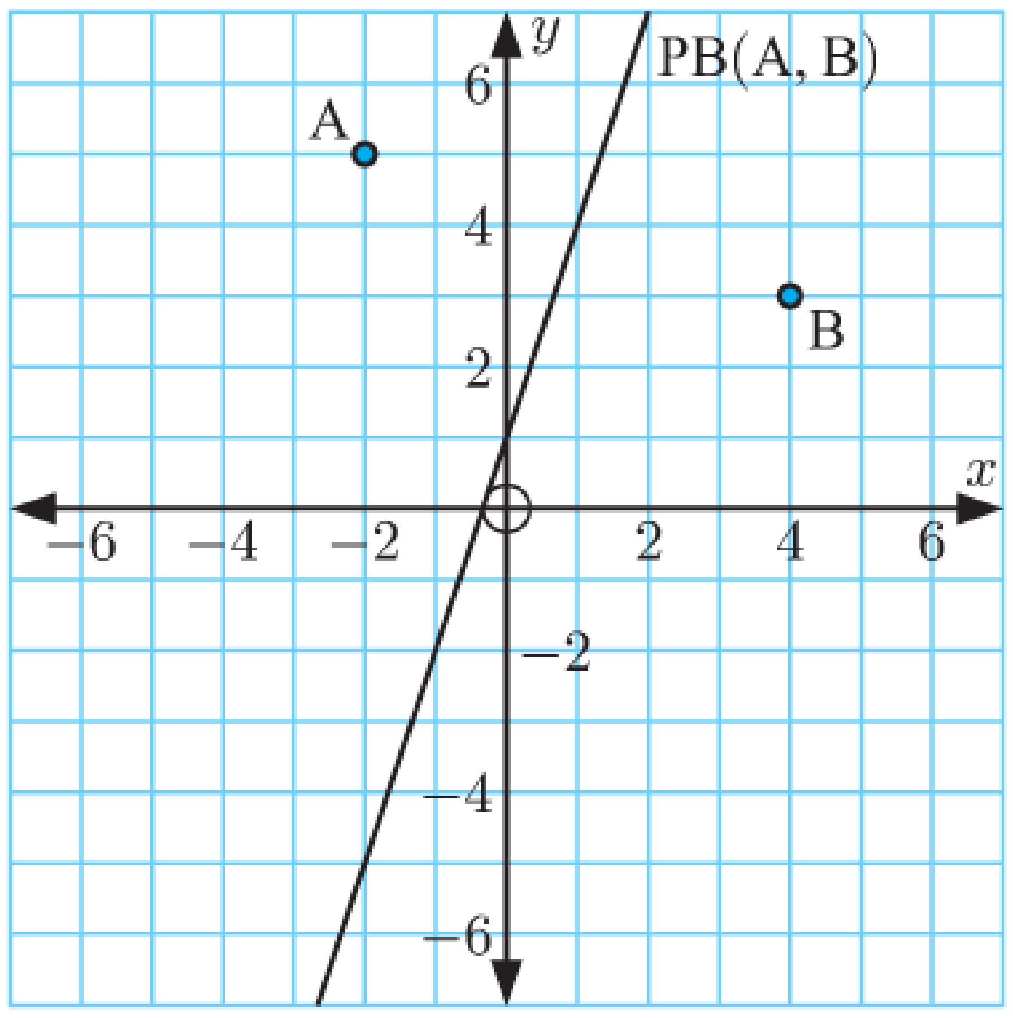
1 **a**



b



2 **a**



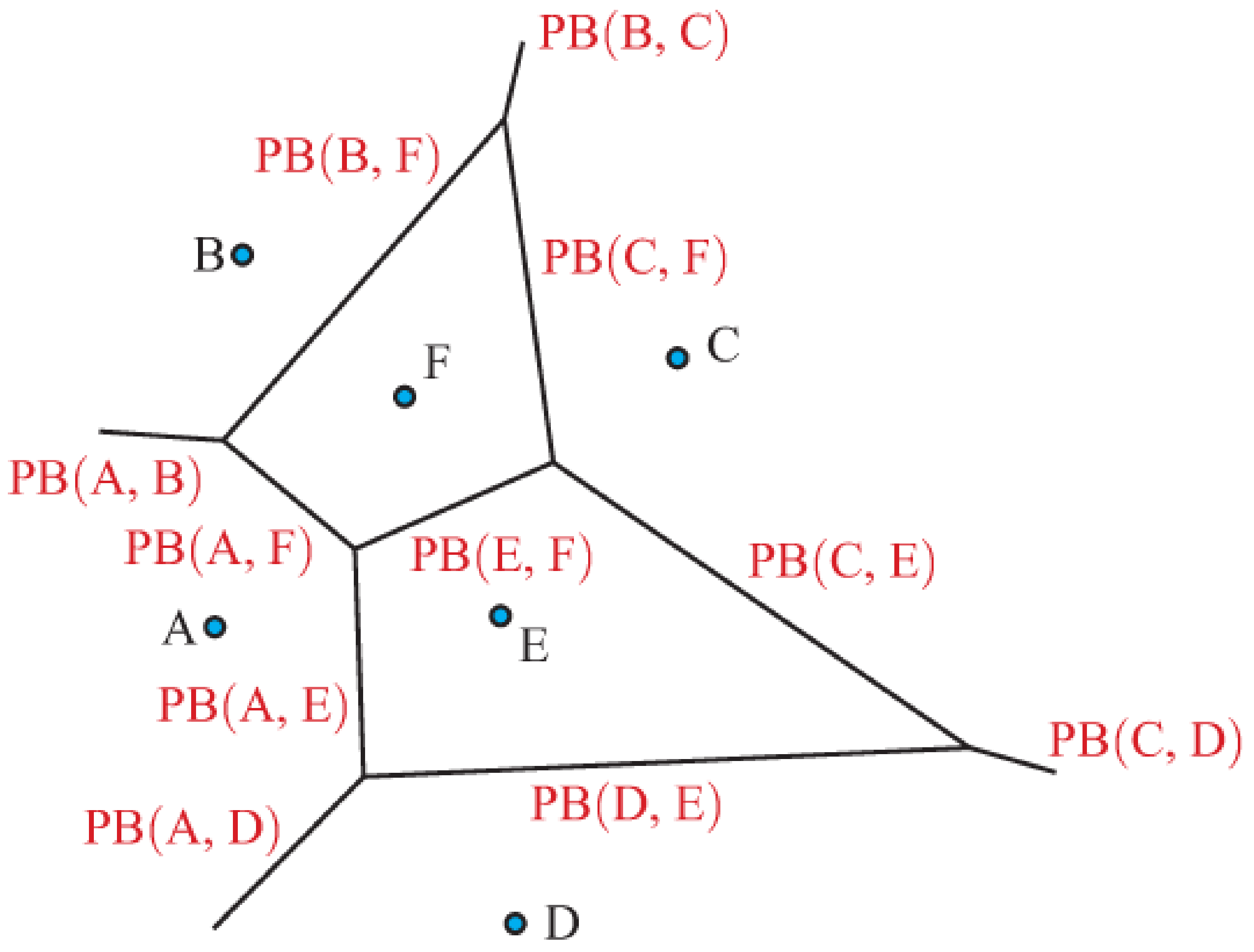
b $y = 3x + 1$

c **i** $-5 = 3(-2) + 1$ ✓

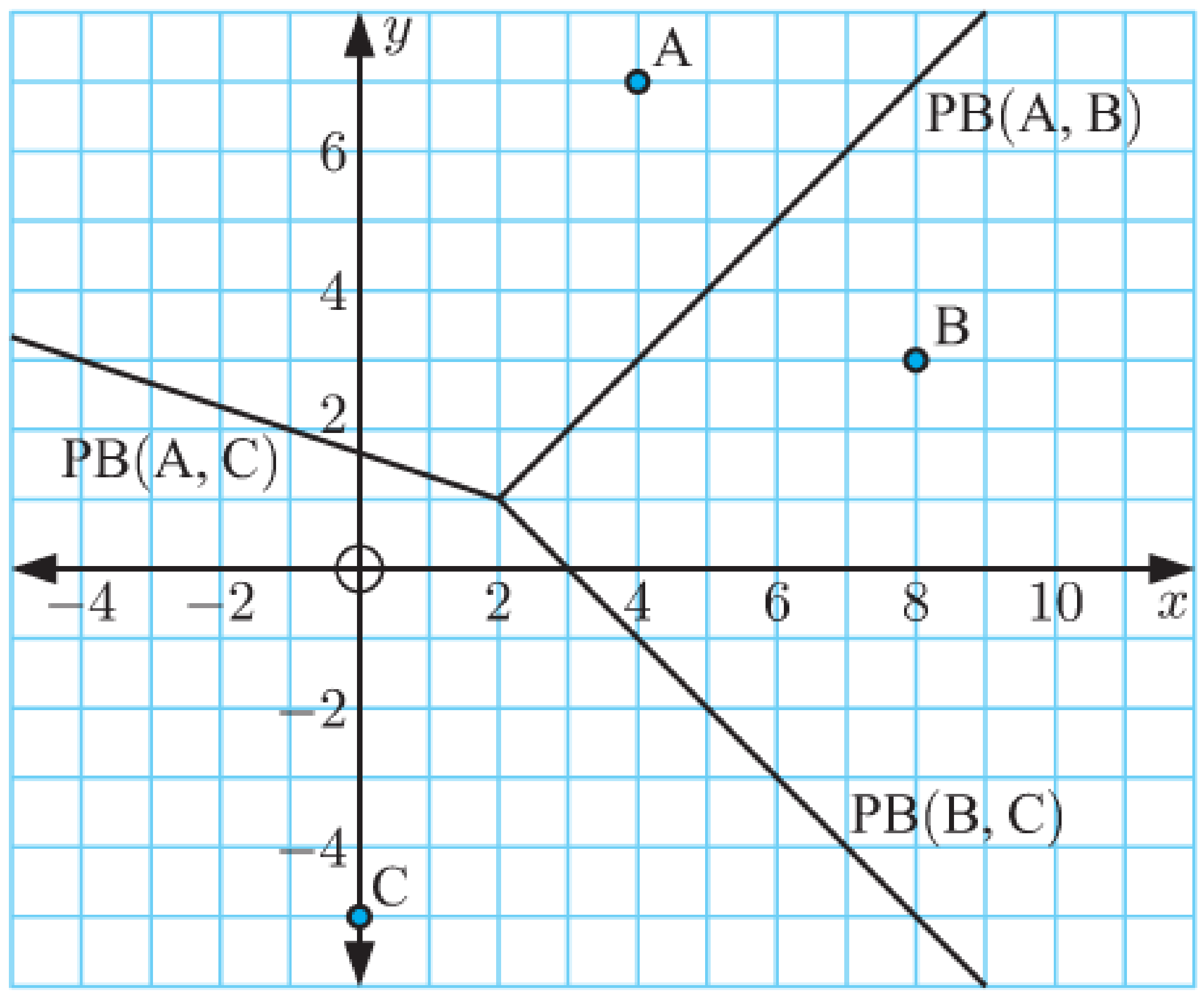
ii distance to site A = 10 units
distance to site B = 10 units

d **i** site B **ii** site B **iii** site A

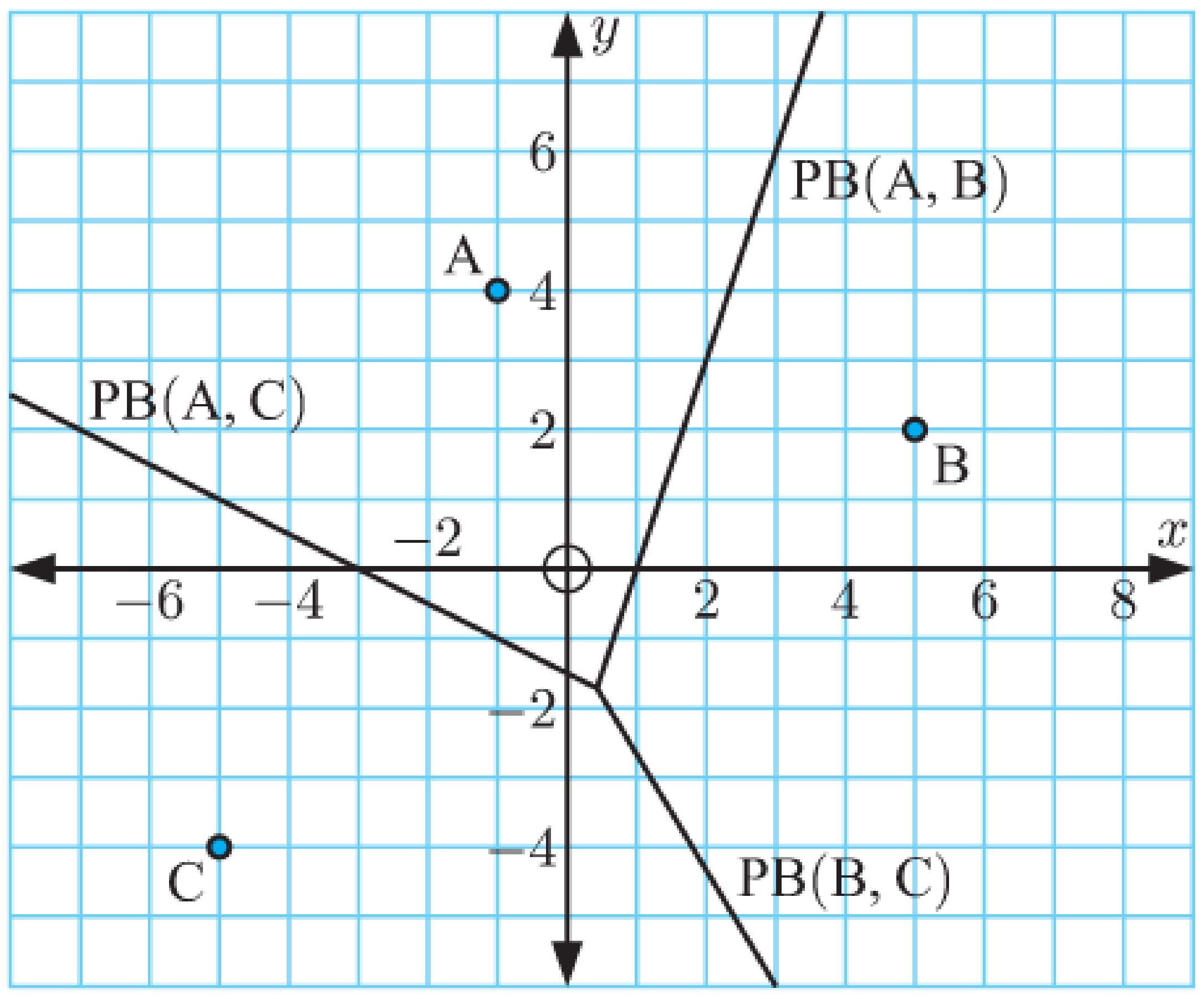
3


PB(B, C)
PB(B, F)
B
PB(C, F)
C
F
PB(A, B)
PB(A, F)
PB(E, F)
PB(C, E)
A
E
PB(A, E)
PB(C, D)
PB(D, E)
PB(A, D)
D

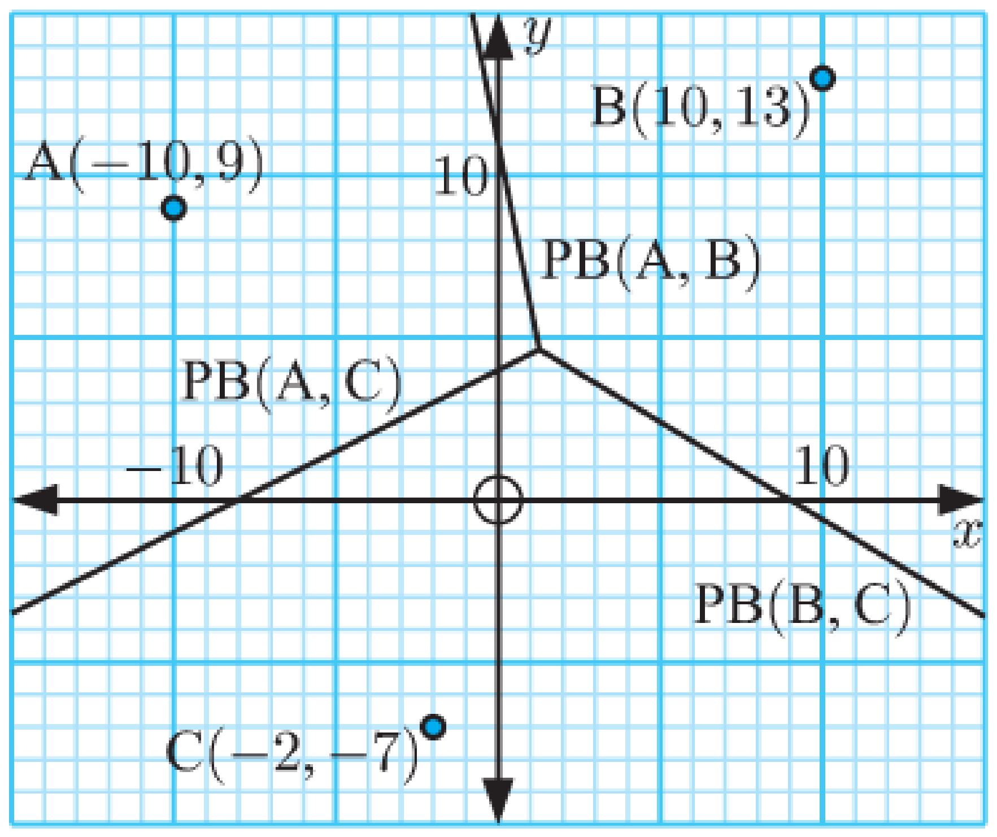
4 **a**



b



5 **a**



b PB(A, B): $y = -5x + 11$

PB(A, C): $y = \frac{1}{2}x + 4$

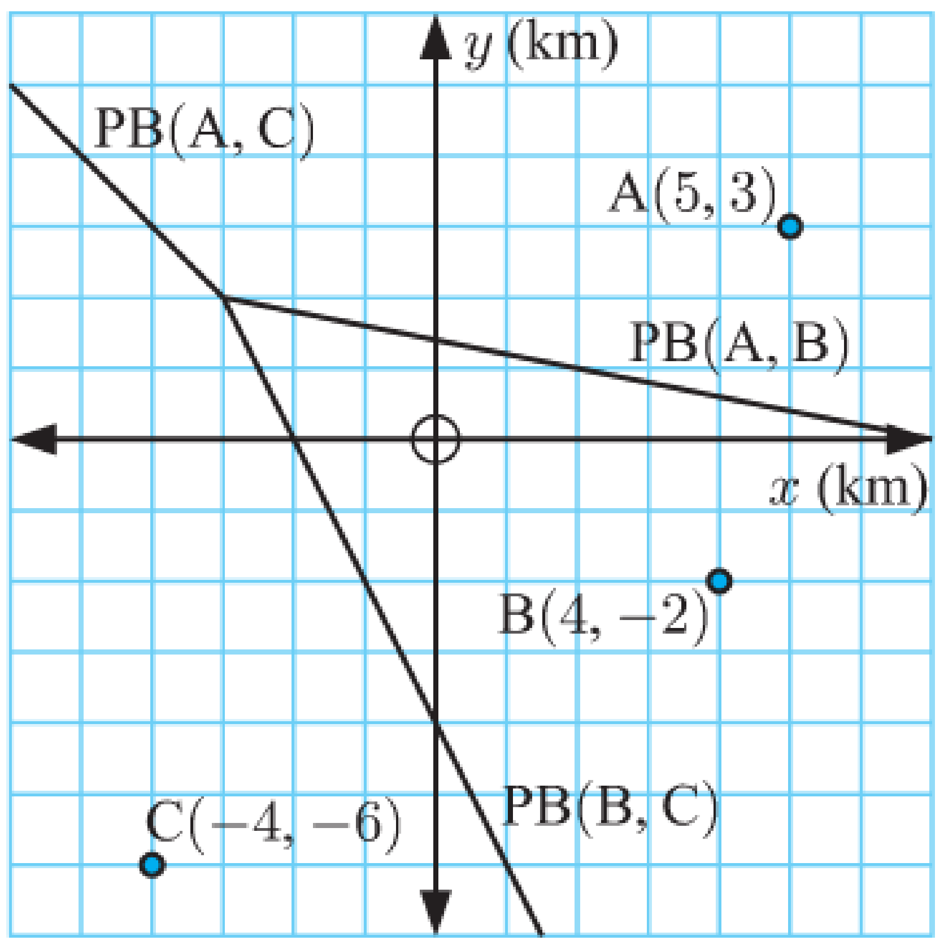
PB(B, C): $y = -\frac{3}{5}x + \frac{27}{5}$

c $\left(\frac{14}{11}, \frac{51}{11}\right)$

distance to A = distance to B = distance to C ≈ 12.1 units

d **i** site A **ii** site B **iii** site C

6 **a**

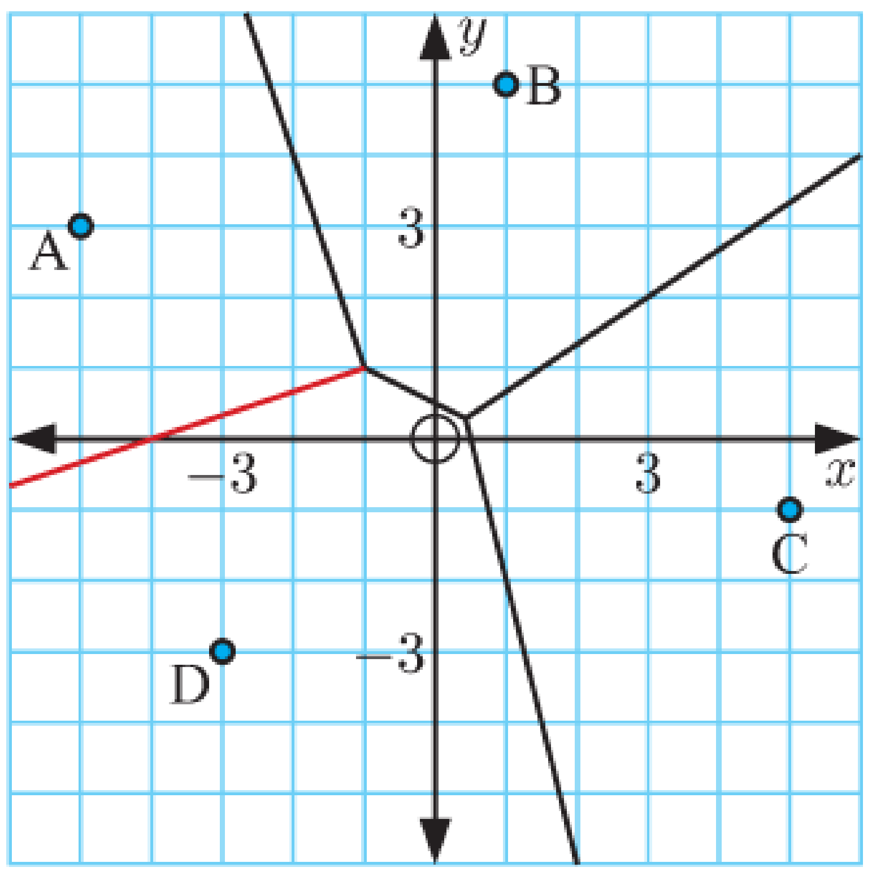


b **i** store A **ii** store A **iii** store C

c **i** $(-3, 2)$ **ii** $\sqrt{65} \approx 8.06$ km

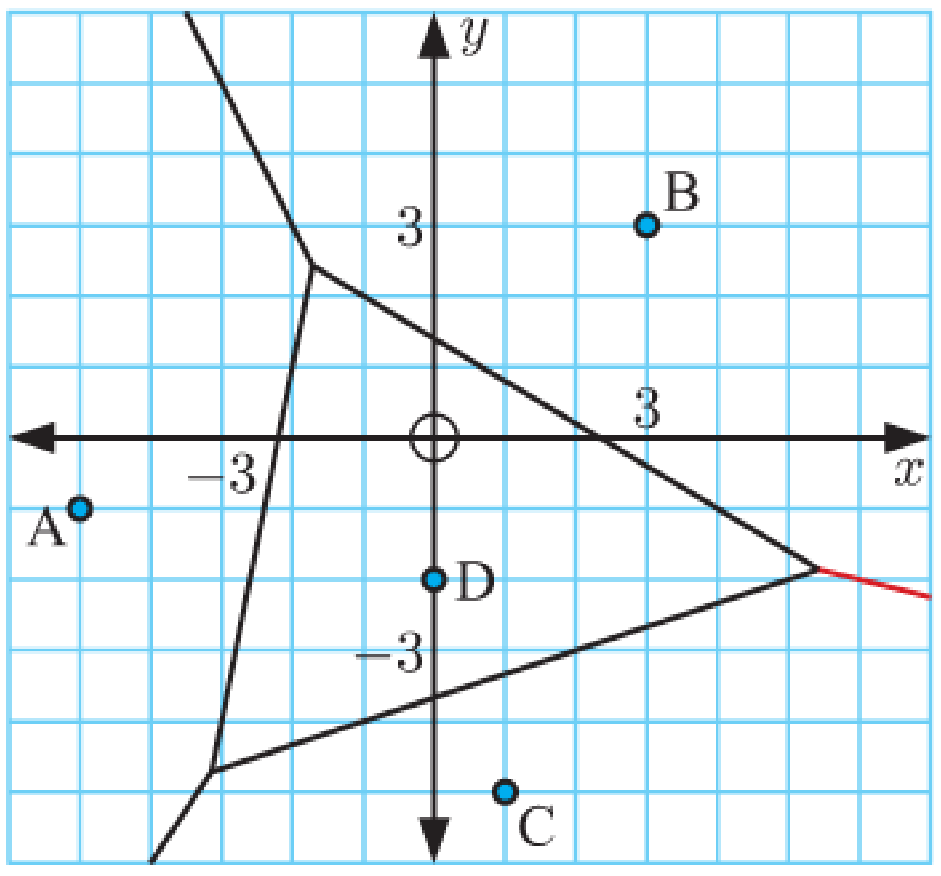
7 **a** Sites A and D are currently in the same cell.

b



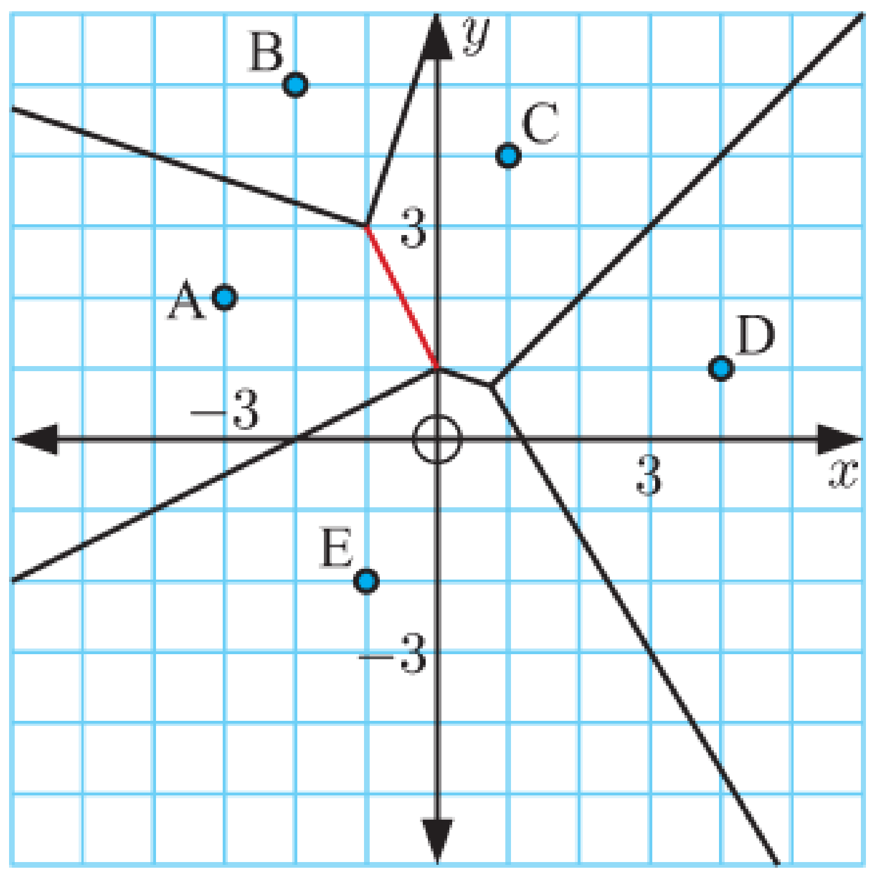
PB(A, D): $y = \frac{1}{3}x + \frac{4}{3}$

8 **a**


y
B
3
3
−3
x
A
D
−3
C

PB(B, C): $x + 4y + 2 = 0$

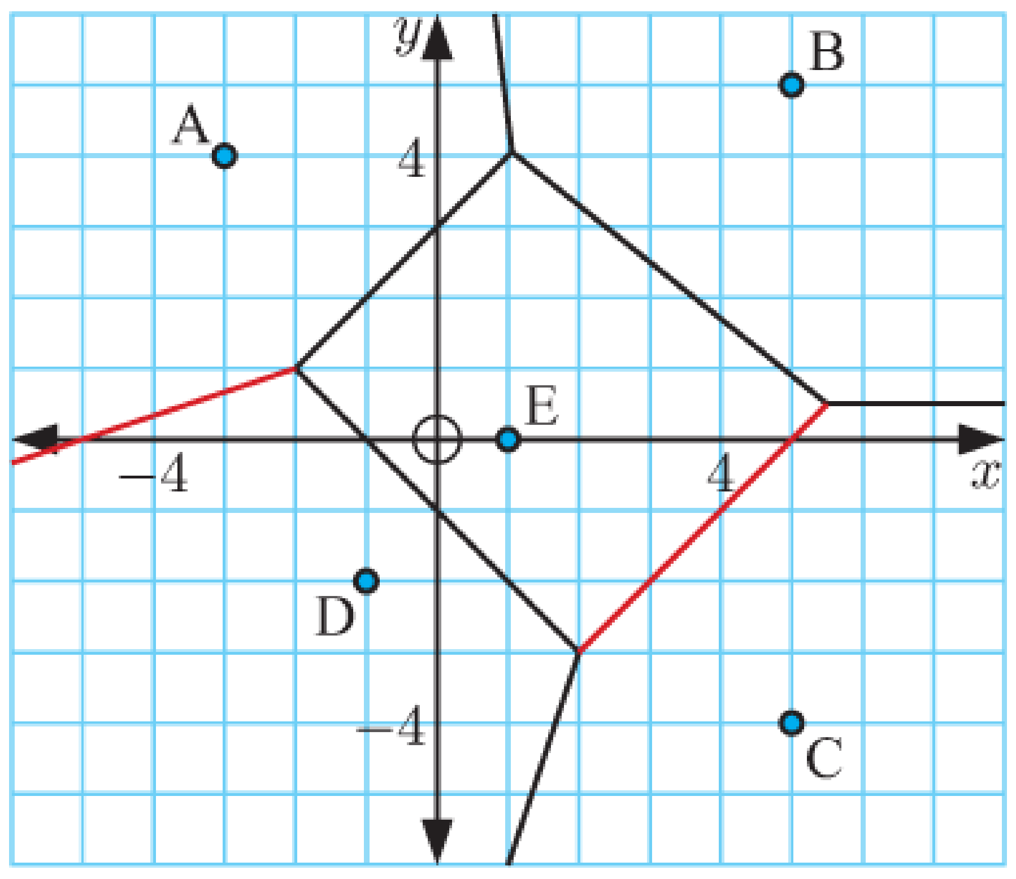
b


B
y
C
3
A
D
−3
3
x
E
−3

PB(A, C): $2x + y - 1 = 0$

9 **a** 2 edges

b


y
B
A
4
E
−4
4
x
D
−4
C

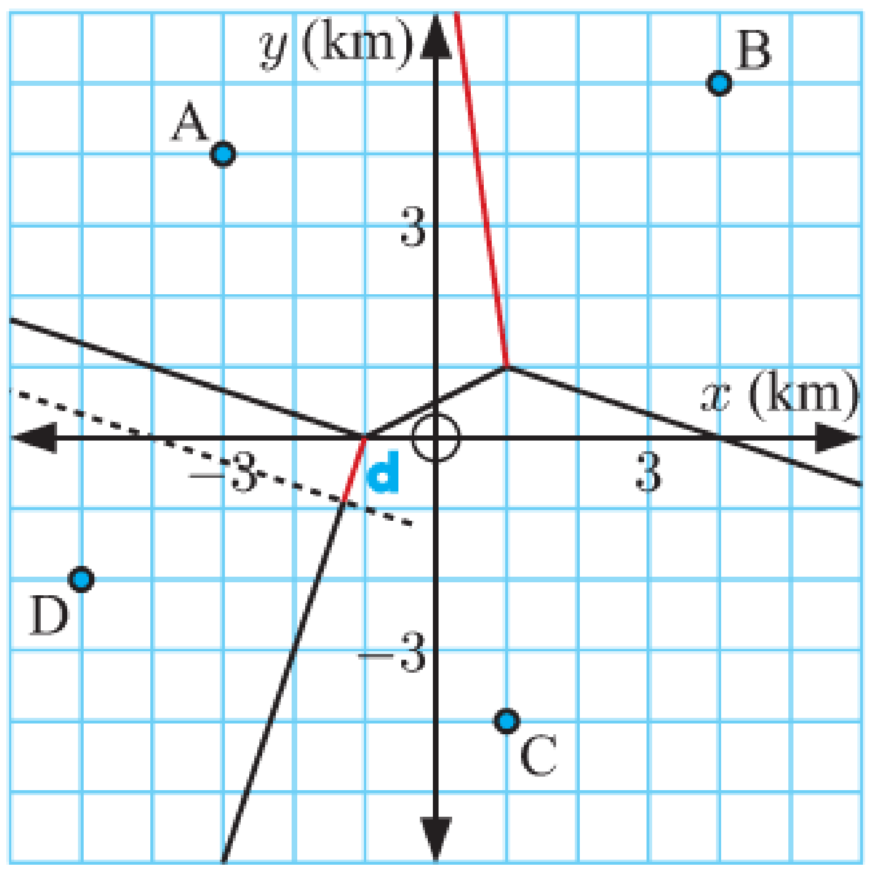
PB(A, D): $y = \frac{1}{3}x + \frac{5}{3}$, PB(C, E): $y = x - 5$

c **i** site A **ii** site E

d **i** sites C and E **ii** sites A, D, and E

10 **a** $y = 3x + 3$

b, d


y (km)
B
A
3
x (km)
−3
d
3
D
−3
C

PB(A, B): $y = -7x + 8$

c **i** camp B **ii** $\sqrt{13}$ km ≈ 3.61 km

11 **a** There is a cell which does not contain a site.

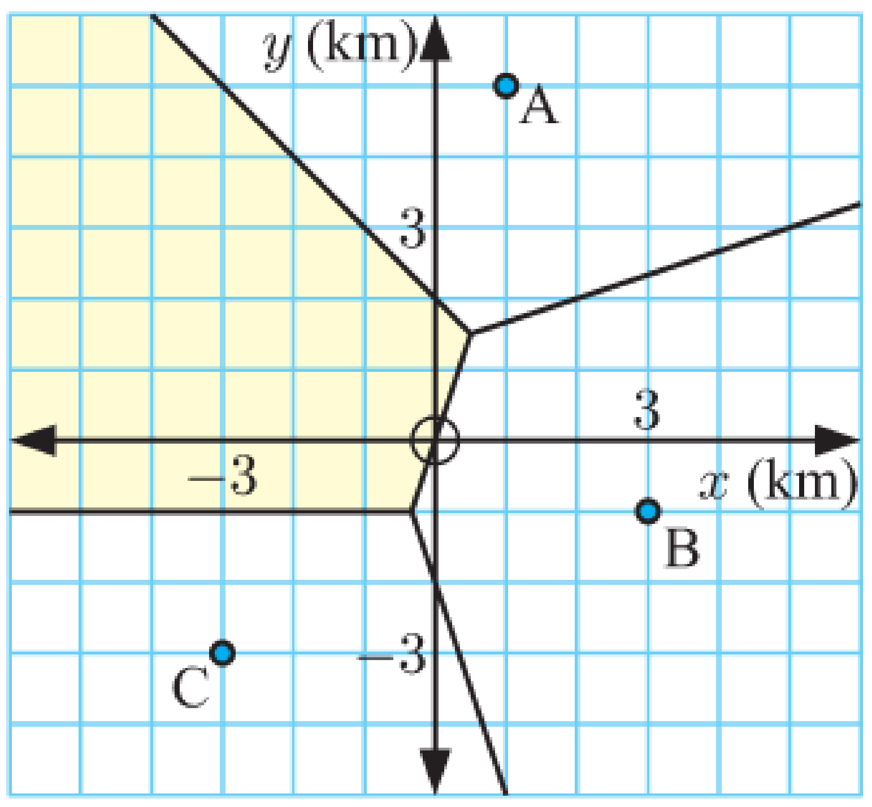
b The edge of the missing site and site B is perpendicular to the x-axis.

$\therefore$ the missing site lies on the x-axis.

c $(-3, 0)$

12 **a** $(-1, -4)$ **b** $(0, -1)$

13 **a**


y (km)
A
3
3
−3
x (km)
B
C
−3

b Distance from $(-2, 1)$ to $(0, -1) = 2\sqrt{2} \approx 2.83$ km

Distance from $(0, -1)$ to $B = 3$ km

If the vet clinic was at $(-2, 1)$ then the cell would include the point $(0, -1)$.

c $(-3, 1)$ **d** **i** vet A **ii** vets B and C

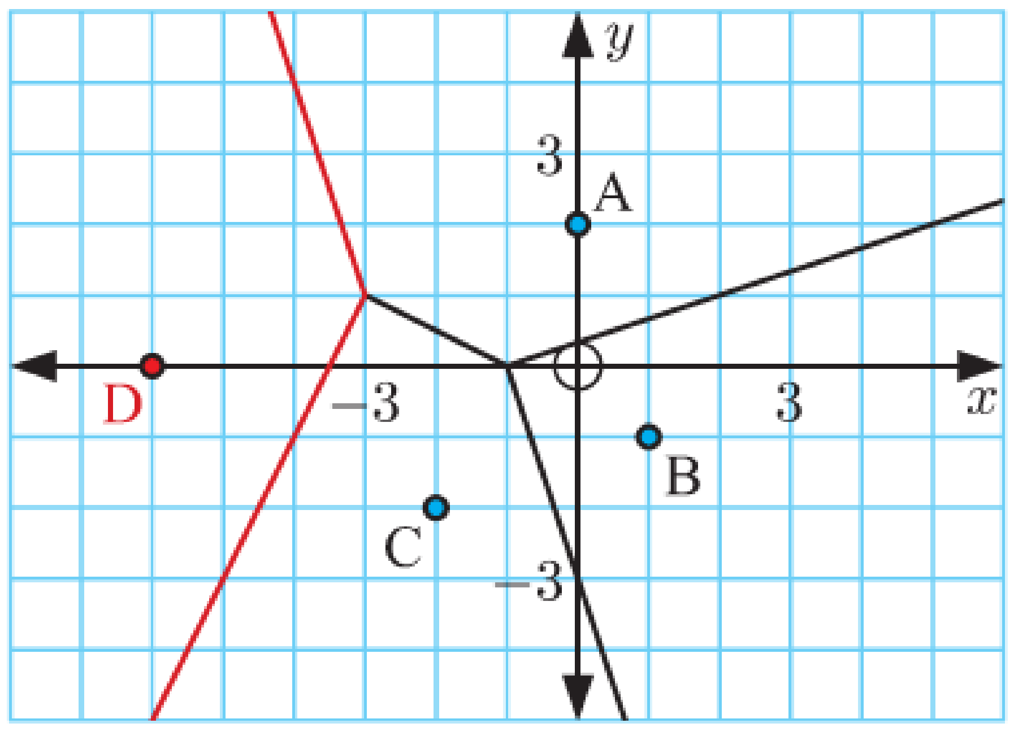
e $2\sqrt{2} \approx 2.83$ km

EXERCISE 17C

1 **a** cell C

b cells A and C as they share an edge

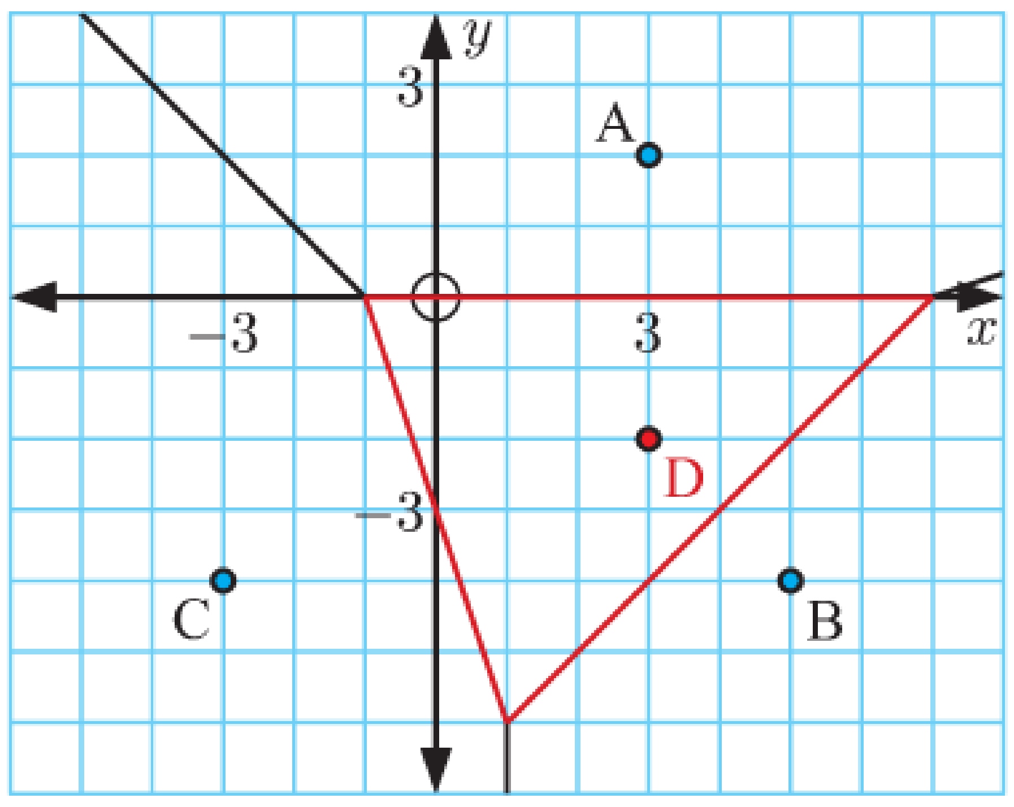
c


y
3
A
D
−3
3
x
B
C
−3

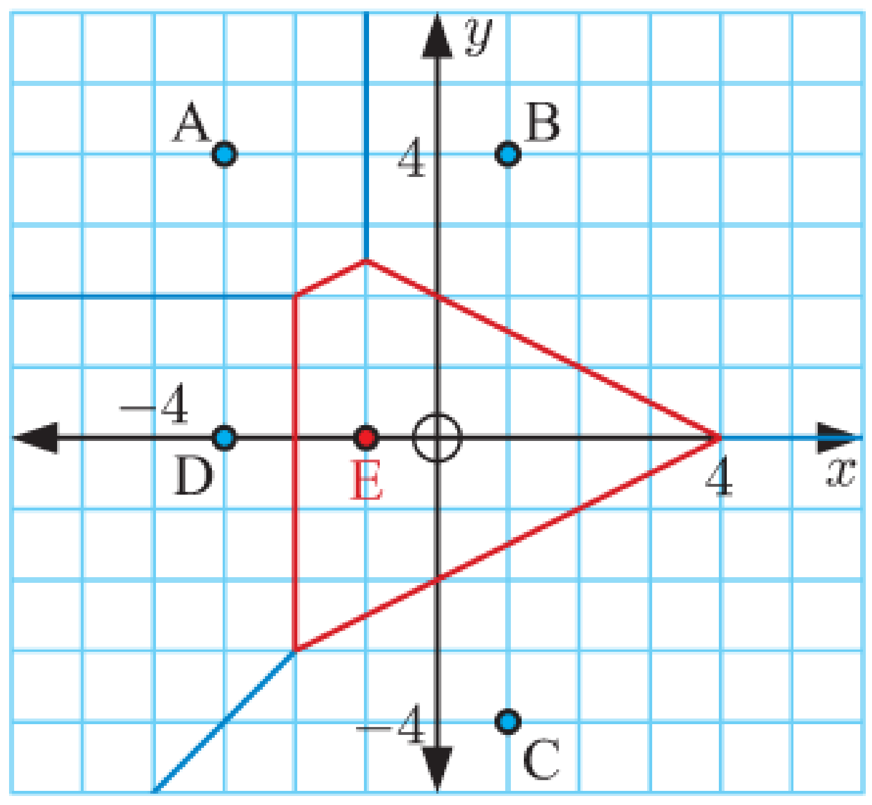
2 **a** Site D is relatively central to the other sites.

c 24 units^2

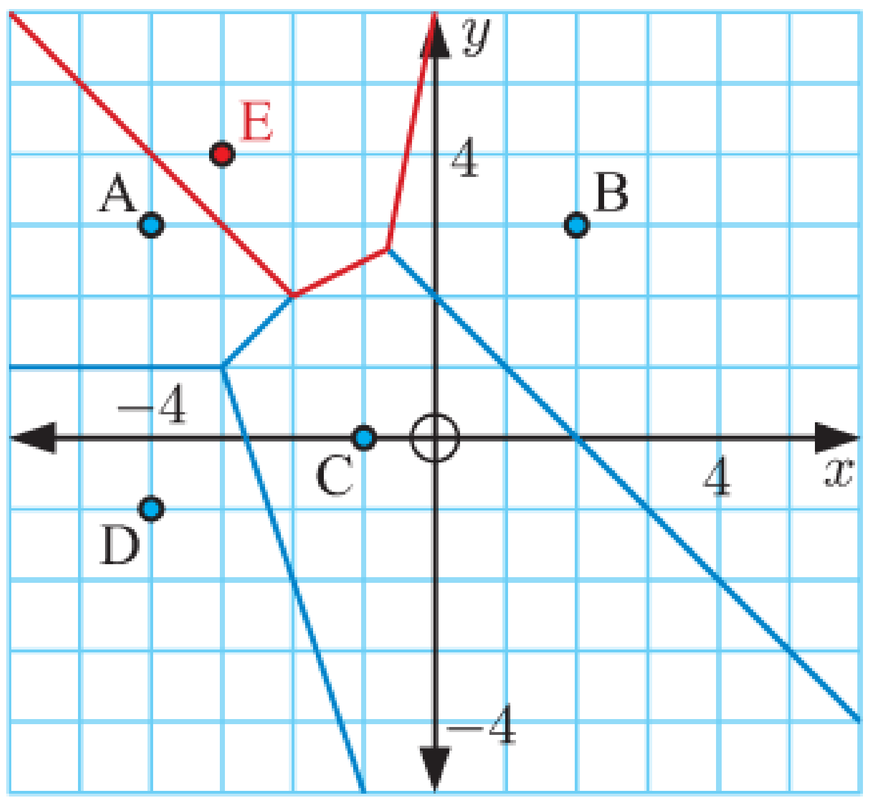
b


y
3
A
−3
3
x
D
−3
C
B

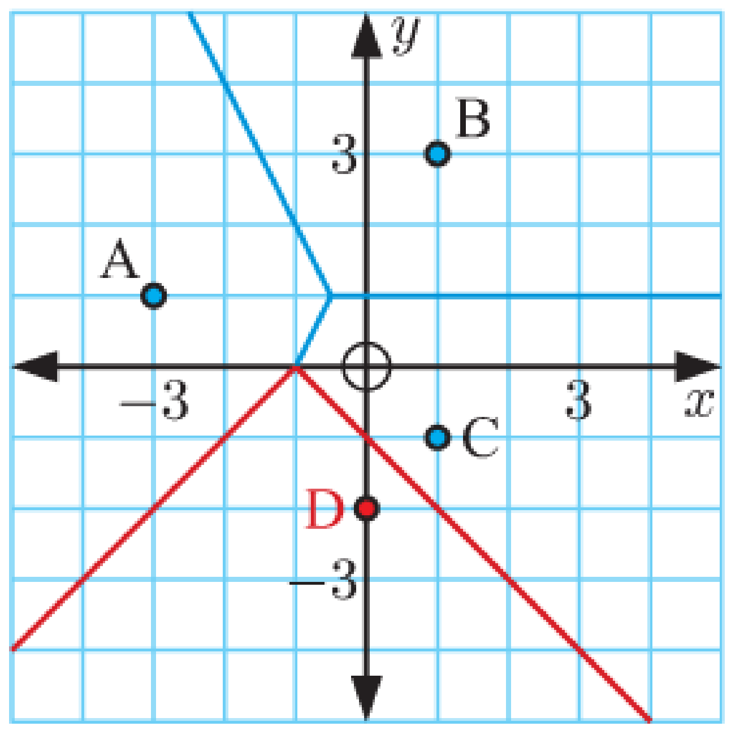
3 **a**


y
A
4
B
−4
D
E
4
x
−4
C

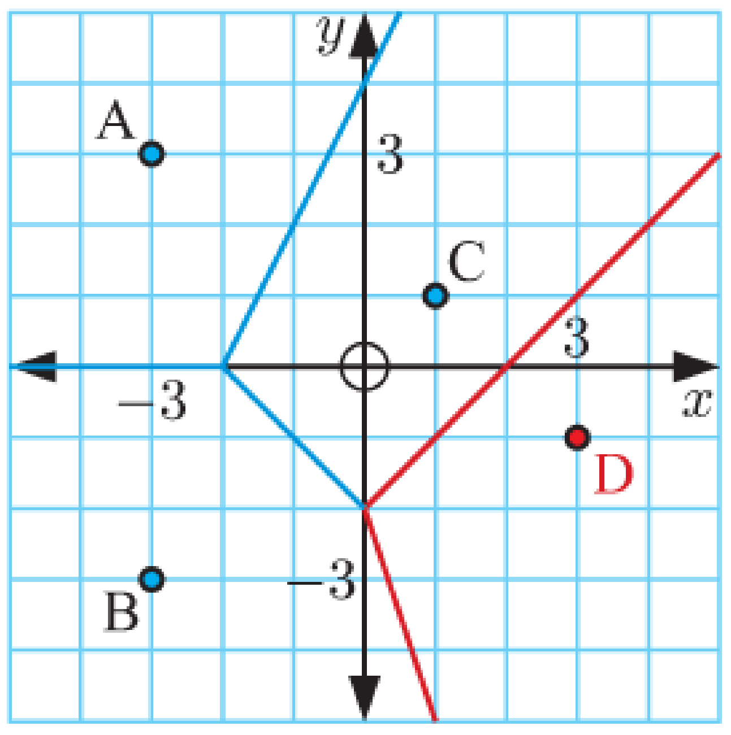
b


y
E
4
A
B
−4
C
4
x
D
−4

4 **a**

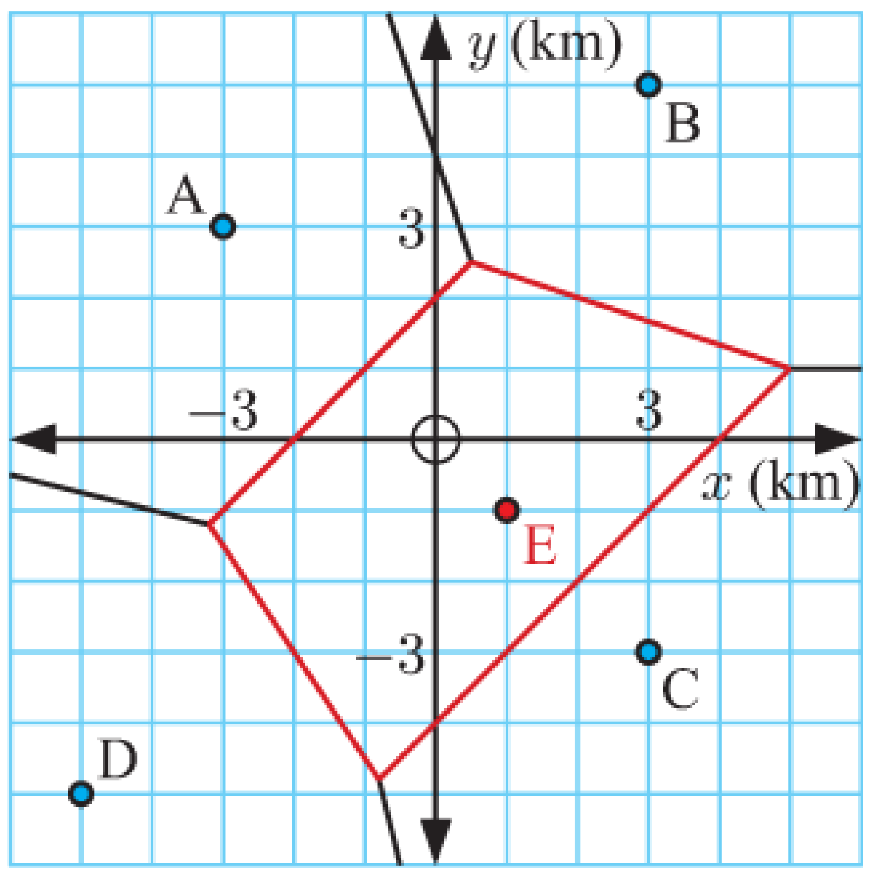


b



5 **a** **i** A **ii** B

b **i**

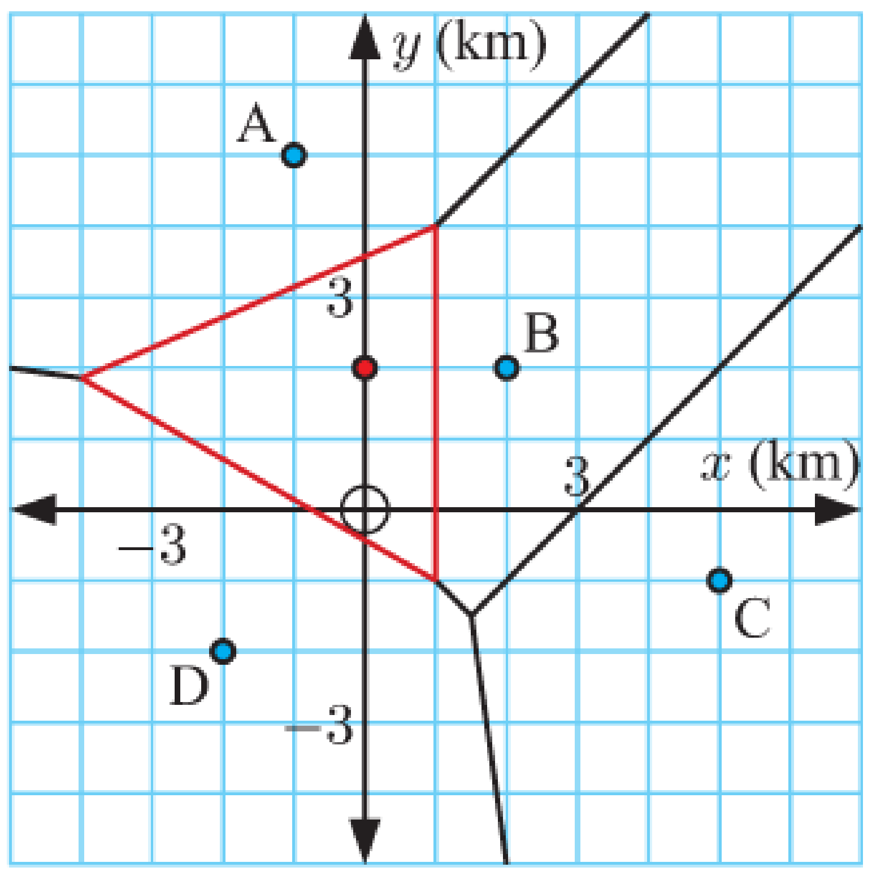


ii Yes, the edge is now closer to ATM D, implying that some people are now closer to ATM E.

iii (5, 1)

6 **a** **i** polling booth A **ii** polling booth D

b **i**

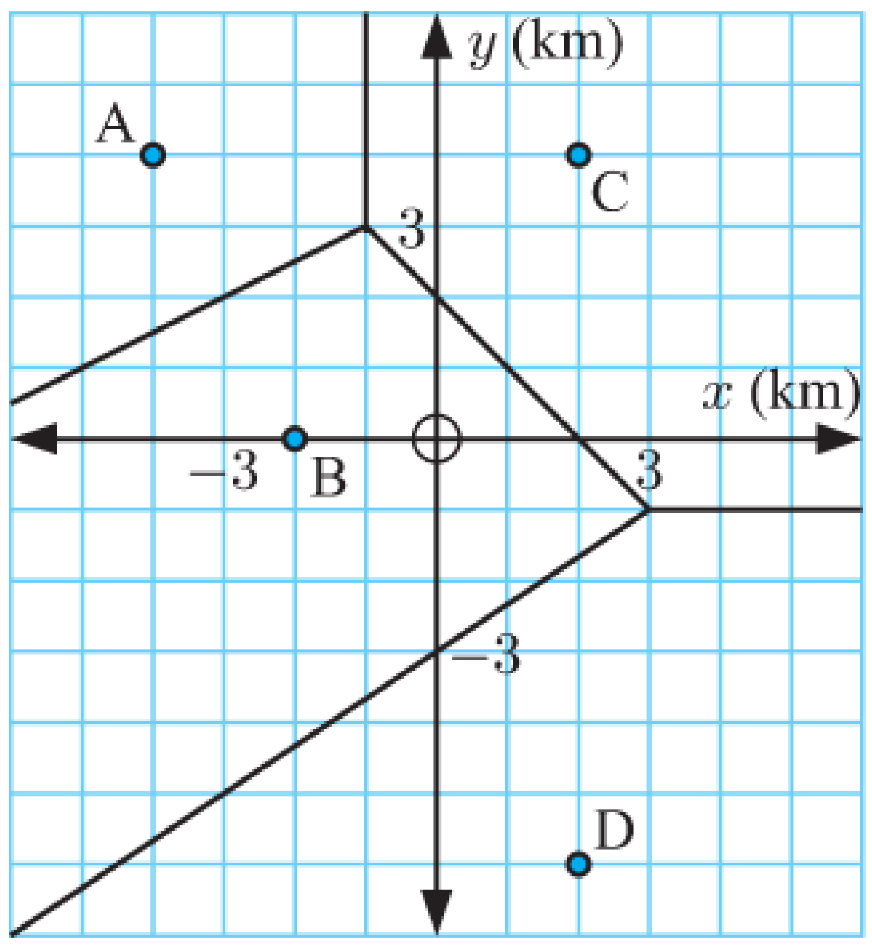


ii cell C **iii** 12.5 km^2

EXERCISE 17D

1 **a** 27.3°C **b** 25.6°C **c** 28.4°C

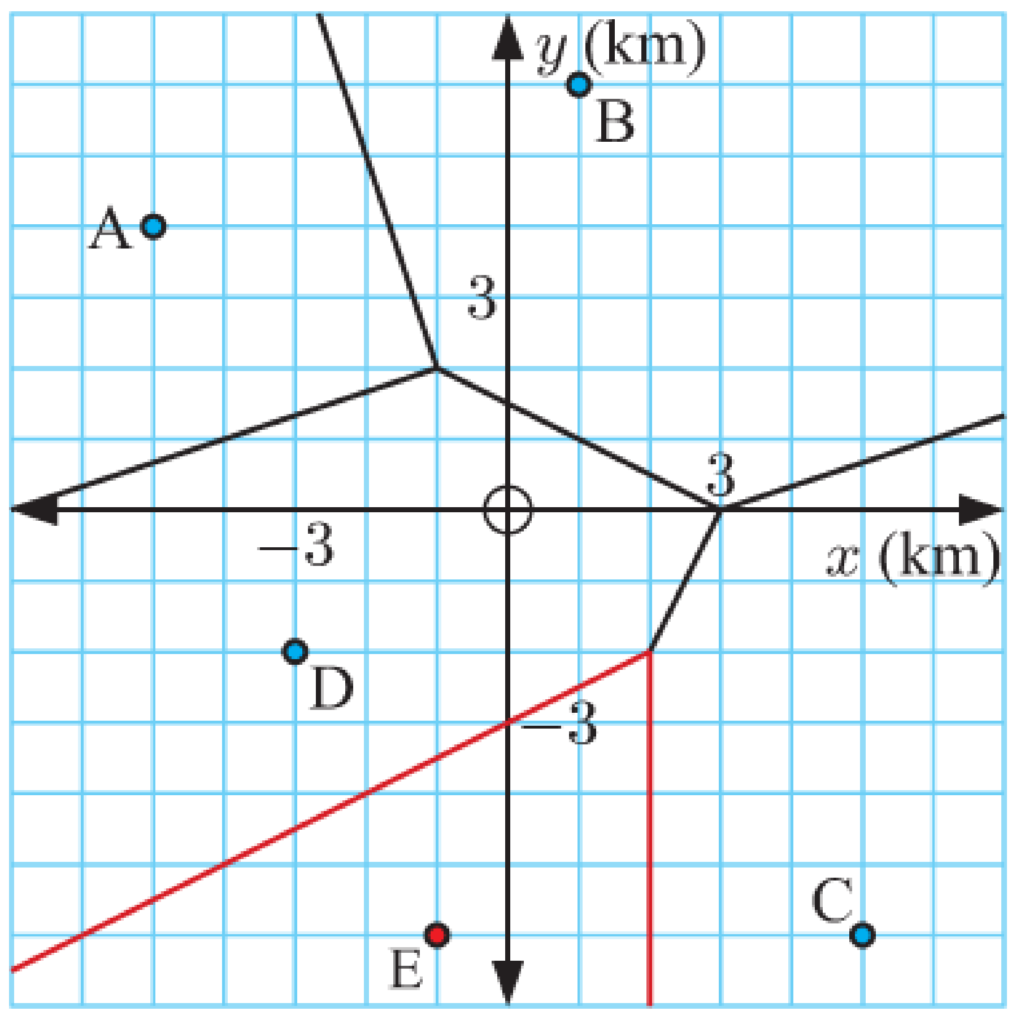
2 **a**



b **i** 48 m **ii** 57 m **iii** 36 m

3 **a** **i** 9.3 inches **ii** 5.5 inches **iii** ≈ 10.8 inches

b **i**



ii Yes, in **i** and **iii** we would now estimate 10.6 inches of snow at $(-1, -4)$ and $(1, -4)$.

EXERCISE 17E

1 **a** $(1, 1)$, radius $\sqrt{13}$ units

b $(0, -1)$, radius $2\sqrt{5}$ units

2 **a** $(20, 10)$ **b** $4\sqrt{65}$ km ≈ 32.2 km

c towns B, C, and E

d Distance from proposed location to B $= 5\sqrt{26} \approx 25.5$ km

$\therefore$ **a** is preferable.

3 Let P lie in cell X. The largest empty circle centred at P would touch X. As P lies within cell X, the largest empty circle does not touch any other site, meaning that a larger empty circle could be created with centre closer to the edge of the cell.

$\therefore$ the circle centred at P is not the largest empty circle.

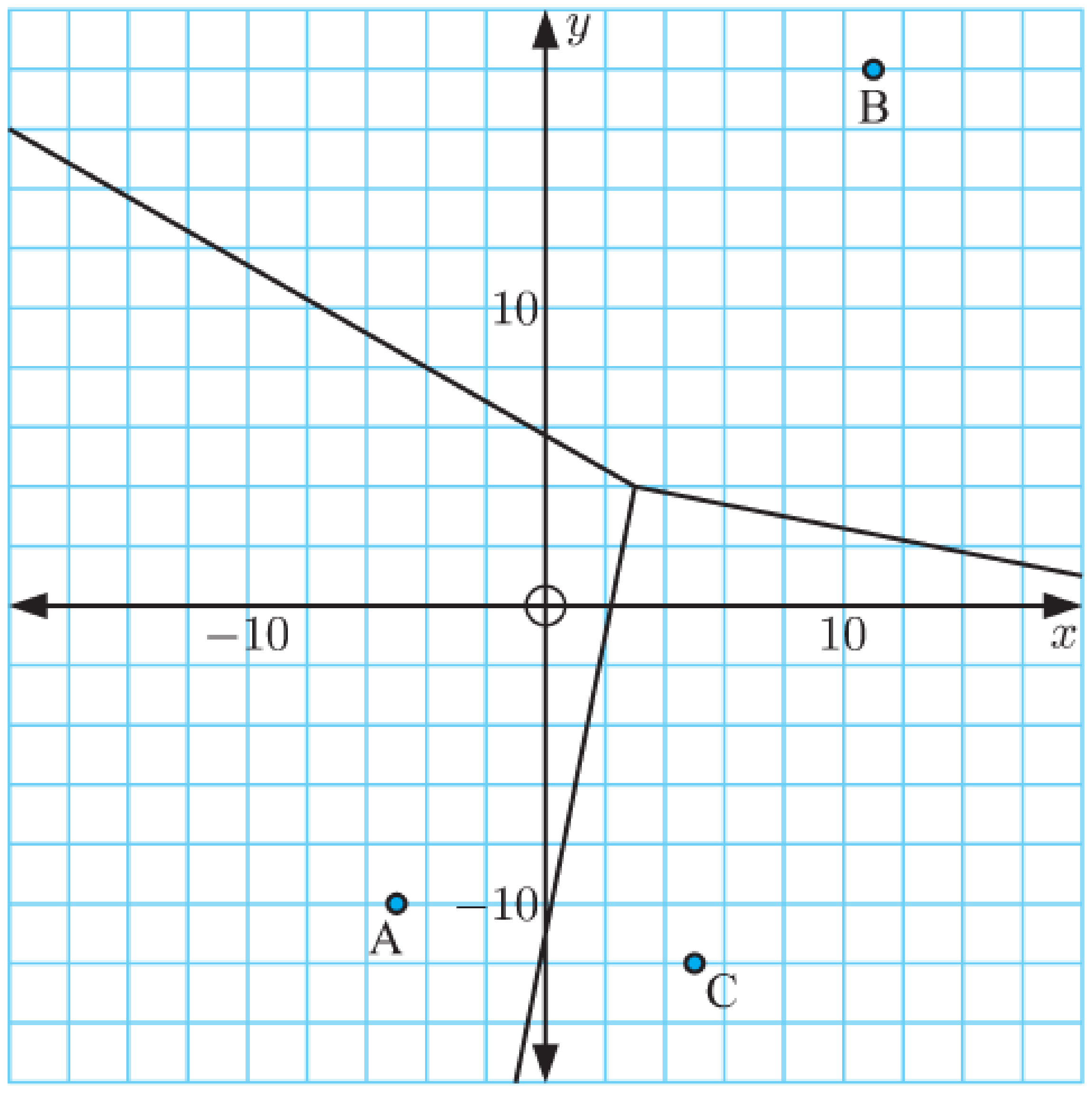
$\therefore$ the largest empty circle cannot lie within a cell.

4 **a** **i** PB(A, D): $y = \frac{3}{7}x - \frac{1}{7}$

ii PB(C, D): $y = -x - 3$

b $V_2(-2, -1)$ **c** $V_1(3, 14)$

5 **a**



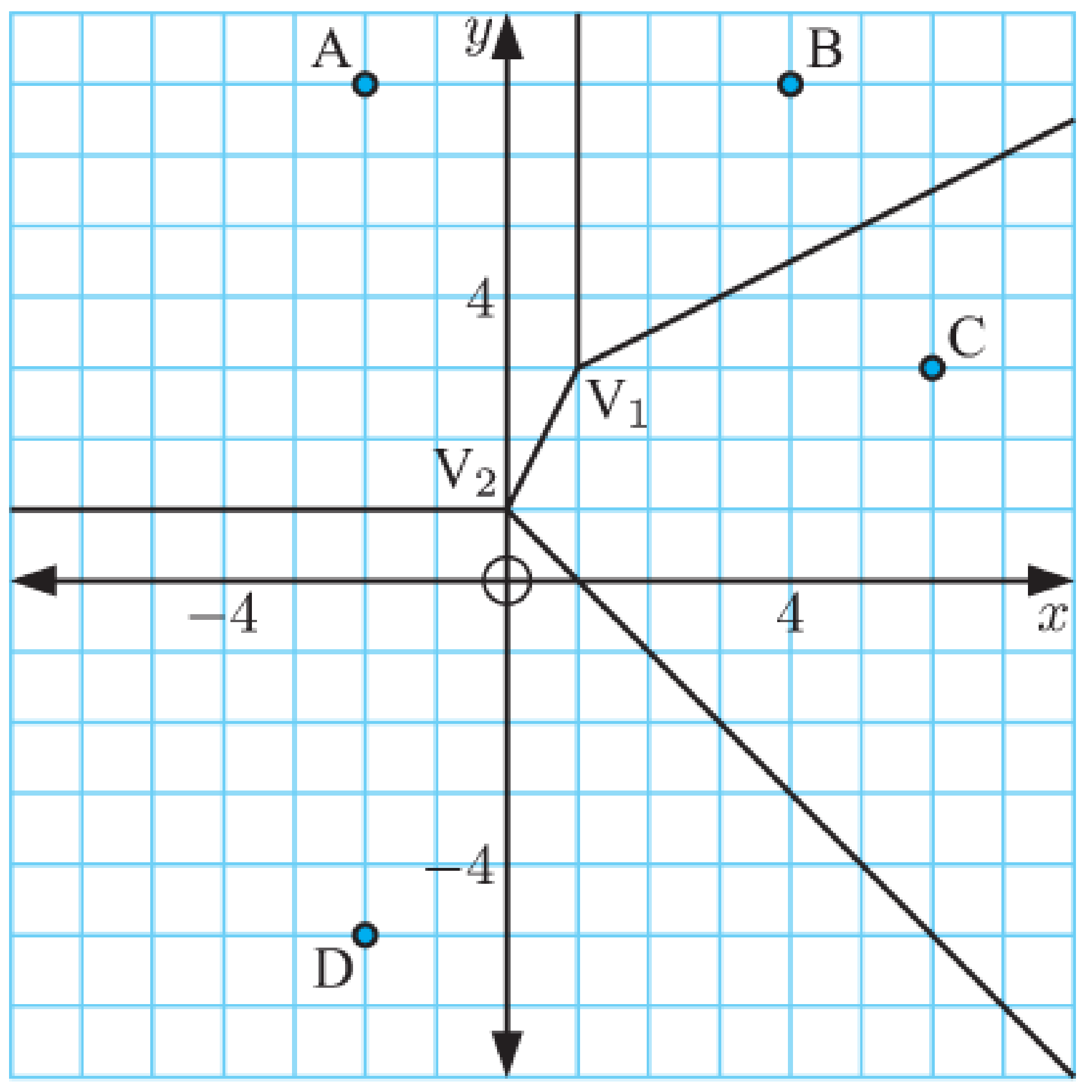
PB(A, B): $y = -\frac{4}{7}x + \frac{40}{7}$, PB(A, C): $y = 5x - 11$,

PB(B, C): $y = -\frac{1}{5}x + \frac{23}{5}$

b $(3, 4)$

c centre $(3, 4)$, radius $= 2\sqrt{65}$ units

6 **a**


A
y
B
4
C
V₁
V₂
−4
4
x
−4
D

b $V_1(1,\ 3)$, $V_2(0,\ 1)$

c centre $(0,\ 1)$, radius $= 2\sqrt{10}$ units

7 **a** **i** station B **ii** station E

b $V_3(-1,\ -\frac{17}{4})$

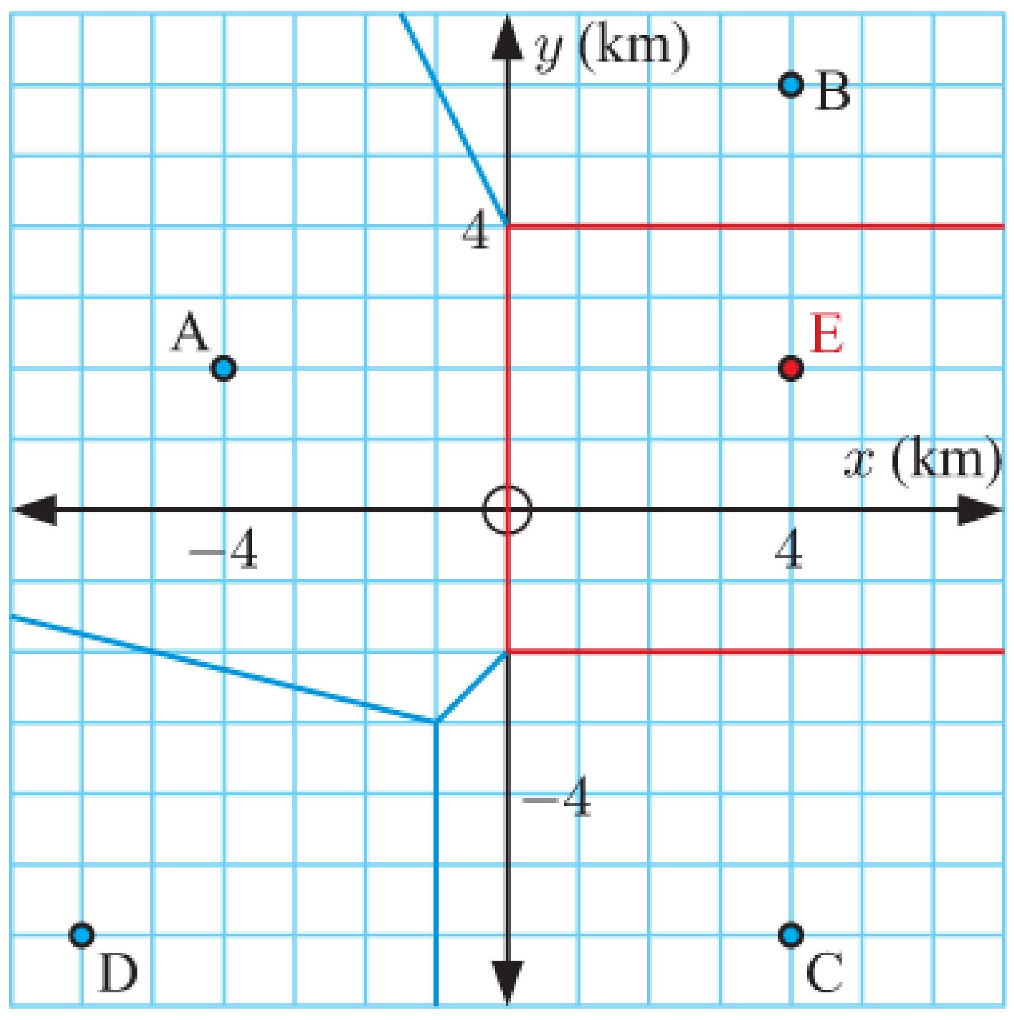
c **i** $V_3(-1,\ -\frac{17}{4})$

ii Distance from $(-3,\ -3)$ to E ≈ 2.83 km

Distance from $(-3,\ -3)$ to $V_3 \approx 2.36$ km

8 **a** $(2,\ 0)$, $2\sqrt{10} \approx 6.32$ km from A, B, and C

b **i**

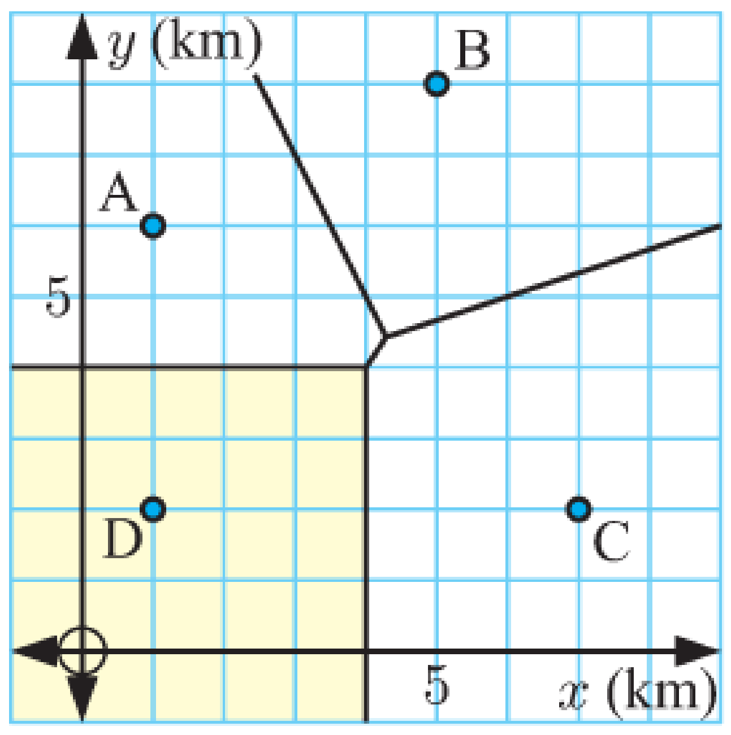

y (km)
B
4
A
E
x (km)
−4
4
−4
D
C

ii $(-1,\ -3)$

9 **a** **i** fire station C

ii fire station B

b

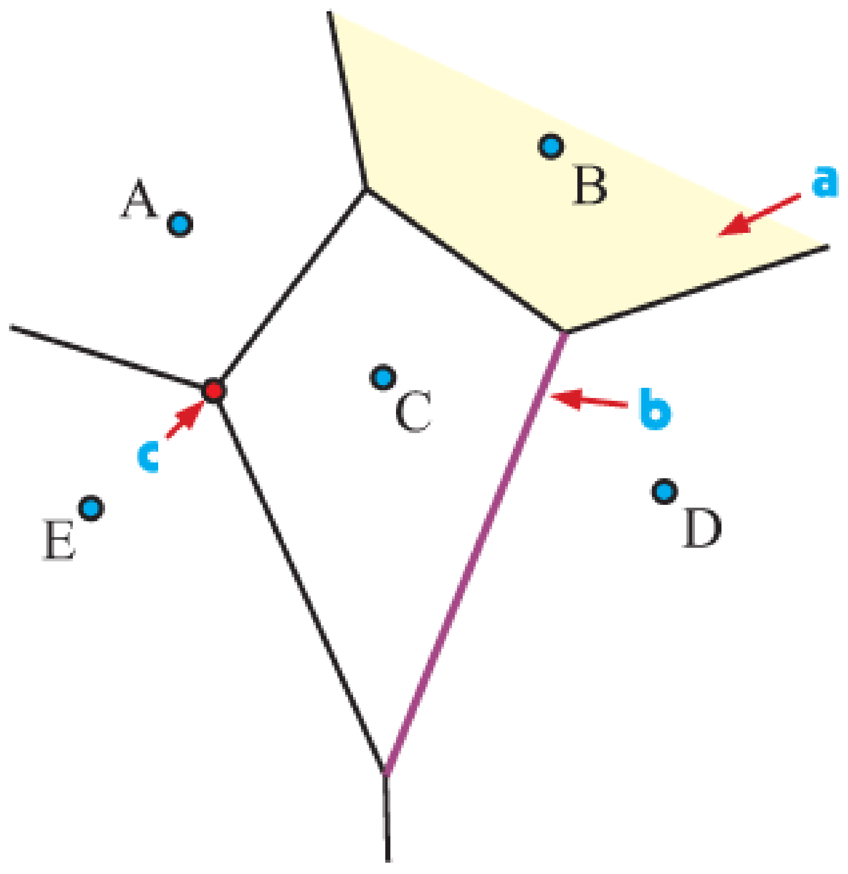

y (km)
B
A
5
D
C
5
x (km)

c We could create a Voronoi diagram like in **b**.

d $(\frac{30}{7},\ \frac{31}{7})$

REVIEW SET 17A

1

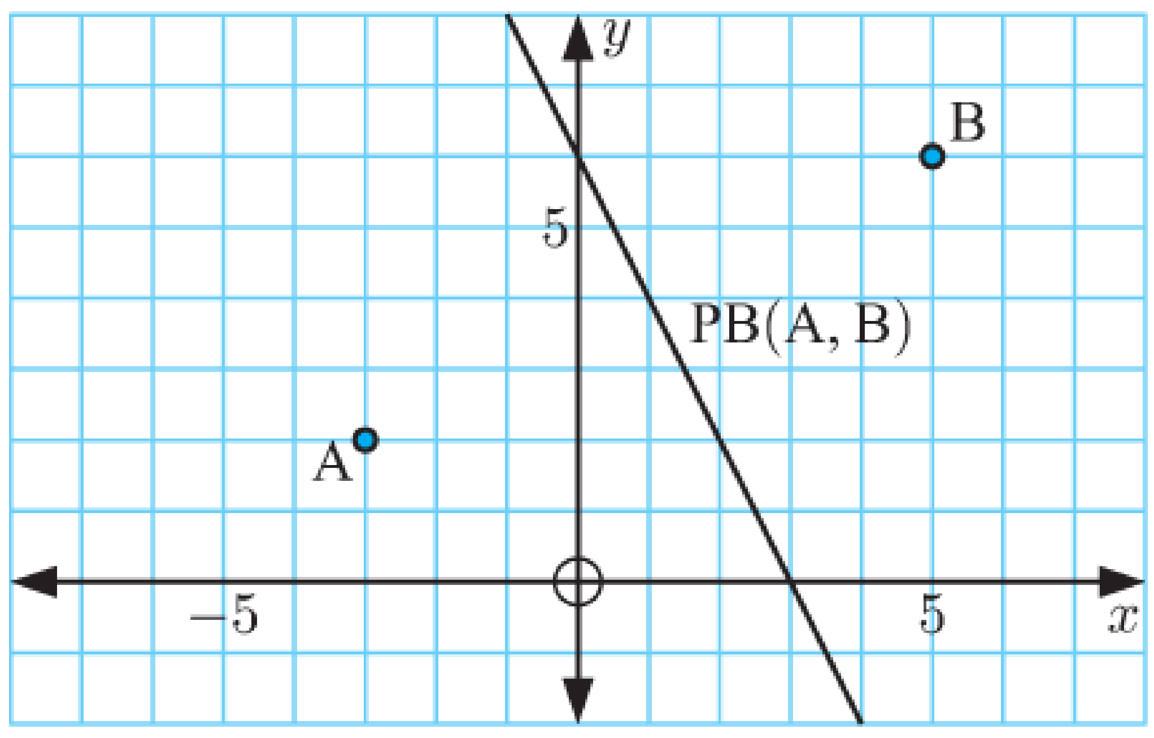


2 **a** **i** airport C **ii** airport A

b **i** The aeroplane is equally closest to airports B and D.

ii ≈ 361 km **iii** 300 km

3 **a**


y
B
5
PB(A, B)
A
−5
5
x

b PB(A, B): $y = -2x + 6$

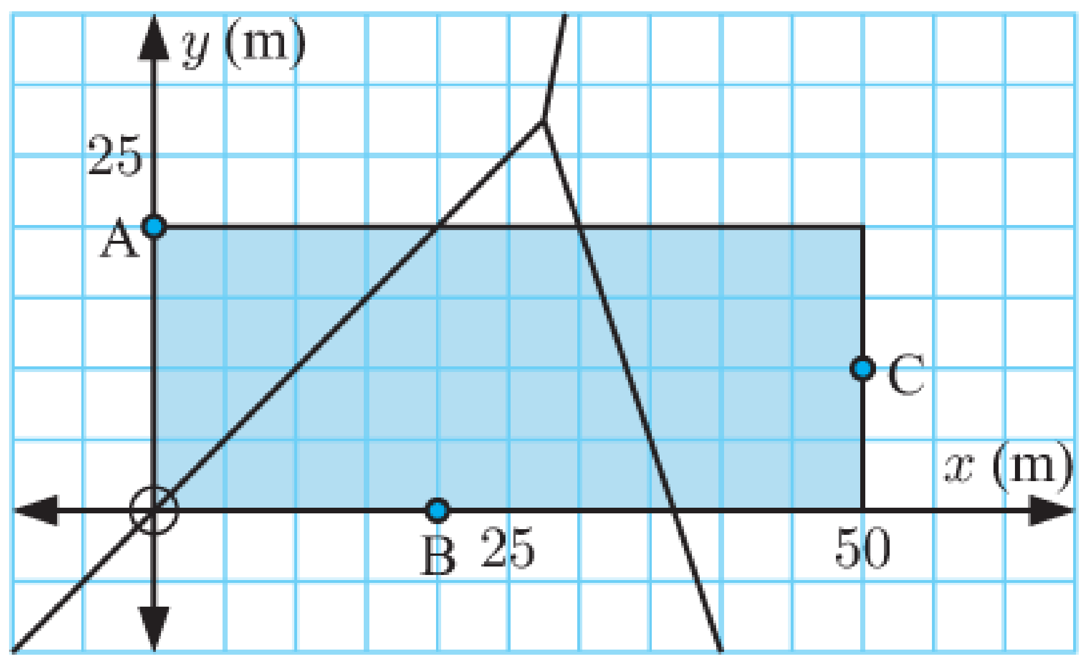
c **i** $0 = -2(3) + 6$ ✓

ii Distance from $(3,\ 0)$ to A $= 2\sqrt{10}$ units

Distance from $(3,\ 0)$ to B $= 2\sqrt{10}$ units

d **i** site A **ii** site A

4 **a**


y (m)
25
A
C
x (m)
B
25
50

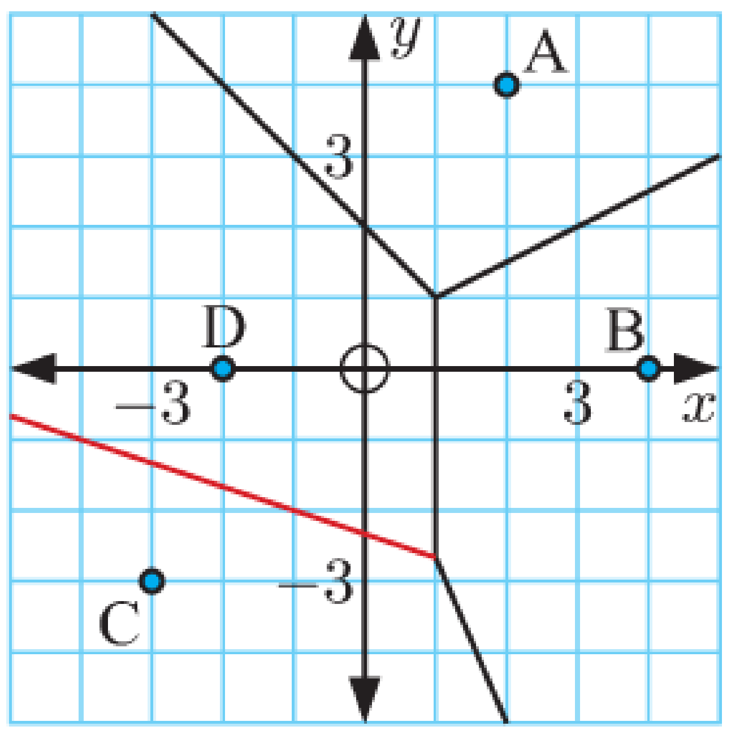
b No, the vertex lies outside the pool.

c **i** exit C **ii** 15 m

d **i** $\frac{1}{5}$ **ii** $\frac{7}{15}$ **iii** $\frac{1}{3}$

5 **a** Sites C and D are currently in the same cell.

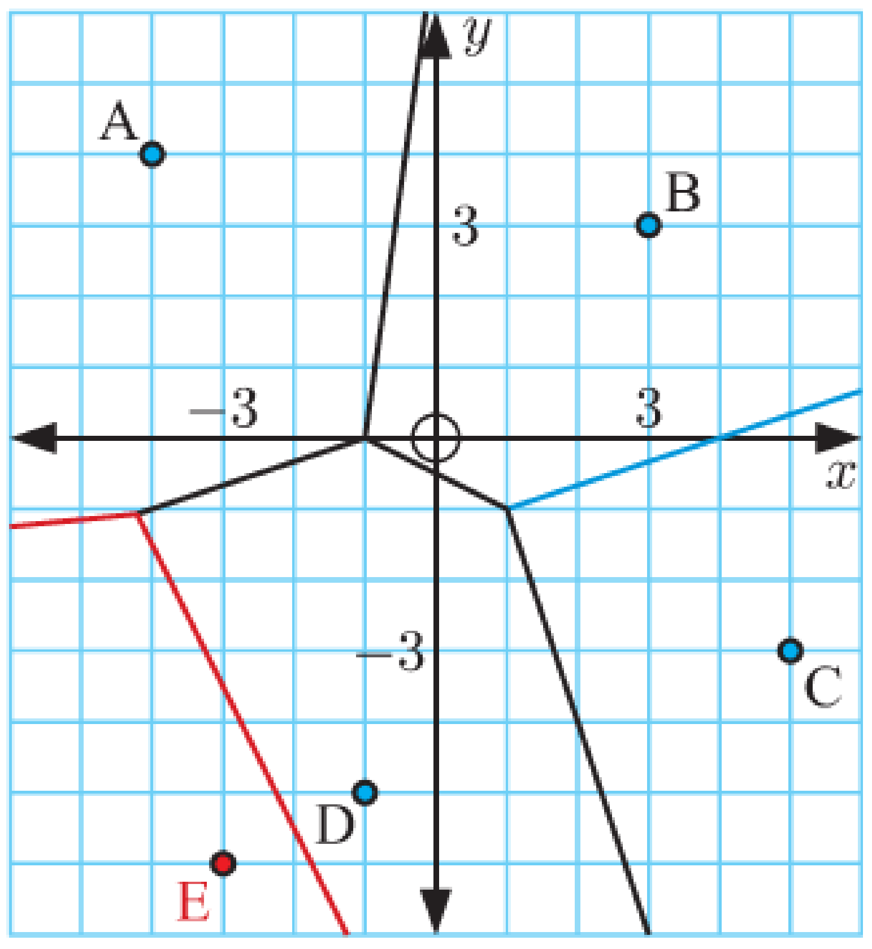
b


y
A
3
D
B
−3
3
x
−3
C

PB(C, D): $x + 3y + 7 = 0$

6 **a** $x - 3y = 4$ **b** **i** site C **ii** site D

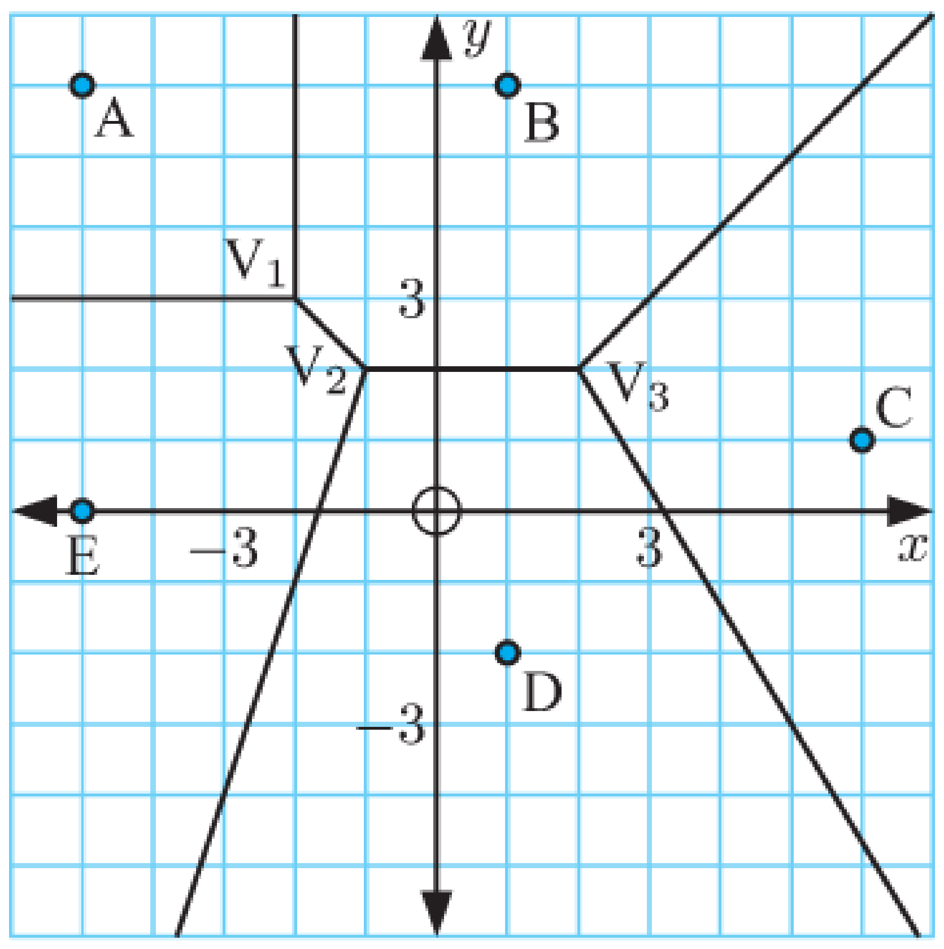
c


y
A
3
B
−3
3
x
−3
C
D
E

d Yes, in **ii**, $(-5,\ -2)$ would now be closest to site E.

7 **a** 11 km h^{-1} **b** 14 km h^{-1} **c** 19 km h^{-1}

8 **a**


y
A
B
V$_1$
3
V$_2$
V$_3$
C
E
−3
3
x
D
−3

b V$_1(-2, 3)$, V$_2(-1, 2)$, and V$_3(2, 2)$

c V$_2(-1, 2)$, radius $= 2\sqrt{5}$ units

9 **a** **i** $y = \frac{1}{2}x - 10$ **ii** $x = 10$

b V$_2(10, -5)$

c **i** V$_1(0, 5)$ **ii** $10\sqrt{2} \approx 14.1$ km

iii towns A, B, and D

REVIEW SET 17B

1 **a** **i** site C **ii** site E **iii** site C

iv sites A, B, and E

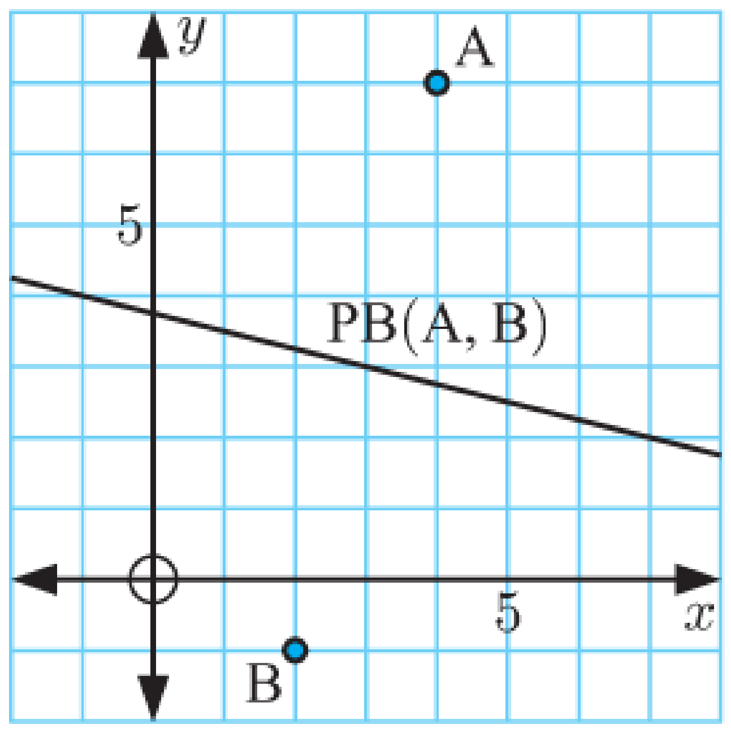
b No, they do not share a common edge. **c** $(1, 2)$

2 A point on an edge is equally closest to two sites. If the circle passes through a site, then the radius of the circle is the distance to that closest site.

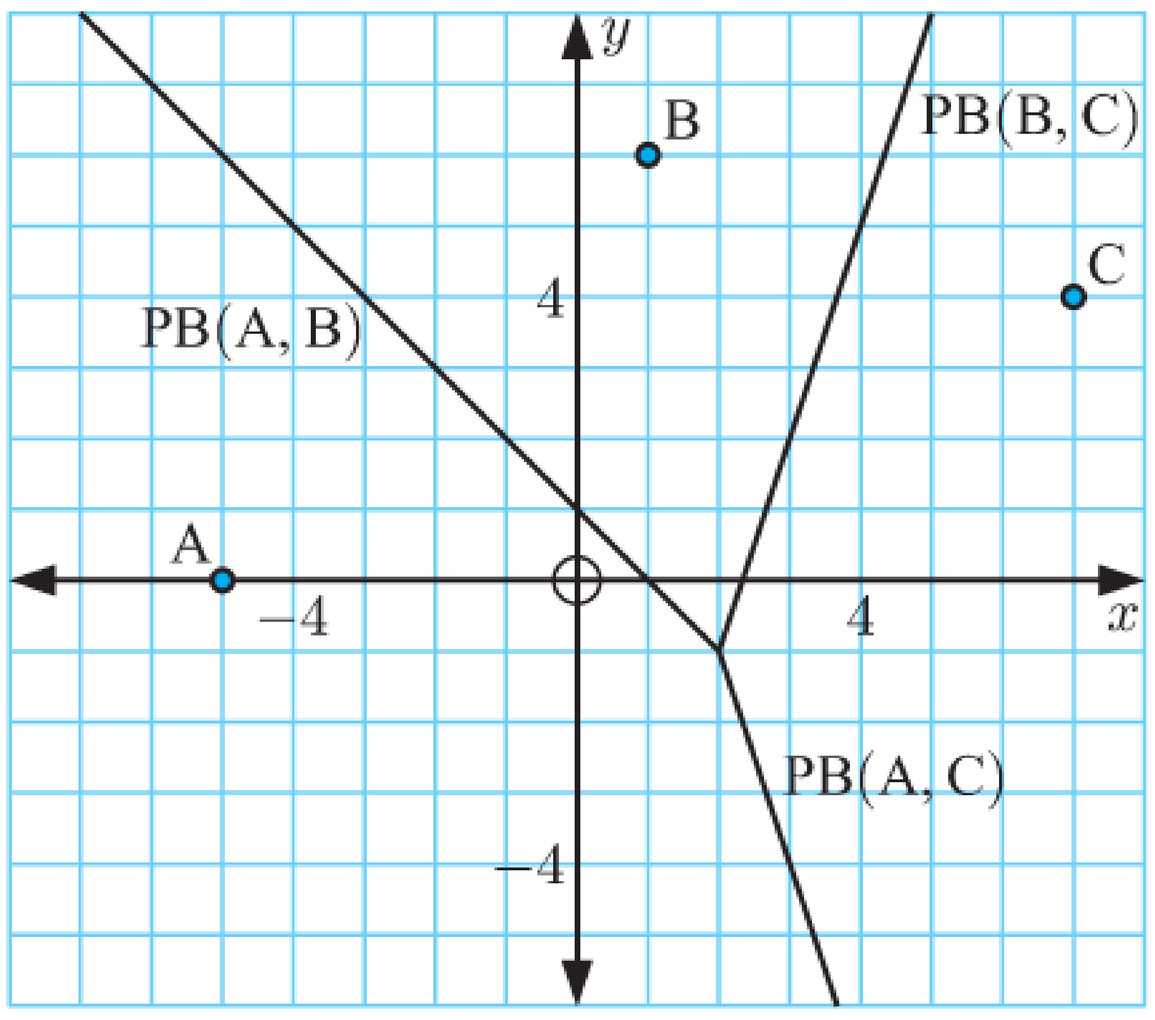
$\therefore$ the radius would also be the distance to the other site.

$\therefore$ the circle passes through two sites.

3 **a**


y
A
5
PB(A, B)
5
x
B

b


y
B
PB(B, C)
C
PB(A, B)
4
A
−4
4
x
PB(A, C)
−4

4 **a** There is an edge with no corresponding site.

b $(3, 1)$

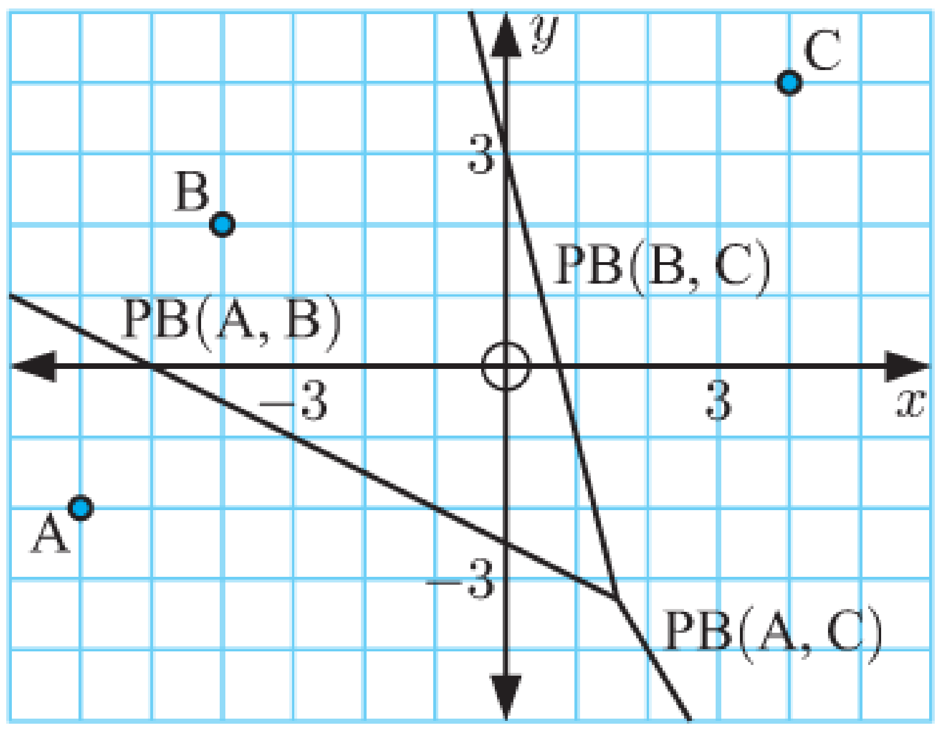
c **i** taxi rank A

ii $\approx$ 26 min 50 s; we are assuming that she can walk there in a straight line, and at a constant speed.

d **i** taxi ranks B and D

ii
- Whether he can walk there in a straight line.
- The price.
- His direction of travel once he gets into the taxi.

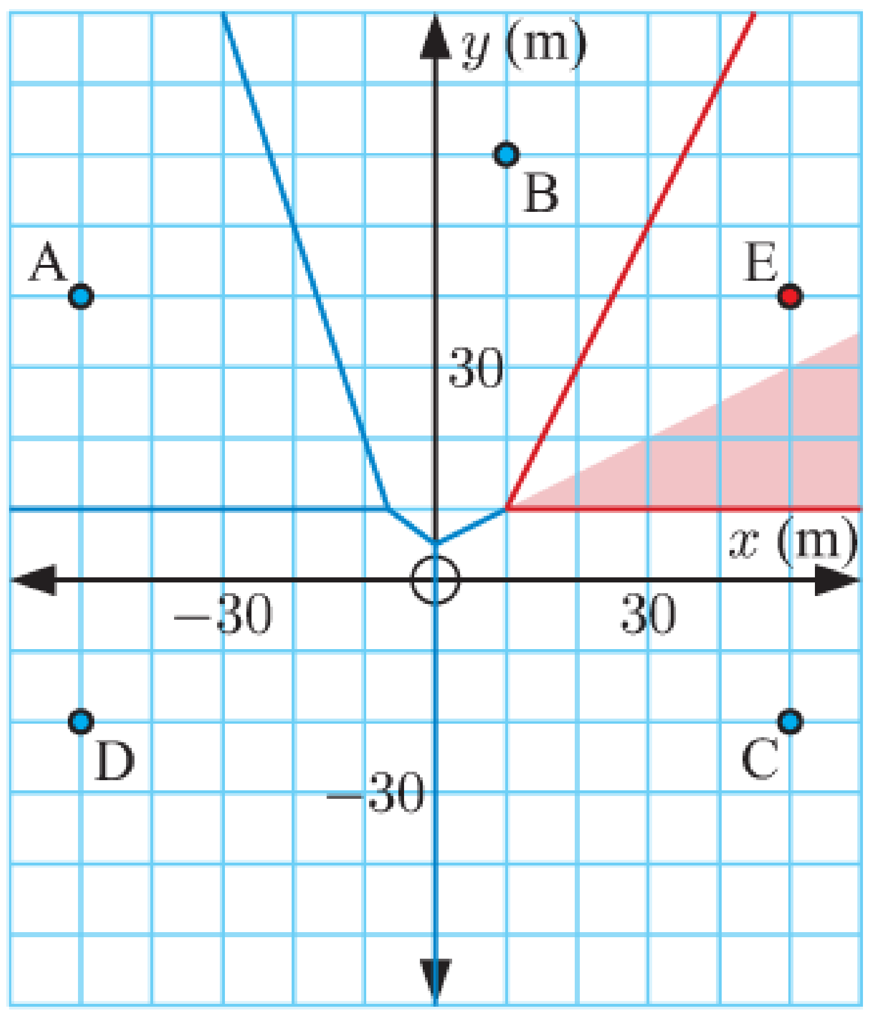
5 **a**


y
C
3
B
PB(B, C)
PB(A, B)
−3
3
x
A
−3
PB(A, C)

b **i** site C **ii** site A **iii** sites A and B

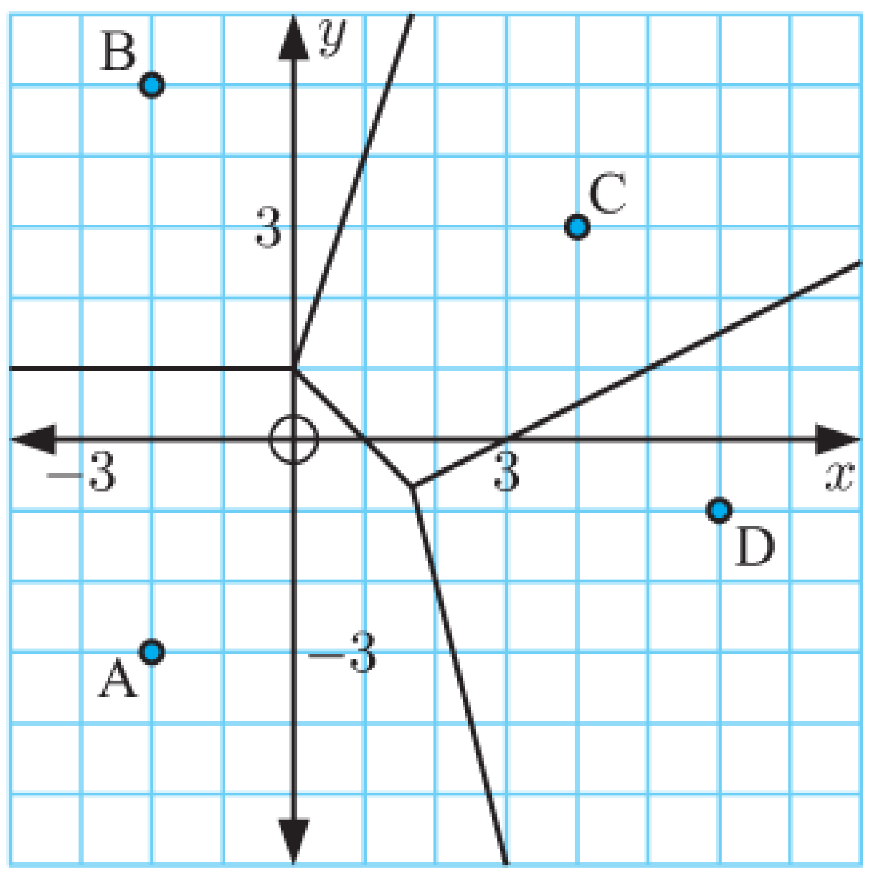
6 **a** ≈ 41.2 m

b **i, iii**


y (m)
B
A
E
30
x (m)
−30
30
D
C
−30

ii cells B and C

7 **a**

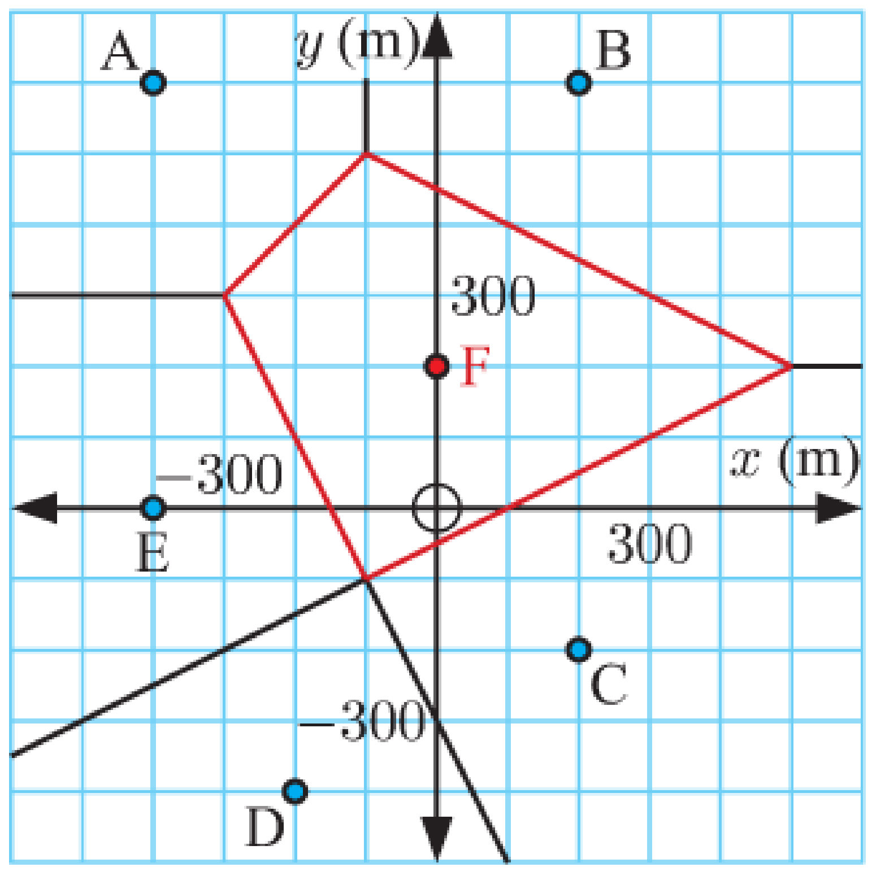

B
y
3
C
−3
3
x
D
A
−3

b **i** 9.2 m **ii** 6.9 m

8 **a** **i** parking lot E **ii** parking lot D

b **i** $(0, 200)$ **ii** ≈ 447 m

iii


A
y (m)
B
300
F
−300
x (m)
E
300
C
−300
D

iv 240 000 m^2

c **i** $(200, 200)$ is in the cell containing the parking lot F.

ii 200 m closer

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